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GREEN'S FUNCTIONS AND EXISTENCE OF SOLUTIONS OF NONLINEAR FRACTIONAL IMPLICIT DIFFERENCE EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract. This article is devoted to deduce the expression of the Green's function related to a general constant coefficients fractional difference equation coupled to Dirichlet conditions. In this case, due to the points where some of the fractional operators are applied, we are in presence of an implicit fractional difference equation. So, due to such a property, it is more complicated to calculate and manage the expression of the Green's function than in the explicit case studied in a previous work of the authors. Contrary to the explicit case, where it is shown that the Green's function is constructed as finite sums, the Green's function constructed here is an infinite series. This fact makes necessary to impose more restrictive assumptions on the parameters that appear in the equation. The expression of the Green's function will be deduced from the Laplace transform on the time scales of the integers. We point out that, despite the implicit character of the considered equation, we can have an explicit expression of the solution by means of the expression of the Green's function. These two facts are not incompatible. Even more, this method allows us to have an explicit expression of the solution of an implicit problem. Finally, we prove two existence results for nonlinear problems, via suitable fixed point theorems.

Keywords: fractional difference, Dirichlet conditions, Green's function, existence of solutions.

Mathematics Subject Classification: 26A33, 39A12, 39A27.

1. INTRODUCTION AND PRELIMINARIES

Considering integrals and derivatives of arbitrary orders allows modeling many real phenomena in which the value that the solution takes at a given instant depends on the value of the solution in all the previous moments of the process. Thus, fractional calculus is an excellent tool when considering several physical phenomena that appears in, among others, viscoelasticity, neurology and control theory [15, 16, 18, 21, 22, 24].

During the last decades, a lot of authors studied fractional difference equations and there has been a progress made in developing the basic theory in this field. We refer to the reader the monographs [13, 20] for more details.

We use the standard notation $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ for $a \in \mathbb{R}$, and

$$[c, c+n_0]_{\mathbb{N}_c} = [c, c+n_0] \cap \mathbb{N}_c,$$

for $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}_1$.

In [4] Atici and Eloe proved that for all $(t,s) \in [\upsilon-2, \upsilon+b+1]_{\mathbb{N}_{\upsilon-2}} \times [0, b+1]_{\mathbb{N}_0}$, the following function

$$G_0(t,s) = \frac{1}{\Gamma(\upsilon)} \begin{cases} \frac{t^{(\upsilon-1)}(\upsilon+b-s)^{(\upsilon-1)}}{(\upsilon+b+1)^{(\upsilon-1)}} - (t-s-1)^{(\upsilon-1)}, & s < t-\upsilon+1, \\ \frac{t^{(\upsilon-1)}(\upsilon+b-s)^{(\upsilon-1)}}{(\upsilon+b+1)^{(\upsilon-1)}}, & t-\upsilon+1 \le s, \end{cases}$$

is the related Green's function to the Dirichlet problem

$$-\Delta_t^{\upsilon} y(t) = h(t+\upsilon-1), \quad t \in [0, b+1]_{\mathbb{N}_0},$$
$$y(\upsilon-2) = y(\upsilon+b+1) = 0,$$

with $\upsilon \in \mathbb{R}$, $1 < \upsilon < 2$ and $b \in \mathbb{N}$. Moreover, they proved that $G_0(t,s) > 0$ for all $(t,s) \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times [0, b+1]_{\mathbb{N}_0}$.

Using previous expression and constructing Green's function as a series of functions, in [8], by using the spectral theory, is ensured, for a suitable range of values of the non-constant function $a(t)$, the positiveness of the Green's function related to the following Dirichlet problem

$$-\Delta_t^{\upsilon} y(t) + a(t+\upsilon-1)\, y(t+\upsilon-1) = h(t+\upsilon-1),$$
$$y(\upsilon-2) = y(\upsilon+b+1) = 0,$$

for $t \in [0, b+1]_{\mathbb{N}_0}$, where $\upsilon \in \mathbb{R}$ with $1 < \upsilon < 2$ and $b \in \mathbb{N}$, $b \ge 5$.

A similar approach has been done in [7] for the following problem with mixed boundary conditions:

$$-\Delta_t^{\upsilon} y(t) + a(t+\upsilon-1)\, y(t+\upsilon-1) = h(t+\upsilon-1),$$
$$y(\upsilon-2) = \Delta^{\beta} y(\upsilon+b+1-\beta) = 0,$$

with $1 < \upsilon \le 2$ and $0 \le \beta \le 1$.

Using another approach in [2, 3, 5] the general expression of several linear n-th order initial value problems is obtained. They use $R_0(f(t))(s)$ the Laplace transform on the time scale of integers [6, 10], which is defined by the following expression:

$$R_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} f(t).$$

Recently, in [9] the authors considered the problem

$$-\Delta^{\upsilon} y(t) + \alpha \Delta^{\mu} y(t+\upsilon-\mu-1) = h(t+\upsilon-1), \tag{1.1}$$
$$y(\upsilon-2) = y(\upsilon+b+1) = 0, \tag{1.2}$$

for $t \in I \equiv [0, b+1]_{\mathbb{N}_0}$, where μ, $\upsilon \in \mathbb{R}$ such that $0 < \mu < 1$ and $1 < \upsilon < 2$; Δ^υ and Δ^μ are the standard υ-th and μ-th order Riemann–Liouville fractional difference operators, respectively; α is a real constant and $h : I \to \mathbb{R}$.

By using the Laplace transform $R_0(f(t))(s)$ they obtained the general expression of equation (1.1) and deduced the explicit expression of the Green's function related to problem (1.1)–(1.2). It was proven that such Green's function has some symmetric properties and is positive on $[\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times [0, b+1]_{\mathbb{N}_0}$ for all $\alpha \geq 0$ and $\upsilon-\mu-1 > 0$, which improved the results given in [4] for the particular case of $\alpha = 0$. Moreover, the authors deduced some strong positiveness conditions on the Green's function that allow them to construct suitable cones where to deduce the existence of solutions of related nonlinear problems. We point out that, in such case, the fact that the fractional operator Δ^μ is defined on the points $t+\upsilon-\mu-1$ gives us a explicit equation. Such property gives us the expression of the Green's function as a combination of finite sums.

The aim of this paper is to continue our work in this direction as we consider the following equation

$$-\Delta^\upsilon y(t) + \alpha \Delta^\mu y(t+\upsilon-\mu) = h(t+\upsilon-1), \quad t \in I \equiv \{0, 1, \ldots, b+1\}, \tag{1.3}$$

coupled to the boundary conditions (1.2).

Here μ, $\upsilon \in \mathbb{R}$ such that $0 < \mu < 1$ and $1 < \upsilon < 2$; Δ^υ and Δ^μ are the standard υ-th and μ-th order Riemann–Liouville fractional difference operators, respectively; α is a constant and $h : I \to \mathbb{R}$. We point out that even if we use Laplace transform $R_0(f(t))(s)$ to equation (1.3), we deduce that the sums are not finite as ones given in [9]. As a result, we study the convergence of the series and we apply some fixed point results to deduce different existence results for a related non linear problem. We remark that problem (1.3) coupled to the Dirichlet conditions (1.2) has been studied in [14] for the particular case of $\alpha = 0$.

The paper is organized as follows: We start with an introduction where we compile the main concepts and properties which we will use along the paper. After showing the expression of the Green's function of Problem (1.3) – (1.2) for $|\alpha| < 1$ (that is the condition that characterizes the convergence of the used series), which is obtained in a similar manner as [9] for Problem (1.1) – (1.2), we deduce, in Section 2, the main properties related to the sign and a priori bounds of the obtained function. In Sections 3 and 4 we deduce two existence (and multiplicity in Section 4) results for a nonlinear problem. Such existence results follow from the expression of the Green's function by constructing an operator whose fixed points coincides with the solutions of the problems that we are looking for. In Section 4, the sign properties obtained in Section 2 allow us to define suitable cones where to apply fixed point theorems for operators defined on cones. We finalize the paper in Section 5 with two examples that point out the applicability of the obtained existence results.

First we recall some basic definitions and lemmas, which will be used till the end of this work.

Definition 1.1. We define $t^{(\upsilon)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\upsilon)}$, for any t and υ for which the right-hand side is well defined. We also appeal to the convention that if $t+1-\upsilon$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{(\upsilon)} = 0$.

Definition 1.2. The υ-th fractional sum of a function f, for $\upsilon > 0$ and $t \in \mathbb{N}_{a+\upsilon}$, is defined as

$$\Delta_a^{-\upsilon} f(t) := \frac{1}{\Gamma(\upsilon)} \sum_{s=a}^{t-\upsilon} (t-s-1)^{(\upsilon-1)} f(s).$$

We also define the υ-th fractional difference for $\upsilon > 0$ by $\Delta_a^{\upsilon} f(t) := \Delta^N \Delta_a^{\upsilon-N} f(t)$, where $t \in \mathbb{N}_{a+N-\upsilon}$ and $N \in \mathbb{N}$ is chosen so that $0 \leq N-1 < \upsilon \leq N$.

The following formula for $\Delta_a^{\upsilon} f(t)$ can be treated as the alternate definition of $\Delta_a^{\upsilon} f(t)$.

Theorem 1.3. *Let $f : \mathbb{N}_a \to \mathbb{R}$, $\upsilon > 0$ and $N \in \mathbb{N}$ is chosen so that $0 \leq N-1 < \upsilon < N$. Then*

$$\Delta_a^{\upsilon} f(t) := \frac{1}{\Gamma(-\upsilon)} \sum_{s=a}^{t+\upsilon} (t-s-1)^{(-\upsilon-1)} f(s),$$

for $t \in \mathbb{N}_{a+N-\upsilon}$.

Remark 1.4. Observe that the value of $\Delta_a^{\upsilon} f(t)$ is a linear combination of $f(a)$, $f(a+1), \ldots, f(t+\upsilon)$.

