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EVERY GRAPH IS LOCAL ANTIMAGIC TOTAL AND ITS APPLICATIONS

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Abstract. Let $G = (V, E)$ be a simple graph of order p and size q. A graph G is called local antimagic (total) if G admits a local antimagic (total) labeling. A bijection $g : E \to \{1, 2, \ldots, q\}$ is called a local antimagic labeling of G if for any two adjacent vertices u and v, we have $g^+(u) \neq g^+(v)$, where $g^+(u) = \sum_{e\in E(u)} g(e)$, and $E(u)$ is the set of edges incident to u. Similarly, a bijection $f : V(G)\cup E(G) \to \{1, 2, \ldots, p+q\}$ is called a local antimagic total labeling of G if for any two adjacent vertices u and v, we have $w_f(u) \neq w_f(v)$, where $w_f(u) = f(u) + \sum_{e\in E(u)} f(e)$. Thus, any local antimagic (total) labeling induces a proper vertex coloring of G if vertex v is assigned the color $g^+(v)$ (respectively, $w_f(u)$). The local antimagic (total) chromatic number, denoted $\chi_{la}(G)$ (respectively $\chi_{lat}(G)$), is the minimum number of induced colors taken over local antimagic (total) labeling of G. We provide a short proof that every graph G is local antimagic total. The proof provides sharp upper bound to $\chi_{lat}(G)$. We then determined the exact $\chi_{lat}(G)$, where G is a complete bipartite graph, a path, or the Cartesian product of two cycles. Consequently, the $\chi_{la}(G \vee K_1)$ is also obtained. Moreover, we determined the $\chi_{la}(G \vee K_1)$ and hence the $\chi_{lat}(G)$ for a class of 2-regular graphs G (possibly with a path). The work of this paper also provides many open problems on $\chi_{lat}(G)$. We also conjecture that each graph G of order at least 3 has $\chi_{lat}(G) \leq \chi_{la}(G)$.

Keywords: local antimagic (total) chromatic number, Cartesian product, join product.

Mathematics Subject Classification: 05C78, 05C15.

1. INTRODUCTION

Consider a (p, q)-graph $G = (V, E)$ of order p and size q. In this paper, all graphs are simple. For positive integers $a < b$, let $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Let $g : E(G) \to [1, q]$ be a bijective edge labeling that induces a vertex labeling $g^+ : V(G) \to \mathbb{N}$ such that $g^+(v) = \sum_{uv\in E(G)} g(uv)$. We say g is a *local antimagic* *labeling* of G if $g^+(u) \neq g^+(v)$ for each $uv \in E(G)$ [1,2]. The number of distinct colors induced by g is called the *color number* of g and is denoted by $c(g)$. The number

$$\chi_{la}(G) = \min\{c(g) \mid g \text{ is a local antimagic labeling of } G\}$$

is called the *local antimagic chromatic number* of G [1]. Clearly, $\chi_{la}(G) \geq \chi(G)$.

Let $f : V(G) \cup E(G) \to [1, p+q]$ be a bijective total labeling that induces a vertex labeling $w_f : V(G) \to \mathbb{N}$, where

$$w_f(u) = f(u) + \sum_{uv \in E(G)} f(uv)$$

and is called the *weight* of u for each vertex $u \in V(G)$. We say f is a *local antimagic total labeling* of G (and G is *local antimagic total*) if $w_f(u) \neq w_f(v)$ for each $uv \in E(G)$. Clearly, w_f corresponds to a proper vertex coloring of G if each vertex v is assigned the color $w_f(v)$. If no ambiguity, we shall drop the subscript f. Let $w(f)$ be the number of distinct vertex weights induced by f. The number

$$\min\{w(f) \mid f \text{ is a local antimagic total labeling of } G\}$$

is called the *local antimagic total chromatic number* of G, denoted $\chi_{lat}(G)$. Clearly, $\chi_{lat}(G) \geq \chi(G)$. It is well known that determining the chromatic number of a graph G is NP-hard [12]. Thus, in general, it is also very difficult to determine $\chi_{la}(G)$ and $\chi_{lat}(G)$.

Let $G \vee H$ be the *join* of G and H with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. The *Cartesian product* of G and H, denoted $G \times H$, has $V(G \times H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$ and two vertices (u, v) and (u', v') are adjacent if and only if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. Let $G + H$ be the disjoint union of G and H with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For convenience, nG denotes the disjoint union of $n \geq 1$ copies of G, and $nK_1 = O_n$, the null graph of order n. If g (respectively f) induces t distinct colors, we say g (respectively f) is a *local antimagic* (*total*) *t-coloring* of G. We refer to [3] for notation not defined in this paper.

In [4], the author proved that every connected graph of order at least 3 is local antimagic. Using this result, we provide in Section 2 a very short proof that every graph is local antimagic total. Sharp bounds of $\chi_{lat}(G)$ are found. We then determined the $\chi_{lat}(G)$ where G is a path P_n of order $n \geq 2$, or $C_n \times C_n$ where C_n is a cycle of order $n \geq 3$. Consequently, we also obtained $\chi_{la}(G \vee K_1)$. Many open problems are also proposed for further research.

2. SHARP BOUNDS

By definition, $\chi_{lat}(G + O_n) \geq n$ and $\chi_{lat}(O_n) = n$. Since $\chi_{la}(K_n) = n$, it is easy to conclude that $\chi_{lat}(K_n) = n$. In what follows, we only consider nonempty graphs.

Theorem 2.1. *Every graph G is local antimagic total.*

Proof. Suppose G is a (p,q)-graph. Let the vertex sets of G and K_1 be $V(G)=\{v_i \mid 1\le i\le p\}$ and $V(K_1)=\{v\}$, respectively. It is obvious that each graph G of order $p\le 3$ are local antimagic total. We now assume G is of order $p\ge 4$. In [4], the author proved that every graph without isolated edges (by definition, necessarily without isolated vertices) admits a local antimagic labeling. Thus, $G\vee K_1$ is local antimagic. Let g be a local antimagic labeling of $G\vee K_1$. Define a total labeling $f: V(G)\cup E(G)\to[1,p+q]$ of G by $f(e)=g(e)$ for each edge $e\in E(G)$ and $f(v_i)=g(vv_i)$. Clearly, $w_f(v_i)=g^+(v_i)$. Thus, $w_f(v_i)=w_f(v_j)$ if and only if $g^+(v_i)=g^+(v_j)$. Therefore, f is a local antimagic total labeling of G. □

The next theorem shows that $\chi_{lat}(G)$ can be arbitrarily large for a graph G with small $\chi(G)$.

Theorem 2.2. *If $G=K_2+O_n, n\ge 1$, then*

$$\chi_{lat}(G)=\begin{cases}2 & \text{for } n=1,2,\\ n & \text{otherwise.}\end{cases}$$

Proof. Let $V(G)=\{u_1,u_2\}\cup\{v_i \mid 1\le i\le n\}$. Define $f(u_i)=i$, $f(u_1u_2)=3$ and $f(v_i)=i+3$, $1\le i\le n$. We now have $w_f(u_1)=4$, $w_f(u_2)=5$ and $w_f(v_i)=i+3$. Thus, $\chi_{lat}(G)\le 2$ for $n=1,2$, and $\chi_{lat}(G)\le n$ for $n\ge 3$. By definition, $\chi_{lat}(G)\ge\chi(G)=2$ and since all the isolated vertices must have distinct weights, this implies that $\chi_{lat}(G)\ge n$. So, the theorem holds. □

Theorem 2.3. *Let G be a graph of order $p\ge 2$ and size q with $V(G)=\{v_i \mid 1\le i\le p\}$.*

(a) $\chi(G)\le\chi_{lat}(G)\le\chi_{la}(G\vee K_1)-1$.
(b) *Suppose f is local antimagic total $\chi_{lat}(G)$-coloring. If $\sum_{i=1}^{p} f(v_i)\ne w_f(v_j)$, $1\le j\le p$, then $\chi_{la}(G\vee K_1)=\chi_{lat}(G)+1$.*

Proof. (a) Suppose $\chi_{la}(G\vee K_1)=c$. From the proof of Theorem 2.1, we know that every local antimagic labeling of $G\vee K_1$ that induces c distinct vertex labels corresponds to a local antimagic total labeling of G that induces $c-1$ distinct vertex weights. Thus, $\chi_{lat}(G)\le c-1$.

