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ON MINIMUM INTERSECTIONS OF CERTAIN SECONDARY DOMINATING SETS IN GRAPHS

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Abstract. In this paper we consider secondary dominating sets, also named as (1,k)-dominating sets, introduced by Hedetniemi *et al.* in 2008. In particular, we study intersections of the $(1,1)$-dominating sets and proper $(1,2)$-dominating sets. We introduce (1,$\overline{2}$)-intersection index as the minimum possible cardinality of such intersection and determine its value for some classes of graphs.

Keywords: dominating set, 2-dominating set, $(1,2)$-dominating set, proper $(1,2)$-dominating set, domination numbers, (1,$\overline{2}$)-intersection index.

Mathematics Subject Classification: 05C69, 05C76.

1. INTRODUCTION AND PRELIMINARY RESULTS

In general we use the standard terminology and notation of graph theory, see [7]. For domination related concepts not defined here see [11].

Let G be a finite, simple, undirected graph with the vertex set $V(G)$ and the edge set $E(G)$. The set of all vertices which are adjacent to $x \in V(G)$ is named *the neighbourhood of* x and denoted by $N_G(x)$. By $d_G(x)$ we denote the degree of the vertex x. A vertex of the degree one is called *a leaf*, every neighbour of a leaf is called *a support vertex.* If a vertex is adjacent to at least two leaves, it is said to be *a strong support vertex.* For a graph G, let $S(G)$ denote the set of support vertices of G and $L(G)$ the set of leaves of G.

By $d_G(x,y)$ we denote the distance between vertices x and y in the graph G. *The eccentricity* of the vertex $x \in V(G)$ is the maximum distance between x and any other vertex in G. Every vertex of the smallest eccentricity in the graph G is called *the central vertex* of G.

Let G be a connected graph and let x be a certain vertex of G such that $d_G(x) = k \geq 2$. By *a branch* B at a vertex x in a graph G we mean a maximal (with respect to inclusion) subtree, which includes x and exactly one edge incident to x. A branch, which is a path, is named *a pendant path.*

By P_n, $n \geq 2$ and C_n, $n \geq 3$ we denote a path and a cycle of order n, respectively. Moreover, $K_{m,n}$ is a complete bipartite graph.

Let $n \geq 3$ and k be positive integers, $n > k$, P_k a path of order k with a leaf u and $K_{1,n-k}$ a star with central vertex v. Then the graph of order n obtained from P_k and $K_{1,n-k}$ by identifying the vertices u and v is called *a broom* and denoted by $B(n,k)$. If $n-k=1$, then the broom is a path, while if $k=1$ or $k=2$, it is a star.

The join of two graphs G and H is the graph $G+H$ such that $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy \colon x \in V(G) \text{ and } y \in V(H)\}$.

The corona of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of the graph G and $|V(G)|$ copies of H and joining the i-th vertex of the graph G to every vertex in the i-th copy of H, $i \in \{1, 2, \ldots, |V(G)|\}$.

A subset $D \subseteq V(G)$ is *a dominating set* of G if every vertex belonging to $V(G) \backslash D$ has at least one neighbour in D. Dominating sets are one of the most intesively studied concepts in graph theory and may be applied in many practical problems, for example facility location problems, monitoring communication and electrical networks, see [10]. The subject of dominating sets has historical roots dating back to 1862, when C.F. de Jaenish in [13] attempted to determine the minimum number of queens which are necessary to cover an $n \times n$ chess board. Over a hundred years later, the concept of dominating sets was introduced by O. Ore in [18].

Apart from classical dominating sets, there are many other kinds of dominating sets, obtained by adding some restrictions to the classical concept of domination. One of them is multiple domination, introduced by J.F. Fink and M.S. Jacobson in [9]. Let $k \geq 1$ be an integer. A subset $D \subseteq V(G)$ is *a k-dominating set* if every vertex from $V(G) \backslash D$ has at least k neighbours in D. Clearly, every k-dominating set is a dominating set of G. If $k=1$, we obtain a dominating set in the classical sense. If $k=2$, we obtain a 2-dominating set of G, which has been broadly studied in literature, see for example [1–5, 15]. Let us see that for large k the vertices of graph need to have large degrees to ensure that the graph have a k-dominating set. This leads to reducing the classes of graphs, in which studying k-dominating sets is interesting.

In 2008 Hedetniemi *et al.* in [12] introduced the (1,k)-dominating sets by weakening the restrictions for 2-domination. More precisely, they considered the situation when a vertex does not have to be adjacent to two vertices from the dominating set, but only to one and the second vertex should not be "too far away". This is how the secondary domination concept was born. Let $k \geq 1$ be an integer. A subset $D \subseteq V(G)$ is a (1,k)-*dominating set of* G if for every vertex $x \in V(G) \backslash D$ there are two vertices $u, v \in D$, $u \neq v$, such that $xv \in E(G)$ and $d_G(x,u) \leq k$. A (1,k)-dominating set is called also as a (1,k)-*dset*. Clearly, every (1,k)-dominating set is a dominating set of G. What is more, it is a (1,$k+1$)-dset, (1,$k+2$)-dset, $\ldots$, (1,diam(G))-dset. Clearly, it is pointless to consider (1,k)-dsets for $k \geq \operatorname{diam}(G)$.

If $k=1$, we obtain a definition of 2-dominating set. Note that from the definition of a $(1,1)$-dset we have that all leaves of a graph G must belong to a $(1,1)$-dset. For $k=2$ we obtain a definition of $(1,2)$-dsets, which were broadly studied in literature, see for example [8, 20], in particular many results for their connections with independent sets have been obtained in [12, 15, 16].