Lemma 1.5 ([11, Lemma 2.3])**.** *Let t and υ be any numbers for which $t^{(\upsilon)}$ and $t^{(\upsilon-1)}$ are defined. Then $\Delta_t t^{(\upsilon)} = \upsilon t^{(\upsilon-1)}$.*

We use the following properties of the Laplace transform to derive the expression for the Green's function related to Problem (1.3)–(1.2).

Lemma 1.6 ([5, Lemma 2.1])**.**

$$R_{\upsilon-1}\left(t^{(\upsilon-1)}\right)(s) = \frac{\Gamma(\upsilon)}{s^{\upsilon}}. \tag{1.4}$$

Lemma 1.7 ([5, Lemma 2.2])**.** *If $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer and f is defined on $\mathbb{N}_{\mu-m}$, then*

$$R_0\left(\Delta_{\mu-m}^{\mu} f\right)(s) = s^{\mu} R_{\mu-m}(f)(s) - \sum_{k=0}^{m-1} s^{m-k-1} \left(\Delta^k \Delta_{\mu-m}^{-(\mu-m)} f\right)\Big|_{t=0}. \tag{1.5}$$

Lemma 1.8 ([5, Lemma 2.4])**.**

$$R_{\upsilon-2}(f *_{\upsilon-2} g)(s) = (s+1)^{\upsilon-1} R_{\upsilon-2}(f)(s)\, R_{\upsilon-2}(g)(s), \tag{1.6}$$

where

$$f *_{\upsilon-2} g(t) = \sum_{s=\upsilon-2}^{t} f(t-s+\upsilon-2)\, g(s) \tag{1.7}$$

is the convolution product of two functions defined on $\mathbb{N}_{\upsilon-2}$.

Definition 1.9. The two parameter delta discrete Mittag–Leffler function is defined by

$$e_{\alpha,\beta}(\lambda, t-a) = \sum_{k=0}^{\infty} \lambda^k \frac{(t-a+k\alpha+\beta-1)^{(k\alpha+\beta-1)}}{\Gamma(k\alpha+\beta)},$$

for $\alpha > 0$, $\beta \in \mathbb{R}$ and $t \in \mathbb{N}_a$.

Observe that

$$e_{\alpha,\beta}(\lambda, -1) = \sum_{k=0}^{\infty} \lambda^k \frac{(k\alpha+\beta-2)^{(k\alpha+\beta-1)}}{\Gamma(k\alpha+\beta)} = 0.$$

Clearly, if $|\lambda| < 1$, then $e_{\alpha,\beta}(\lambda, 0) = \frac{1}{1-\lambda}$. Also,

$$e_{\alpha,\beta}(\lambda, 1) = \sum_{k=0}^{\infty} \lambda^k \frac{(k\alpha+\beta)^{(k\alpha+\beta-1)}}{\Gamma(k\alpha+\beta)} = \sum_{k=0}^{\infty} \lambda^k (k\alpha+\beta) = \frac{\alpha\lambda}{(1-\lambda)^2} + \frac{\beta}{(1-\lambda)}.$$

Remark 1.10. Using D'Alembert's Ratio test, one can easily check that the above function converges for all $|\lambda| < 1$ and diverges for $|\lambda| > 1$ ([23, Theorem 6]).

2. CONSTRUCTION OF THE GREEN'S FUNCTION

In this section we will construct the Green's function related to Problem (1.3)–(1.2), following the approach given in [9]. We point out that, despite the sequence of steps is the same in both papers, the content in each step is quite different. In particular, since the delta discrete Mittag-Leffler function will be used, along all the section we assume that $|\alpha| < 1$, in order to ensure its convergence by the characterization given in Remark 1.10. Such restriction is not assumed on [9], where the expression of the Green's function is valid for any real α that is not an eigenvalue of problem (1.1)–(1.2). Moreover, on the contrary to reference [9], in which the expression of the Green's function comes from finite sums, the obtained expression of the Green's function for problem (1.3), (1.2), as we will see in the sequel, is given as a combination of series of infinitely non zero terms. At any case, it is important to point out that, despite the implicit character of equation (1.3), we can have an explicit expression of the solution by means of the expression of the Green's function. These two facts are not incompatible. Even more, this method of solving equation (1.3), (1.2) allows us to have an explicit expression of the solution of an implicit problem.

First, from (1.5), we have

$$R_0\left[\Delta^\upsilon y(t)\right](s) = s^\upsilon R_{\upsilon-2}\left[y(t)\right](s) - sA - B, \tag{2.1}$$

where

$$A = y(\upsilon-2) \tag{2.2}$$

and

$$B = (1-\upsilon)y(\upsilon-2) + y(\upsilon-1).$$

Denote by $Y_1(t) = y(t + \upsilon - \mu)$. Then,

$$\begin{aligned} R_{\mu-1}\left[Y_1(t)\right](s) &= \sum_{t=\mu-1}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} Y_1(t) \\ &= \sum_{t=\upsilon-1}^{\infty} \left(\frac{1}{s+1}\right)^{t-\upsilon+\mu+1} y(t) \\ &= (s+1)^{\upsilon-\mu} \sum_{t=\upsilon-1}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} y(t) \\ &= (s+1)^{\upsilon-\mu} \left[\sum_{t=\upsilon-2}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} y(t) - \left(\frac{1}{s+1}\right)^{\upsilon-2+1} y(\upsilon-2)\right] \\ &= (s+1)^{\upsilon-\mu} R_{\upsilon-2}\left[y(t)\right](s) - (s+1)^{1-\mu} y(\upsilon-2). \end{aligned} \tag{2.3}$$

Next, from (1.5) we have

$$R_0\left[\Delta^{\mu} y(t)\right](s) = s^{\mu} R_{\mu-1}\left[y(t)\right](s) - \left[\Delta^{\mu-1} y(t)\right]_{t=0}. \tag{2.4}$$

Using (2.3) and (2.4), we obtain

$$\begin{aligned} &R_0\left[\Delta^{\mu} Y_1(t)\right](s) \\ &= s^{\mu} R_{\mu-1}\left[Y_1(t)\right](s) - \left[\Delta^{\mu-1} Y_1(t)\right]_{t=0} \\ &= s^{\mu}\left[(s+1)^{\upsilon-\mu} R_{\upsilon-2}\left[y(t)\right](s) - (s+1)^{1-\mu} y(\upsilon-2)\right] - \left[\Delta^{\mu-1} Y_1(t)\right]_{t=0}. \end{aligned} \tag{2.5}$$

Now, consider

$$\begin{aligned} \left[\Delta^{\mu-1} Y_1(t)\right]_{t=0} &= \left[\Delta^{-(1-\mu)} Y_1(t)\right]_{t=0} \\ &= \left[\frac{1}{\Gamma(1-\mu)} \sum_{s=\mu-1}^{t-(1-\mu)} (t-s-1)^{(1-\mu-1)} Y_1(s)\right]_{t=0} \\ &= \left[\frac{1}{\Gamma(1-\mu)} \sum_{s=\mu-1}^{t-(1-\mu)} (t-s-1)^{(1-\mu-1)} y(s+\upsilon-\mu)\right]_{t=0} \\ &= \frac{1}{\Gamma(1-\mu)} (-\mu)^{(-\mu)} y(\upsilon-1) \\ &= y(\upsilon-1) = B - (1-\upsilon)A. \end{aligned} \tag{2.6}$$

Using (2.2) and (2.6) in (2.5), we deduce

$$\begin{aligned} &R_0\left[\Delta^{\mu} Y_1(t)\right](s) \\ &= s^{\mu}(s+1)^{\upsilon-\mu} R_{\upsilon-2}\left[y(t)\right](s) - s^{\mu}(s+1)^{1-\mu} A - B + (1-\upsilon)A \\ &= s^{\mu}(s+1)^{\upsilon-\mu} R_{\upsilon-2}\left[y(t)\right](s) + \left[(1-\upsilon) - s^{\mu}(s+1)^{1-\mu}\right] A - B. \end{aligned} \tag{2.7}$$

Denote $H_1(t) = h(t+\upsilon-1)$. Then, from (2.3), we obtain that

$$R_0\left[H_1(t)\right](s) = (s+1)^{\upsilon-1}R_{\upsilon-2}\left[h(t)\right](s) - h(\upsilon-2). \tag{2.8}$$

By applying R_0 to each side of (1.3) and employing (2.1), (2.7) and (2.8), we obtain

$$\begin{aligned}
&-\left[s^{\upsilon}R_{\upsilon-2}\left[y(t)\right](s) - sA - B\right] + \alpha\Big[s^{\mu}(s+1)^{\upsilon-\mu}R_{\upsilon-2}[y(t)](s)\\
&+\left[(1-\upsilon) - s^{\mu}(s+1)^{1-\mu}\right]A - B\Big]\\
&\quad = (s+1)^{\upsilon-1}R_{\upsilon-2}\left[h(t)\right](s) - h(\upsilon-2).
\end{aligned}$$

Rearranging the terms gives us

$$\begin{aligned}
&\left(s^{\upsilon} - \alpha s^{\mu}(s+1)^{\upsilon-\mu}\right)R_{\upsilon-2}\left[y(t)\right](s)\\
&= \left(s + \alpha(1-\upsilon) - \alpha s^{\mu}(s+1)^{1-\mu}\right)A\\
&\quad + (1-\alpha)B - (s+1)^{\upsilon-1}R_{\upsilon-2}\left[h(t)\right](s) + h(\upsilon-2).
\end{aligned}$$