(b) Let $\chi_{lat}(G)=a$. Define $g: E(G\vee K_1)\to[1,p+q]$ by $g(e)=f(e)$ if $e\in E(G)$, and $g(vv_i)=f(v_i)$ for each $v_i\in V(G)$. Clearly, $g^+(v)=\sum_{i=1}^{p} f(v_i)$ and $g^+(v_i)=w_f(v_i)$. Since $w_f(v_i)\ne w_f(v_j)$ if $v_iv_j\in E(G)$ and $g^+(v)\ne w_f(v_j)$ for $1\le j\le p$, g is a local antimagic $(a+1)$-coloring of G. Hence, $\chi_{la}(G\vee K_1)\le a+1$. By (a), $\chi_{la}(G\vee K_1)\ge\chi_{lat}(G)+1$. Thus, we have $\chi_{la}(G\vee K_1)=\chi_{lat}(G)+1$. □

Theorem 2.4. *Let G be a (p,q)-graph, $p\ge 2$ and $q\ge 1$. Let $V(G)=\{v_i \mid 1\le i\le p\}$, $V(K_1)=\{u_1\}$ and $V(2K_1)=\{u_1,u_2\}$. Suppose g is a local antimagic labeling of $G\vee 2K_1$ that induces a minimum number of vertex labels and $2p+q+1+g^+(u_1)\ne g^+(v_i)$, $1\le i\le p$, then*

$$\chi(G\vee K_1)\le\chi_{lat}(G\vee K_1)\le\begin{cases}\chi_{la}(G\vee 2K_1) & \text{if } g^+(u_1)=g^+(u_2),\\ \chi_{la}(G\vee 2K_1)-1 & \text{if } g^+(u_1)\ne g^+(u_2).\end{cases}$$

Proof. Define a bijection $f : V(G \vee K_1) \cup E(G \vee K_1) \to [1, 2p + q + 1]$ such that for $1 \le i < j \le p$, $f(v_iv_j) = g(v_iv_j)$ if $v_iv_j \in E(G)$, $f(u_1v_i) = g(u_1v_i)$, $f(v_i) = g(u_2v_i)$, and $f(u_1) = 2p + q + 1$. Clearly, $w_f(v_i) = g^+(v_i)$ and $w_f(u_1) = 2p + q + 1 + g^+(u_1)$. Since $2p + q + 1 + g^+(u_1) \neq g^+(v_i)$ for $1 \le i \le p$, f is a local antimagic total labeling of $G \vee K_1$ that induces $\chi_{la}(G \vee 2K_1)$ distinct vertex weights if $g^+(u_1) = g^+(u_2)$, and induces $\chi_{la}(G \vee 2K_1) - 1$ distinct vertex weights if $g^+(u_1) \neq g^+(u_2)$. □

The following lemmas are analogous to Lemmas 2.2–2.5 in [9].

Lemma 2.5. *Suppose G is a d-regular graph of order p and size q with an edge e. If f is a local antimagic total labeling of G, then $g = p + q + 1 - f$ is also a local antimagic total labeling of G with $w(g) = w(f)$. Moreover, suppose $f(e) = 1$ or $f(e) = p + q$, then $\chi(G - e) \le \chi_{lat}(G - e) \le \chi_{lat}(G)$*

Proof. Let $x, y \in V(G)$. Here,

$$w_g(x) = (d + 1)(p + q + 1) - w_f(x) \quad \text{and} \quad w_g(y) = (d + 1)(p + q + 1) - w_f(y).$$

Therefore, $w_f(x) = w_f(y)$ if and only if $w_g(x) = w_g(y)$. Thus, g is also a local antimagic total labeling of G with $w(g) = w(f)$.

If $f(e) = p+q$, then we may consider $g = p+q+1-f$. So without loss of generality, we may assume that $f(e) = 1$. Define $h : V(G - e) \cup E(G - e) \to [1, p + q - 1]$ such that $h(x) = f(x) - 1$ and $h(xy) = f(xy) - 1$ for $xy \neq e$. So, $w_h(x) = w_f(x) - d - 1$ for each vertex x of $G - e$. Therefore, $w_f(x) = w_f(y)$ if and only if $w_h(x) = w_h(y)$. Thus, h is also a local antimagic total labeling of G with $w(h) = w(f)$. Consequently, $\chi(G - e) \le \chi_{lat}(G - e) \le \chi_{lat}(G)$. The theorem holds. □

Note that if G is a regular edge-transitive graph, then $\chi_{lat}(G - e) \le \chi_{lat}(G)$.

Lemma 2.6. *Suppose G is a graph of order p and size q and f is a local antimagic total labeling of G. For any $x, y \in V(G)$, if*

(i) *$w_f(x) = w_f(y)$ implies that $\deg(x) = \deg(y)$, and*
(ii) *$w_f(x) \neq w_f(y)$ implies that $(p + q + 1)(\deg(x) - \deg(y)) \neq w_f(x) - w_f(y)$,*

then $g = p + q + 1 - f$ is also a local antimagic total labeling of G with $w(g) = w(f)$.

Proof. For any $x, y \in V(G)$, we have

$$w_g(x) = (\deg(x)+1)(p+q+1)-w_f(x) \quad \text{and} \quad w_g(y) = (\deg(y)+1)(p+q+1)-w_f(y).$$

If $w_f(x) = w_f(y)$, then condition (i) implies that $w_g(x) = w_g(y)$. If $w_f(x) \neq w_f(y)$, then condition (ii) implies that $w_g(x) \neq w_g(y)$. Thus, g is also a local antimagic total labeling of G with $w(g) = w(f)$. □

For $t \geq 2$, consider the following conditions for a graph G.

(i) $\chi_{lat}(G) = t$ and f is a local antimagic total labeling of G that induces a t-independent partition $\bigcup_{i=1}^{t} V_i$ of $V(G)$.
(ii) For each $x \in V_k$, $1 \leq k \leq t$, $\deg(x) = d_k$ satisfying $w_f(x) - d_a \neq w_f(y) - d_b$, where $x \in V_a$ and $y \in V_b$ for $1 \leq a < b \leq t$.
(iii) There exist two non-adjacent vertices u, v with $u \in V_i$, $v \in V_j$ for some $1 \leq i \neq j \leq t$ such that
  (a) $|V_i| = |V_j| = 1$ and $\deg(x) = d_k$ for $x \in V_k, 1 \leq k \leq t$; or
  (b) $|V_i| = 1$, $|V_j| \geq 2$ and $\deg(x) = d_k$ for $x \in V_k, 1 \leq k \leq t$ except that $\deg(v) = d_j - 1$; or
  (c) $|V_i|, |V_j| \geq 2$ and $\deg(x) = d_k$ for $x \in V_k$, $1 \leq k \leq t$ except that $\deg(u) = d_i - 1$, $\deg(v) = d_j - 1$,

  each satisfying $w_f(x) + d_a \neq w_f(y) + d_b$, where $x \in V_a$ and $y \in V_b$ for $1 \leq a \neq b \leq t$.

Lemma 2.7. *Let H be obtained from G with an edge e deleted. If G satisfies conditions* (i) *and* (ii) *and $f(e) = 1$, then $\chi(H) \leq \chi_{lat}(H) \leq t$.*

Proof. By definition, we have the lower bound. Define $g : E(H) \to [1, |E(H)|]$ such that $g(e') = f(e') - 1$ for each $e' \in E(H)$. Observe that g is a bijection with $w_g(x) = w_f(x) - d_k - 1$ for each $x \in V_k, 1 \leq k \leq t$. Thus, $w_g(x) = w_g(y)$ if and only if $x, y \in V_k, 1 \leq k \leq t$. Therefore, g is a local antimagic total labeling of H with $w(g) = w(f)$. Thus, $\chi_{lat}(H) \leq t$. □

Lemma 2.8. *Suppose $uv \in E(G)$. Let H be obtained from G with an edge uv added. If G satisfies conditions* (i) *and* (iii)*, then $\chi(H) \leq \chi_{lat}(H) \leq t$.*

Proof. By definition, we have the lower bound. Define $g : E(H) \to [1, |E(H)|]$ such that $g(uv) = 1$ and $g(e) = f(e) + 1$ for $e \in E(G)$. Observe that g is a bijection with $w_g(x) = w_f(x) + d_k + 1$ for each $x \in V_k, 1 \leq k \leq t$. Thus, $w_g(x) = w_g(y)$ if and only if $x, y \in V_k, 1 \leq k \leq t$. Therefore, g is a local antimagic total labeling of H with $w(g) = w(f)$. Thus, $\chi_{lat}(H) \leq t$. □

3. COMPLETE BIPARTITE GRAPHS, PATHS AND CYCLES

In [8, Theorems 2.7, 2.8, 2.10, 2.11], the authors showed that $(1, p, q)$-board is tri-magic for all $1 \leq p < q$. This is equivalent to $\chi_{la}(K_{1,p,q}) = 3$ for $1 \leq p < q$, where $K_{1,p,q}$ is the complete tripartite graph, i.e., $K_{p,q} \vee K_1$. By Theorem 2.3, we have $\chi_{lat}(K_{p,q}) = 2$ for $1 \leq p < q$.

Theorem 3.1. *For $1 \leq p \leq q$, $\chi_{lat}(K_{p,q}) = 2$.*

Proof. We only need to consider $p = q$. If $p = 1$, then $\chi_{lat}(K_{1,1}) = 2$ is obvious.

For $p = 2n$, we may make use of the matrix constructed at the proof of [7, Theorem 11]. For easy reading, we copy the construction here.

Suppose $p + 1 = 2n + 1 \geq 3$. Consider the $(2n + 1) \times (2n + 1)$ magic square A constructed by Siamese method.

Starting from the $(1, n+1)$-entry (i.e. $A_{1,n+1}$) with the number 1, the fundamental movement for filling the entries is diagonally up and right, one step at a time. When a move would leave the matrix, it is wrapped around to the last row or first column, respectively. If a filled entry is encountered, one moves vertically down one box instead, then continuing as before. One may find the detail in [5].

For convenience, let $k = p + 1$. Note that each of the ranges $[1, k]$, $[k + 1, 2k]$, ..., $[k^2 - k + 1, k^2]$ occupies a diagonal of the matrix, wrapping at the edges. Namely, the range $[1, k]$ starts at $A_{1,n+1}$ and ends at $A_{2,n}$; the range $[k+1, 2k]$ starts at $A_{3,n}$ ends at $A_{4,n-1}$; the range $[2k + 1, 3k]$ starts at $A_{5,n-1}$ and ends at $A_{6,n-2}$, etc. In general, the range $[ik+1, (i+1)k]$ starts at $A_{2i+1,n+1-i}$ and ends at $A_{2i+2,n-i}$, where $0 \le i \le k-1$ and the indices are taken modulo k. It is easy to see that the $(n + 1)$-st column of A is $(1, k + 2, \dots, k^2)$ which is an arithmetic sequence with common difference $k + 1$.