Note that every $(1,1)$-dset of a graph G is a $(1,2)$-dset of this graph. The converse implication does not hold. Hence, Michalski introduced in [14] *proper* $(1,2)$-*dominating sets* as a natural distinguishment between $(1,1)$-dsets and $(1,2)$-dsets. A $(1,2)$-dominating set D is called a *proper* $(1,2)$-*dominating set* of the graph G if D is a $(1,2)$-dset but it is not a $(1,1)$-dset. For convenience, instead of "proper $(1,2)$-dominating set" we will write also "$(1,\overline{2})$-*dset*".

Clearly, every $(1,\overline{2})$-dset is a $(1,2)$-dset and a dominating set and it must have at least two vertices. The properties of $(1,\overline{2})$-dsets were studied for example in [17]. In particular, a necessary and sufficient condition for the existence of such sets in a connected graph was given.

Theorem 1.1. *A connected graph G has a $(1,\overline{2})$-dset if and only if G is not a complete graph.*

While $(1,1)$-dsets and $(1,\overline{2})$-dsets are different sets, an interesting question is to determine what is the minimum number of common vertices in the intersection of these sets in graphs. In other words, we want to establish how much "disjoint" these sets may be. Therefore, we introduce the parameter measuring the "level of disjointness" of $(1,\overline{2})$-dsets and $(1,1)$-dsets. Let $\mathcal{F}$ be a family of all $(1,1)$-dsets of a graph G and let $\mathcal{F}^*$ be a family of all $(1,\overline{2})$-dsets of G. Then let us denote

$$\sigma(G) = \min_{D\in\mathcal{F}, D^*\in\mathcal{F}^*} |D \cap D^*|.$$

The number $\sigma(G)$ is called a $(1,\overline{2})$-*intersection index* of a graph G.

2. MAIN RESULTS

In this section we determine the $(1,\overline{2})$-intersection index in some classes of graphs. First we give useful lemmas.

Lemma 2.1. *Let G be a connected graph containing a pendant path $v_1v_2\ldots v_n$, $n \geq 3$, $d_G(v_n) = 1$. Then each pair of a $(1,1)$-dset and a $(1,\overline{2})$-dset of G has at least one common vertex from the set $\{v_{n-2}, v_{n-1}, v_n\}$.*

Proof. Let G be a connected graph containing a pendant path $v_1v_2\ldots v_n$, $n \geq 3$, where v_n is a leaf of G. For the contradiction suppose that there exist a $(1,1)$-dset D and a $(1,\overline{2})$-dset D^* of G such that $D \cap D^* \cap \{v_{n-2}, v_{n-1}, v_n\} = \emptyset$. Since D is a $(1,1)$-dset then $v_n \in D$, which implies that $v_n \notin D^*$ and $v_{n-1}, v_{n-2} \in D^*$, because v_n must be $(1,2)$-dominated by D^*. Therefore, $v_{n-1}, v_{n-2} \notin D$, so v_{n-1} is not $(1,1)$-dominated by D, which is a contradiction. □

Corollary 2.2. *Let G be a connected graph containing a pendant path $v_1v_2\ldots v_n$, $n \geq 3$, $d_G(v_n) = 1$. Then $\sigma(G) \geq 1$.*

Lemma 2.3. *Let G be a connected graph with $\sigma(G) = 0$ and let D and D^* be disjoint $(1,1)$-dset and $(1,\overline{2})$-dset of G, respectively. Then $L(G) \subseteq D$ and $S(G) \subseteq D^*$.*

Proof. The inclusion $L(G) \subseteq D$ follows from the definition of $(1,1)$-dset of a graph. To prove that $S(G) \subseteq D^*$, assume that G has disjoint $(1,1)$-dset and $(1,\overline{2})$-dset denoted by D and D^*, respectively. Let $v \in S(G)$ and $v \notin D^*$. All leaves, adjacent to the vertex v, belong to D, so they do not belong to D^*. Then these leaves are not $(1,2)$-dominated by the set D^*, a contradiction. □

Now we will show that while in paths $n \geq 3$, there are no disjoint $(1,1)$-dset and $(1,\overline{2})$-dset, almost all cycles have such property.

Theorem 2.4. *Let $n \geq 3$ be an integer. Then*

$$\sigma(P_n) = \begin{cases} 1 & \text{for } n \in \{3,4,5\}, \\ 2 & \text{for } n \geq 6. \end{cases}$$

Proof. For $n \in \{3,4,5\}$ the result is obvious. Let $n \geq 6$ be an integer. First, we will prove that $\sigma(P_n) \leq 2$. Let us consider the following cases.

Case 1. Let $n = 2p+1$. Then the set

$$D_1 = \{x_2, x_4, x_6, \ldots, x_{2p-2}, x_{2p}\} \cup \{x_1, x_{2p+1}\}$$

is a $(1,1)$-dset of P_{2p+1} and the set

$$D_1^* = (V(P_{2p+1}) \backslash D_1) \cup \{x_2, x_{2p}\}$$

is a $(1,\overline{2})$-dset. Note that $|D_1 \cap D_1^*| = |\{x_2, x_{2p}\}| = 2$.

Case 2. Let $n = 2p$. Then the set

$$D_2 = \{x_1\} \cup \{x_2, x_4, x_6, \ldots, x_{2p-2}, x_{2p}\}$$

is $(1,1)$-dominating in P_{2p}. Moreover, the set

$$D_2^* = (V(P_{2p}) \backslash D_2) \cup \{x_2, x_{2p-2}\}$$

is a $(1,\overline{2})$-dset of P_{2p}. Hence,

$$|D_2 \cap D_2^*| = |\{x_2, x_{2p-2}\}| = 2.$$

Now we will prove that $\sigma(P_n) \geq 2$. By Lemma 2.1, for every $(1,1)$-dset D and $(1,\overline{2})$-dset D^* of P_n we have

$$|D \cap D^* \cap \{x_1, x_2, x_3\}| \geq 1 \quad \text{and} \quad |D \cap D^* \cap \{x_{n-2}, x_{n-1}, x_n\}| \geq 1,$$

which ends the proof. □

Theorem 2.5. *Let $n \geq 4$ be an integer. Then*

$$\sigma(C_n) = \begin{cases} 1 & \text{for } n = 4, \\ 0 & \text{for } n \geq 5. \end{cases}$$

Proof. For $n = 4$ the result is obvious. Let $n \geq 5$ be an integer. We consider the following cases.

