WEAK SIGNED ROMAN $k$-DOMINATION IN DIGRAPHS

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Abstract. Let $k \geq 1$ be an integer, and let $D$ be a finite and simple digraph with vertex set $V(D)$. A weak signed Roman $k$-dominating function (WSR$k$DF) on a digraph $D$ is a function $f : V(D) \to \{−1, 1, 2\}$ satisfying the condition that $\sum_{x \in N^-[v]} f(x) \geq k$ for each $v \in V(D)$, where $N^-[v]$ consists of $v$ and all vertices of $D$ from which arcs go into $v$. The weight of a WSR$k$DF $f$ is $w(f) = \sum_{v \in V(D)} f(v)$. The weak signed Roman $k$-domination number $\gamma^k_{wsR}(D)$ is the minimum weight of a WSR$k$DF on $D$. In this paper we initiate the study of the weak signed Roman $k$-domination number of digraphs, and we present different bounds on $\gamma^k_{wsR}(D)$. In addition, we determine the weak signed Roman $k$-domination number of some classes of digraphs. Some of our results are extensions of well-known properties of the weak signed Roman domination number $\gamma^1_{wsR}(D)$ and the signed Roman $k$-domination number $\gamma^k_{sR}(D)$.

Keywords: digraph, weak signed Roman $k$-dominating function, weak signed Roman $k$-domination number, signed Roman $k$-dominating function, signed Roman $k$-domination number.

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1. TERMINOLOGY AND INTRODUCTION

In this paper we continue the study of signed Roman dominating functions in graphs and digraphs (see for example the survey article [2]). Let $k \geq 1$ be an integer, $G$ a simple graph with vertex set $V(G)$, and $N[v] = N_G[v]$ the closed neighborhood of the vertex $v$. A weak signed Roman $k$-dominating function (WSR$k$DF) on a graph $G$ is defined in [16] as a function $f : V(G) \to \{−1, 1, 2\}$ such that $\sum_{x \in N_G[v]} f(x) \geq k$ for every $v \in V(G)$. A weak signed Roman $k$-dominating function $f$ on a graph $G$ is called a signed Roman $k$-dominating function (SR$k$DF) on $G$ if every vertex $u$ for which $f(u) = −1$ is adjacent to a vertex $v$ for which $f(v) = 2$ (see [6]). The weight of a WSR$k$DF or an SR$k$DF $f$ on a graph $G$ is $w(f) = \sum_{v \in V(G)} f(v)$. The weak signed Roman $k$-domination number $\gamma^k_{wsR}(G)$ or signed Roman $k$-domination number $\gamma^k_{sR}(G)$ of $G$ is the minimum weight of a WSR$k$DF or an SR$k$DF on $G$, respectively. The
special case $\gamma_{sR}(G) = \gamma_{sR}^1(G)$ was investigated by Ahangar, Henning, Löwenstein, Zhao and Samodivkin [1].

Let now $D$ be a finite and simple digraph with vertex set $V(D)$ and arc set $A(D)$. The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the order and the size of the digraph $D$, respectively. The sets $N_D^+(v) = N^+(v) = \{x \mid (v, x) \in A(D)\}$ and $N_D^-(v) = N^-(v) = \{x \mid (x, v) \in A(D)\}$ are called the out-neighborhood and in-neighborhood of the vertex $v$. Likewise, $N_D^+\{v\} = N^+\{v\} = N^+(v) \cup \{v\}$ and $N_D^-\{v\} = N^-(v) \cup \{v\}$. We write $d_D^+(v) = d^+(v) = |N^+(v)|$ for the out-degree of a vertex $v$ and $d_D^-(v) = d^-(v) = |N^-(v)|$ for its in-degree. The minimum and maximum in-degree are $\delta^- = \delta^-(D)$ and $\Delta^- = \Delta^-(D)$ and the minimum and maximum out-degree are $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by $X$. For an arc $(x, y) \in A(D)$, the vertex $y$ is an out-neighbor of $x$ and $x$ is an in-neighbor of $y$, and we also say that $x$ dominates $y$ or $y$ is dominated by $x$. For a real-valued function $f : V(D) \to \mathbb{R}$, the weight of $f$ is $w(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V(D))$. Consult [4] and [5] for notation and terminology which are not defined here.

For an integer $p \geq 1$, we define a set $S \subseteq V(D)$ to be a $p$-dominating set of $D$ if for all $v \not\in S$, $v$ is dominated by $p$ vertices in $S$. The $p$-domination number $\gamma_p(D)$ of a digraph $D$ is the minimum cardinality of a $p$-dominating set of $D$.