We now perform the following steps:

(1) Move each entry of the $(n+1)$-st column one position down (the $(p+1, n+1)$-entry becomes the $(1, n+1)$-entry). Note that each column sum is still the magic number $\frac{1}{2}k(k^2 + 1)$. The first row sum is now $\frac{1}{2}k(k^2 + 1) + k^2 - 1$ while each remaining row sum is $\frac{1}{2}k(k^2 + 1) - k - 1$.
(2) Exchange the $(n+1)$-st column and the $(p+1)$-st column. Now, the $(1, p+1)$-entry of B is $(p + 1)^2$.
(3) Replace the entry $(p + 1)^2$ by $*$. Let this matrix be B.
(4) Move every row of B one position up (the first row becomes the last row). Let this matrix be M.

Thus, M is an augmented bipartite labeling matrix of a local antimagic total labeling of $K_{p,p}$ with local antimagic total chromatic number 2.

For $p = 2n + 1$, let A be the magic square of order p constructed by Siamese method (as above). So, the anti-diagonal of A is

$$(A_{p,1}, A_{p-1,2}, \dots, A_{1,p}) = \left(\frac{p(p-1)}{2} + 1, \frac{p(p-1)}{2} + 2, \dots, \frac{p(p-1)}{2} + p\right).$$

Let B be a matrix obtained from A by exchanging the $(p - i + 1, i)$-entry with $(i, p - i + 1)$-entry, for $1 \le i \le \frac{p-1}{2}$. Then the i-row sum is $\frac{p(p^2+1)}{2} - (p + 1) + 2i$, $1 \le i \le p$; and the j-column sum is $\frac{p(p^2+1)}{2} + (p + 1) - 2j$, $1 \le j \le p$.

Let

$$R = (p^2 + 1, p^2 + 3, \dots, p^2 + 2p - 1, *)$$

be a row vector of length $p + 1$ and

$$C = (p^2 + 2p, p^2 + 2p - 2, \dots, p^2 + 2, *)^T$$

be a column vector of length $p + 1$. Now let M be a $(p + 1) \times (p + 1)$ matrix obtained from B by adding C at the rightmost of B and R at the bottom of B. Now, each row sum is $\frac{p(p^2+1)}{2} + p^2 + p + 1$ and each column sum is $\frac{p(p^2+1)}{2} + p^2 + p$. Hence, M is an augmented bipartite labeling matrix of a local antimagic total labeling of $K_{p,p}$ with local antimagic total chromatic number 2. □

Example 3.2. Suppose $p = 4$. We have the following magic square of order 5:

$$A = \begin{pmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{pmatrix} \to \begin{pmatrix} 17 & 24 & 25 & 8 & 15 \\ 23 & 5 & 1 & 14 & 16 \\ 4 & 6 & 7 & 20 & 22 \\ 10 & 12 & 13 & 21 & 3 \\ 11 & 18 & 19 & 2 & 9 \end{pmatrix}$$

$$\xrightarrow{C_3 \leftrightarrow C_5} \begin{pmatrix} 17 & 24 & 15 & 8 & 25 \\ 23 & 5 & 16 & 14 & 1 \\ 4 & 6 & 22 & 20 & 7 \\ 11 & 18 & 3 & 2 & 13 \\ 10 & 12 & 9 & 21 & 19 \end{pmatrix}.$$

Now

$$B = \begin{pmatrix} 17 & 24 & 15 & 8 & * \\ 23 & 5 & 16 & 14 & 1 \\ 4 & 6 & 22 & 20 & 7 \\ 11 & 18 & 3 & 2 & 13 \\ 10 & 12 & 9 & 21 & 19 \end{pmatrix} \quad \text{and} \quad M = \left(\begin{array}{cccc|c} 23 & 5 & 16 & 14 & 1 \\ 4 & 6 & 22 & 20 & 7 \\ 11 & 18 & 3 & 2 & 13 \\ 10 & 12 & 9 & 21 & 19 \\ \hline 17 & 24 & 15 & 8 & * \end{array}\right)$$

is an augmented bipartite labeling matrix of a local antimagic total labeling of $K_{4,4}$ with local antimagic total chromatic number 2. Note that the first 4 rows sum are 59 and first 4 column sums are 65.

Suppose $p = 5$. We still use the magic square of order 5:

$$A = \begin{pmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{pmatrix} \to \begin{pmatrix} 17 & 24 & 1 & 8 & 11 \\ 23 & 5 & 7 & 12 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 14 & 19 & 21 & 3 \\ 15 & 18 & 25 & 2 & 9 \end{pmatrix} = B.$$

$$M = \left(\begin{array}{ccccc|c} 17 & 24 & 1 & 8 & 11 & 35 \\ 23 & 5 & 7 & 12 & 16 & 33 \\ 4 & 6 & 13 & 20 & 22 & 31 \\ 10 & 14 & 19 & 21 & 3 & 29 \\ 15 & 18 & 25 & 2 & 9 & 27 \\ \hline 26 & 28 & 30 & 32 & 34 & * \end{array}\right)$$

is an augmented bipartite labeling matrix of a local antimagic total labeling of $K_{5,5}$ with local antimagic total chromatic number 2. Note that the first 5 rows sum are 96 and first 5 column sums are 95.

Let $P_n = v_1 v_2 \ldots v_n$ be the path of order $n \geq 2$. Let $F_n = P_n \vee K_1$ be the fan graph, $n \geq 2$. Obviously $\chi_{la}(F_2) = 3$. Combining with the results in [9, Theorems 3.5, 3.6 and 3.7] we have

Theorem 3.3. *For $n \geq 2$, $\chi_{la}(F_n) = 3$ for even n with $n \neq 4$, $\chi_{la}(F_4) = 4$ and $3 \leq \chi_{la}(F_n) \leq 4$ for odd n.*

We shall improve this theorem in Corollary 3.6. We note that in [9, Theorem 3.7], the authors also stated that $\chi_{la}(W_m - e) = 4$ for odd $m \geq 9$ and $e \notin E(C_m)$. However, the proof for $\chi_{la}(W_m - e) \geq 4$ was incomplete. We make a supplement here: Since $m \geq 9$ is odd, vertices of C_m must consist of at least 3 distinct induced vertex labels under any local antimagic labeling f of $U_m = W_m - e$. Let v be the central vertex of U_m that has degree $m-1$. So its induced vertex label is at least $m(m-1)/2$. Now, the only degree 2 vertex in C_m, say x, has induced vertex label at most $2q - 1 = 4m - 3$. Since $m(m-1) - 2(4m-3) = m^2 - 9m + 6 > 0$ for $m \geq 9$, we have $f^+(v) > f^+(x)$. Since f^+ is a coloring, $f^+(v) \neq f^+(u)$ for every vertex $u \in V(C_m) \setminus \{x\}$. Thus, any local antimagic labeling of U_m must induce at least 4 distinct vertex labels. Consequently, $\chi_{la}(U_m) \geq 4$.

Theorem 3.4. *For $n \geq 2$, $\chi_{lat}(P_n) = 2$ except that $\chi_{lat}(P_4) = 3$.*

Proof. We first consider odd n. Suppose $n = 4k + 1$. For $k = 1$, a required labeling sequence that labeled the vertices and edges of P_5 alternately is 6, 4, 7, 3, 2, 5, 8, 1, 9 with distinct vertex weights 10 and 14. For $k \geq 2$, define $f : V(P_{4k+1}) \cup E(P_{4k+1}) \to [1, 8k + 1]$ as follows:

(i) $f(v_1) = 8k$, $f(v_{4i+1}) = 6k + i$ for $i \in [1, k - 1]$ and $f(v_{4k+1}) = 8k + 1$,
(ii) $f(v_{4i-1}) = 3k + i$ for $i \in [1, k - 1]$ and $f(v_{4k-1}) = 6k$,
(iii) $f(v_{4i+2}) = 7k + i$ for $i \in [0, k - 1]$,
(iv) $f(v_{4i+4}) = 4k + 1 + i$ for $i \in [0, k - 1]$,
(v) $f(v_{2i}v_{2i+1}) = 2k - i$ for $i \in [1, 2k - 1]$ and $f(v_{4k}v_{4k+1}) = 2k$,
(vi) $f(v_{4i+1}v_{4i+2}) = 2k + 1 + i$ for $i \in [0, k - 1]$,
(vii) $f(v_{4i-1}v_{4i}) = 5k + i$ for $i \in [1, k - 1]$ and $f(v_{4k-1}v_{4k}) = 4k$.

It is not difficult to check that

$$w(v_i) = \begin{cases} 10k + 1 & \text{for odd } i, \\ 11k & \text{for even } i. \end{cases}$$

Thus, $\chi_{lat}(P_{4k+1}) = 2$.

Suppose $n = 4k + 3$. For $k = 0$, a required labeling sequence that labeled the vertices and edges of P_3 alternately is 5, 1, 3, 4, 2 with distinct vertex weights 6 and 8. For $k \geq 1$, we define $f : V(P_{4k+3}) \cup E(P_{4k+3}) \to [1, 8k + 5]$ as follows:

(i) $f(v_1) = 3k + 2$, $f(v_3) = 4k + 3$ and $f(v_{2i+3}) = 3k + 3 + i$ for $i \in [1, k - 1]$ if $k \geq 2$,
(ii) $f(v_{2k+2i+1}) = 2k + 1 + i$ for $i \in [1, k]$ and $f(v_{4k+3}) = 3k + 3$,
(iii) $f(v_2) = 1$ and $f(v_{2i+2}) = 5k + 4 + i$ for $i \in [1, k]$,
(iv) $f(v_{2k+2i+2}) = 4k + 3 + i$ for $i \in [1, k]$,
(v) $f(v_1v_2) = 8k + 5$ and $f(v_{2i+1}v_{2i+2}) = 2k + 2 - 2i$ for $i \in [1, k]$,
(vi) $f(v_{2k+2i+1}v_{2k+2i+2}) = 2k + 3 - 2i$ for $i \in [1, k]$,
(vii) $f(v_2v_3) = 5k + 4$ and $f(v_{2i+2}v_{2i+3}) = 6k + 4 + i$ for $i \in [1, 2k]$.