Case 1. Let $n = 2p + 1$. Then the set

$$D_1 = \{x_1, x_3, x_5, \ldots, x_{2p+1}\}$$

is a $(1, 1)$-dset and the set $V(C_n) \backslash D_1$ is a $(1, \overline{2})$-dset of the cycle C_n.

Case 2. Let $n = 2p$. Then the set

$$D_2 = \{x_1, x_2, x_4, x_5\} \cup \{x_7, x_9, \ldots, x_{2p-3}, x_{2p-1}\}$$

is a $(1, 1)$-dset and $V(C_n) \backslash D_2$ is a $(1, \overline{2})$-dset of C_n. □

Now we will give the value of the $(1,\overline{2})$-intersection index for spiders, which were considered for example in [6, 19].

A spider graph is a tree with at most one vertex of degree greater than 2 and this vertex is called *the central vertex*. *A leg* of a spider graph is a path from the central vertex to a leaf of the tree. Let $SP(l_1, l_2, \ldots, l_n)$ denote a spider of n legs of lengths l_1, l_2, $\ldots$, l_n, where $l_1, l_2, \ldots, l_n, n \in \mathbb{N}$. An example of a spider is given in Figure 1.

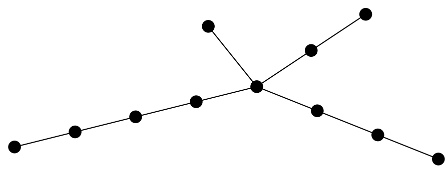

Fig. 1. The spider $SP(1, 2, 3, 4)$

Note that if there is no vertex of the degree greater than 2, then the spider graph is isomorphic to the path P_n. Then any vertex may serve as the central vertex of the spider. Moreover, if $l_j = 1$ for every $j \in \{1, 2, \ldots, n\}$, then the spider graph is isomorphic to the star $K_{1,n}$. Finally, if $l_j \geq 2$ for exactly one $j \in \{1, 2, \ldots, n\}$ and $l_i = 1$ for every $i \in \{1, 2, \ldots, n\}$, $i \neq j$, then the spider $SP(l_1, l_2, \ldots, l_n)$ is isomorphic to the broom $B(n + l_j, l_j + 1)$.

Theorem 2.6. *Let s be the number of legs of length* 1 *or* 2 *in $SP(l_1, l_2, \ldots, l_n)$, where $n \geq 3$. Then*

(1) $\sigma(SP(l_1, l_2, \ldots, l_n)) = n - s$, *if $l_i \geq 3$ for every $i \in \{1, 2, \ldots, n\}$ or there exist j, k such that $l_j = 1$ and $l_k = 3$ and $l_m \neq 2$ for every $m \in \{1, 2, \ldots, n\}$,*
(2) $\sigma(SP(l_1, l_2, \ldots, l_n)) = n - s + 1$ *otherwise.*

Proof. For the proof, we denote vertices in $SP(l_1, l_2, \ldots, l_n)$ as in the Figure 2. Let s be a number of legs of length 1 or 2.

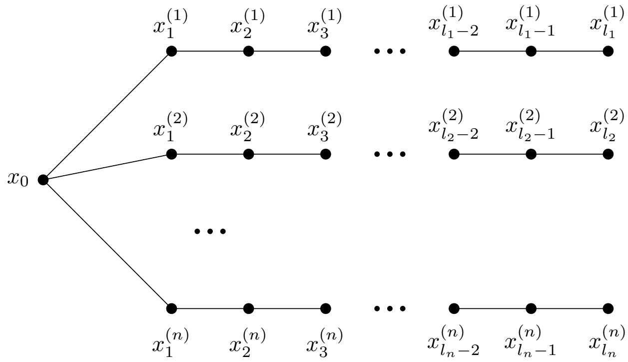

Fig. 2. The spider $SP(l_1,l_2,\dots,l_n)$

First, we start with proving (1). Let $l_j \geq 3$ for every $j \in \{1,2,\dots,n\}$. Then, $s=0$. We will show that $\sigma(SP(l_1,l_2,\dots,l_n)) \leq n$. Let D_1 and D_1^* be a $(1,1)$-dset and a $(1,\overline{2})$-dset of the spider graph, respectively. Suppose that $x_0 \in D_1^*$. Let us consider an arbitrary leg of the length l_j, where $j \in \{1,2,\dots,n\}$. We consider the following cases.

Case 1. Let $l_j = 2p_j$. Vertices $x_1^{(j)}, x_3^{(j)}, \dots, x_{l_j-3}^{(j)}$ and $x_{l_j-2}^{(j)}$, $x_{l_j}^{(j)}$ belong to D_1. Moreover, vertices $x_2^{(j)}, x_4^{(j)}, \dots, x_{l_j-2}^{(j)}$ and the vertex $x_{l_j-1}^{(j)}$ belong to D_1^*. Then

$$D_1 \cap D_1^* \cap \bigcup_{i=1}^{l_j} \{x_i^{(j)}\} = \{x_{l_j-2}^{(j)}\}.$$

Case 2. Let $l_j = 2p_j + 1$. Then vertices $x_1^{(j)}, x_3^{(j)}, \dots, x_{l_j}^{(j)}$ belong to D_1. What is more, vertices $x_2^{(j)}, x_4^{(j)}, \dots, x_{l_j-1}^{(j)}$ and the vertex $x_{l_j-2}^{(j)}$ belong to D_1^*. We obtain

$$D_1 \cap D_1^* \cap \bigcup_{i=1}^{l_j} \{x_i^{(j)}\} = \{x_{l_j-2}^{(j)}\}.$$

In view of the arbitrariness of choice of the leg, since there exist n legs in spider graph, we conclude that the sets D_1 and D_1^* have n common vertices.