This implies that

$$\begin{aligned}
R_{\upsilon-2}\left[y(t)\right](s) = {} & \frac{\left(s + \alpha(1-\upsilon) - \alpha s^{\mu}(s+1)^{1-\mu}\right)}{\left(s^{\upsilon} - \alpha s^{\mu}(s+1)^{\upsilon-\mu}\right)}A\\
&+ \frac{(1-\alpha)}{\left(s^{\upsilon} - \alpha s^{\mu}(s+1)^{\upsilon-\mu}\right)}B\\
&- \frac{(s+1)^{\upsilon-1}}{\left(s^{\upsilon} - \alpha s^{\mu}(s+1)^{\upsilon-\mu}\right)}R_{\upsilon-2}\left[h(t)\right](s)\\
&+ \frac{1}{\left(s^{\upsilon} - \alpha s^{\mu}(s+1)^{\upsilon-\mu}\right)}h(\upsilon-2).
\end{aligned} \tag{2.9}$$

Denote $Z(t) = y(t + n(\upsilon-\mu))$. Then,

$$\begin{aligned}
R_{\upsilon-2}\left[Z(t)\right](s) &= \sum_{t=\upsilon-2}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}Z(t)\\
&= \sum_{t=(n+1)\upsilon-n\mu-2}^{\infty}\left(\frac{1}{s+1}\right)^{t-n\upsilon+n\mu+1}y(t)\\
&= (s+1)^{n\upsilon-n\mu}\sum_{t=(n+1)\upsilon-n\mu-2}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}y(t)\\
&= (s+1)^{n\upsilon-n\mu}R_{(n+1)\upsilon-n\mu-2}\left[y(t)\right](s).
\end{aligned} \tag{2.10}$$

Note that using (1.4) and (2.10), we obtain

$$\begin{aligned}
\frac{s}{s^{\upsilon}-\alpha s^{\mu}(s+1)^{\upsilon-\mu}} &= \frac{1}{s^{\upsilon-1}}\frac{1}{\left[1-\alpha\left(\frac{s+1}{s}\right)^{\upsilon-\mu}\right]} = \frac{1}{s^{\upsilon-1}}\sum_{k=0}^{\infty}\alpha^{k}\left(\frac{s+1}{s}\right)^{k\upsilon-k\mu} \\
&= \sum_{k=0}^{\infty}\alpha^{k}(s+1)^{k\upsilon-k\mu}\frac{1}{s^{(k+1)\upsilon-k\mu-1}} \\
&= \sum_{k=0}^{\infty}\alpha^{k}(s+1)^{k\upsilon-k\mu}\frac{R_{(k+1)\upsilon-k\mu-2}\left[t^{((k+1)\upsilon-k\mu-2)}\right](s)}{\Gamma((k+1)\upsilon-k\mu-1)} \\
&= \sum_{k=0}^{\infty}\alpha^{k}\frac{R_{\upsilon-2}\left[(t+k(\upsilon-\mu))^{((k+1)\upsilon-k\mu-2)}\right](s)}{\Gamma((k+1)\upsilon-k\mu-1)} \\
&= R_{\upsilon-2}\left[\sum_{k=0}^{\infty}\alpha^{k}\frac{(t+k(\upsilon-\mu))^{((k+1)\upsilon-k\mu-2)}}{\Gamma((k+1)\upsilon-k\mu-1)}\right](s) \\
&= R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon-1}(\alpha,t-\upsilon+2)\right](s).
\end{aligned} \tag{2.11}$$

Similar to (2.10), we have

$$R_{\upsilon-1}\left[Z(t)\right](s) = (s+1)^{n\upsilon-n\mu}R_{(n+1)\upsilon-n\mu-1}\left[y(t)\right](s). \tag{2.12}$$

Moreover, using (1.4) and (2.12) we obtain

$$\begin{aligned}
\frac{1}{s^{\upsilon}-\alpha s^{\mu}(s+1)^{\upsilon-\mu}} &= \frac{1}{s^{\upsilon}}\sum_{k=0}^{\infty}\alpha^{k}\left(\frac{s+1}{s}\right)^{k\upsilon-k\mu} = \sum_{k=0}^{\infty}\alpha^{k}(s+1)^{k\upsilon-k\mu}\frac{1}{s^{(k+1)\upsilon-k\mu}} \\
&= \sum_{k=0}^{\infty}\alpha^{k}(s+1)^{k\upsilon-k\mu}\frac{R_{(k+1)\upsilon-k\mu-1}\left[t^{((k+1)\upsilon-k\mu-1)}\right](s)}{\Gamma((k+1)\upsilon-k\mu)} \\
&= \sum_{k=0}^{\infty}\alpha^{k}\frac{R_{\upsilon-1}\left[(t+k(\upsilon-\mu))^{((k+1)\upsilon-k\mu-1)}\right](s)}{\Gamma((k+1)\upsilon-k\mu)} \\
&= R_{\upsilon-1}\left[\sum_{k=0}^{\infty}\alpha^{k}\frac{(t+k(\upsilon-\mu))^{((k+1)\upsilon-k\mu-1)}}{\Gamma((k+1)\upsilon-k\mu)}\right](s) \\
&= R_{\upsilon-2}\left[\sum_{k=0}^{\infty}\alpha^{k}\frac{(t+k(\upsilon-\mu))^{((k+1)\upsilon-k\mu-1)}}{\Gamma((k+1)\upsilon-k\mu)}\right](s) \\
&\quad -(s+1)^{1-\upsilon}\left[\sum_{k=0}^{\infty}\alpha^{k}\frac{(t+k(\upsilon-\mu))^{((k+1)\upsilon-k\mu-1)}}{\Gamma((k+1)\upsilon-k\mu)}\right]_{t=\upsilon-2} \\
&= R_{\upsilon-2}\left[\sum_{k=0}^{\infty}\alpha^{k}\frac{(t+k(\upsilon-\mu))^{((k+1)\upsilon-k\mu-1)}}{\Gamma((k+1)\upsilon-k\mu)}\right](s) \\
&= R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\right](s).
\end{aligned} \tag{2.13}$$

Denote $Z_1(t) = y(t + n(\upsilon - \mu) - \mu + 1)$. Then,

$$\begin{aligned} R_{\upsilon-2}\left[Z_1(t)\right](s) &= \sum_{t=\upsilon-2}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} Z_1(t) \\ &= \sum_{t=(n+1)(\upsilon-\mu)-1}^{\infty} \left(\frac{1}{s+1}\right)^{t-n(\upsilon-\mu)+\mu-1+1} y(t) \\ &= (s+1)^{n\upsilon-(n+1)\mu+1} \sum_{t=(n+1)(\upsilon-\mu)-1}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} y(t) \\ &= (s+1)^{n\upsilon-(n+1)\mu+1} R_{(n+1)(\upsilon-\mu)-1}\left[y(t)\right](s). \end{aligned} \tag{2.14}$$

Note that using (1.4) and (2.14), we obtain

$$\begin{aligned} &\frac{s^{\mu}(s+1)^{1-\mu}}{s^{\upsilon} - \alpha s^{\mu}(s+1)^{\upsilon-\mu}} \\ &= \frac{s^{\mu}(s+1)^{1-\mu}}{s^{\upsilon}} \frac{1}{1 - \alpha s^{\mu-\upsilon}(s+1)^{\upsilon-\mu}} \\ &= \frac{s^{\mu}(s+1)^{1-\mu}}{s^{\upsilon}} \frac{1}{\left[1 - \alpha\left(\frac{s+1}{s}\right)^{\upsilon-\mu}\right]} \\ &= \frac{s^{\mu}(s+1)^{1-\mu}}{s^{\upsilon}} \sum_{k=0}^{\infty} \alpha^k \left(\frac{s+1}{s}\right)^{k\upsilon-k\mu} \\ &= \sum_{k=0}^{\infty} \alpha^k (s+1)^{k\upsilon-(k+1)\mu+1} \frac{1}{s^{(k+1)\upsilon-(k+1)\mu}} \\ &= \sum_{k=0}^{\infty} \alpha^k (s+1)^{k\upsilon-(k+1)\mu+1} \frac{R_{(k+1)\upsilon-(k+1)\mu-1}\left[t^{((k+1)\upsilon-(k+1)\mu-1)}\right](s)}{\Gamma((k+1)\upsilon-(k+1)\mu)} \\ &= \sum_{k=0}^{\infty} \alpha^k \frac{R_{\upsilon-2}\left[(t+k(\upsilon-\mu)-\mu+1)^{((k+1)\upsilon-k\mu-1)}\right](s)}{\Gamma((k+1)\upsilon-k\mu)} \\ &= R_{\upsilon-2}\left[\sum_{k=0}^{\infty} \alpha^k \frac{(t+k(\upsilon-\mu)-\mu+1)^{((k+1)\upsilon-k\mu-1)}}{\Gamma((k+1)\upsilon-k\mu)}\right](s) \\ &= R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon-\mu}(\alpha, t-\upsilon+2)\right](s). \end{aligned} \tag{2.15}$$

Using (2.11), (2.13) and (2.15) in (2.9), we deduce

$$\begin{aligned}R_{\upsilon-2}\left[y(t)\right](s) = \Big[&R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon-1}(\alpha,t-\upsilon+2)\right](s)\\&+\alpha(1-\upsilon)R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\right](s)\\&-\alpha R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon-\mu}(\alpha,t-\upsilon+2)\right](s)\Big]A\\&+(1-\alpha)R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\right](s)B\\&-(s+1)^{\upsilon-1}R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\right](s)R_{\upsilon-2}\left[h(t)\right](s)\\&+R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\right](s)h(\upsilon-2),\end{aligned}$$

which, by (1.6), can be written as

$$\begin{aligned}R_{\upsilon-2}\left[y(t)\right](s) = \Big[&R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon-1}(\alpha,t-\upsilon+2)\right](s)\\&+\alpha(1-\upsilon)R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\right](s)\\&-\alpha R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon-\mu}(\alpha,t-\upsilon+2)\right](s)\Big]A\\&+(1-\alpha)R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\right](s)B\\&-R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)*_{\upsilon-2}h\right](s)\\&+R_{\upsilon-2}\left[e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\right](s)h(\upsilon-2).\end{aligned}$$