A weak signed Roman $k$-dominating function (abbreviated WSRkDF) on $D$ is defined as a function $f : V(D) \to \{-1, 1, 2\}$ such that $f(N^-[v]) = \sum_{x \in N^-\{v\}} f(x) \geq k$ for every $v \in V(D)$. A weak signed Roman $k$-dominating function $f$ on $D$ is called a signed Roman $k$-dominating function on $D$ if every vertex $u$ for which $f(u) = -1$ has an in-neighbor for which $f(v) = 2$ (see [8]). The weight of a WSRkDF or an SRkDF $f$ on a digraph $D$ is $w(f) = \sum_{v \in V(D)} f(v)$. The weak signed Roman $k$-domination number $\gamma_{WSR}^k(D)$ or signed Roman $k$-domination number $\gamma_{SR}^k(D)$ of $D$ is the minimum weight of a WSRkDF or an SRkDF on $D$, respectively. A $\gamma_{WSR}^k(D)$-function or a $\gamma_{SR}^k(D)$-function is a weak signed Roman $k$-dominating function or a signed Roman $k$-dominating function on $D$ of weight $\gamma_{WSR}^k(D)$ or $\gamma_{SR}^k(D)$, respectively. For a WSRkDF or an SRkDF $f$ on $D$, let $V_i = V_i(f) = \{v \in V(D) : f(v) = i\}$. A weak signed Roman $k$-dominating function or a signed Roman $k$-dominating function $f : V(D) \to \{-1, 1, 2\}$ can be represented by the ordered partition $(V_{-1}, V_1, V_2)$ of $V(D)$. The special cases $k = 1$ were introduced and investigated by Sheikholeslami and Volkmann [7] and Volkmann [11].

The weak signed Roman $k$-domination number exists when $\delta^- \geq \frac{k}{2} - 1$. The definitions lead to $\gamma_{WSR}^k(D) \leq \gamma_{SR}^k(D)$. Therefore each lower bound on $\gamma_{WSR}^k(D)$ is also a lower bound on $\gamma_{SR}^k(D)$.

Our purpose in this work is to initiate the study of the weak signed Roman $k$-domination number in digraphs. We present basic properties and sharp bounds on $\gamma_{WSR}^k(D)$. In particular we show that many lower bounds on $\gamma_{SR}^k(D)$ are also valid for $\gamma_{WSR}^k(D)$. In addition, we determine the weak signed Roman $k$-domination number of some classes of digraphs. Some of our results are extensions of known properties of the signed Roman domination number $\gamma_{sR}(D) = \gamma_{sR}^1(D)$ by Sheikholeslami and
Volkmann [7] and the signed Roman $k$-domination number $\gamma^k_{wsR}(G)$ of graphs $G$, given by Henning and Volkmann in [6].

The associated digraph $D(G)$ of a graph $G$ is the digraph obtained from $G$ when each edge $e$ of $G$ is replaced by two oppositely oriented arcs with the same ends as $e$. Since $N_{D(G)}[v] = N_D[v]$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 1.1. If $D(G)$ is the associated digraph of a graph $G$, then we have $\gamma^k_{wsR}(D(G)) = \gamma^k_{wsR}(G)$.

Let $K_n$ and $K^*_n$ be the complete graph and complete digraph of order $n$, respectively. In [9] and [10], the author determines the weak signed Roman $k$-domination number of complete graphs $K_n$ for $n \geq k \geq 1$.

Proposition 1.2 ([9,10]). If $n \geq k \geq 1$, then $\gamma^k_{wsR}(K_n) = k$.

Using Observation 1.1 and Proposition 1.2, we obtain the weak signed Roman $k$-domination number of complete digraphs.

Corollary 1.3. If $n \geq k \geq 1$, then $\gamma^k_{wsR}(K^*_n) = k$.

Proposition 1.4 ([10]). Let $k \geq 1$ be an integer, and let $K_{p,p}$ be the complete bipartite graph of order $2p$. If $p \geq k + 3$, then $\gamma^k_{wsR}(K_{p,p}) = 2k + 2$. If $k + 1 \leq p \leq k + 2$, then $\gamma^k_{wsR}(K_{p,p}) = p + k - 1$. If $k \geq 2$, then $\gamma^k_{wsR}(K_{k,k}) = 2k$ and $\gamma^k_{wsR}(K_{1,1}) = 1$. If $k \geq 2$, then $\gamma^k_{wsR}(K_{k-1,k-1}) = 2k - 2$.

Using Observation 1.1 and Proposition 1.4, we obtain the weak signed Roman $k$-domination number of complete bipartite digraphs $K^*_{p,p}$.

Corollary 1.5. If $p \geq k + 3$, then $\gamma^k_{wsR}(K^*_{p,p}) = 2k + 2$. If $k + 1 \leq p \leq k + 2$, then $\gamma^k_{wsR}(K^*_{p,p}) = p + k - 1$. If $k \geq 2$, then $\gamma^k_{wsR}(K^*_{k,k}) = 2k$ and $\gamma^k_{wsR}(K^*_{1,1}) = 1$. If $k \geq 2$, then $\gamma^k_{wsR}(K^*_{k-1,k-1}) = 2k - 2$.

2. PRELIMINARY RESULTS

In this section we present basic properties of the weak signed Roman $k$-dominating functions and the weak signed Roman $k$-domination numbers of digraphs.

Lemma 2.1. If $f = (V_-, V_1, V_2)$ is a WSRkDF on a digraph $D$ of order $n$ and minimum in-degree $\delta^+(D) \geq \frac{k}{2} - 1$, then

(a) $|V_-| + |V_1| + |V_2| = n$,
(b) $\omega(f) = |V_1| + 2|V_2| - |V_-|$,
(c) $V_1 \cup V_2$ is a $\lceil \frac{k+1}{2} \rceil$-dominating set of $D$.