Now we will prove that $\sigma(SP(l_1,l_2,\dots,l_n)) \geq n$. Let us consider an arbitrary leg of the length l_j, where $j \in \{1,2,\dots,n\}$. From Lemma 2.1 we obtain that every $(1,1)$-dset and $(1,\overline{2})$-dset of the spider graph have at least one common vertex from the set $\{x_{l_j-2}^{(j)}, x_{l_j-1}^{(j)}, x_{l_j}^{(j)}\}$. In view of arbitrariness of choice of the leg we obtain that every $(1,1)$-dset and $(1,\overline{2})$-dset of the spider graph have at least n common vertices, which ends the proof of this case.

Now, let there exist legs of length 1 and 3 in spider graph. Moreover, $l_j \neq 2$ for every $j \in \{1,2,\dots,n\}$. Let D_2 and D_2^* be a $(1,1)$-dset and a $(1,\overline{2})$-dset of this graph, respectively. Let $x_0 \in D_2^*$. If $l_j = 1$, where $j \in \{1,2,\dots,n\}$, then $x_1^{(j)} \in D_2$.

If $l_j \geq 3$, then we construct the sets D_2 and D_2^* in the same way as the sets D_1 and D_1^*, respectively. Note that the sets D_2 and D_2^* have $n-s$ common vertices, so $\sigma(SP(l_1,l_2,\ldots,l_n)) \leq n-s$.

Now we will prove that $\sigma(SP(l_1,l_2,\ldots,l_n)) \geq n-s$. Let us consider an arbitrary leg of the length $l_j \geq 3$, where $j \in \{1,2,\ldots,n\}$. From Lemma 2.1 we obtain that every $(1,1)$-dset and $(1,\overline{2})$-dset have one common vertex from the set $\{x_{l_j-2}^{(j)}, x_{l_j-1}^{(j)}, x_{l_j}^{(j)}\}$. Note that there exist $n-s$ legs of length greater than or equal to 3 in spider graph. Hence, the sets D_2 and D_2^* have at least $n-s$ common vertices. This means that $\sigma(SP(l_1,l_2,\ldots,l_n)) = n-s$, which ends the proof of (1).

Now, we will prove (2). Therefore let us assume that there exists $l_j \leq 2$ for $j \in \{1,2,\ldots,n\}$. Moreover, let us suppose that if there exist i,k such that $l_i = 1$ and $l_k = 3$, then there exists at least one leg of the length 2 in the spider graph. This assumption is true in some disjoint cases, which we will now consider.

First, let $l_j = 1$ for every $j \in \{1,2,\ldots,n\}$. Then $SP(l_1,l_2,\ldots,l_n) \cong K_{1,n}$. The sets

$$D_3 = \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)}\} \quad \text{and} \quad D_3^* = \{x_0, x_1^{(1)}\}$$

are a $(1,1)$-dset and a $(1,\overline{2})$-dset of the graph, respectively. We have $|D_3 \cap D_3^*| = 1$. Hence, $\sigma(SP(l_1,l_2,\ldots,l_n)) \leq 1$. To show that $\sigma(SP(l_1,l_2,\ldots,l_n)) \geq 1$, for the contradiction let us assume that there exist disjoint $(1,1)$-dset and $(1,\overline{2})$-dset. By Lemma 2.3, we know that all leaves of the graph belong to $(1,1)$-dset. We have $|V(G)\backslash D_3| = 1$. Consequently, the set $V(G)\backslash D_3$ is not the $(1,\overline{2})$-dset, a contradiction. Thus $\sigma(SP(l_1,l_2,\ldots,l_n)) = 1$. Since $n = s$ we obtain $\sigma(SP(l_1,l_2,\ldots,l_n)) = n-s+1$.

Second, let $l_j \leq 2$ for every $j \in \{1,2,\ldots,n\}$ and let there exists $i \in \{1,2,\ldots,n\}$ such that $l_i = 2$. Then the set

$$D_4 = \{x_0\} \cup L(SP(l_1,l_2,\ldots,l_n))$$

is a $(1,1)$-dset of the graph. Moreover,

$$D_4^* = \{x_0\} \cup \{x \in V(SP(l_1,l_2,\ldots,l_n)) : d_{SP}(x) = 2\}$$

is the $(1,\overline{2})$-dset. Clearly, $|D_4 \cap D_4^*| = 1$. Hence $\sigma(SP(l_1,l_2,\ldots,l_n)) \leq 1$. Let us assume that there exist disjoint $(1,1)$-dset and $(1,\overline{2})$-dset of the spider graph. We consider an arbitrary leg of the length 2. From Lemma 2.1 we obtain that every $(1,1)$-dset and $(1,\overline{2})$-dset have at least one common vertex, a contradiction. Hence $\sigma(SP(l_1,l_2,\ldots,l_n)) = 1 = n-s+1$.

Now, assume that there exist legs of length less than or equal to 2 and legs of length greater than or equal to 3 in the spider graph. Moreover, we know that if there exist legs of length 1 and 3, then there exists at least one leg of length 2 at the same time. Let D_5 and D_5^* be $(1,1)$-dset and $(1,\overline{2})$-dset, respectively. Let $x_0 \in D_5$ and $x_0 \in D_5^*$. Clearly, if there exists $j \in \{1,2,\ldots,n\}$ such that $l_j = 1$, then $x_1^{(j)} \in D_5$. If $l_j = 2$, then $x_1^{(j)} \in D_5^*$ and $x_2^{(j)} \in D_5$. Let us consider an arbitrary leg of the length $l_j \geq 3$, where $j \in \{1,2,\ldots,n\}$. We consider the following cases.