Apply to each side the inverse of $R_{\upsilon-2}$, we obtain

$$\begin{aligned}y(t) = \Big[&e_{\upsilon-\mu,\upsilon-1}(\alpha,t-\upsilon+2)+\alpha(1-\upsilon)e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\\&-\alpha e_{\upsilon-\mu,\upsilon-\mu}(\alpha,t-\upsilon+2)\Big]A\\&+(1-\alpha)e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)B\\&-e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)*_{\upsilon-2}h+e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)h(\upsilon-2).\end{aligned}$$

Thus, using (1.7), we have

$$\begin{aligned}y(t) = \Big[&e_{\upsilon-\mu,\upsilon-1}(\alpha,t-\upsilon+2)+\alpha(1-\upsilon)e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\\&-\alpha e_{\upsilon-\mu,\upsilon-\mu}(\alpha,t-\upsilon+2)\Big]A\\&+(1-\alpha)e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)B\\&-\sum_{s=\upsilon-2}^{t}e_{\upsilon-\mu,\upsilon}(\alpha,t-s+\upsilon-2-\upsilon+1)h(s)\\&+e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)h(\upsilon-2).\end{aligned}$$

That is,

$$
\begin{aligned}
y(t) = \Big[&e_{\upsilon-\mu,\upsilon-1}(\alpha, t-\upsilon+2) + \alpha(1-\upsilon)e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1) \\
&- \alpha e_{\upsilon-\mu,\upsilon-\mu}(\alpha, t-\upsilon+2)\Big]A \\
&+ (1-\alpha)e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)B - \sum_{s=\upsilon-1}^{t} e_{\upsilon-\mu,\upsilon}(\alpha, t-s-1)h(s).
\end{aligned}
\tag{2.16}
$$

Using $y(\upsilon-2)=0$ in (2.16), we have

$$
\begin{aligned}
0 = &\Big[e_{\upsilon-\mu,\upsilon-1}(\alpha,0) + \alpha(1-\upsilon)e_{\upsilon-\mu,\upsilon}(\alpha,-1) - \alpha e_{\upsilon-\mu,\upsilon-\mu}(\alpha,0)\Big]A \\
&+ (1-\alpha)e_{\upsilon-\mu,\upsilon}(\alpha,-1)B - \sum_{s=\upsilon-1}^{\upsilon-2} e_{\upsilon-\mu,\upsilon}(\alpha, t-s-1)h(s).
\end{aligned}
$$

That is,

$$
0 = \left[\frac{1}{(1-\alpha)} - \frac{\alpha}{(1-\alpha)}\right] A.
$$

Using $y(\upsilon+b+1)=0$ in (2.16) and taking $A=0$, we have

$$
0 = (1-\alpha)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)B - \sum_{s=\upsilon-1}^{\upsilon+b+1} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)h(s),
$$

or

$$
\begin{aligned}
B &= \frac{1}{(1-\alpha)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)} \sum_{s=\upsilon-1}^{\upsilon+b+1} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)h(s) \\
&= \frac{1}{(1-\alpha)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)} \sum_{s=\upsilon-1}^{\upsilon+b} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)h(s).
\end{aligned}
\tag{2.17}
$$

Using (2.17) and $A=0$ in (2.16), we obtain

$$
\begin{aligned}
y(t) = (1-\alpha)e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)\Bigg[&\frac{1}{(1-\alpha)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)} \\
&\times \sum_{s=\upsilon-1}^{\upsilon+b} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)h(s)\Bigg] \\
- \sum_{s=\upsilon-1}^{t} e_{\upsilon-\mu,\upsilon}(\alpha, t-s-1)h(s).&
\end{aligned}
$$

Rearranging the terms, we obtain

$$y(t) = \frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+2)} \sum_{s=\upsilon-1}^{\upsilon+b} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)h(s) - \sum_{s=\upsilon-1}^{t} e_{\upsilon-\mu,\upsilon}(\alpha, t-s-1)h(s).$$

That is,

$$y(t) = \sum_{s=\upsilon-1}^{t} \left[\frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+2)} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s) - e_{\upsilon-\mu,\upsilon}(\alpha, t-s-1)\right] h(s) - \sum_{s=t+1}^{\upsilon+b} \left[\frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+2)} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)\right] h(s).$$

Denote by

$$I_1 = \{(t, s) : \upsilon-1 \leq s \leq t \leq \upsilon+b+1\},$$

and

$$I_2 = \{(t, s) : \upsilon-1 \leq t+1 \leq s \leq \upsilon+b\}.$$

Theorem 2.1. *Assuming that* $|\alpha| < 1$*, we have that Problem* (1.3)–(1.2) *has a unique solution if and only if*

$$e_{\upsilon-\mu,\upsilon}(\alpha, b+2) \neq 0.$$

Moreover, the expression for the related Green's function, when $|\alpha| < 1$*, is given by*

$$G(t, s) = \begin{cases} \frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+2)} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s) - e_{\upsilon-\mu,\upsilon}(\alpha, t-s-1), & (t, s) \in I_1, \\ \frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+2)} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s), & (t, s) \in I_2. \end{cases} \tag{2.18}$$

Further, the unique solution of the Problem (1.3)–(1.2) *is given by*

$$y(t) = \sum_{s=\upsilon-1}^{\upsilon+b} G(t, s)h(s), \quad t \in [\upsilon-2, \upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}.$$

Remark 2.2. Consider (1.1). From Theorem 1.3, we have

$$\Delta^{\upsilon} y(t) = \Delta^{\upsilon}_{\upsilon-2} f(t) = \frac{1}{\Gamma(-\upsilon)} \sum_{s=\upsilon-2}^{t+\upsilon} (t-s-1)^{(-\upsilon-1)} y(s), \quad t \in \mathbb{N}_0, \tag{2.19}$$

and

$$\Delta^{\mu} y(t+\upsilon-\mu-1) = \Delta^{\mu}_{\upsilon-2} y(t+\upsilon-\mu-1) = \frac{1}{\Gamma(-\mu)} \sum_{s=\upsilon-2}^{t+\upsilon-1} (t+\upsilon-\mu-s-2)^{(-\upsilon-1)} y(s), \quad t \in \mathbb{N}_0. \tag{2.20}$$

Clearly, the value of $\Delta^{\upsilon}y(t)$ is a linear combination of $y(\upsilon-2)$, $y(\upsilon-1)$, ..., $y(t+\upsilon)$ and the value of $\Delta^{\mu}y(t+\upsilon-\mu-1)$ is a linear combination of $y(\upsilon-2)$, $y(\upsilon-1)$, ..., $y(t+\upsilon-1)$. Thus, the unknown $y(t+\upsilon)$ is present only in the first term of LHS of (1.1). Using (2.19) and (2.20) in (1.1) and rearranging the terms, we notice that the unknown $y(t+\upsilon)$ can be explicitly expressed as a linear combination of $y(\upsilon-2)$, $y(\upsilon-1)$, ..., $y(t+\upsilon-1)$ and $h(t+\upsilon-1)$.

Now, consider (1.3). From Theorem 1.3, we have

$$\begin{aligned}\Delta^{\mu}y(t+\upsilon-\mu) &= \Delta^{\mu}_{\upsilon-2}y\,(t+\upsilon-\mu)\\ &= \frac{1}{\Gamma(-\mu)}\sum_{s=\upsilon-2}^{t+\upsilon}(t+\upsilon-\mu-s-1)^{(-\upsilon-1)}\,y(s), \quad t\in\mathbb{N}_0.\end{aligned} \tag{2.21}$$

Clearly, the value of $\Delta^{\mu}y(t+\upsilon-\mu)$ is a linear combination of $y(\upsilon-2)$, $y(\upsilon-1)$, ..., $y(t+\upsilon)$. Since the unknown $y(t+\upsilon)$ is present in both the terms of LHS of (1.3), the fractional difference equation (1.3) becomes an implicit difference equation in the unknown $y(t+\upsilon)$ unlike the fractional difference equation (1.1).

3. PROPERTIES OF THE GREEN'S FUNCTION

In this section, we derive some sign properties of the Green's function related to Problem (1.3)–(1.2), which is given by expression (2.18). For this purpose, we will prove the following preliminary results.

Lemma 3.1. *Assume $0<\mu<1<\upsilon<2$ such that $\upsilon-\mu-1\geq 0$ and $t\in[\upsilon-1,\upsilon+b-2]_{\mathbb{N}_{\upsilon-1}}$. For each $0\leq\alpha\leq 1$, denote*

$$g(t,\alpha)=(\upsilon-2)\frac{t^{(\upsilon-3)}}{\Gamma(\upsilon-1)}+\sum_{k=1}^{\infty}\alpha^k\frac{(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-3)}}{\Gamma(k\upsilon-k\mu+\upsilon-2)}. \tag{3.1}$$

Then, there exists a unique $\bar{\alpha}(t)\in(0,1)$ such that

$$g(t,\bar{\alpha}(t))=0,\quad t\in[\upsilon-1,\upsilon+b-2]_{\mathbb{N}_{\upsilon-1}}.$$

Proof. First, we point out that since the function $e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)$ converges for all $|\alpha|<1$ and

$$\begin{aligned}\Delta_t^2 e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1) &= \sum_{k=0}^{\infty}\alpha^k\frac{\Delta_t^2(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-1)}}{\Gamma(k\upsilon-k\mu+\upsilon)}\\ &= \sum_{k=0}^{\infty}\alpha^k\frac{(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-3)}}{\Gamma(k\upsilon-k\mu+\upsilon-2)}\\ &= \frac{t^{(\upsilon-3)}}{\Gamma(\upsilon-2)}+\sum_{k=1}^{\infty}\alpha^k\frac{(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-3)}}{\Gamma(k\upsilon-k\mu+\upsilon-2)}\\ &= g(t,\alpha),\end{aligned}$$

it follows that the function $g(t,\alpha)$ also converges for all $|\alpha|<1$.