It is not difficult to check that

$$w(v_i) = \begin{cases} 11k + 7 & \text{for odd } i, \\ 13k + 10 & \text{for even } i. \end{cases}$$

Thus, $\chi_{lat}(P_{4k+3}) = 2$.

Now, we consider even n. Obviously $\chi_{lat}(P_2) = 2$.

Assume $n \geq 6$. By Theorem 3.3, $\chi_{la}(P_n \vee K_1) = 3$. By Theorem 2.3 (a), we have $\chi_{lat}(P_n) \leq 2$. Since $\chi_{lat}(P_n) \geq \chi(P_n) = 2$, the theorem holds.

Thus, we are left with $n = 4$. Label the vertices and edges of P_4 alternately by 7, 1, 6, 4, 2, 3, 5 to get distinct vertex weights 8, 9, 11. Thus, $\chi_{lat}(P_4) \leq 3$.

Suppose there were a local antimagic total 2-coloring of P_4. Suppose the labels of P_4 are a, x, b, y, c, z, d for vertex and edge alternately. We have (1) $a + x = y + c + z$ and (2) $x + b + y = z + d$. Moreover, $a + b + c + d$ must equal (1) or (2). If not, it corresponds to a local antimagic labeling of F_4 with 3 induced vertex colors, which is impossible.

By symmetry, we only need to consider $a+b+c+d = a+x$. Now $b+c+d = x \in \{6, 7\}$. From (2) we have $z = 2b + c + y \geq 7$. Thus, $z = 7$ and $b = 1$. Hence, $x = 6$ and $\{c, d\} = \{2, 3\}$. So, $y \geq 4$. This implies $z \geq 8$ which is impossible. Thus, $\chi_{lat}(P_4) \geq 3$. Hence, $\chi_{lat}(P_4) = 3$. This completes the proof. □

Example 3.5. The labeling sequence for P_{14} is 24, 13, 16, 1, 27, 9, 19, 2, 23, 12, 15, 3, 26, 8, 18, 4, 22, 11, 14, 5, 25, 7, 17, 6, 21, 10, 20 with 2 distinct vertex weights 37 and 30.

The labeling sequence for P_{16} is 22, 13, 24, 5, 16, 14, 25, 3, 17, 15, 26, 1, 27, 7, 23, 12, 21, 2, 30, 10, 19, 6, 28, 8, 18, 9, 29, 4, 20, 11, 31 with 2 distinct vertex weights 35 and 42.

The labeling sequence for P_{13} is 24, 7, 21, 5, 10, 16, 13, 4, 19, 8, 22, 3, 11, 17, 14, 2, 20, 9, 23, 1, 18, 12, 15, 6, 25 with 2 distinct vertex weights 31 and 33.

The labeling sequence for P_{15} is 11, 29, 1, 19, 15, 6, 20, 23, 13, 4, 21, 24, 14, 2, 22, 25, 8, 7, 16, 26, 9, 5, 17, 27, 10, 3, 18, 28, 12 with 2 distinct vertex weights 40 and 49.

Corollary 3.6. *For $n \geq 2$,*

$$\chi_{la}(F_n) = \begin{cases} 3 & \text{if } n \neq 4, \\ 4 & \text{if } n = 4. \end{cases}$$

Proof. From the proof of Theorem 3.4, we have

$$\sum_{u \in V(P_5)} f(u) = 32 \notin \{10, 14\},$$
$$\sum_{u \in V(P_{4k+1})} f(u) = 22k^2 + 12k + 1 \notin \{10k + 1, 11k\} \quad \text{for } k \geq 2,$$
$$\sum_{u \in V(P_3)} f(u) = 10 \notin \{6, 8\} \quad \text{for } n = 3,$$
$$\sum_{u \in V(P_{4k+3})} f(u) = 16k^2 + 19k + 6 \notin \{11k + 7, 13k + 10\} \quad \text{for } k \geq 1.$$

By Theorems 3.3, 3.4 and 2.3 (a) or (b), we have the corollary. □

We note that the concept of local super antimagic total chromatic number of a graph G, denoted $\chi_{lsat}(G)$, was introduced in [10]. By definition, we must have $\chi_{lat}(G) \le \chi_{lsat}(G)$ if $\chi_{lsat}(G)$ exists. In [11, Theorem 2], the authors proved that for $n \ge 3$,

$$\chi_{lsat}(C_n) = \begin{cases} 3 & \text{if } n \text{ is odd or } n = 4, \\ 2 & \text{otherwise.} \end{cases}$$

This result implies that

$$\chi_{lat}(C_n) = \begin{cases} 3 & \text{if } n \text{ is odd}, \\ 2 & \text{if } n \ge 6 \text{ is even.} \end{cases}$$

The following theorem completely determines $\chi_{lat}(C_n)$ and the proof is short.

Theorem 3.7. *For $n \ge 3$,*

$$\chi_{lat}(C_n) = \begin{cases} 2 & \textit{if } n \textit{ is even}, \\ 3 & \textit{otherwise.} \end{cases}$$

Proof. It is obvious that $\chi_{lat}(C_3) = 3$. Assume $n \ge 4$. In [1, 7], the authors showed that

$$\chi_{la}(W_n) = \begin{cases} 3 & \text{if } n \text{ is even}, \\ 4 & \text{otherwise.} \end{cases}$$

Since

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even}, \\ 3 & \text{otherwise,} \end{cases}$$

by Theorem 2.3 (a), we conclude that the theorem holds. □

For a wheel graph $W_n = K_1 \vee C_n$, $n \ge 3$, the vertex of K_1 is called its core. In [7, Theorem 5], the authors constructed a local antimagic 3-coloring for W_{4k}. By a similar approach we can construct a local antimagic 3-coloring for $r(K_1 \vee sC_{4k+2})$ for $r \ge 1$, $k \ge 1$ and for some s.

Theorem 3.8. *Suppose $k \ge 1$. Then:*

(a) $\chi_{la}(K_1 \vee sC_{4k+2}) = 3$ *for* $s \ge 1$,
(b) $\chi_{la}(r(K_1 \vee sC_{4k+2})) = 3$ *for* $r \ge 2$ *and even* $s \ge 2$.

Proof. Let $G = rH$ and $H = K_1 \vee sC_{4k+2}$. Observe that each copy of H can be obtained from s copies of W_{4k+2} by merging their cores.

We consider the following Tables 1 and 2 whose column sum is $3k + 3$.

Table 1

$S_1 =$

C_1	C_2	C_3	C_4	$\dots$	C_{2i-1}	C_{2i}	$\dots$	C_{2k-1}	C_{2k}	C_{2k+1}
1	2	3	4	$\dots$	$2i-1$	$2i$	$\dots$	$2k-1$	$2k$	$2k+1$
$2k+1$	k	$2k$	$k-1$	$\dots$	$2k+2-i$	$k+1-i$	$\dots$	$k+2$	1	$k+1$
$k+1$	$2k+1$	k	$2k$	$\dots$	$k+2-i$	$2k+2-i$	$\dots$	2	$k+2$	1

Table 2

$S_2 =$

C_1	C_2	C_3	C_4	$\dots$	C_{2i-1}	C_{2i}	$\dots$	C_{2k-1}	C_{2k}	C_{2k+1}
$k+1$	$2k+1$	k	$2k$	$\dots$	$k+2-i$	$2k+2-i$	$\dots$	2	$k+2$	1
1	2	3	4	$\dots$	$2i-1$	$2i$	$\dots$	$2k-1$	$2k$	$2k+1$
$2k+1$	k	$2k$	$k-1$	$\dots$	$2k+2-i$	$k+1-i$	$\dots$	$k+2$	1	$k+1$

For $r, s \geq 1$ and $1 \leq i \leq rs$, we define a table T_{2i-1} from S_1 by the following way.

1. Add each entry of Row 1 by $(i-1)(2k+1)$. So the set of entries of Row 1 is

$$[(i-1)(2k+1)+1, i(2k+1)].$$

2. Add each entry of Row 2 by $(rs+i-1)(2k+1)$. So the set of entries of Row 2 is

$$[(rs+i-1)(2k+1)+1, (rs+i)(2k+1)].$$

3. Add each entry of Row 3 by $(4rs-2i)(2k+1)$. So the set of entries of Row 3 is

$$[(4rs-2i)(2k+1)+1, (4rs-2i+1)(2k+1)].$$

Note that, the column sum of T_{2i-1} is $s_1 = (5rs-2)(2k+1)+3k+3$, and the row sum of Row 3 is $r_{2i-1} = (4rs-2i)(2k+1)^2 + (2k+1)(k+1)$.

For $r, s \geq 1$ and $1 \leq i \leq rs$, we define a table T_{2i} from S_2 by the following way.