Case 1. Let $l_j = 2p_j$. Then vertices $x_2^{(j)}, x_4^{(j)}, \ldots, x_{l_j-2}^{(j)}, x_{l_j}^{(j)}$ belong to D_5. Moreover, vertices $x_1^{(j)}, x_3^{(j)}, \ldots, x_{l_j-1}^{(j)}$ and the vertex $x_{l_j-2}^{(j)}$ belong to D_5^*. We have

$$D_5 \cap D_5^* \cap \bigcup_{i=1}^{l_j} \{x_i^{(j)}\} = \{x_{l_j-2}^{(j)}\}.$$

Case 2. Let $l_j = 2p_j + 1$. Then vertices $x_2^{(j)}, x_4^{(j)}, \ldots, x_{l_j-3}^{(j)}$ and $x_{l_j-2}^{(j)}, x_{l_j}^{(j)}$ belong to D_5. What is more, vertices $x_1^{(j)}, x_3^{(j)}, \ldots, x_{l_j-2}^{(j)}$ and the vertex $x_{l_j-1}^{(j)}$ belong to D_5^*. We obtain

$$D_5 \cap D_5^* \cap \bigcup_{i=1}^{l_j} \{x_i^{(j)}\} = \{x_{l_j-2}^{(j)}\}.$$

Since there are $n-s$ legs of length greater or equal to 3, in view of the arbitrariness of choice the leg we conclude that the sets D_5 and D_5^* have $n-s+1$ common vertices. Hence $\sigma(SP(l_1, l_2, \ldots, l_n)) \leq n-s+1$.

Now we will prove that all $(1,1)$-dsets and $(1,\overline{2})$-dsets of this graph have at least $n-s+1$ common vertices. Let D_6 and D_6^* be $(1,1)$-dset and $(1,\overline{2})$-dset of the spider graph, respectively. We consider an arbitrary leg such that $l_j \geq 3$, where $j \in \{1, 2, \ldots, n\}$. From Lemma 2.1 we obtain

$$|D_6 \cap D_6^* \cap \{x_{l_j-2}^{(j)}, x_{l_j-1}^{(j)}, x_{l_j}^{(j)}\}| \geq 1.$$

Note that there exist $n-s$ legs of the length greater or equal 3 in the spider. Moreover, there exist legs of the length less or equal that 2 in this graph.

If there exists $j \in \{1, 2, \ldots, n\}$ such that $l_j = 2$, then from Lemma 2.1

$$|D_6 \cap D_6^* \cap \{x_0, x_1^{(j)}, x_2^{(j)}\}| \geq 1.$$

Consequently, we obtain that the sets D_6 and D_6^* has at least $n-s+1$ common vertices.

Otherwise, we have $l_j = 1$ or $l_j \geq 4$ for every $j \in \{1, 2, \ldots, n\}$. Let us assume that the sets D_6 and D_6^* do not have any common vertex in the set $\{x_0, x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(j)}\}$. If $l_j = 1$, then $x_1^{(j)} \in D_6$. Hence $x_1^{(j)} \notin D_6^*$. The set D_6^* is $(1,\overline{2})$-dset, so $x_0 \in D_6^*$. Then $x_0 \notin D_6$. Since the vertex $x_1^{(j)}$ must be $(1,2)$-dominated, we obtain $x_1^{(i)} \in D_6^*$ for some i such that $l_i \geq 4$. Then $x_1^{(i)} \notin D_6$. The vertex $x_1^{(i)}$ is not $(1,1)$-dominated, a contradiction. Hence $(1,1)$-dsets and $(1,\overline{2})$-dsets of this graph have at least $n-s+1$ common vertices, which ends the proof. $\square$

Example 2.7. Let us consider the following spider graphs.

For the graph $SP(1,3,5,3,4)$ we have $n = 5$, $s = 1$, so by Theorem 2.6 we obtain $\sigma(SP(1,3,5,3,4)) = 5 - 1 = 4$. Examples of the $(1,1)$-dset and the $(1,\overline{2})$-dset of this graph, which have four common vertices, are denoted by circle and square, respectively, in Figure 3.

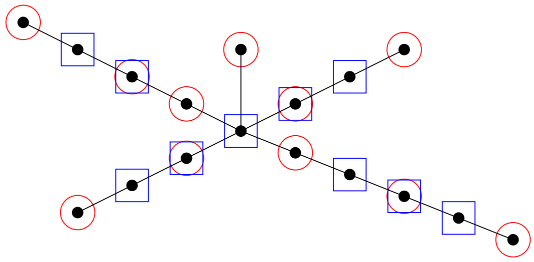

Fig. 3. The spider $SP(1,3,5,3,4)$

For the graph $SP(1,2,4,2,3)$ we have $n = 5$, $s = 3$, so by Theorem 2.6 we obtain $\sigma(SP(1,2,4,2,3)) = 5 - 3 + 1 = 3$. Examples of the $(1,1)$-dset and the $(1,\overline{2})$-dset of this graph, which have three common vertices, are denoted by circle and square, respectively, in Figure 4.

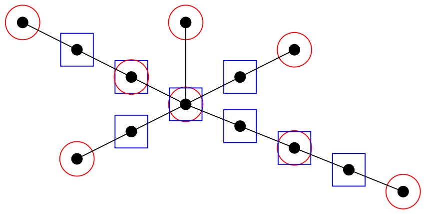

Fig. 4. The spider $SP(1,2,4,2,3)$

From Theorem 2.6 we have the following corollaries concerning special classes of spiders.

Corollary 2.8. *Let $n \geq 2$ be an integer. Then $\sigma(K_{1,n}) = 1$.*

Corollary 2.9. *Let $n \geq 3$ and k be positive integers, $n > k$. Then*

$$\sigma(B(n,k)) = \begin{cases} 1 & \text{for } k \in \{1,2,3,4\}, \\ 2 & \text{for } k \geq 5. \end{cases}$$

As we see from previous results, in some trees there are no disjoint $(1,1)$-dsets and $(1,\overline{2})$-dsets. Now we give necessary conditions for the existence of such disjoint sets in trees.

