Opuscula Math. 43, no. 3 (2023), 393–407
https://doi.org/10.7494/OpMath.2023.43.3.393 Opuscula Mathematica

AXIOMATIC CHARACTERIZATIONS OF PTOLEMAIC AND CHORDAL GRAPHS

Manoj Changat, Lekshmi Kamal K. Sheela,
and Prasanth G. Narasimha-Shenoi

Communicated by Andrzej Żak

Abstract. The interval function and the induced path function are two well studied class of set functions of a connected graph having interesting properties and applications to convexity, metric graph theory. Both these functions can be framed as special instances of a general set function termed as a transit function defined on the Cartesian product of a non-empty set V to the power set of V satisfying the expansive, symmetric and idempotent axioms. In this paper, we propose a set of independent first order betweenness axioms on an arbitrary transit function and provide characterization of the interval function of Ptolemaic graphs and the induced path function of chordal graphs in terms of an arbitrary transit function. This in turn gives new characterizations of the Ptolemaic and chordal graphs.

Keywords: interval function, betweenness axioms, Ptolemaic graphs, transit function, induced path transit function.

Mathematics Subject Classification: 05C12, 05C75.

1. INTRODUCTION

Transit functions were introduced by Mulder [15] to generalize the three classical notions in mathematics, namely, convexity, interval and betweenness in an axiomatic approach.

Given a non-empty set V, a *transit function* is defined as a function $R : V \times V \longrightarrow 2^V$ satisfying the following three axioms:

($t1$) $u \in R(u, v)$ for all $u, v \in V$,
($t2$) $R(u, v) = R(v, u)$ for all $u, v \in V$,
($t3$) $R(u, u) = \{u\}$ for all $u \in V$.

We refer to R as a transit function on V. If V is the vertex set of a graph G, then we say that R is a transit function on G. Throughout this paper, we consider only finite, simple and connected graphs. Given a transit function R on V, one can define the underlying graph G_R of a transit function R on V as the graph with vertex set V, where two distinct vertices u and v are joined by an edge if and only if $R(u,v) = \{u,v\}$.

A u,v-*shortest path* in a connected graph $G = (V,E)$ is a u,v-path in G containing the minimum number of edges. The length $d(u,v)$ of the shortest u,v-path P (that is, the number of edges in P) is the standard distance in G. The interval function I_G of a connected graph G is the function $I_G : V \times V \longrightarrow 2^V$ defined with respect to the standard distance d in G as

$$\begin{aligned} I_G(u,v) &= \{w \in V : d(u,w) + d(w,v) = d(u,v)\} \\ &= \{w \in V : w \text{ lies on some shortest } u,v\text{-path in } G\}. \end{aligned}$$

The interval function I_G is a classical example of a transit function on a graph (we sometimes denote I_G by I). It is easy to see that the underlying graph G_{I_G} of I_G is isomorphic to G. The interval function of a connected graph is extensively used in metric graph theory. Mulder in [14] has systematically studied I_G by an axiomatic approach.

A u,v-*induced path* P in a connected graph $G = (V,E)$ is a u,v-path in G containing no shortcuts in the sense that it contains no chords (a *chord* in a path P is an edge between two non-consecutive vertices in P). It follows that the shortest path is always an induced path. This observation defines another interesting transit function on a connected graph $G = (V,E)$ as a natural generalization of the interval function, named as the induced path function, also known as monophonic function or minimal path function which is the function $J_G : V \times V \longrightarrow 2^V$:

$$J_G(u,v) = \{w \in V\colon w \text{ lies on some induced } u,v\text{-path in } G\}.$$

Nebeský addressed an interesting problem on the interval function I of a connected graph $G = (V,E)$ during the 1990s as follows: Is it possible to give a characterization of the interval function I_G of a connected graph G by a set of simple axioms (first-order axioms) defined on an arbitrary transit function R on V? Nebeský [17, 18] proved that there exists such a characterization for the interval function $I(u,v)$ in terms of a set of first-order axioms on a transit function R. More such characterizations are described in [16, 19–22]. The axiomatic characterization of I_G is extended to disconnected graphs in [3]. Nebeský also proved an interesting result on the induced path function J_G of G, that a first order axiomatic characterization similar to the interval function I_G is not possible for J_G in [23]. The following three axioms denoted as $(b2)$, $(b3)$, and $(b4)$ together with the defining transit axioms $(t1)$, $(t2)$ are essential in all the characterizations of the function I.