1. Add each entry of Row 1 by $(rs+i-1)(2k+1)$. Note that, this row is the same as Row 2 of T_{2i-1} by right shifting one entry. So the set of entries of Row 1 is

$$[(rs+i-1)(2k+1)+1, (rs+i)(2k+1)].$$

2. Add each entry of Row 2 by $(i-1)(2k+1)$. Note that, this row is the same as Row 1 of T_{2i-1}. So the set of entries of Row 2 is

$$[(i-1)(2k+1)+1, i(2k+1)].$$

3. Add each entry of Row 3 by $(4rs-2i+1)(2k+1)$. So the set of entries of Row 3 is

$$[(4rs-2i+1)(2k+1)+1, (4rs-2i+2)(2k+1)].$$

Note that, the column sum of T_{2i} is

$$s_2 = (5rs - 1)(2k + 1) + 3k + 3$$

and the row sum of Row 3 is

$$r_{2i} = (4rs - 2i + 1)(2k + 1)^2 + (2k + 1)(k + 1).$$

By exactly the same approach as in [7, Theorem 5], we can obtain a W_{4k+2} that admits a bijective edge labeling using all the integers in T_{2i-1} and T_{2i}, denoted G_i for $1 \le i \le rs$, such that the edge labels of the C_{4k+2} are given by $(i-1)(2k+1)+1$, $(rs+i)(2k+1)$, $(i-1)(2k+1)+2$, $(rs+i-1)(2k+1)+k$, $(i-1)(2k+1)+3$, $(rs+i)(2k+1)-1$, $(i-1)(2k+1)+4$, $(rs+i-1)(2k+1)+k-1$, …, $i(2k+1)-3$, $(rs+i-1)(2k+1)+2$, $i(2k+1)-2$, $(rs+i-1)(2k+1)+k+2$, $i(2k+1)-1$, $(rs+i-1)(2k+1)+1$, $i(2k+1)$, $(rs+i-1)(2k+1)+k+1$ consecutively. Moreover, all the Row 3 integers of T_{2i-1} and T_{2i} are assigned to the spokes of G_i so that the incident edge labels sum of the core is

$$r_{2i-1} + r_{2i} = (8rs - 4i + 1)(2k + 1)^2 + 2(k + 1)(2k + 1) = R_i,$$

and the incident edge labels sum of the vertices of C_{4k+2} are s_1 and s_2 alternately. One may easily check that all labels in $[1, 4rs(2k + 1)]$ have been used.

(a) When $r = 1$. From the above construction, it is clear that we have a local antimagic 3-coloring for $K_1 \vee sC_{4k+2}$ with induced vertex labels s_1, s_2 and $L = \sum_{i=1}^{s} R_i$ for $s \ge 1$. Thus, $\chi_{la}(G) \le 3$. Since $\chi_{la}(G) \ge \chi(G) = 3$, $\chi_{la}(G) = 3$.

(b) Suppose $r \ge 2$ and $s = 2n \ge 2$. We group G_1 to G_{rs} into sets

$$A_t = \{G_i \mid i \in [tn - n + 1, tn] \cup [(2r - t)n + 1, (2r - t)n + n]\},$$

for $t = 1, 2, \dots, r$. Finally, for all the wheels in each A_t, we merge their cores into a vertex to get a $K_1 \vee sC_{4k+2}$, denoted $H_t = H$. The common core of each H_t has the label

$$\begin{aligned} L &= \sum_{i=tn-n+1}^{tn} R_i + \sum_{j=(2r-t)n+1}^{(2r-t)n+n} R_j \\ &= 2n(16rn + 1)(2k + 1)^2 + 4n(k + 1)(2k + 1) \\ &\quad - 4(2k + 1)^2 \left[\sum_{i=tn-n+1}^{tn} i + \sum_{j=(2r-t)n+1}^{(2r-t)n+n} j \right] \\ &= 2n(16rn + 1)(2k + 1)^2 + 4n(k + 1)(2k + 1) - 2n(2k + 1)^2(4rn + 2) \\ &= 2n(12rn - 1)(2k + 1)^2 + 4n(k + 1)(2k + 1) \end{aligned}$$

which is a constant.

Clearly, $L > s_2 > s_1$. Thus, $H_1 + H_2 + \ldots + H_r = rH = G$ admits a local antimagic labeling that induces three distinct colors so that $\chi_{la}(G) \le 3$. Hence, $\chi_{la}(G) = 3$. □

Example 3.9. Let us consider the graph $K_1 \vee 2C_{10}$. According to the proof of Theorem 3.8 we have

$$T_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 15 & 12 & 14 & 11 & 13 \\ 33 & 35 & 32 & 34 & 31 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 13 & 15 & 12 & 14 & 11 \\ 1 & 2 & 3 & 4 & 5 \\ 40 & 37 & 39 & 36 & 38 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 6 & 7 & 8 & 9 & 10 \\ 20 & 17 & 19 & 16 & 18 \\ 23 & 25 & 22 & 24 & 21 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 18 & 20 & 17 & 19 & 16 \\ 6 & 7 & 8 & 9 & 10 \\ 30 & 27 & 29 & 26 & 28 \end{pmatrix}.$$

So we have a local antimagic 3-coloring of $K_1 \vee 2C_{10}$ with the induced colors 49, 54, 610 as in Figure 1.

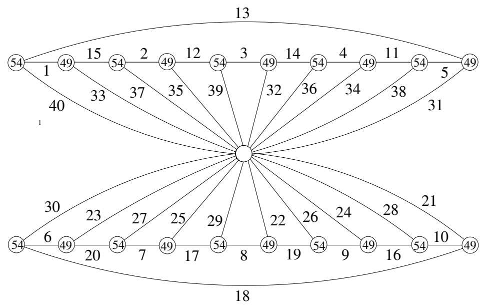

Fig. 1. A local antimagic 3-coloring of $K_1 \vee 2C_{10}$

The induced label of the core is 610.

Example 3.10. Let us consider the graph $2(K_1 \vee 2C_6)$. According to the proof of Theorem 3.8 we have

$$T_1 = \begin{pmatrix} 1 & 2 & 3 \\ 15 & 13 & 14 \\ 44 & 45 & 43 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 14 & 15 & 13 \\ 1 & 2 & 3 \\ 48 & 46 & 47 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 4 & 5 & 6 \\ 18 & 16 & 17 \\ 38 & 39 & 37 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 17 & 18 & 16 \\ 4 & 5 & 6 \\ 42 & 40 & 41 \end{pmatrix},$$

$$T_5 = \begin{pmatrix} 7 & 8 & 9 \\ 21 & 19 & 20 \\ 32 & 33 & 31 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 20 & 21 & 19 \\ 7 & 8 & 9 \\ 36 & 34 & 35 \end{pmatrix},$$

$$T_7 = \begin{pmatrix} 10 & 11 & 12 \\ 24 & 22 & 23 \\ 26 & 27 & 25 \end{pmatrix}, \quad T_8 = \begin{pmatrix} 23 & 24 & 22 \\ 10 & 11 & 12 \\ 30 & 28 & 29 \end{pmatrix}.$$

$A_1 = \{G_1, G_4\}$ and $A_2 = \{G_2, G_3\}$. So we have a local antimagic 3-coloring of $2(K_1 \vee 2C_6)$ with the induced colors 60, 63, 438 as in Figure 2.

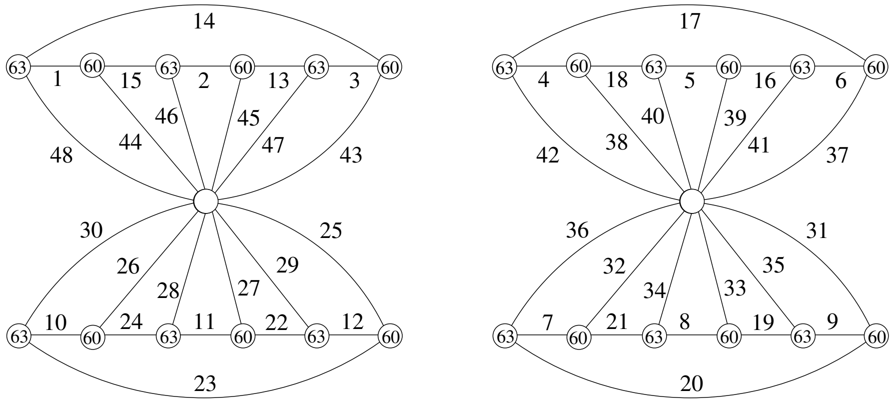

Fig. 2. A local antimagic 3-coloring of $2(K_1 \vee 2C_6)$

The induced label of each core is 438.

Example 3.11. Let us consider the graph $K_1 \vee 3C_6$. According to the proof of Theorem 3.8 we have

$$T_1 = \begin{pmatrix} 1 & 2 & 3 \\ 12 & 10 & 11 \\ 32 & 33 & 31 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 11 & 12 & 10 \\ 1 & 2 & 3 \\ 36 & 34 & 35 \end{pmatrix},$$
$$T_3 = \begin{pmatrix} 4 & 5 & 6 \\ 15 & 13 & 14 \\ 26 & 27 & 25 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 14 & 15 & 13 \\ 4 & 5 & 6 \\ 30 & 28 & 29 \end{pmatrix},$$
$$T_5 = \begin{pmatrix} 7 & 8 & 9 \\ 18 & 16 & 17 \\ 20 & 21 & 19 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 17 & 18 & 16 \\ 7 & 8 & 9 \\ 24 & 22 & 23 \end{pmatrix}.$$

So we have a local antimagic 3-coloring of $K_1 \vee 3C_6$ with the induced colors 45, 48, 495 as in Figure 3.

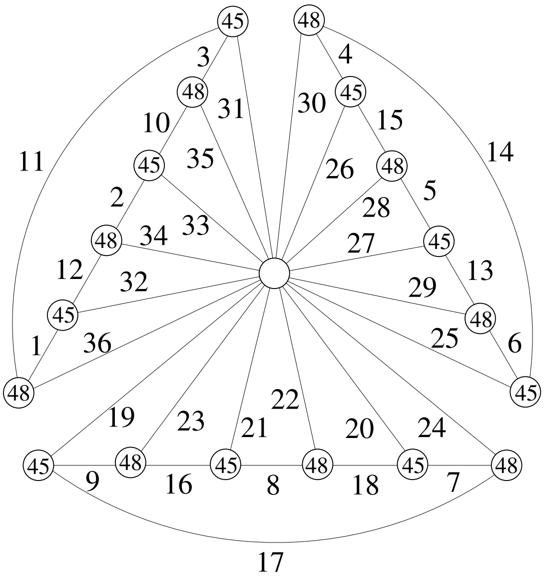

Fig. 3. A local antimagic 3-coloring of $K_1 \vee 3C_6$

The induced label of each core is 495.