Theorem 2.10. *Let T be a tree. If $\sigma(T) = 0$, then*

(1) $|V(T)| \geq 6$ *and* $\operatorname{diam}(T) \geq 3$, *and*
(2) *T does not contain a pendant path P_n, where $n \geq 3$, and*
(3) *if $x \in S(T)$ and x is not a strong support vertex of T, then there exists $y \in V(T) \backslash (L(T) \cup S(T))$ such that $xy \in E(T)$, and*
(4) *if $x \in S(T)$ and $N_T(x) \cap S(T) = \emptyset$, then there exists $y \in V(T) \backslash (L(T) \cup S(T))$ such that $xy \in E(T)$ and $d_T(y) \geq 3$.*

Proof. Let T be an arbitrary tree and let D and D^* be respectively the $(1,1)$-dset and $(1,\overline{2})$-dset of the tree. Let assume that the sets D and D^* are disjoint. We will show that all conditions (1)–(4) must hold.

First, assume that the condition (1) does not hold. It is easy to check that for every tree T such that $V(T) < 6$ or $\mathrm{diam}(T) < 3$ we have that no $(1,\overline{2})$-set exists or $\sigma(T) > 0$.

Second, assume that the condition (2) does not hold, i.e. the tree T contains a pendant path $x_1x_2x_3\ldots x_n$, where $n \geq 3$. From Lemma 2.1 we obtain that the sets D and D^* are not disjoint, a contradiction.

Next, assume that the condition (3) does not hold. Then the vertex $x \in S(T)$ is adjacent only to exactly one leaf $v \in V(T)$ and other support vertices. By Lemma 2.3, we know that $v \in D$ and $x \in D^*$. Moreover, all support vertices which are adjacent to the vertex x, belong to the set D^*. Hence the vertex x is not $(1,1)$-dominated, a contradiction.

Finally, let us suppose that the condition (4) does not hold. It means that $x \in S(T)$ and $N_T(x) \cap S(T) = \emptyset$ and every vertex from the set $V(T)\backslash(L(T) \cup S(T))$ is not adjacent to x or has degree less than 3. If the former is true we obtain either $T \cong K_{1,n}$, a contradiction with Corollary 2.8, or T is not connected, a contradiction with the fact T is a tree. Therefore let us assume that all vertices from the set $V(T)\backslash(L(T) \cup S(T))$ which are adjacent to x have degree less than 3. Moreover, at least one of them, say y, must belong to D^* to ensure that the set $N_T(x) \cap L(T)$ is $(1,2)$-dominated by D^*. Hence $y \notin D$ and it may be adjacent to at most one vertex from D. This means that D is not a $(1,1)$-dset, a contradicton.

We obtain that if the tree T has disjoint $(1,1)$-dset and $(1,\overline{2})$-dset, then the tree T satisfies the conditions (1)–(4), which ends the proof. □

However, it turns out that the conditions (1)–(4) from Theorem 2.10 are not sufficient. There are trees, which satisfy them all and do not have disjoint $(1,1)$-dset and $(1,\overline{2})$-dset. In Figure 5 we can see an example of such tree.

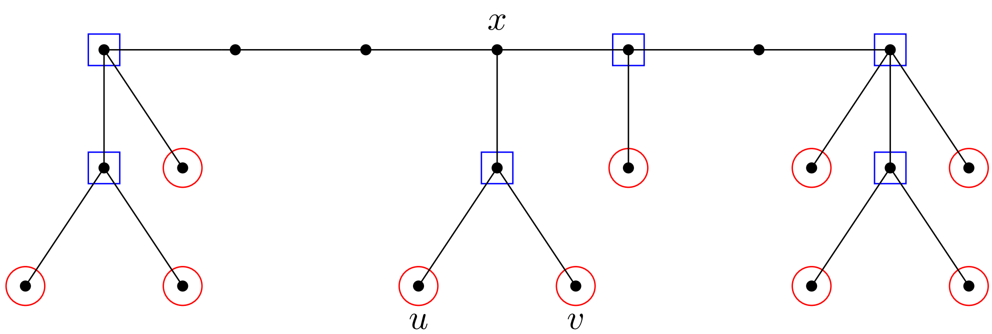

Fig. 5. An example of the tree which does not have disjoint $(1,1)$-dset and $(1,\overline{2})$-dset

From Lemma 2.3 we know that if the graph has disjoint $(1,1)$-dset and $(1,\overline{2})$-dset, then all leaves (denoted by circle in Figure 5) belong to $(1,1)$-dset and all support vertices (denoted by square in Figure 5) belong to $(1,\overline{2})$-dset. If the vertex x belonged to the $(1,1)$-dset and did not belong to $(1,\overline{2})$-dset, then the leaves v and u would not be $(1,2)$-dominated. If the vertex x belonged to $(1,\overline{2})$-dset and did not belong to $(1,1)$-dset, then it would not be $(1,1)$-dominated.

Now we will give the value of the parameter σ for the complete bipartite graph $K_{m,n}$.

Theorem 2.11. *Let m,n be integers. Then for $m+n \geq 3$ holds*

$$\sigma(K_{m,n}) = \begin{cases} 0 & \textit{for } m \geq 3 \textit{ and } n \geq 3, \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. Let $K_{m,n}$ be the complete bipartite graph and let $V_1 = \{x_1, x_2, \ldots, x_m\}$ and $V_2 = \{y_1, y_2, \ldots, y_n\}$.

Firsty, assume that $m \geq 3$ and $n \geq 3$. Then the set $D^* = \{x_1, y_1\}$ is a $(1,\overline{2})$-dset of $K_{m,n}$ and the set $D = V(K_{m,n}) \backslash D^*$ is a $(1,1)$-dset of this graph. Hence, $\sigma(K_{m,n}) = 0$.

Now, let us assume that one of the sets V_1, V_2 has less than 3 vertices. Without loss of generality, let $m \leq 2$.