$(b2)$ If $x \in R(u,v)$, then $R(u,x) \subseteq R(u,v)$, for all $u,v,x \in V$,
$(b3)$ If $x \in R(u,v)$ and $y \in R(u,x)$, then $x \in R(y,v)$, for all $u,v,x,y \in V$,
$(b4)$ If $x \in R(u,v)$, then $R(u,x) \cap R(x,v) = \{x\}$, for all $u,v,x \in V$.

We have a weaker axiom than $(b3)$, named as $(b1)$, and is defined as:

$(b1)$ If $x \in R(u,v)$ and $x \neq v$, then $v \notin R(u,x)$, for all $u,v,x \in V$.

The following implications can be easily verified for a function $R : V \times V \longrightarrow 2^V$ among axioms $(t1)$, $(t2)$, $(t3)$, $(b1)$, $(b3)$ and $(b4)$.

(i) Axioms $(t1)$ and $(b4)$ implies axiom $(t3)$.
(ii) Axioms $(t1), (t2), (t3)$ and $(b3)$ implies axiom $(b4)$ which implies axiom $(b1)$ (that is, for a transit function $R, (b3)$ implies $(b4)$ implies $(b1)$).

The converse of the above implications does not hold in general. A transit function R satisfying axioms $(b2)$ and $(b3)$ is known as a *geometric transit function*.

The problem of characterizing the interval function of an arbitrary connected graph can be adopted for different graph classes; viz., characterizing the interval function of special graph classes using a set of first-order axioms on an arbitrary transit function. Such a problem was first attempted by Sholander in [24] with a partial proof for characterizing the interval function of trees under the name *tree betweenness*. Chvátal *et al.* [10] obtained the completion of this proof. Further new characterizations of the interval function of trees and block graphs are discussed in [1]. Axiomatic characterization of the interval function of median graphs, modular graphs, geodetic graphs, (claw, paw)-free graphs and bipartite graphs are respectively described in [4,5,14,16,19]. In [24], Sholander termed the interval function of a graph as segment function and considered ternary relation B on the vertex set V of the graph G to study tree betweenness. We can easily translate such a ternary relation into a function $R : V \times V \longrightarrow 2^V$ by defining $R(u,v)$ to be the set of all x for which (u,x,v) is in B. Sholander introduced two more betweenness axioms, which can be translated into our terminologies as a function R satisfying axiom $(t3)$ and the following additional axiom.

(C) $x \in R(u,v)$ and $y \in R(x,z)$, then $x \in R(v,y)$ or $x \in R(z,u)$ for all $u,v,x,y,z \in V$.

It turns out that this axiom is quite strong: Sholander proved that axioms $(t3)$ and (C) imply axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$ and the following axiom.

$(J0)$ If $x \in R(u,y)$ and $y \in R(x,v)$, then $x \in R(u,v)$, for distinct $u,v,x,y \in V$.

One can show that both I_G and J_G of a connected graph G does not satisfy $(J0)$ in general. For example, consider the 3-fan and the induced cycle of length at least four C_n, $n \geq 4$ in Figure 1. For the 3-fan, $x \in I(u,y)$, $y \in I(x,v)$, but $x \notin I(u,v)$. For the induced cycle, C_n, $n \geq 4$, $x \in J(u,y)$, $y \in J(x,v)$, but $x \notin J(u,v)$. In [6], it is proved that I_G or J_G of a connected graph G satisfies $(J0)$ if and only if G is Ptolemaic graph or a chordal graph, respectively.