In each of the local antimagic labeling in Theorem 3.8, an edge in a cycle is labeled 1. By Theorem 2.3(a) and Lemma 2.5, we immediately have the following two theorems.

Theorem 3.12. *For $k \geq 1$, $s \geq 2$, $\chi_{lat}(sC_{4k+2}) = 2$.*

Theorem 3.13. *For $k \geq 1$, $s \geq 1$, $\chi_{lat}(sC_{4k+2} + P_{4k+2}) = 2$.*

Theorem 3.14. *For odd $n \geq 3$, $4 \leq \chi_{lat}(C_n \vee 2K_1) \leq 5$, and for even $n \geq 6$, $3 \leq \chi_{lat}(C_n \vee 3K_1) \leq 5$.*

Proof. Here we let $C_n = u_1u_2 \dots u_nu_1$ and $V(sK_1) = \{v_j \mid 1 \leq j \leq s\}$.

Suppose $n \geq 3$ is odd. Clearly, $\chi_{lat}(C_n \vee 2K_1) \geq \chi(C_n \vee 2K_1) = 4$. In [9, Theorem 3.1], the authors provided a local antimagic 4-coloring f of $C_n \vee 3K_1$ which induces $f^+(v_1) = f^+(v_2) = f^+(v_3) = n(5n+1)/2$, $f^+(u_1) = 8n+3$, $f^+(u_i) = (17n+7)/2$ for odd $i \geq 3$, and $f^+(u_i) = (17n+5)/2$ for even $i \geq 2$.

Define $g : V(C_n \vee 2K_1) \cup E(C_n \vee 2K_1) \to [1, 4n+2]$ by $g(u_i) = f(u_iv_3)$, $g(e) = f(e)$ for $e \in E(C_n)$ or $e = u_iv_j$, and $g(v_j) = 4n+j$ for $1 \leq i \leq n$ and $j = 1, 2$. Now, $w_g(u_i) = f^+(u_i)$ and $w_g(v_j) = f^+(v_j) + 4n + j$ for $1 \leq i \leq n$ and $j = 1, 2$. Thus, g induces 5 distinct vertex weights and $\chi_{lat}(C_n \vee 2K_1) \leq 5$.

Suppose $n \geq 6$ is even. Clearly, $\chi_{lat}(C_n \vee 3K_1) \geq 3$. In [9, Theorem 3.3], the authors provided a local antimagic 3-coloring f of $C_n \vee 4K_1$ which induces $f^+(u_i) = 9n+3$ for odd i, $f^+(u_i) = 17n+3$ for even i, and $f^+(v_j) = n(6n+1)/2$ for $1 \leq j \leq 4$.

Define $g : V(C_n \vee 3K_1) \cup E(C_n \vee 3K_1) \to [1, 5n+3]$ by $g(u_i) = f(u_iv_4)$, $g(e) = f(e)$ for $e \in E(C_n)$ or $e = u_iv_j$, and $g(v_j) = 5n+j$ for $1 \leq i \leq n$ and $j = 1, 2, 3$. Now

$w_g(u_i) = f^+(u_i)$ and $w_g(v_j) = f^+(v_j) + 5n + i$ for $1 \le i \le n$ and $j = 1, 2, 3$. Thus, g induces 5 distinct vertex weights and $\chi_{lat}(C_n \vee 3K_1) \le 5$. □

Problem 3.15. Determine $\chi_{lat}(C_n \vee 2K_1)$ for odd $n \ge 3$, and $\chi_{lat}(C_n \vee 3K_1)$ for even $n \ge 4$.

In [9, Theorem 3.9], the authors proved that for $n, m \ge 3$,

$$\chi_{la}(K_m \vee C_n) = \begin{cases} m+2 & \text{if } m, n \text{ are even,} \\ m+3 & \text{if } m, n \text{ are odd.} \end{cases}$$

By Theorem 2.3, the following theorem holds.

Theorem 3.16. *For $m, n \ge 3$,*

$$\chi_{lat}(K_{m-1} \vee C_n) = \begin{cases} m+1 & \textit{if } m, n \textit{ are even,} \\ m+2 & \textit{if } m, n \textit{ are odd.} \end{cases}$$

4. CARTESIAN PRODUCT OF CYCLES

Let $C_{2k-1} = u_1 u_2 \dots u_{2k-1} u_1$ be the $(2k-1)$-cycle. We let $e_i = u_i u_{i+1}$, $1 \le i \le 2k-1$, the index taken modulus $2k-1$. We define two edge labelings g_1 and g_2 and one vertex labeling g for C_{2k-1} as follows. Define $g_1, g_2 : E(C_{2k-1}) \to [1, 2k-1]$ by

$$g_1(e_i) = 2k - i,$$
$$g_2(e_i) = \begin{cases} k + \frac{i-1}{2} & \text{if } i \text{ is odd,} \\ \frac{i}{2} & \text{if } i \text{ is even,} \end{cases}$$

and define $g : V(C_{2k-1}) \to [1, 2k-1]$ by

$$g(u_i) = \begin{cases} 1 & \text{if } i = 1, \\ i-1 & \text{if } i \text{ is odd and } i \ne 1, \\ i+1 & \text{if } i \text{ is even,} \end{cases}$$

where $i \in [1, 2k-1]$.

Now $g_1^+(u_1) = 2k$ and $g_1^+(u_i) = 4k + 1 - 2i$ for $i \in [2, 2k-1]$; $g_2^+(u_1) = 3k - 1$ and $g_2^+(u_i) = k - 1 + i$ for $i \in [2, 2k-1]$. By direct computation we have the following lemma.

Lemma 4.1. *Keeping all the notation used above, we have*

$$s_g(u_i) = g_1^+(u_i) + g_2^+(u_i) + g(u_i) = \begin{cases} 5k & \textit{if } i = 1, \\ 5k-1 & \textit{if } i \textit{ is odd and } i \ne 1, \\ 5k+1 & \textit{if } i \textit{ is even.} \end{cases}$$

Example 4.2. Figure 4 shows labelings g_1, g_2 and g for $C_5 = u_1u_2u_3u_4u_5u_1$.

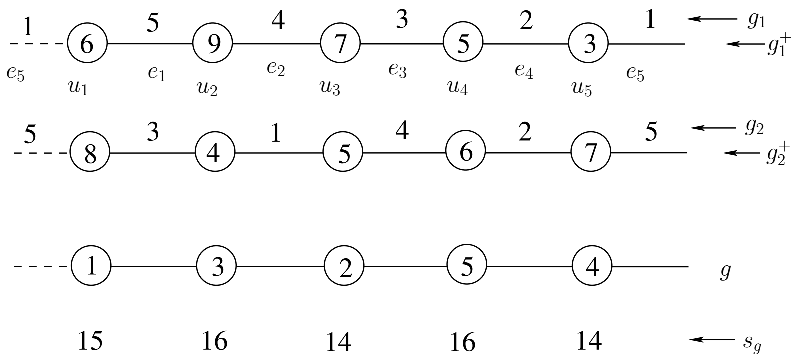

Fig. 4. Labelings g_1, g_2 and g for $C_5 = u_1u_2u_3u_4u_5u_1$

Similar to the definitions of g_1, g_2 and g, we define another 3 labelings for C_{2k-1}. Define $h_1, h_2 : E(C_{2k-1}) \to [0, 2k-2]$ by

$$h_1(e_i) = i - 1,$$
$$h_2(e_i) = \begin{cases} k - 1 - \frac{i}{2} & \text{if } i \text{ is even,} \\ 2k - 2 - \frac{i-1}{2} & \text{if } i \text{ is odd,} \end{cases}$$

and define $h : V(C_{2k-1}) \to [0, 2k-2]$ by

$$h(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ 2k - i & \text{if } i \neq 1, \end{cases}$$

where $i \in [1, 2k-1]$.

Now $h_1^+(u_1) = 2k - 2$ and $h_1^+(u_i) = 2i - 3$ for $i \in [2, 2k-1]$; $h_2^+(u_i) = 3k - 2 - i$ for $i \in [1, 2k-1]$. By direct computation, we have the following lemma.

Lemma 4.3. *Keeping all the notation defined above, we have*

$$s_h(u_i) = h_1^+(u_i) + h_2^+(u_i) + h(u_i) = 5k - 5,$$

for $i \in [1, 2k-1]$.

Let $G = C_n \times C_n$. Then

$$V(G) = \{(u_i, u_j) = v_{i,j} \mid 1 \leq i, j \leq n\}.$$

Let

$$H_i = \{v_{i,j} \mid 1 \leq j \leq n\} \quad \text{and} \quad V_j = \{v_{i,j} \mid 1 \leq i \leq n\}.$$

Edges in $G[H_i]$ and $G[V_j]$ are called *horizontal edges* and *vertical edges*, respectively. The edges in $G[H_i]$ are denoted by $x_{i,j} = v_{i,j}v_{i,j+1}$ and the edges in $G[V_j]$ are denoted by $y_{i,j} = v_{i,j}v_{i+1,j}$.

We will keep all notation defined above in this section. Note that the labelings below of $C_{2k-1} \times C_{2k-1}$ use constructions that incorporate pairs of orthogonal Latin squares.