First, if $m = 1$, then the complete bipartite graph is isomorphic to the star $K_{1,n}$, $n \geq 2$. From Corollary 2.8 we know that $\sigma(K_{1,n}) = 1$.

Next, assume that $m = 2$ and $n = 2$. Then $K_{2,2} \cong C_4$ and from Theorem 2.5 we obtain that $\sigma(K_{2,2}) = 1$.

Finally, let $m = 2$ and $n \geq 3$. We will prove that $\sigma(K_{2,n}) \leq 1$. The set $D^* = \{x_1, y_1\}$ is the $(1,\overline{2})$-dset and the set $D = (V(K_{2,n}) \backslash D^*) \cup \{x_1\}$ is the $(1,1)$-dset of this graph. We have $|D \cap D^*| = 1$.

Now we will prove that $\sigma(K_{2,n}) \geq 1$. Suppose that there exist $(1,1)$-dset D and $(1,\overline{2})$-dset D^* of $K_{m,n}$ which are disjoint. Note that every set which contains V_1 or V_2 is not a $(1,\overline{2})$-dset of $K_{2,n}$ (because it is the $(1,1)$-dset of this graph). This means that there exists a vertex $x_1 \in V_1$ such that $x_1 \notin D^*$. Then, there exists at least one vertex $y_1 \in V_2$ such that $y_1 \in D^*$. Moreover, there exists $y_2 \in V_2$, $y_2 \neq y_1$ such that $y_2 \notin D^*$. Hence, there exists $x_2 \in V_1$, $x_2 \neq x_1$ such that $x_2 \in D^*$. Note that $x_2 \notin D$ and $y_1 \notin D$. We obtain that the vertex y_1 is not $(1,1)$-dominated by the set D, a contradiction. □

The next result gives a complete characterization of a join of graphs with the $(1,\overline{2})$-intersection index equal to 0.

Theorem 2.12. *Let H_1, H_2 be arbitrary, nonempty graphs. Then $\sigma(H_1 + H_2) = 0$ if and only if*

(1) *$H_1 \ncong K_n$ or $H_2 \ncong K_n$, $n \geq 1$, and*
(2) *if $H_1 \cong K_1$ and H_2 has an isolated vertex then there exists a vertex $v \in V(H_2)$ such that $d_{H_2}(v) \geq 2$, and*
(3) *if $|V(H_1)| = 2$ then $H_2 \ncong N_n$, $n \geq 1$.*

Proof. Let H_1 and H_2 be arbitrary graphs such that $V(H_1) = \{x_1, x_2, \ldots, x_n\}$ and $V(H_2) = \{y_1, y_2, \ldots, y_m\}$, $n, m \geq 1$.

First we will prove the necessary condition. Let us assume that $\sigma(H_1 + H_2) = 0$, i.e. there exist a $(1,1)$-dset D and $(1,\overline{2})$-dset D^* such that $D \cap D^* = \emptyset$. We will show that all condictions (1)–(3) must hold.

Assume first that the condition (1) does not hold. Then $H_1 + H_2$ is complete and from Theorem 1.1 we obtain it does not have any $(1,\overline{2})$-dset.

Second, let us suppose that the condition (2) does not hold, i.e. $H_1 \cong K_1$ and H_2 has an isolated vertex and $\Delta(H_2) \leq 1$. If for every vertex $y \in V(H_2)$ we have $d_{H_2}(y) = 0$, then $H_1 + H_2 \cong K_{1,l}$, $l \geq 1$. From Corollary 2.8 we obtain a contradiction. Now, let there exist a vertex $y_j \in V(H_2)$ such that $d_{H_2}(y_j) = 0$ and at least two vertices $y_i, y_k \in V(H_2)$ such that $d_{H_2}(y_i) = d_{H_2}(y_k) = 1$. Note that $d_{H_1+H_2}(y_j) = 1$. Then the vertex $x \in V(H_1)$ is a support vertex of $H_1 + H_2$. By Lemma 2.3 we obtain that $y_j \in D$ and $x \in D^*$. We have $x \notin D$ and $d_{H_1+H_2}(y_i) = d_{H_1+H_2}(y_k) = 2$, so the vertices y_i, y_k must belong to D, otherwise they would not be $(1,1)$-dominated by D. Extending this reasoning to all vertices of degree 1 in graph H_2 we obtain $V(H_2) \subseteq D$. But the set $V(H_1 + H_2) \backslash V(H_2) = \{x_1\}$ is not the $(1,\overline{2})$-dset of the graph $H_1 + H_2$, a contradiction.

Finally, assume that the condition (3) does not hold. It means that $|V(H_1)| = 2$ and $H_2 \cong N_n$, $n \geq 1$. If $H_1 \cong N_2$, then $H_1 + H_2 \cong K_{m,n}$, $m = 2$, $n \geq 1$. From Theorem 2.11 we obtain that the graph $H_1 + H_2$ does not have disjoint $(1,1)$-dset and $(1,\overline{2})$-dset. Now, let us suppose that $H_1 \cong P_2$. If $|V(H_2)| = 1$, then $H_1 + H_2 \cong K_3$. By Theorem 1.1 we know it does not have a $(1,\overline{2})$-dset. Let assume that $|V(H_2)| \geq 2$. We consider the following cases.

Case 1. Let at least one of vertices of H_1, say x_1, belong to D^*. Note that if $x_2 \in D^*$, then the set D^* would not be a proper $(1,2)$-dset. Since the vertex x_2 must be $(1,2)$-dominated, there must exist a vertex $y \in V(H_2) \cap D^*$. Since $N_{H_1+H_2}(y) = \{x_1, x_2\}$ and $x_1 \notin D$, we obtain that the vertex y is not $(1,1)$-dominated by D, a contradiction.