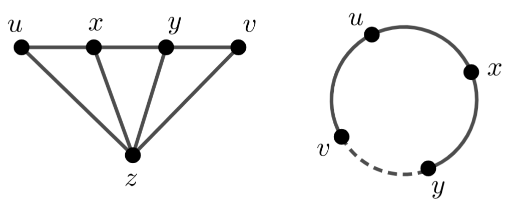

Fig. 1. 3-fan, C_{n+4}

The standard shortest path distance (d) of a graph G is an α_0-*metric* if for any edge vw of G and any two vertices u, x such that $v \in I(u,w)$ and $w \in I(v,x)$, the equality $d(u,x) = d(u,v) + d(v,w) + d(w,x)$ holds. In [9], Chepoi proved that a connected graph G is Ptolemaic if and only if the graph metric in G is α_0-metric. We can easily see that the graph metric in G is an α_0-metric if its interval function I_G satisfies axiom $(J0)$.

In the rest of this section, we fix some of the graph theoretical notations and terminology used in this paper. Let G be a graph and H a subgraph of G. H is called an *isometric* subgraph of G if the distance $d_H(u,v)$ between any pair of vertices, u, v in H coincides with that of the distance $d_G(u,v)$. H is called an *induced* subgraph if u, v are vertices in H such that uv is an edge in G, then uv must be an edge in H also. A path in G which is induced as a subgraph is an *induced path*. A graph G is said to be H*-free*, if G has no induced subgraph isomorphic to H. Let $G_1, G_2, \ldots, G_k$ be graphs. For a graph G, we say that G is $G_1, G_2, \ldots, G_k$*-free* if G has no induced subgraph isomorphic to G_i, $i \in \{1, \ldots, k\}$.

Chordal graph is an example of a graph G which is defined by an infinite number of forbidden induced subgraphs (G is chordal if G have no induced cycles C_n for $n \geq 4$). There are several graphs that can be defined or characterized by a list of forbidden induced subgraphs or isometric subgraphs. See the survey by Brandstädt *et al.* [2] and the information system [11], for such graph families. *Ptolemaic graphs* were introduced by Kay and Chartrand in [13] as graphs in which the distances obey the Ptolemy inequality. That is, for every four vertices u, v, w and x the inequality $d(u,v)d(w,x) + d(u,x)d(v,w) \geq d(u,w)d(v,x)$ holds. Howorka in [12] proved that a graph is Ptolemaic if and only if it is both chordal and distance-hereditary (a graph G is distance hereditary, if every induced path in G is isometric) so that it is a chordal graph which is 3-fan-free.

Thus, we observe that Ptolemaic graphs possess a characterization in terms of a list of forbidden induced subgraphs.

In this paper, we prove that an arbitrary transit function R satisfying axiom $(J0)$, forbids induced C_n, $n \geq 4$ in the underlying graph G_R. We further explore the property of axiom $(J0)$ on an arbitrary transit function R. That is, our approach is more general in the sense that we consider an arbitrary transit function R defined on a non-empty set V. We pose a set of betweenness axioms on R including axiom $(J0)$ so that R will be the interval function or the induced path transit function of its underlying graph G_R. We prove that in the case of R being an interval function of G_R, if R satisfies axiom $(J0)$, then G_R is a Ptolemaic graph. Similarly, we prove that when R become the induced path transit function of G_R and if R satisfies axiom $(J0)$, then G_R is a chordal graph.

We organize the results of the paper as follows. In Section 2, the relation between the axioms that we consider in this paper, namely the geometric axioms $(b2)$, $(b3)$, $(J0)$ and $(J2)$ will be discussed. In Section 3, we provide a characterization of the interval function of a Ptolemaic graph using the axioms $(J0), (b3), (J2)$ on an arbitrary transit function. In Section 4, we prove that the induced path transit function of a chordal graph is possible using first order axioms.