Theorem 4.4. *For $k \geq 2$, $\chi_{lat}(C_{2k-1} \times C_{2k-1}) = 3$.*

Proof. It is known that $\chi(C_{2k-1} \times C_{2k-1}) = 3$, so we have $\chi_{lat}(C_n \times C_n) \geq 3$.

Following we shall define two total labelings f_1 and f_2 for $G = C_{2k-1} \times C_{2k-1}$ using the labelings g_1, g_2 and g defined above. In this proof, all addition and subtraction of indices are taken modulo $2k-1$.

Define $f_1 : V(G) \cup E(G) \to [1, 2k-1]$ by $f_1(y_{i,j}) = g_1(e_{j-i-1})$, $f_1(x_{i,j}) = g_2(e_{j-i})$ and $f_1(v_{i,j}) = g(u_{j-i})$. Thus,

$$\begin{aligned} w_{f_1}(v_{i,j}) &= f_1(y_{i,j}) + f_1(y_{i-1,j}) + f_1(x_{i,j}) + f_1(x_{i,j-1}) + f_1(v_{i,j}) \\ &= g_1(e_{j-i-1}) + g_1(e_{j-i}) + g_2(e_{j-i}) + g_2(e_{j-1-i}) + g(u_{j-i}) \\ &= g_1^+(u_{j-i}) + g_2^+(u_{j-i}) + g(u_{j-i}) = s_g(u_{j-i}). \end{aligned}$$

Define $f_2 : V(G) \cup E(G) \to [0, 6k-4]$ by $f_2(y_{i,j}) = h_1(e_{i+j})$, $f_2(x_{i,j}) = h_2(e_{i+j}) + 2k-1$ and $f_2(v_{i,j}) = h(u_{i+j}) + 4k - 2$. Thus,

$$\begin{aligned} w_{f_2}(v_{i,j}) &= f_2(y_{i,j}) + f_2(y_{i-1,j}) + f_2(x_{i,j}) + f_2(x_{i,j-1}) + f_2(v_{i,j}) \\ &= h_1(e_{i+j}) + h_1(e_{i+j-1}) + [h_2(e_{i+j}) + 2k - 1] + [h_2(e_{i+j-1}) + 2k - 1] \\ &\quad + [h(u_{i+j}) + 4k - 2] \\ &= h_1^+(u_{i+j}) + h_2^+(u_{i+j}) + h(u_{i+j}) + 8k - 4 = s_h(u_{i+j}) + 8k - 4 = 13k - 9. \end{aligned}$$

Note that, the images of all vertical edges are in $[0, 2k-2]$, those of all horizontal edges are in $[2k-1, 4k-3]$ and those of all vertices are in $[4k-2, 6k-4]$.

Now define $f : V(G) \cup E(G) \to [1, 3(2k-1)^2]$ by $f(x) = f_1(x) + (2k-1)f_2(x)$ for $x \in V(G) \cup E(G)$. Suppose $f(x) = f(y)$, then $f_1(x) + (2k-1)f_2(x) = f_1(y) + (2k-1)f_2(y)$ or equivalently $f_1(x) - f_1(y) = (2k-1)[f_2(y) - f_2(x)]$. Hence, $f_1(x) = f_1(y)$ and $f_2(x) = f_2(y)$ (since $0 \leq |f_1(x) - f_1(y)| \leq 2k-2$). By the definition of f_2, $f_2(x) = f_2(y)$ implies that x and y both are vertices, vertical edges or horizontal edges. Since g_1, g_2 and g are bijective, $x = y$. Thus, f is injective and hence is bijective.

Next,

$$\begin{aligned} w_f(v_{i,j}) &= f(y_{i,j}) + f(y_{i-1,j}) + f(x_{i,j}) + f(x_{i,j-1}) + f(v_{i,j}) \\ &= w_{f_1}(v_{i,j}) + (2k-1)w_{f_2}(v_{i,j}) = s_g(u_{j-i}) + (2k-1)(13k-9) \\ &= \begin{cases} 5k + c & \text{if } j - i \equiv 1 \pmod{2k-1} \\ 5k + 1 + c & \text{if } j + i \equiv 0 \pmod{2} \\ 5k - 1 + c & \text{if } j + i \equiv 1 \pmod{2}, \quad j - i \not\equiv 1 \pmod{2k-1}. \end{cases} \end{aligned} \tag{4.1}$$

where $c = (2k-1)(13k-9)$. Note that $v_{i,j}$ and $v_{i',j'}$ are adjacent only if $i + j \not\equiv i' + j' \pmod{2}$. Thus, f is a local antimagic total 3-coloring of G. So $\chi_{lat}(G) = 3$. □

Example 4.5. Figure 5 shows labelings f_1 and f_2 for $C_5 \times C_5$ (the lowest left corner is the vertex $v_{1,1}$, the lowest right corner is the vertex $v_{1,5}$).

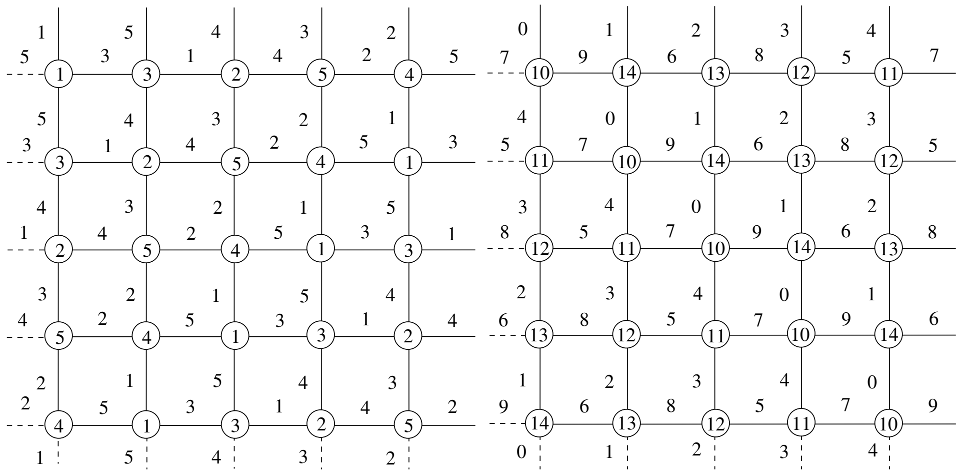

Fig. 5. Labelings f_1 and f_2 for $C_5 \times C_5$

One may see that the w_{f_1}-value is 15, 16 or 14; and w_{f_2}-value is 30.

Figure 6 shows labelings $f = f_1 + 5f_2$ and w_f.

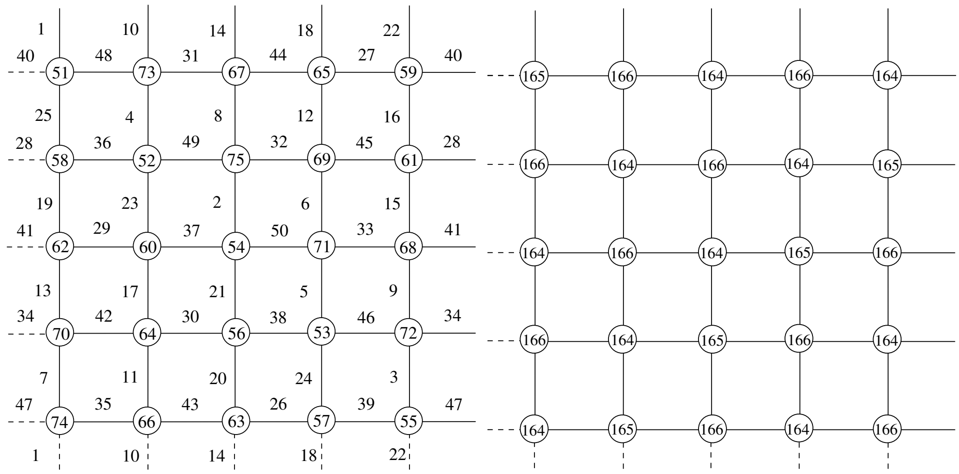

Fig. 6. Labelings $f = f_1 + 5f_2$ and w_f

One may see that the w_f-value is 165, 166 or 164. Thus, f is a local antimagic 3-coloring for $C_5 \times C_5$.

Similarly, we define a labeling ϕ for C_{2k-1}. Define $\phi : E(C_{2k-1}) \to [1, 2k-1]$ by

$$\phi(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd,} \\ 2k - \frac{i}{2} & \text{if } i \text{ is even,} \end{cases}$$

where $i \in [1, 2k-1]$.

Now $\phi^+(u_1) = k+1$, $\phi^+(u_i) = 2k+1$ for odd i and $\phi^+(u_i) = 2k$ for even i, where $i \in [2, 2k-1]$.

Theorem 4.6. *For $k \geq 2$, $\chi_{la}(C_{2k-1} \times C_{2k-1}) = 3$.*

Proof. It is known that $\chi(C_{2k-1} \times C_{2k-1}) = 3$, so we have $\chi_{la}(C_n \times C_n) \geq 3$.

In the following we shall define two labelings ρ_1 and ρ_2 for $G = C_{2k-1} \times C_{2k-1}$ using the labelings ϕ and h_1.