Proof. Since (a) and (b) are immediate, we only prove (c). If $|V_-| = 0$, then $V_1 \cup V_2 = V(D)$ is a $\lceil \frac{k+1}{2} \rceil$-dominating set of $D$. Let now $|V_-| \geq 1$, and let $v \in V_-$
an arbitrary vertex. Assume that $v$ has $j$ in-neighbors in $V_1$ and $q$ in-neighbors in $V_2$. The condition $f(N^-[v]) \geq k$ leads to $j + 2q - 1 \geq k$ and so $q \geq \frac{k+1-j}{2}$. This implies
\[ j + q \geq j + \frac{k+1-j}{2} = \frac{k + j + 1}{2} \geq \frac{k+1}{2}. \]
Therefore $v$ has at least $j + q \geq \lceil \frac{k+1}{2} \rceil$ in-neighbors in $V_1 \cup V_2$. Since $v$ was an arbitrary vertex in $V_1$, we deduce that $V_1$ is a $(k+1)$-dominating set and thus $\gamma_{k}(D) \geq \min \{2\gamma_{\lceil \frac{k+1}{2} \rceil}(D) + 2 - n, 2\gamma_{k}(D) + 1 - n, 2\gamma_{k+1}(D) - n\}.$

\[ \gamma_{\lceil \frac{k+1}{2} \rceil}(D) = 2\gamma_{k+1}(D) - n \geq 2|V_1| - n \geq 2\gamma_{k+1}(D) - n, \]
and the proof is complete.

**Corollary 2.2.** If $D$ is a digraph of order $n$ and minimum in-degree $\delta^-(D) \geq \frac{k}{2} - 1$, then $\gamma_{k}(D) \geq \min \{2\gamma_{\lceil \frac{k+1}{2} \rceil}(D) + 2 - n, 2\gamma_{k}(D) + 1 - n, 2\gamma_{k+1}(D) - n\}.$

**Proof.** Let $f = (V_1, V_1, V_2)$ be a $\gamma_{k}(D)$-function. Then it follows from Lemma 2.1 that
\[ \gamma_{\lceil \frac{k+1}{2} \rceil}(D) = 2\gamma_{k+1}(D) - n \]

The digraph without arcs and the digraph $qK^*_2$ show that Corollary 2.2 is sharp for $k = 1$ and $k = 2$. For the case $\Delta^-(D) \geq \frac{k+1}{2}$, we can improve Corollary 2.2 slightly.

**Theorem 2.3.** If $D$ is a digraph of order $n$ with $\delta^-(D) \geq \frac{k}{2} - 1$ and $\Delta^-(D) \geq \frac{k+1}{2}$, then
\[ \gamma_{k}(D) \geq \min \{2\gamma_{\lceil \frac{k+1}{2} \rceil}(D) + 2 - n, 2\gamma_{k}(D) + 1 - n, 2\gamma_{k+1}(D) - n\} \]

**Proof.** Let $f = (V_1, V_1, V_2)$ be a $\gamma_{k}(D)$-function. If $|V_2| \geq 2$, then it follows from Lemma 2.1 that
\[ \gamma_{\lceil \frac{k+1}{2} \rceil}(D) = 2|V_1| + 3|V_2| - n = 2|V_1 \cup V_2| + |V_2| - n \geq 2\gamma_{\lceil \frac{k+1}{2} \rceil}(D) + 2 - n. \]

If $|V_2| = 1$ and $v \in V_1$ is an arbitrary vertex, then we deduce from the condition $f(N^-[v]) \geq k$ that $v$ has at least $k$ in-neighbors in $V_1 \cup V_2$. Hence $V_1 \cup V_2$ is a $k$-dominating set and thus
\[ \gamma_{\lceil \frac{k+1}{2} \rceil}(D) = 2|V_1 \cup V_2| + |V_2| - n \geq 2\gamma_{k}(D) + 1 - n. \]

Let now $|V_2| = 0$. If $|V_1| = 0$, then $V_1 = V(D)$ and therefore $\gamma_{\lceil \frac{k+1}{2} \rceil}(D) = |V_1| = n$. If $v$ is a vertex with $d^-(v) = \Delta^-(D)$, then the condition $\Delta^-(D) \geq \frac{k+1}{2}$ implies that $V(D) \setminus \{v\}$ is a $\lceil \frac{k+1}{2} \rceil$-dominating set of $D$. Thus, $\gamma_{\lceil \frac{k+1}{2} \rceil}(D) \leq n - 1$, and we obtain
\[ \gamma_{\lceil \frac{k+1}{2} \rceil}(D) = n \geq 2(n - 1) + 2 - n \geq 2\gamma_{k+1}(D) + 2 - n. \]

Finally, let $|V_2| = 0$ and $|V_1| \geq 1$. If $v \in V_1$ is an arbitrary vertex, then we deduce from the condition $f(N^-[v]) \geq k$ that $v$ has at least $k + 1$ in-neighbors in $V_1$. Hence $V_1$ is a $(k+1)$-dominating set and thus
\[ \gamma_{\lceil \frac{k+1}{2} \rceil}(D) = 2|V_1| - n \geq 2\gamma_{k+1}(D) - n, \]
and the proof is complete.
The proof of the next proposition is identically with the proof of Proposition 7 in [8] and is therefore omitted.