Case 2. Let $x_1 \notin D^*$ and $x_2 \notin D^*$. Then there exist two vertices $y_i \in V(H_2)$ and $y_j \in V(H_2)$, $i \neq j$ such that $y_i \in D^*$ and $y_j \in D^*$. If for every $y \in V(H_2)$ we have $y \in D^*$, then the set D^* is the $(1,1)$-dset of the graph, so it is not the $(1,\overline{2})$-dset, a contradiction. If there exists $y \notin D^*$, then the vertex y is not dominated by the set D^*, which ends the proof in this case.

We obtain that if the join $H_1 + H_2$ has disjoint $(1,1)$-dset and $(1,\overline{2})$-dset, then the graphs H_1, H_2 satisfy the conditions (1)–(3). Now, we will prove the sufficient condition.

First, let $|V(H_1)| \geq 3$ and $|V(H_2)| \geq 3$. From the condition (1) we conclude that at least one of the graphs H_1, H_2, say H_1, is not a complete graph. It means that there exists the vertex $x \in V(H_1)$ such that $d_{H_1}(x) \leq |V(H_1)| - 2$. Let $x \in D^*$. Moreover, without loss of generality we choose some vertex y from the set $V(H_2)$. Suppose that $y \in D^*$. Then the set $D^* = \{x, y\}$ is the $(1,\overline{2})$-dset of $H_1 + H_2$, and the set $D = V(H_1 + H_2) \backslash D^*$ is the $(1,1)$-dset. Hence, $\sigma(H_1 + H_2) = 0$.

Second, let us suppose that $|V(H_1)| \leq 2$ or $|V(H_2)| \leq 2$. Without loss of generality assume that $|V(H_1)| \leq |V(H_2)|$. We consider the following cases.

Case 1. Let $|V(H_1)| = 1$. Let us consider two subcases.

Subcase 1.1. Let assume that there exists the vertex $y \in V(H_2)$ such that $d_{H_2}(y) \geq 2$. Note that $|V(H_2)| \geq 3$. Then the set $D^* = V(H_1) \cup \{y\}$ is the (1,$\overline{2}$)-dset of $H_1 + H_2$. Moreover, the set $D = V(H_1 + H_2)\backslash D^*$ is the $(1,1)$-dset of the graph.

Subcase 1.2. Now, let $d_{H_2}(y) \leq 1$ for every vertex from the set $V(H_2)$. By condition (2) we know that H_2 does not have any isolated vertex. Hence for every $y \in V(H_2)$ we have $d_{H_2}(y) = 1$. Since $H_1 + H_2$ is not a complete graph, we obtain that $|V(H_2)| \geq 4$. Then the graph H_2 is not connected graph and consists of s disjoint paths P_2, $s \geq 2$. Without loss of generality we choose exactly one vertex from each path. The union of these vertices is the (1,$\overline{2}$)-dset D^* of $H_1 + H_2$. Then the set $D = V(H_1 + H_2)\backslash D^*$ is the $(1,1)$-dset, which ends the proof of case 1.

Case 2. Let $|V(H_1)| = 2$. From the condition (3) we know that $H_2 \ncong N_n$, $n \geq 2$. Since, $H_1 + H_2$ is not a complete graph, we have $|V(H_2)| \geq 3$. Moreover, in view of the fact that $H_2 \ncong N_n$ we obtain that there exists the vertex $y \in V(H_2)$ of degree greater or equal to 1 in graph H_2. Let $y \in D^*$. Additionally, let one of the vertices of H_1, say x_1, belong to D^*. We have $D^* = \{y, x_1\}$ is the (1,$\overline{2}$)-dset of $H_1 + H_2$. The set $D = V(H_1 + H_2)\backslash D^*$ is the $(1,1)$-dset of this graph.

We obtain that if the conditions (1)–(3) hold, the join $H_1 + H_2$ has a $(1,1)$-dset and a (1,$\overline{2}$)-dset, which are disjoint, which ends the proof. □

The final result of the paper gives the complete characterization of a corona of graph with the (1,$\overline{2}$)-intersection index equal to 0.

Theorem 2.13. *Let $G \ncong K_1$ be a connected graph. Then $\sigma(G \circ H) = 0$ if and only if $H \ncong K_1$.*

Proof. Let G be a connected graph such that $G \ncong K_1$. First, assume that $H \ncong K_1$. Note that $|V(G)| \geq 2$ and $|V(H)| \geq 2$. Then $V(G)$ is the (1,$\overline{2}$)-dset of $G \circ H$. Moreover, the set $V(G \circ H)\backslash V(G)$ is $(1,1)$-dset of the graph. We obtain that $\sigma(G \circ H) = 0$.

Now, for the contradiction suppose that $H \cong K_1$ and $\sigma(G \circ H) = 0$ i.e. $G \circ H$ has disjoint $(1,1)$-dset and (1,$\overline{2}$)-dset denoted by D and D^*, respectively. We have $S(G \circ H) = V(G)$ and $L(G \circ H) = V(G \circ H)\backslash V(G)$. By Lemma 2.3 we obtain that all vertices of G must belong to D^* and all vertices from the set $V(G \circ H)\backslash V(G)$ belong to D. Then the vertices of G are not $(1,1)$-dominated by D, a contradiction. □

3. CLOSING REMARKS

In this paper we determined the (1,$\overline{2}$)-intersection index for some classes of graphs such as cycles, complete bipartite graphs and spiders. We also completely characterized the join of graphs and the corona of two graphs, for which the (1,$\overline{2}$)-intersection index is equal to 0. Moreover, we gave the neccessary conditions for trees T such that $\sigma(T) = 0$. However, finding a complete characterization of trees with the (1,$\overline{2}$)-intersection index equal to 0 we leave as an open problem. Other interesting questions related to our topic were suggested by one of the referees:

1. What is the maximum possible value of $\sigma(G)$?
2. Is $\sigma(G) = 0$ providing $\delta(G) > 2$ and $|V(G)|$ is large?
3. What is the computational complexity of the problem of determining $\sigma(G)$?

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