Consider

$$g(t,0) = (\upsilon - 2)\frac{t^{(\upsilon-3)}}{\Gamma(\upsilon-1)} = (\upsilon - 2)\frac{\Gamma(t+1)}{\Gamma(t-\upsilon+4)\Gamma(\upsilon-1)}.$$

Clearly, for each $t \in [\upsilon - 1, \upsilon + b - 2]_{\mathbb{N}_{\upsilon-1}}$,

$$\frac{\Gamma(t+1)}{\Gamma(t-\upsilon+4)\Gamma(\upsilon-1)} > 0,$$

implying that $g(t,0) < 0$.

Moreover,

$$g(t,1) = (\upsilon - 2)\frac{\Gamma\,(t+1)}{\Gamma\,(t-\upsilon+4)\,\Gamma(\upsilon-1)} + \sum_{k=1}^{\infty}\frac{\Gamma(t+k\upsilon-k\mu+1)}{\Gamma\,(t-\upsilon+4)\,\Gamma(k\upsilon-k\mu+\upsilon-2)}.$$

Note that since $t \geq \upsilon - 1$, then

$$\begin{aligned}\frac{\Gamma(t+k\upsilon-k\mu+1)}{\Gamma(k\upsilon-k\mu+\upsilon-2)} &\geq \frac{\Gamma(k\upsilon-k\mu+\upsilon)}{\Gamma(k\upsilon-k\mu+\upsilon-2)}\\ &= (k\upsilon-k\mu+\upsilon-2)(k\upsilon-k\mu+\upsilon-1).\end{aligned}$$

Thus,

$$\begin{aligned}g(t,1) &= (\upsilon - 2)\frac{\Gamma\,(t+1)}{\Gamma\,(t-\upsilon+4)\,\Gamma(\upsilon-1)} + \sum_{k=1}^{\infty}\frac{\Gamma(t+k\upsilon-k\mu+1)}{\Gamma\,(t-\upsilon+4)\,\Gamma(k\upsilon-k\mu+\upsilon-2)}\\ &\geq (\upsilon - 2)\frac{\Gamma\,(t+1)}{\Gamma\,(t-\upsilon+4)\,\Gamma(\upsilon-1)} + \sum_{k=1}^{\infty}\frac{(k\upsilon-k\mu+\upsilon-2)(k\upsilon-k\mu+\upsilon-1)}{\Gamma\,(t-\upsilon+4)}\\ &> 0.\end{aligned}$$

Consider

$$\frac{\partial g}{\partial \alpha}(t,\alpha) = \sum_{k=1}^{\infty} k\alpha^{k-1}\frac{(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-3)}}{\Gamma(k\upsilon-k\mu+\upsilon-2)}.$$

For each $t \in [\upsilon - 1, \upsilon + b - 2]_{\mathbb{N}_{\upsilon-1}}$, $\alpha > 0$ and $k \in \mathbb{N}_1$, we have $k\alpha^{k-1} > 0$, and

$$\frac{(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-3)}}{\Gamma(k\upsilon-k\mu+\upsilon-2)} = \frac{\Gamma(t+k\upsilon-k\mu+1)}{\Gamma(t-\upsilon+4)\Gamma(k\upsilon-k\mu+\upsilon-2)} > 0,$$

implying that

$$\frac{\partial g}{\partial \alpha}(t,\alpha) > 0, \quad \text{for all } \alpha > 0 \text{ and } t \in [\upsilon - 1, \upsilon + b - 2]_{\mathbb{N}_{\upsilon-1}}.$$

Then, there exists a unique $\bar{\alpha}(t) \in (0,1)$ such that

$$g(t,\bar{\alpha}(t)) = 0, \quad t \in [\upsilon - 1, \upsilon + b - 2]_{\mathbb{N}_{\upsilon-1}}.$$

The proof is complete. □

Now, we define

$$\alpha^* = \min_{t\in[\upsilon-1,\upsilon+b-2]_{\mathbb{N}_{\upsilon-1}}} \bar{\alpha}(t).$$

Clearly, $\alpha^* > 0$. Throughout, we assume that the parameter α satisfies $\alpha \in (0, \alpha^*)$.

Lemma 3.2. *Assume $0 < \mu < 1 < \upsilon < 2$ such that $\upsilon - \mu - 1 \geq 0$. Then the following assertions hold.*

(i)

$$0 < \frac{t^{(\upsilon-1)}}{\Gamma(\upsilon)} < e_{\upsilon-\mu,\upsilon}(\alpha, t - \upsilon + 1)$$

for all $\alpha > 0$ and $t \in [\upsilon - 1, \upsilon + b]_{\mathbb{N}_{\upsilon-1}}$.

(ii)

$$0 < \frac{t^{(\upsilon-2)}}{\Gamma(\upsilon - 1)} < \Delta_t e_{\upsilon-\mu,\upsilon}(\alpha, t - \upsilon + 1)$$

for all $0 \leq \alpha < 1$ and $t \in [\upsilon-1, \upsilon+b-1]_{\mathbb{N}_{\upsilon-1}}$. Consequently, $e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)$ is an increasing function with respect to t for all $0 \leq \alpha < 1$ and $t \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$.

(iii) *$\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)$ is a decreasing function with respect to t for all $0 \leq \alpha < \alpha^*$ and $t \in [\upsilon - 1, \upsilon + b - 1]_{\mathbb{N}_{\upsilon-1}}$.*

(iv)

$$e_{\upsilon-\mu,\upsilon}(\alpha, t - s - 1) \leq e_{\upsilon-\mu,\upsilon}(\alpha, t - \upsilon)$$

for all $0 \leq \alpha < 1$ and $(t, s) \in I_1$.

(v)

$$\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha, t - s - 1) \geq \Delta_t e_{\upsilon-\mu,\upsilon}(\alpha, t - \upsilon)$$

for all $0 \leq \alpha < \alpha^*$ and $(t, s) \in I_1$.

Proof. For each $t \in [\upsilon - 1, \upsilon + b]_{\mathbb{N}_{\upsilon-1}}$, consider

$$\begin{aligned} e_{\upsilon-\mu,\upsilon}(\alpha, t - \upsilon + 1) &= \sum_{k=0}^{\infty} \alpha^k \frac{(t + k\upsilon - k\mu)^{(k\upsilon-k\mu+\upsilon-1)}}{\Gamma(k\upsilon - k\mu + \upsilon)} \\ &= \frac{t^{(\upsilon-1)}}{\Gamma(\upsilon)} + \sum_{k=1}^{\infty} \alpha^k \frac{(t + k\upsilon - k\mu)^{(k\upsilon-k\mu+\upsilon-1)}}{\Gamma(k\upsilon - k\mu + \upsilon)} \\ &= \frac{\Gamma(t + 1)}{\Gamma(t - \upsilon + 2)\Gamma(\upsilon)} \\ &\quad + \sum_{k=1}^{\infty} \alpha^k \frac{\Gamma(t + k\upsilon - k\mu + 1)}{\Gamma(t - \upsilon + 2)\Gamma(k\upsilon - k\mu + \upsilon)}. \end{aligned}$$

Clearly, for each $t \in [\upsilon - 1, \upsilon + b]_{\mathbb{N}_{\upsilon-1}}$ and $k \in \mathbb{N}_1$, we have $\alpha^k > 0$,

$$\frac{\Gamma(t + 1)}{\Gamma(t - \upsilon + 2)\Gamma(\upsilon)} > 0,$$

and

$$\frac{\Gamma(t+k\upsilon-k\mu+1)}{\Gamma(t-\upsilon+2)\Gamma(k\upsilon-k\mu+\upsilon)}>0,$$

implying that

$$0<\frac{t^{(\upsilon-1)}}{\Gamma(\upsilon)}<e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1).$$

The proof of (i) is complete.

Now, for each $t\in[\upsilon-1,\upsilon+b-1]_{\mathbb{N}_{\upsilon-1}}$, consider

$$\begin{aligned}
&\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)\\
&=\sum_{k=0}^{\infty}\alpha^k\frac{\Delta_t(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-1)}}{\Gamma(k\upsilon-k\mu+\upsilon)}\\
&=\sum_{k=0}^{\infty}\alpha^k\frac{(k\upsilon-k\mu+\upsilon-1)(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-2)}}{\Gamma(k\upsilon-k\mu+\upsilon)}\\
&=\sum_{k=0}^{\infty}\alpha^k\frac{(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-2)}}{\Gamma(k\upsilon-k\mu+\upsilon-1)}\\
&=\frac{t^{(\upsilon-2)}}{\Gamma(\upsilon-1)}+\sum_{k=1}^{\infty}\alpha^k\frac{(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-2)}}{\Gamma(k\upsilon-k\mu+\upsilon-1)}\\
&=\frac{\Gamma(t+1)}{\Gamma(t-\upsilon+3)\Gamma(\upsilon-1)}+\sum_{k=1}^{\infty}\alpha^k\frac{\Gamma(t+k\upsilon-k\mu+1)}{\Gamma(t-\upsilon+3)\Gamma(k\upsilon-k\mu+\upsilon-1)}.
\end{aligned}$$

Clearly, for each $t\in[\upsilon-1,\upsilon+b-1]_{\mathbb{N}_{\upsilon-1}}$ and $k\in\mathbb{N}_1$, we have $\alpha^k>0$,

$$\frac{\Gamma(t+1)}{\Gamma(t-\upsilon+3)\Gamma(\upsilon-1)}>0,$$

and

$$\frac{\Gamma(t+k\upsilon-k\mu+1)}{\Gamma(t-\upsilon+3)\Gamma(k\upsilon-k\mu+\upsilon-1)}>0,$$

implying that

$$0<\frac{t^{(\upsilon-2)}}{\Gamma(\upsilon-1)}<\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1).$$

Thus, $e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)$ is an increasing function with respect to t for $t\in[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}$ and the proof of (ii) is complete.