**Proposition 2.4.** Assume that \( f = (V_1, V_2) \) is a WSRDF on a digraph \( D \) of order \( n \) with \( \delta^-(D) \geq \frac{k}{2} - 1 \). If \( \Delta^+(D) = \Delta^+ \) and \( \delta^+(D) = \delta^+ \), then

- (i) \((2\Delta^+ + 2 - k)|V_2| + (\Delta^+ + 1 - k)|V_1| \geq (\delta^+ + k + 1)|V_1| - |V_1| - 1\),
- (ii) \((2\Delta^+ + \delta^+ + 3)|V_2| + (\Delta^+ + \delta^+ + 2)|V_1| \geq (\delta^+ + k + 1)n\),
- (iii) \((\Delta^+ + \delta^+ + 2)\omega(f) \geq (\delta^+ - \Delta^+ + 2k)n + (\delta^+ - \Delta^+)\omega(f)\),
- (iv) \(\omega(f) \geq (\delta^+ - 2\Delta^+ + 2k - 1)n/(2\Delta^+ + \delta^+ + 3) + |V_2|\).

3. BOUNDS ON THE WEAK SIGNED ROMAN \( k \)-DOMINATION NUMBER

We start with a general upper bound, and we characterize all extremal digraphs.

**Theorem 3.1.** Let \( D \) be a digraph of order \( n \) with \( \delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1 \). Then \( \gamma_{wsR}^k(D) \leq 2n \) with equality if and only if \( k \) is even, \( \delta^-(D) = \frac{k}{2} - 1 \), and each vertex of \( D \) is of minimum in-degree or has an out-neighbor of minimum in-degree.

**Proof.** Define the function \( g : V(D) \to \{-1, 1, 2\} \) by \( g(x) = 2 \) for each vertex \( x \in V(D) \). Since \( \delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1 \), the function \( g \) is a WSRDF on \( D \) of weight \( 2n \) and thus \( \gamma_{wsR}^k(D) \leq 2n \).

Now let \( k \) be even, \( \delta^-(D) = \frac{k}{2} - 1 \), and assume that each vertex of \( D \) is of minimum in-degree or has an out-neighbor of minimum in-degree. Let \( f \) be a \( \gamma_{wsR}^k(D) \)-function, and let \( x \in V(D) \) be an arbitrary vertex. If \( d^-(x) = \frac{k}{2} - 1 \), then \( f(N^-[x]) \geq k \) implies \( f(x) = 2 \). If \( x \) is not of minimum in-degree, then \( x \) has an out-neighbor \( w \) of minimum in-degree. Now the condition \( f(N^-[w]) \geq k \) leads to \( f(x) = 2 \). Thus \( f \) is of weight \( 2n \), and we obtain \( \gamma_{wsR}^k(D) = 2n \) in this case.

Conversely, assume that \( \gamma_{wsR}^k(D) = 2n \). If \( k = 2p + 1 \) is odd, then \( \delta^-(D) \geq p \).

Define the function \( h : V(D) \to \{-1, 1, 2\} \) by \( h(w) = 1 \) for an arbitrary vertex \( w \) and \( h(x) = 2 \) for each vertex \( x \in V(D) \setminus \{w\} \). Then

\[
h(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq 1 + 2\delta^+(D) \geq 1 + 2p = k
\]

for each \( v \in V(D) \). Thus the function \( h \) is a WSRDF on \( D \) of weight \( 2n - 1 \), a contradiction to the assumption \( \gamma_{wsR}^k(D) = 2n \).

Let now \( k \) be even and assume that there exists a vertex \( w \) such that \( d^-(w) \geq \frac{k}{2} \) and \( d^-(x) \geq \frac{k}{2} \) for each out-neighbor of \( w \). Define the function \( h_1 : V(D) \to \{-1, 1, 2\} \) by \( h_1(w) = 1 \) and \( h_1(x) = 2 \) for each vertex \( x \in V(D) \setminus \{w\} \). Then \( h_1(N^-[w]) \geq k + 1 \) for each vertex \( v \in N^-[w] \) and \( h_1(N^-[x]) \geq k \) for each vertex \( x \notin N^-[w] \). Hence the function \( h_1 \) is a WSRDF on \( D \) of weight \( 2n - 1 \), and we obtain the contradiction \( \gamma_{wsR}^k(D) \leq 2n - 1 \). This completes the proof. \( \square \)

The proof of Theorem 3.1 also leads to the next result.
Theorem 3.2. Let \( D \) be a digraph of order \( n \) with \( \delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1 \). Then \( \gamma_k^sR(D) \leq 2n \) with equality if and only if \( k \) is even, \( \delta^-(D) = \frac{k}{2} - 1 \), and each vertex of \( D \) is of minimum in-degree or has an out-neighbor of minimum in-degree.

Proposition 3.3. If \( D \) is a digraph of order \( n \) with minimum in-degree \( \delta^- \geq k - 1 \), then \( \gamma_k^sR(D) \leq 2n \) with equality if and only if \( n \geq k \geq 1 \), then it follows from Corollary 1.3 that \( \gamma_k^sR(K^*_n) = k \). Therefore, the bound given in Theorem 3.7 is sharp.

Example 3.6. If \( H \) is a \( (k - 1) \)-regular digraph of order \( n \), then it follows from Corollary 3.4 that \( \gamma_k^sR(H) \geq \gamma_k^sR(H) \geq n \) and so \( \gamma_k^sR(H) = \gamma_k^sR(H) = n \), according to Proposition 3.3.