Define $\rho_1 : E(G) \to [1, 2k-1]$ by $\rho_1(y_{i,j}) = \phi(e_{j-i-1})$ and $\rho_1(x_{i,j}) = \phi(e_{j-i})$. Then

$$\begin{aligned} \rho_1^+(v_{i,j}) &= \rho_1(y_{i,j}) + \rho_1(y_{i-1,j}) + \rho_1(x_{i,j}) + \rho_1(x_{i,j-1}) \\ &= \phi(e_{j-i-1}) + \phi(e_{j-i}) + \phi(e_{j-i}) + \phi(e_{j-1-i}) \\ &= 2\phi^+(u_{j-i}). \end{aligned}$$

Define $\rho_2 : E(G) \to [0, 2k-2]$ by $\rho_2(y_{i,j}) = h_1(e_{i+j})$ and $\rho_2(x_{i,j}) = h_1(e_{2k-i-j}) + 2k-1$. Thus,

$$\begin{aligned} \rho_2^+(v_{i,j}) &= \rho_2(y_{i,j}) + \rho_2(y_{i-1,j}) + \rho_2(x_{i,j}) + \rho_2(x_{i,j-1}) \\ &= h_1(e_{i+j}) + h_1(e_{i+j-1}) + [h_1(e_{2k-i-j}) + 2k-1] \\ &\quad + [h_1(e_{2k-i-j+1}) + 2k-1] \\ &= h_1^+(u_{i+j}) + h_1^+(u_{2k-i-j+1}) + 4k-2. \end{aligned}$$

Let us consider $h_1^+(u_{i+j}) + h_1^+(u_{2k-i-j+1})$. Note that, $i+j \equiv 1 \pmod{2k-1}$ if and only if $2k-i-j+1 \equiv 1 \pmod{2k-1}$. Thus, $u_{i+j} = u_{2k-i-j+1} = u_1$ and $h_1^+(u_{i+j}) + h_1^+(u_{2k-i-j+1}) = 2h_1^+(u_1) = 4k-4$.

Suppose $i+j \not\equiv 1 \pmod{2k-1}$. If $i+j \in [2, 2k-1]$, then $2k-i-j+1 \in [2, 2k-1]$. Hence,

$$h_1^+(u_{i+j}) + h_1^+(u_{2k-i-j+1}) = [2(i+j)-3] + [2(2k-i-j+1)-3] = 4k-4.$$

If $i+j \in [2k+1, 4k-2]$, then $4k-i-j \in [2, 2k-1]$. Hence, $u_{2k-i-j+1} = u_{4k-i-j}$ and $u_{i+j} = u_{i+j-2k+1}$. Then

$$\begin{aligned} h_1^+(u_{i+j}) + h_1^+(u_{2k-i-j+1}) &= h_1^+(u_{i+j-2k+1}) + h_1^+(u_{4k-i-j}) \\ &= [2(i+j-2k+1)-3] + [2(4k-i-j)-3] \\ &= 4k-4. \end{aligned}$$

Thus,

$$\rho_2^+(v_{i,j}) = 8k-6$$

for $i, j \in [1, 2k-1]$.

Now define $F : E(G) \to [1, 2(2k-1)^2]$ by $F(x) = \rho_1(x) + (2k-1)\rho_2(x)$ for $x \in E(G)$. By a similar argument to the proof of Theorem 4.4, we can show that F is bijective.

Next,

$$\begin{aligned} F^+(v_{i,j}) &= F(y_{i,j}) + F(y_{i-1,j}) + F(x_{i,j}) + F(x_{i,j-1}) \\ &= \rho_1^+(v_{i,j}) + (2k-1)\rho_2^+(v_{i,j}) \\ &= 2\phi^+(u_{j-i}) + (2k-1)(8k-6) \\ &= \begin{cases} 2k+2+d & \text{if } j-i \equiv 1 \pmod{2k-1} \\ 4k+2+d & \text{if } j-i \equiv 1 \pmod{2}, j-i \not\equiv 1 \pmod{2k-1} \\ 4k+d & \text{if } j-i \equiv 0 \pmod{2} \end{cases} \end{aligned}$$

where $d = (2k-1)(8k-6)$. Note that $v_{i,j}$ and $v_{i',j'}$ are adjacent only if $i+j \not\equiv i'+j' \pmod{2}$. Thus, F is a local antimagic 3-coloring of G. So $\chi_{lat}(G) = 3$. □

Example 4.7. Figure 7 shows labelings ρ_1 and ρ_2 for $C_5 \times C_5$ with their induced vertex labelings.

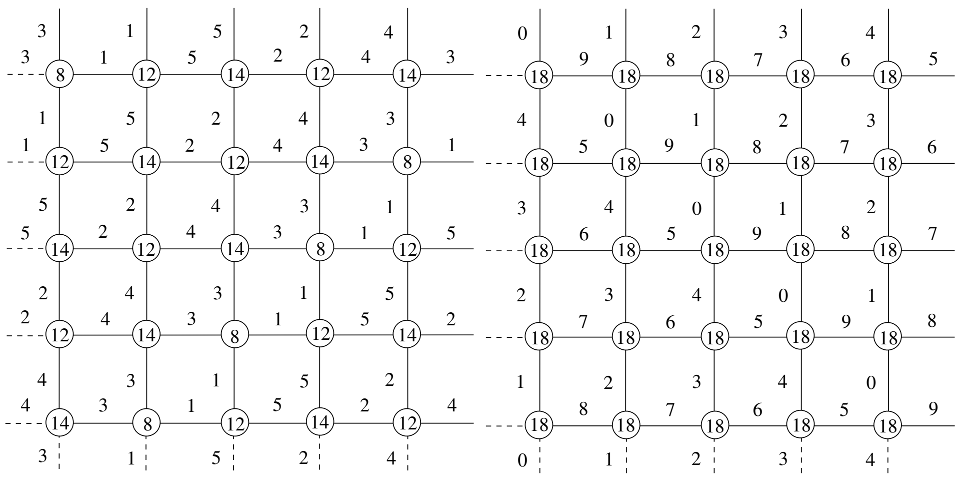

Fig. 7. Labelings ρ_1 and ρ_2 for $C_5 \times C_5$

Figure 8 shows the labelings $F = \rho_1 + 5\rho_2$ and F^+ for $C_5 \times C_5$.

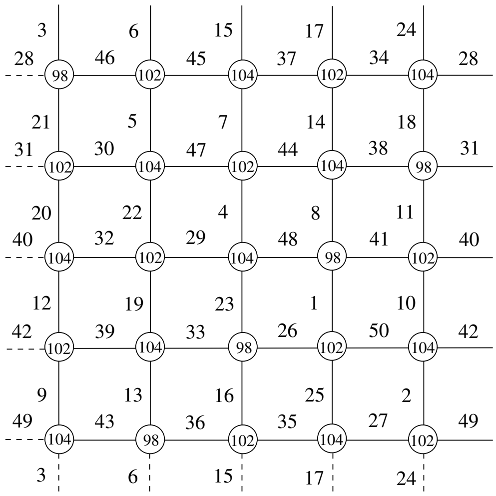

Fig. 8. The labelings $F = \rho_1 + 5\rho_2$ and F^+ for $C_5 \times C_5$

Theorem 4.8. *For $k \geq 2$, $\chi_{la}((C_{2k-1} \times C_{2k-1}) \vee K_1) = 4$.*

Proof. Let f be the local antimagic total labeling of $C_{2k-1} \times C_{2k-1}$ in the proof of Theorem 4.4. Since

$$\sum_{i=1}^{2k-1} g(u_i) = k(2k-1) \quad \text{and} \quad \sum_{i=1}^{2k-1} h(u_i) = \frac{1}{2}(2k-1)^2,$$

we have

$$\begin{aligned}\sum_{i=1}^{2k-1}\sum_{j=1}^{2k-1} f(v_{i,j}) &= \sum_{i=1}^{2k-1}\sum_{j=1}^{2k-1} g(u_{i+j}) + (2k-1)\sum_{i=1}^{2k-1}\sum_{j=1}^{2k-1}[h(u_{i+j}) + (4k-2)] \\ &= k(2k-1)^2 + \frac{1}{2}(2k-1)^4 + 2(2k-1)^4 \\ &= (2k-1)^2\left[k + \frac{5}{2}(2k-1)^2\right] > w_f(v_{i,j}). \qquad \text{(by (4.1))}\end{aligned}$$

By Theorem 2.3(b), we immediately have $\chi_{la}((C_{2k-1} \times C_{2k-1}) \vee K_1) = 4$. □

5. CONCLUSION AND OPEN PROBLEMS

In this paper, we first proved that every graph is local antimagic. The proof gives a sharp bound for us to determine $\chi_{lat}(G)$ (or $\chi_{la}(G \vee K_1)$) using a local antimagic labeling of $G \vee K_1$ (or a local antimagic total labeling of G). The local antimagic (total) chromatic number of many family of graphs are determined. The following problems arise naturally.

Problem 5.1. Determine $\chi_{lat}(sC_n)$ for $s \geq 2$ and $n \not\equiv 2 \pmod 4$.

Problem 5.2. For (i) $m \neq n \geq 3$ and (ii) $m = n \geq 4$ are even, determine $\chi_{la}(C_m \times C_n)$ and $\chi_{lat}(C_m \times C_n)$.

Problem 5.3. For $m, n \geq 2$, determine $\chi_{la}(P_m \times P_n)$ and $\chi_{lat}(P_m \times P_n)$.

Problem 5.4. Characterize G such that $\chi(G) = \chi_{lat}(G) = \chi_{la}(G) - 1$.

In [6, Theorem 3.4], the authors showed that there are infinitely many circulant graphs (with at most an edge deleted) of $\chi_{la} = 3$. Since cycles are the simplest circulant graphs with $\chi_{lat} = 2$, we have

Problem 5.5. Determine the exact values of $\chi_{lat}(C)$ and $\chi_{lat}(C - e)$ for each circulant graph $C \not\cong C_n, C_{2n}(1, n), n \geq 3$.

Since every known result has $\chi_{lat}(G) \leq \chi_{la}(G)$, we end this paper with the following.

Conjecture 5.6. *For each graph G of order at least 3, $\chi_{lat}(G) \leq \chi_{la}(G)$.*

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