Now, for each $t \in [\upsilon-1, \upsilon+b-2]_{\mathbb{N}_{\upsilon-1}}$, consider

$$\begin{aligned}
&\Delta_t^2 e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)\\
&= \sum_{k=0}^{\infty} \alpha^k \frac{\Delta_t^2 (t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-1)}}{\Gamma(k\upsilon-k\mu+\upsilon)}\\
&= \sum_{k=0}^{\infty} \alpha^k \frac{(k\upsilon-k\mu+\upsilon-1)(k\upsilon-k\mu+\upsilon-2)(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-3)}}{\Gamma(k\upsilon-k\mu+\upsilon)}\\
&= \sum_{k=0}^{\infty} \alpha^k \frac{(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-3)}}{\Gamma(k\upsilon-k\mu+\upsilon-2)}\\
&= \frac{t^{(\upsilon-3)}}{\Gamma(\upsilon-2)} + \sum_{k=1}^{\infty} \alpha^k \frac{(t+k\upsilon-k\mu)^{(k\upsilon-k\mu+\upsilon-3)}}{\Gamma(k\upsilon-k\mu+\upsilon-2)}\\
&= g(t,\alpha) \le g(t,\alpha^*) = 0,
\end{aligned}$$

implying that $\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)$ is a decreasing function of t for $t \in [\upsilon-1, \upsilon+b-1]_{\mathbb{N}_{\upsilon-1}}$. Thus, the proof of (iii) is complete.

The proofs of (iv) and (v) follow from (ii) and (iii), respectively. □

Theorem 3.3. *Assume* $0 < \mu < 1$, $1 < \upsilon < 2$ *such that* $\upsilon-\mu-1 \ge 0$ *and* $\alpha \in (0, \alpha^*)$. *The Green's function* $G(t,s)$ *defined in* (2.18) *satisfies*

1. $G(\upsilon-2, s) = 0$ *for each* $s \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$.
2. $G(\upsilon+b+1, s) = 0$ *for each* $s \in [\upsilon-1, \upsilon+b+1]_{\mathbb{N}_{\upsilon-1}}$.
3. $G(t,s) > 0$ *for each* $(t,s) \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$.
4. $\max\limits_{t \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}} G(t,s) = G(s,s)$ *for each* $s \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$.

Proof. For each $s \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$,

$$G(\upsilon-2, s) = \frac{e_{\upsilon-\mu,\upsilon}(\alpha,-1)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s) = 0.$$

The proof of (1) is complete. For each $s \in [\upsilon-1, \upsilon+b+1]_{\mathbb{N}_{\upsilon-1}}$,

$$G(\upsilon+b+1, s) = \frac{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s) - e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s) = 0.$$

The proof of (2) is complete. Assume $(t,s) \in I_2$. It follows from Lemma 3.2(i) that

$$G(t,s) = \frac{e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s) > 0,$$

and from Lemma 3.2(iii) that for the first order backward difference of $G(t,s)$ with respect to t we have

$$\begin{aligned}
\Delta_t G(t,s) &= \Delta_t \left[\frac{e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s) \right]\\
&= \frac{\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)} e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s) > 0,
\end{aligned}$$

implying that $G(t,s)$ is an increasing function of t from $t=\upsilon-2$ to $t=s-1$. Assume $(t,s)\in I_1$. It follows from Lemma 3.2(iv) that for the first order backward difference of $G(t,s)$ with respect to t we have

$$\begin{aligned}\Delta_t G(t,s) &= \Delta_t\left[\frac{e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)}e_{\upsilon-\mu,\upsilon}(\alpha,\upsilon+b-s)-e_{\upsilon-\mu,\upsilon}(\alpha,t-s-1)\right]\\ &= \frac{\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)}e_{\upsilon-\mu,\upsilon}(\alpha,\upsilon+b-s)-\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha,t-s-1)\\ &= \frac{e_{\upsilon-\mu,\upsilon}(\alpha,\upsilon+b-s)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)}\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)-\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha,t-s-1)\\ &\le \Delta_t e_{\upsilon-\mu,\upsilon}(\alpha,t-\upsilon+1)-\Delta_t e_{\upsilon-\mu,\upsilon}(\alpha,t-s-1)<0,\end{aligned}$$

implying that $G(t,s)$ is a decreasing function of t from $t=s$ to $t=\upsilon+b+1$. Since $G(\upsilon+b+1,s)=0$ for each $s\in[\upsilon-1,\upsilon+b+1]_{\mathbb{N}_{\upsilon-1}}$, it follows that $G(t,s)>0$ for each $(t,s)\in[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}\times[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}$. The proof of (3) is complete. It follows from (3) that

$$\max_{t\in[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}} G(t,s)=\max\{G(s-1,s),G(s,s)\} \tag{3.2}$$

for all $s\in[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}$. Moreover,

$$G(s-1,s)=\frac{e_{\upsilon-\mu,\upsilon}(\alpha,s-\upsilon)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)}e_{\upsilon-\mu,\upsilon}(\alpha,\upsilon+b-s), \tag{3.3}$$

and

$$\begin{aligned}G(s,s) &= \frac{e_{\upsilon-\mu,\upsilon}(\alpha,s-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)}e_{\upsilon-\mu,\upsilon}(\alpha,\upsilon+b-s)-e_{\upsilon-\mu,\upsilon}(\alpha,-1)\\ &= \frac{e_{\upsilon-\mu,\upsilon}(\alpha,s-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)}e_{\upsilon-\mu,\upsilon}(\alpha,\upsilon+b-s)-0\\ &= \frac{e_{\upsilon-\mu,\upsilon}(\alpha,s-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha,b+2)}e_{\upsilon-\mu,\upsilon}(\alpha,\upsilon+b-s),\end{aligned}$$

for $s\in[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}$. It follows from Lemma 3.2(ii) that

$$G(s-1,s)<G(s,s),\quad s\in[\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}}.$$

The proof is complete. □

Lemma 3.4 ([19]). *Let $\{a_n\}_{n\in\mathbb{N}_0}$ and $\{b_n\}_{n\in\mathbb{N}_0}$ be real numbers and let the power series*

$$A(x)=\sum_{n=0}^{\infty}a_nx^n \quad \text{and} \quad B(x)=\sum_{n=0}^{\infty}b_nx^n$$

be convergent for $|x|<r$. Then, if $b_n>0$, $n=0,1,2,\ldots$, and the sequence $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}_0}$ is (strictly) increasing (decreasing), then the function $\frac{A(x)}{B(x)}$ is also (strictly) increasing (decreasing) on $[0,r)$.

Lemma 3.5. *Assume* $0 < \mu < 1$, $1 < \upsilon < 2$ *such that* $\upsilon - \mu - 1 \geq 0$, $\alpha \in (0, \alpha^*)$ *and* a, b *be two real numbers such that* $\upsilon < a \leq b$. *Then,*

$$\frac{e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon - 1 - s)}{e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon + b - s)}$$

is a decreasing function with respect to $s \in [0, \upsilon - 1)_{\mathbb{N}}$.

Proof. Take $s \in [0, \upsilon - 1)_{\mathbb{N}}$. We have

$$e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon - 1 - s) = \sum_{k=0}^{\infty} \alpha^k \frac{(\upsilon - 1 - s + k(\upsilon - \mu) + \upsilon - 1)^{(k(\upsilon-\mu)+\upsilon-1)}}{\Gamma(k(\upsilon - \mu) + \upsilon)},$$

and

$$e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon + b - s) = \sum_{k=0}^{\infty} \alpha^k \frac{(\upsilon + b - s + k(\upsilon - \mu) + \upsilon - 1)^{(k(\upsilon-\mu)+\upsilon-1)}}{\Gamma(k(\upsilon - \mu) + \upsilon)},$$

are convergent for $|\alpha| < 1$. Clearly,

$$\alpha^k \frac{(\upsilon + b - s + k(\upsilon - \mu) + \upsilon - 1)^{(k(\upsilon-\mu)+\upsilon-1)}}{\Gamma(k(\upsilon - \mu) + \upsilon)} > 0, \quad k = 0, 1, 2, \ldots$$

One can check that the sequence

$$\frac{(\upsilon - 1 - s + k(\upsilon - \mu) + \upsilon - 1)^{(k(\upsilon-\mu)+\upsilon-1)}}{(\upsilon + b - s + k(\upsilon - \mu) + \upsilon - 1)^{(k(\upsilon-\mu)+\upsilon-1)}}$$

is a decreasing function for $s \in [0, \upsilon - 1)_{\mathbb{N}}$. Then, by Lemma 3.4,

$$\frac{e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon - 1 - s)}{e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon + b - s)}$$

is a decreasing function with respect to $s \in [0, \upsilon - 1)_{\mathbb{N}}$. The proof is complete. □

Theorem 3.6. *Assume* $0 < \mu < 1$, $1 < \upsilon < 2$ *such that* $\upsilon - \mu - 1 \geq 0$ *and* $\alpha \in (0, \alpha^*)$. *Then, there exits* $\gamma \in (0, 1)$ *such that*

$$\min_{c \leq t \leq d} G(t, s) \geq \gamma \max_{t \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}} G(t, s) = \gamma G(s, s), \tag{3.4}$$

for $s \in [\upsilon - 1, \upsilon + b]_{\mathbb{N}_{\upsilon-1}}$. *Here* $c = \frac{b+\upsilon}{4}$ *and* $d = \frac{3(b+\upsilon)}{4}$.