Example 3.6 demonstrates that Proposition 3.3 and Corollary 3.4 are both sharp. If \( k \geq 2 \), then Corollary 1.5 implies that \( \gamma_k^sR(K^*_k,k) = 2k \). This is a further example showing the sharpness of Proposition 3.3.

Theorem 3.7. If \( D \) is a digraph of order \( n \) with \( \delta^-(D) \geq \frac{k}{2} - 1 \), then

\[
\gamma_k^sR(D) \geq k + 1 + \Delta^-(D) - n.
\]

Proof. Let \( w \in V(D) \) be a vertex of maximum in-degree, and let \( f \) be a \( \gamma_k^sR(D) \)-function. Then the definitions imply

\[
\gamma_k^sR(D) = \sum_{x \in V(D)} f(x) = \sum_{x \in N^-[w]} f(x) + \sum_{x \in V(D) - N^-[w]} f(x) \geq k + \sum_{x \in V(D) - N^-[w]} f(x) \geq k - (n - (\Delta^-(D) + 1)) = k + 1 + \Delta^-(D) - n,
\]

and the proof of the desired lower bound is complete. \( \Box \)

If \( n \geq k \geq 1 \), then it follows from Corollary 1.3 that \( \gamma_k^sR(K^*_n) = k \). Therefore, the bound given in Theorem 3.7 is sharp.
A digraph $D$ is *out-regular* or *r-out-regular* if $\Delta^+(D) = \delta^+(D) = r$. If $D$ is not out-regular, then the next lower bound on the weak signed Roman $k$-domination number holds.

**Corollary 3.8.** Let $D$ be a digraph of order $n$, minimum in-degree $\delta^- \geq \frac{k}{2} - 1$, minimum out-degree $\delta^+$ and maximum out-degree $\Delta^+$. If $\delta^+ < \Delta^+$, then

$$\gamma_{w,sR}^k(D) \geq \left( \frac{2\delta^+ + 3k - 2\Delta^+}{2\Delta^+ + \delta^+ + 3} \right)^n.$$  

**Proof.** Multiplying both sides of the inequality in Proposition 2.4 (iv) by $\Delta^+ - \delta^+$ and adding the resulting inequality to the inequality in Proposition 2.4 (iii), we obtain the desired lower bound. \hfill \Box

**Corollary 3.9** ([8]). Let $D$ be a digraph of order $n$, minimum in-degree $\delta^- \geq \frac{k}{2} - 1$, minimum out-degree $\delta^+$ and maximum out-degree $\Delta^+$. If $\delta^+ < \Delta^+$, then

$$\gamma_{sR}^k(D) \geq \left( \frac{2\delta^+ + 3k - 2\Delta^+}{2\Delta^+ + \delta^+ + 3} \right)^n.$$  

Since the bound given in Corollary 3.9 is sharp (see [8]), the bound given in Corollary 3.8 is sharp too. Since $\Delta^+(D(G)) = \Delta(G)$ and $\delta^+(D(G)) = \delta(G)$, Corollary 3.8 and Observation 1.1 lead to the next known result.

**Corollary 3.10** ([6,10]). Let $G$ be a graph of order $n$, minimum degree $\delta \geq \frac{k}{2} - 1$ and maximum degree $\Delta$. If $\delta < \Delta$, then

$$\gamma_{sR}^k(G) \geq \gamma_{w,sR}^k(G) \geq \left( \frac{2\delta + 3k - 2\Delta}{2\Delta + \delta + 3} \right)^n.$$  

The special case $k = 1$ of Corollary 3.10 can be found in [1,9].

The complement $\overline{D}$ of a digraph $D$ is the digraph with vertex set $V(D)$ such that for any two distinct vertices $u$ and $v$ the arc $uv$ belongs to $\overline{D}$ if and only if $uv$ does not belong to $D$. Using Corollary 3.5 one can prove the following Nordhaus–Gaddum type inequality analogously to Theorem 17 in [8].

**Theorem 3.11.** If $D$ is an $r$-regular digraph of order $n$ such that $r \geq \frac{k}{2} - 1$ and $n - r - 1 \geq \frac{k}{2} - 1$, then

$$\gamma_{w,sR}^k(D) + \gamma_{w,sR}^k(\overline{D}) \geq \frac{4kn}{n + 1}.$$  

If $n$ is even, then $\gamma_{w,sR}^k(D) + \gamma_{w,sR}^k(\overline{D}) \geq \frac{4k(n+1)}{n+2}.$

**Example 3.12.** Let $k \geq 1$ be an integer, and let $H$ and $\overline{H}$ be $(k-1)$-regular digraphs of order $n = 2k - 1$. In view of Example 3.6, we have $\gamma_{w,sR}^k(H) + \gamma_{w,sR}^k(\overline{H}) = 2n$. This leads to

$$\gamma_{w,sR}^k(H) + \gamma_{w,sR}^k(\overline{H}) = 2n = \frac{4kn}{n + 1}.$$  

Example 3.12 shows that the Nordhaus-Gaddum bound in Theorem 3.11 is sharp.
4. SPECIAL FAMILIES OF DIGRAPHS

Example 4.1. If \( k \geq 1 \) and \( n \geq \frac{k}{2} \) are integers, then \( \gamma^k_{\omega wsR}(K^*_n) = k \).

Proof. If \( n \geq k \), then Corollary 1.3 leads to the desired result. Let now \( k > n \geq \frac{k}{2} \).