Proof. It follows from Theorem 3.3 that

$$\max_{t \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}} G(t, s) = G(s, s),$$

for each $s \in [\upsilon - 1, \upsilon + b]_{\mathbb{N}_{\upsilon-1}}$. Consider now

$$\frac{G(t, s)}{G(s, s)} = \begin{cases} \frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, s-\upsilon+1)} - \frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-s-1) e_{\upsilon-\mu,\upsilon}(\alpha, b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha, s-\upsilon+1) e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)}, & (t, s) \in I_1, \\ \frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, s-\upsilon+1)}, & (t, s) \in I_2. \end{cases}$$

For $(t,s) \in I_2$ and $c \le t \le d$, we have

$$\frac{G(t,s)}{G(s,s)} = \frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, s-\upsilon+1)} \ge \frac{e_{\upsilon-\mu,\upsilon}(\alpha, c-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+1)}.$$

For $(t,s) \in I_1$, we know that $G(t,s)$ is decreasing with respect to t, hence we have that

$$\begin{aligned}\frac{G(t,s)}{G(s,s)} &= \frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, s-\upsilon+1)} - \frac{e_{\upsilon-\mu,\upsilon}(\alpha, t-s-1)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha, s-\upsilon+1)e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)} \\ &\ge \frac{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, s-\upsilon+1)} - \frac{e_{\upsilon-\mu,\upsilon}(\alpha, d-s-1)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha, s-\upsilon+1)e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)}.\end{aligned}$$

Now, for any $s \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}$, we define

$$\begin{aligned}\gamma(s) = &\frac{1}{e_{\upsilon-\mu,\upsilon}(\alpha, s-\upsilon+1)} \\ &\cdot \left[e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1) - \frac{e_{\upsilon-\mu,\upsilon}(\alpha, d-s-1)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)} \right].\end{aligned}$$

By Lemma 3.5,

$$\frac{e_{\upsilon-\mu,\upsilon}(\alpha, d-s-1)}{e_{\upsilon-\mu,\upsilon}(\alpha, \upsilon+b-s)}$$

is decreasing of with respect to s for $0 \le s < d-1$. Hence

$$\begin{aligned}\gamma(s) &\ge \frac{1}{e_{\upsilon-\mu,\upsilon}(\alpha, s-\upsilon+1)} \left[e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1) - \frac{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+1)} \right] \\ &\ge \frac{1}{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1)} \left[e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1) - \frac{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+1)} \right].\end{aligned}$$

So this implies

$$\begin{aligned}\frac{G(t,s)}{G(s,s)} \ge &\frac{1}{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1)} \\ &\cdot \left[e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1) - \frac{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+1)} \right].\end{aligned}$$

Thus, we have

$$\min_{c \le t \le d} G(t,s) \ge \gamma \max_{t \in [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}}} G(t,s) = \gamma G(s,s),$$

for $s \in [\upsilon - 1, \upsilon + b]_{\mathbb{N}_{\upsilon-1}}$, where

$$\gamma = \min\left\{\frac{1}{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1)}\left[e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1) - \frac{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+1)}\right], \frac{e_{\upsilon-\mu,\upsilon}(\alpha, c-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+1)}\right\}$$

$$= \min\left\{\left[1 - \frac{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1)e_{\upsilon-\mu,\upsilon}(\alpha, b+1)}\right], \frac{e_{\upsilon-\mu,\upsilon}(\alpha, c-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+1)}\right\}.$$

Since

$$\frac{e_{\upsilon-\mu,\upsilon}(\alpha, c-\upsilon+1)}{e_{\upsilon-\mu,\upsilon}(\alpha, b+1)} < 1$$

and

$$\frac{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon)e_{\upsilon-\mu,\upsilon}(\alpha, b+2)}{e_{\upsilon-\mu,\upsilon}(\alpha, d-\upsilon+1)e_{\upsilon-\mu,\upsilon}(\alpha, b+1)} > 0,$$

it is immediate to verify that $0 < \gamma < 1$. □

4. EXISTENCE OF SOLUTIONS OF NONLINEAR PROBLEMS

In this section we will apply the following Krasnosel'skii–Zabreiko fixed point theorem to obtain nontrivial solutions of the following nonlinear equation

$$-\Delta^{\upsilon} y(t) + \alpha\Delta^{\mu} y(t+\upsilon-\mu) = f(t+\upsilon-1, y(t+\upsilon-1)), \quad t \in I, \tag{4.1}$$

coupled to the boundary conditions (1.2).

Here we assume that $f : [\upsilon - 1, \upsilon + b]_{\mathbb{N}_{\upsilon-1}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Theorem 4.1 ([17]). *Let X be a Banach space and $F : X \to X$ be a completely continuous operator. If there exists a bounded linear operator $A : X \to X$ such that 1 is not an eigenvalue and*

$$\lim_{\|y\|\to\infty} \frac{\|F(y) - A(y)\|}{\|y\|} = 0,$$

then F has a fixed point in X.

We will apply Theorem 4.1 to a nonlinear summation operator whose kernel is $G(t, s)$. The arguments are in the line to the ones used in [14].

In this context, let the Banach space $(X, \|\cdot\|)$ be defined by

$$X := \{h : [\upsilon - 2, \upsilon + b + 1]_{\mathbb{N}_{\upsilon-2}} \to \mathbb{R}\}, \tag{4.2}$$

with norm

$$\|h\| := \max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}} |h(t)|\,. \tag{4.3}$$

Clearly, $y \in X$ is a fixed point of the completely continuous operator $F : X \to X$ defined by

$$(Fy)\,(t) := \sum_{s=\upsilon-1}^{\upsilon+b} G(t,s) f(s,y(s)), \quad t \in [\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}. \tag{4.4}$$

In order to ensure the existence of solutions of (4.1)–(1.2), we now apply Theorem 4.1 to operator F defined in (4.4) and to an associated linear operator.

Theorem 4.2. *Assume that $|\alpha| < 1$, $f : [\upsilon-1,\upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous and that for any $t \in I$ the following property holds:*

$$\lim_{|r|\to\infty} \frac{f(t+\upsilon-1,r)}{r} = m(t+\upsilon-1).$$

If

$$|m(t+\upsilon-1)| < d := \frac{1}{\displaystyle\max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}} \sum_{s=\upsilon-1}^{\upsilon+b} |G(t,s)|}, \quad \text{for all } t \in I,$$

then the boundary value problem (4.1)–(1.2) *has a solution y, and moreover, $y \not\equiv 0$ on $[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}$, when $f(t,0) \neq 0$ for at least one $t \in I$.*

Proof. Let be the Banach space $(X, \|\cdot\|)$ and the completely continuous operator $F : X \to X$ defined as above in (4.2), (4.3), and (4.4), respectively.

Corresponding to (4.1)–(1.2), we consider the following linear equation

$$-\Delta^{\upsilon} y(t) + \alpha \Delta^{\mu} y(t+\upsilon-\mu) = m(t+\upsilon-1)\, y(t+\upsilon-1), \quad t \in I, \tag{4.5}$$

coupled to the boundary conditions (1.2).

We define a completely continuous linear operator $A : X \to X$ by

$$(Ay)\,(t) := \sum_{s=\upsilon-1}^{\upsilon+b} G(t,s)\, m(s)\, y(s), \quad t \in [\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}.$$

Clearly, solutions of (4.5)–(1.2) are fixed points of A, and conversely.

First, we show that 1 is not an eigenvalue of A.

To see this, we consider two cases:

(a) $m(t+\upsilon-1) = 0$ for all $t \in I$, and
(b) $m(t+\upsilon-1) \neq 0$ for at least one $t \in I$.

For (a), if $m(t+\upsilon-1) = 0$ for all $t \in I$, since the boundary value problem (4.5)–(1.2) has only the trivial solution, it is immediate to verify that 1 is not an eigenvalue of A.

For (b), if $m(t+\upsilon-1)\neq 0$ for at least one $t\in I$ and (4.5)–(1.2) has a nontrivial solution, then $\|y\|>0$. And so, since $G(\cdot,s)$ is not identically zero on $[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}$, we have

$$\begin{aligned}\|y\| = \|Ay\| &= \max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}} \left|\sum_{s=\upsilon-1}^{\upsilon+b} G(t,s)\,m(s)\,y(s)\right| \\ &< d \max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}} \sum_{s=\upsilon-1}^{\upsilon+b} |G(t,s)|\,|y(s)| \\ &\leq d\,\|y\| \max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}} \sum_{s=\upsilon-1}^{\upsilon+b} |G(t,s)| = \|y\|\,,\end{aligned}$$

a contradiction.

Again, 1 is not an eigenvalue of A.