Corollary 3.5 implies \( \gamma^k_{\omega wsR}(K^*_n) \geq k \). For the converse inequality, let the function \( f : V(K^*_n) \to \{-1, 1, 2\} \) assign to \( k - n \) vertices the value 2 and to the remaining \( 2n - k \) vertices the value 1. Then \( f \) is a WSRkDF on \( K^*_n \) of weight \( \omega(f) = k \) and so \( \gamma^k_{\omega wsR}(K^*_n) \leq k \). This leads to \( \gamma^k_{\omega wsR}(K^*_n) = k \) also in this case. \( \square \)

Let \( C_n \) be an oriented cycle of order \( n \). In [7] and [11] it was shown that \( \gamma_{\omega R}(C_n) = \gamma_{\omega wsR}(C_n) = \frac{n + 1}{2} \) when \( n \) is even and \( \gamma_{\omega R}(C_n) = \gamma_{\omega wsR}(C_n) = \frac{n + 3}{2} \) when \( n \) is odd. Now we determine \( \gamma^k_{\omega wsR}(C_n) \) and \( \gamma^k_{\omega R}(C_n) \) for \( 2 \leq k \leq 4 \).

Theorems 3.1 and 3.2 immediately lead to \( \gamma^k_{\omega R}(C_n) = \gamma^k_{\omega wsR}(C_n) = 2n \). In addition, according to Example 3.6, we have \( \gamma^2_{\omega R}(C_n) = \gamma^2_{\omega wsR}(C_n) = n \).

Example 4.2. For \( n \geq 2 \), we have \( \gamma^3_{\omega wsR}(C_n) = \gamma^3_{\omega R}(C_n) = \lceil \frac{3n}{2} \rceil \).

Proof. Corollary 3.5 implies \( \gamma^3_{\omega R}(C_n) \geq \gamma^3_{\omega wsR}(C_n) \geq \lceil \frac{3n}{2} \rceil \). For the converse inequality we distinguish two cases.

Case 1. Assume that \( n = 2t \) is even for an integer \( t \geq 1 \). Let \( C_{2t} = v_0v_1 \ldots v_{2t-1}v_0 \). Define \( f : V(C_{2t}) \to \{-1, 1, 2\} \) by \( f(v_i) = 1 \) and \( f(v_{2i+1}) = 2 \) for \( 0 \leq i \leq t - 1 \). Then \( f(N^{-}[v_i]) = 3 \) for each \( 0 \leq j \leq 2t - 1 \), and therefore \( f \) is an SR3DF on \( C_{2t} \) of weight \( \omega(f) = 3t \). Thus \( \gamma^3_{\omega wsR}(C_n) \leq \gamma^3_{\omega R}(C_n) \leq 3t \). Consequently, \( \gamma^3_{\omega wsR}(C_n) = \gamma^3_{\omega R}(C_n) = 3t = \lceil \frac{3n}{2} \rceil \) in this case.

Case 2. Assume that \( n = 2t + 1 \) is odd for an integer \( t \geq 1 \). Let \( C_{2t+1} = v_0v_1 \ldots v_{2t}v_0 \). Define \( f : V(C_{2t+1}) \to \{-1, 1, 2\} \) by \( f(v_i) = 1 \), \( f(v_{2i+1}) = 2 \) for \( 0 \leq i \leq t - 1 \) and \( f(v_{2t}) = 2 \). Then \( f(N^{-}[v_i]) \geq 3 \) for each \( 0 \leq j \leq 2t \), and therefore \( f \) is an SR3DF on \( C_{2t+1} \) of weight \( \omega(f) = 3t + 2 \). Thus \( \gamma^3_{\omega wsR}(C_n) \leq \gamma^3_{\omega R}(C_n) \leq 3t + 2 \).

Consequently, \( \gamma^3_{\omega wsR}(C_n) = \gamma^3_{\omega R}(C_n) = 3t + 2 = \lceil \frac{3n}{2} \rceil \) in the second case. \( \square \)

A digraph is connected if its underlying graph is connected. A rooted tree is a connected digraph with a vertex \( r \) of in-degree 0, called the root, such that every vertex different from the root has in-degree 1.

Proposition 4.3. If \( T \) is a rooted tree of order \( n \geq 1 \), then \( \gamma^2_{\omega wsR}(T) = \gamma^2_{\omega R}(T) = n + 1 \).

Proof. Let \( f \) be a \( \gamma^2_{\omega wsR}(T) \)-function, and let \( r \) be the root of \( T \). Since \( d^-(r) = 0 \) and \( d^-(x) = 1 \) for \( x \in V(T) \setminus \{r\} \), we note that \( f(r) = 2 \) and \( f(x) \geq 1 \) for \( x \in V(T) \setminus \{r\} \). Thus \( \gamma^2_{\omega wsR}(T) \geq \gamma^2_{\omega wsR}(T) \geq n + 1 \). On the other hand, the function \( g : V(T) \to \{-1, 1, 2\} \) defined by \( g(r) = 2 \) and \( f(x) = 1 \) for \( x \in V(T) \setminus \{r\} \), is an SR2DF on \( T \) of weight \( \omega(g) = n + 1 \). Hence \( \gamma^2_{\omega wsR}(T) \leq \gamma^2_{\omega R}(T) \leq n + 1 \) and thus \( \gamma^2_{\omega wsR}(T) = \gamma^2_{\omega R}(T) = n + 1 \). \( \square \)

Corollary 4.4. If \( P_n \) is an oriented path of order \( n \geq 1 \), then \( \gamma^2_{\omega wsR}(P_n) = \gamma^2_{\omega R}(P_n) = n + 1 \).
5. FURTHER LOWER BOUNDS

Let $S_1$ be an orientation of the star $K_{1,n-1}$ such that the center $w$ has out-degree $n - 1$. In addition, let $S_2$ consists of $S_1$ together with an arc $vw$ for an arbitrary leaf $v$ of $K_{1,n-1}$.

Theorem 5.1. Let $D$ be a digraph of order $n \geq 2$. Then $\gamma_{wsR}(D) \geq 3 - n$, with equality if and only if $D \in \{S_1, S_2\}$.

Proof. If $\Delta^-(D) \geq 1$, then Theorem 3.7 implies $\gamma_{wsR}(D) \geq 3 - n$. Clearly, this remains valid for $\Delta^-(D) = 0$, and the lower bound is proved.

If $D \in \{S_1, S_2\}$, then define $g : V(D) \to \{-1, 1, 2\}$ by $g(w) = 2$ and $g(x) = -1$ for $x \in V(D) \setminus \{w\}$. Then $g$ is a weak signed Roman dominating function on $D$ of weight $3 - n$ and thus $\gamma_{wsR}(D) = 3 - n$.

Assume now that $\gamma_{wsR}(D) = 3 - n$, and let $f$ be a $\gamma_{wsR}(D)$-function. This implies that $D$ has exactly one vertex $w$ with $f(w) = 2$ and $n - 1$ vertices $y_1, y_2, \ldots, y_{n-1}$ such that $f(y_i) = -1$ for $1 \leq i \leq n - 1$. By the definition, $w$ dominates $y_i$ for $1 \leq i \leq n - 1$. If there exists an arc $y_i y_j$ for $i \neq j$, then $f(N^-[y_i]) \leq 0$, a contradiction. If $y_i$ and $y_j$ dominate $w$ for $i \neq j$, then $f(N^-[w]) \leq 0$, a contradiction. Thus, $D \in \{S_1, S_2\}$, and the proof is complete.

Theorem 5.2. Let $D$ be a digraph of order $n \geq 2$. Then $\gamma_{wsR}^2(D) \geq 4 - n$, with equality if and only if $D = K^*_2$.

Proof. If $\Delta^-(D) = 0$, then $\gamma_{wsR}^2(D) = 2n > 4 - n$. If $\Delta^-(D) \geq 1$, then Theorem 3.7 implies $\gamma_{wsR}(D) \geq 4 - n$, and the lower bound is proved. If $D = K^*_2$, then it follows from Example 4.1 that $\gamma_{wsR}^2(D) = 2 = 4 - n$.

Assume now that $\gamma_{wsR}^2(D) = 4 - n$, and let $f$ be a $\gamma_{wsR}^2(D)$-function. This implies that $D$ has exactly two vertices $u$ and $v$ with $f(u) = f(v) = 1$ and $n - 2$ vertices $x_1, x_2, \ldots, x_{n-2}$ such that $f(x_i) = -1$ for $1 \leq i \leq n - 2$. It follows that $n = 2$, $u$ dominates $v$ and $v$ dominates $u$ and thus $D = K^*_2$.

Theorem 5.3. Let $k \geq 3$ be an integer, and let $D$ be a digraph of order $n$ with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$. Then

$$\gamma_{wsR}^k(D) \geq k + \left\lceil \frac{k}{2} \right\rceil - n,$$

with equality if and only if $D = K^*_k$.

Proof. Since $\Delta^-(D) \geq \delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$, it follows from Theorem 3.7 that

$$\gamma_{wsR}^k(D) \geq k + 1 + \Delta^-(D) - n \geq k + 1 + \left\lceil \frac{k}{2} \right\rceil - 1 - n = k + \left\lceil \frac{k}{2} \right\rceil - n,$$

and the desired lower bound is proved. If $D = K^*_k$, then Example 4.1 shows that

$$\gamma_{wsR}^k(D) = k = k + \left\lceil \frac{k}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil.$$
Conversely, assume that \( \gamma_{wsR}(D) = k + \left\lceil \frac{k}{2} \right\rceil - n \), and let \( f \) be \( \gamma_{wsR}(D) \)-function. If \( \Delta^-(D) \geq \left\lceil \frac{k}{2} \right\rceil \), then Theorem 3.7 implies \( \gamma_{wsR}(D) \geq k + \left\lceil \frac{k}{2} \right\rceil + 1 - n \), a contradiction. Thus, \( \Delta^-(D) = \delta^-(D) = \left\lceil \frac{k}{2} \right\rceil - 1 \). If there exists a vertex \( w \) with \( f(w) = -1 \), then we obtain the contradiction

\[
k \leq f(N^-[w]) \leq -1 + 2\Delta^-(D) = -1 + 2 \left( \left\lceil \frac{k}{2} \right\rceil - 1 \right) \leq k - 2.
\]

So \( f(x) \geq 1 \) for each \( x \in V(D) \). Next we distinguish two cases.