Our next claim is that

$$\lim_{\|y\|\to\infty} \frac{\|F(y)-A(y)\|}{\|y\|} = 0.$$

In this direction, let $\varepsilon>0$ be given. Now, for any $t\in I$ be given, we know that

$$\lim_{|r|\to\infty} \frac{f(t+\upsilon-1,r)}{r} = m(t+\upsilon-1),$$

which implies that there exists an $M(t+\upsilon-1)>0$ such that, for all $|r|>M(t+\upsilon-1)$,

$$|f(t+\upsilon-1,r)-m(t+\upsilon-1)\,r| < \varepsilon\,|r|\,. \tag{4.6}$$

Now, let

$$N_1 = \max_{t\in I}\{M(t+\upsilon-1)\} > 0,$$

$$N = \max_{|r|\leq N_1,\, t\in I}\{|f(t+\upsilon-1,r)|\},$$

and let $L\geq N_1$ be such that

$$\frac{N+d\,N_1}{L} < \varepsilon.$$

Next, choose $y\in X$ with $\|y\|>L$. Now, for $s\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}$, if $|y(s)|\leq N_1$, we have

$$|f(s,y(s))-m(s)\,y(s)| \leq |f(s,y(s))| + d\,|y(s)| \leq N + d\,N_1 < \varepsilon L < \varepsilon\,\|y\|\,.$$

On the other hand, if $|y(s)|>N_1$, we have from (4.6) that

$$|f(s,y(s))-m(s)\,y(s)| < \varepsilon|y(s)| \leq \varepsilon\,\|y\|\,.$$

Thus, for all $s \in [\upsilon-2, \upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}$, we deduce that

$$|f(s, y(s)) - m(s)\, y(s)| \leq \varepsilon \,\|y\| \,. \tag{4.7}$$

It follows from (4.7) that, for $y \in X$ with $\|y\| > L$,

$$\begin{aligned}\|F(y) - A(y)\| &= \max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}} \left| \sum_{s=\upsilon-1}^{\upsilon+b} G(t,s)[f(s,y(s)) - m(s)\,y(s)] \right| \\ &\leq \max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}} \sum_{s=\upsilon-1}^{\upsilon+b} |G(t,s)|\,|f(s,y(s)) - m(s)\,y(s)| \\ &\leq \varepsilon \,\|y\| \max_{t\in[\upsilon-2,\upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}} \sum_{s=\upsilon-1}^{\upsilon+b} |G(t,s)| = \varepsilon\,\|y\|\,\frac{1}{d}.\end{aligned}$$

Therefore,

$$\lim_{\|y\|\to\infty} \frac{\|F(y) - A(y)\|}{\|y\|} = 0.$$

By Theorem 4.1, F has a fixed point $y \in X$, and y is a desired solution of (4.1), (1.2). Moreover $y \not\equiv 0$ on $[\upsilon-2, \upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}$, when $f(t,0) \neq 0$ for at least one $t \in I$ and the proof is complete. □

Remark 4.3. Notice that under the hypotheses of previous existence result we cannot ensure that the Green's function G has constant sign on its square of definition.

5. MULTIPLICITY RESULTS

In this section, for completeness of our work, we will briefly establish some results concerning the existence of at least two nontrivial solutions of the nonlinear Dirichlet Problem (4.1)–(1.2), based on Krasnosel'skii's fixed point theorem.

Assume that $f : [\upsilon-1, \upsilon+b]_{\mathbb{N}_{\upsilon-1}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f \not\equiv 0$.

Let B represents all maps from $[\upsilon-2, \upsilon+b]_{\mathbb{N}_{\upsilon-2}}$ into $\mathbb{R}$, equipped with the standard maximum norm $\|\cdot\|$. Clearly, B is a Banach space. Define the cone

$$K := \left\{ y \in B \,:\, y(t) \geq 0, \;\; \min_{t\in[c,d]} y(t) \geq \gamma \,\|y\| \right\},$$

where c, d and γ are defined in Theorem 3.6 and operator T is given by

$$(Ty)\,(t) := \sum_{s=0}^{b+1} G(t,s) f(s+\upsilon-1, y(s+\upsilon-1)), \quad t \in [\upsilon-2, \upsilon+b+1]_{\mathbb{N}_{\upsilon-2}}.$$

It is trivial to show that $T : K \to K$. Denote

$$f_0 = \lim_{y \to 0^+} \min_{t \in [\upsilon-2,\upsilon+b]_{\mathbb{N}_{\upsilon-2}}} \frac{f(t,y)}{y}, \quad f^0 = \lim_{y \to 0^+} \max_{t \in [\upsilon-2,\upsilon+b]_{\mathbb{N}_{\upsilon-2}}} \frac{f(t,y)}{y},$$

$$f_\infty = \lim_{y \to \infty} \min_{t \in [\upsilon-2,\upsilon+b]_{\mathbb{N}_{\upsilon-2}}} \frac{f(t,y)}{y}, \quad f^\infty = \lim_{y \to \infty} \max_{t \in [\upsilon-2,\upsilon+b]_{\mathbb{N}_{\upsilon-2}}} \frac{f(t,y)}{y}.$$

Set

$$\eta = \frac{1}{\sum\limits_{s=\upsilon-1}^{\upsilon+b} G(s,s)} \quad \text{and} \quad \mu = \frac{1}{\gamma \sum_{s=\lceil \frac{\upsilon+b}{4} - \upsilon + 1 \rceil}^{\lfloor \frac{3(\upsilon+b)}{4} - \upsilon + 1 \rfloor} G\left(\lfloor \frac{b+1}{2} \rfloor + \upsilon, s\right)},$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the usual so-called floor and ceiling functions. Moreover, assume that the following conditions hold:

(H1) There exists $p > 0$ such that $f(t,y) < \eta p$ for all $0 \le y \le p$ and $t \in [\upsilon-2,\upsilon+b]_{\mathbb{N}_{\upsilon-2}}$.
(H2) There exists $p > 0$ such that $f(t,y) > \mu p$ for all $\gamma p \le y \le p$ and $t \in [c,d]$.
(H3) $f_0 > \mu$ and $f_\infty > \mu$.
(H4) $f^0 < \eta$ and $f^\infty < \eta$.

Our main result in this section is as follows:

Theorem 5.1. *If f satisfies* (H1) *and* (H3), *then the nonlinear Dirichlet Problem* (4.1)–(1.2) *has at least two positive solutions y_1 and y_2 with* $0 < \|y_1\| < p \le \|y_2\|$.

Proof. We omit the proof since the arguments are standard for Green's function G, satisfying (3.4). We refer [12] to the reader for more details. □

Theorem 5.2. *Suppose that f satisfies* (H2), (H4) *and $f > 0$ for $t \in [\upsilon-2,\upsilon+b]_{\mathbb{N}_{\upsilon-2}}$. Then, the nonlinear Dirichlet Problem* (4.1)–(1.2) *has at least two positive solutions y_1 and y_2 with* $0 < \|y_1\| < p < \|y_2\|$.

Proof. Again, we omit the proof. One can check [12] for more details. □

Remark 5.3. We point out that one can take weaker conditions, namely $f^0 = 0$ and $f^\infty = 0$ and the above results still hold.

As a direct consequence, one can obtain the following results

Theorem 5.4. *Suppose that $f_0 > \mu$ and $f^\infty < \eta$, then the nonlinear Dirichlet Problem* (4.1)–(1.2) *has at least one positive solution.*

Theorem 5.5. *Suppose that $f_\infty > \mu$ and $f^0 < \eta$, then the nonlinear Dirichlet Problem* (4.1)–(1.2) *has at least one positive solution.*

6. EXAMPLES

In this section, we provide two examples to demonstrate the applicability of established results.

Example 6.1. *Consider the boundary value problem* (4.1), (1.2) *with* $a = 0$, $b = 5$, $\upsilon = 1.5$, $\mu = 0.5$, $\alpha = 0.5$ *and*

$$f(t+\upsilon-1, r) = C\, e^{-(t+\upsilon-1)^2}\, r\ \left|\tan^{-1}\left((t+\upsilon-1)^2 (r+1)^3\right)\right| + e^{(t+\upsilon-1)^2}\sqrt{|r+1|},$$

with $C > 0$. *Clearly,*

$$m(t+\upsilon-1) = \lim_{|r|\to\infty} \frac{f(t+\upsilon-1, r)}{r} = \frac{C\,\pi}{2}\, e^{-(t+\upsilon-1)^2} \qquad \text{for all } t \in I.$$

The Green's function associated with the boundary value problem is given by

$$G(t,s) = \begin{cases} \frac{e_{1,1.5}(0.5,t-0.5)}{e_{1,1.5}(0.5,7)} e_{1,1.5}(0.5, 6.5-s) - e_{1,1.5}(0.5, t-s-1), & (t,s) \in I_1, \\ \frac{e_{1,1.5}(0.5,t-0.5)}{e_{1,1.5}(0.5,7)} e_{1,1.5}(0.5, 6.5-s), & (t,s) \in I_2, \end{cases} \tag{6.1}$$

where

$$I_1 = \{(t,s) : 0.5 \le s \le t \le 7.5\},$$

and

$$I_2 = \{(t,s) : 0.5 \le t+1 \le s \le 6.5\}.$$

Since

$$d = \frac{1}{\max\limits_{t\in[-0.5,7.5]_{\mathbb{N}_{-0.5}}} \sum\limits_{s=0.5}^{6.5} |G(t,s)|} = 0.241342,$$

by Theorem 4.2, we can ensure that the boundary value problem has a nontrivial solution defined on $[-0.5, 7.5]_{\mathbb{N}_{-0.5}}$ *for all*

$$0 < C < \frac{2\,d}{\pi}\, e^{0.25} \approx 0.197282.$$

Example 6.2. *Consider the boundary value problem* (4.1), (1.2) *with* $a = 0$, $b = 5$, $\upsilon = 1.5$, $\mu = 0.5$, $\alpha = 0.5$ *and*

$$f(t+\upsilon-1, y) = c\left[[y(t+\upsilon-1)]^{\frac{1}{2}} + [y(t+\upsilon-1)]^{\frac{3}{2}}\right]$$

with $c < \frac{\eta}{2}$. *Here*

$$\eta = \frac{1}{\sum\limits_{s=0.5}^{6.5} G(s,s)}.$$

Clearly, $f_0 = f_\infty = +\infty$. *Taking* $p = 1$, *we have that*

$$f(t,y) = c\left[y^{\frac{1}{2}} + y^{\frac{3}{2}}\right] \le c\left[p^{\frac{1}{2}} + p^{\frac{3}{2}}\right] < \eta p$$

for all $0 \leq y \leq p$ *and* $t \in [-0.5, 7.5]_{\mathbb{N}_{-0.5}}$. *Thus, all conditions in Theorem 5.1 are satisfied. Therefore, the nonlinear Dirichlet Problem* (4.1)–(1.2) *has at least two positive solutions* y_1 *and* y_2 *with* $0 < \|y_1\| < p \leq \|y_2\|$.

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