Case 1. Assume that \( k \) is even. If there exists a vertex \( w \) with \( f(w) = 1 \), then we arrive at the contradiction

\[
k \leq f(N^-[w]) \leq 1 + 2\Delta^-(D) = 1 + 2 \left( \left\lceil \frac{k}{2} \right\rceil - 1 \right) = k - 1.
\]

Therefore \( f(x) = 2 \) for all \( x \in V(D) \). We deduce that \( \omega(f) = 2n = k + \frac{k}{2} - n \) and thus \( n = \frac{k}{2} \). Consequently, \( D = K[*\frac{k}{2}] \) in this case.

Case 2. Assume that \( k \) is odd. If there exists a vertex \( w \) with \( f(w) = 1 \), then \( w \) has exactly \( \frac{k+1}{2} \) in-neighbors of weight 2. Suppose that \( D \) has \( t \) ≥ 0 further vertices of weight 1 and \( s \) ≥ 0 further vertices of weight 2. Then \( n = 1 + \frac{k-1}{2} + s + t \) and hence

\[
2n = 2s + 2t + k + 1. \tag{5.1}
\]

On the other hand we observe that \( \omega(f) = 2n - (t + 1) = k + \frac{k+1}{2} - n \) and thus

\[
6n = 3k + 2t + 3. \tag{5.2}
\]

Combining (5.1) and (5.2), we find that \( 6s + 4t = 0 \) and therefore \( s = t = 0 \). It follows that \( n = \frac{k+1}{2} \) and so \( D = K[*\frac{k}{2}] \).

Finally, assume that \( f(x) = 2 \) for each \( x \in V(D) \). Then \( \omega(f) = 2n = k + \frac{k+1}{2} - n \), and we obtain the contradiction \( 6n = 3k + 1 \).

Let \( \{u, v, x_1, x_2, \ldots, x_{n-2}\} \) be the vertex set of the digraph \( B \) of order \( n \geq 2 \) such that \( u \) and \( v \) dominate \( x_i \) for \( 1 \leq i \leq n - 2 \). In addition, let \( B_1 = B \cup \{vu\} \), \( B_2 = B_1 \cup \{w\} \), \( B_3 = B_1 \cup \{x_1u\} \), \( B_4 = B_2 \cup \{x_1u\} \), \( B_5 = B_2 \cup \{x_1u, x_1v\} \) and \( B_6 = B_2 \cup \{x_1u, x_2v\} \).

**Theorem 5.4.** Let \( D \) be a digraph of order \( n \geq 2 \). If \( D \not\in \{S_1, S_2\} \), then \( \gamma_{wsR}(D) \geq 4 - n \), with equality if and only if \( D \in \{B, B_1, B_2, B_3, B_4, B_5, B_6\} \).

**Proof.** Theorem 5.1 implies \( \gamma_{wsR}(D) \geq 4 - n \). If \( D \in \{B, B_1, B_2, B_3, B_4, B_5, B_6\} \), then define the function \( g : V(D) \to \{-1, 1, 2\} \) by \( g(u) = g(v) = 1 \) and \( g(x_i) = -1 \) for \( 1 \leq i \leq n - 2 \). Then \( g \) is a weak signed Roman dominating function on \( D \) of weight \( 4 - n \) and thus \( \gamma_{wsR}(D) = 4 - n \).
Assume now that $\gamma_{w,R}(D) = 4 - n$, and let $f$ be a $\gamma_{w,R}(D)$-function. This implies that $D$ has exactly two vertices $u$ and $v$ with $f(u) = f(v) = 1$ and $n - 2$ vertices $x_1, x_2, \ldots, x_{n-2}$ such that $f(x_i) = -1$ for $1 \leq i \leq n - 2$. By the definition, $u$ and $v$ dominate $x_i$ for $1 \leq i \leq n - 2$. If there exists an arc $x_ix_j$ for $i \neq j$, then $f(N^-[x_i]) \leq 0$, a contradiction. If $x_i$ and $x_j$ dominate $u$ or $v$ for $i \neq j$, then $f(N^-[u]) \leq 0$ or $f(N^-[v]) \leq 0$, a contradiction. If $x_1$ dominates $u$, then $v$ dominates $u$ and $D = B_3$ or $D = B_4$. If $x_1$ dominates $u$ and $v$, then $v$ dominates $u$ and $u$ dominates $v$ and $D = B_5$. If $x_1$ dominates $u$ and $x_2$ dominates $v$, the $D = B_6$. Finally, if there is no arc from $x_i$ to $\{u,v\}$, then $D \in \{B, B_1, B_2\}$.

Let $\{u,v,x_1,x_2,\ldots,x_{n-2}\}$ be the vertex set of the digraph $L$ of order $n \geq 2$ such that $u$ and $v$ dominate $x_i$ for $1 \leq i \leq n - 2$ and $u$ dominates $v$. In addition, let $L_1 = L \cup \{vu\}, L_2 = L_1 \cup \{x_1u\}, L_3 = L \cup \{x_1v\}, L_4 = L_1 \cup \{x_1u,x_1v\}$, and $L_5 = L_2 \cup \{x_1u,x_2v\}$. Using Theorem 5.2 instead of Theorem 5.1, one can prove the next result analogously to Theorem 5.4.

**Theorem 5.5.** Let $D$ be a digraph of order $n \geq 3$. Then $\gamma_{w,R}^2(D) \geq 5 - n$, with equality if and only if $D \in \{L, L_1, L_2, L_3, L_4, L_5\}$.

**REFERENCES**