2. RELATIONS BETWEEN THE BETWEENNESS AXIOMS

In this section, we discuss the relationship between the betweenness axioms that we consider in this paper. In addition to the geometric axioms $(b2)$ and $(b3)$, and the axiom $(J0)$, we consider the following betweenness axioms $(J2)$ for a transit function R on V for proving the mentioned characterizations. Let V be a nonempty set and R be a transit function on V,

$(J2)$ If $R(u,x) = \{u,x\}$, $R(x,v) = \{x,v\}$ and $R(u,v) \neq \{u,v\}$, then $x \in R(u,v)$, for distinct $u,x,v \in V$.

From the definition of the axiom, we observe the following. Axiom $(J2)$ is a simple betweenness axiom which is always satisfied by the interval function I and the induced path transit function J of a graph G.

Theorem 2.1. *Let R be any transit function defined on a non-empty set V. If R satisfies $(J0)$ and $(J2)$, then the underlying graph G_R of R is a chordal graph.*

Proof. Let R be a transit function satisfying $(J0)$ and $(J2)$. Let G_R contains an induced cycle, say $C_n = u_1u_2\dots u_nu_1$ for $n \geq 4$. Without loss of generality assume C_n is a minimum such cycle (in the sense that the length of an induced cycle is as small as possible).

Since $R(u_1,u_2) = \{u_1,u_2\}$ and $R(u_2,u_3) = \{u_2,u_3\}$, then by $(J2)$ we have $u_2 \in R(u_1,u_3)$. By a similar argument, we can prove that $u_3 \in R(u_2,u_4)$. By $(J0)$ axiom we have $u_2 \in R(u_1,u_4)$ and by $(t2)$ we have $u_3 \in R(u_1,u_4)$. By continuing these steps we get $u_2,\dots,u_{n-2} \in R(u_1,u_{n-1})$ and $u_3,\dots,u_{n-1} \in R(u_2,u_n)$. Since $u_2 \in R(u_1,u_{n-1})$ and $u_{n-1} \in R(u_2,u_n)$, by $(J0)$, we have $u_2 \in R(u_1,u_n)$, which is a contradiction to the fact that $R(u_1,u_n) = \{u_1,u_n\}$.

Hence, G_R does not contain C_n where $n \geq 4$ as an induced subgraph. This completes the proof. □

The following straightforward Lemma for the connectedness of the underlying graph G_R of a transit function R is proved in [7].

Lemma 2.2 ([7])**.** *If the transit function R on a non-empty set V satisfies axioms $(b1)$ and $(b2)$, then the underlying graph G_R of R is connected.*

We have the following proposition.

Proposition 2.3. *If R is a transit function on V satisfying the axioms $(J0)$ and $(b3)$, then R satisfies axiom $(b2)$ and G_R is connected.*

Proof. Let R satisfies axioms $(J0)$ and $(b3)$. To prove that R satisfies $(b2)$ let $x \in R(u,v)$, and $y \in R(u,x)$ for any $u,v,x,y \in V$. Since R satisfies $(b3)$, we have $x \in R(y,v)$. Now $y \in R(u,x)$, $x \in R(y,v)$ and so by axiom $(J0)$, we have $y \in R(u,v)$, which implies that R satisfies $(b2)$. Connectedness of G_R follows from Lemma 2.2, since R satisfies axioms $(b1)$ and $(b2)$ as axiom $(b3)$ implies axiom $(b1)$. □

Example 2.4 (There exists a transit function R that satisfies $(J0),(J2)$ and $(b2)$ but not $(b3)$)**.** Let $V = \{u, v, w, x, y\}$. Let $R : V \times V \to 2^V$ be defined by $R(u, v) = V$, $R(u, x) = \{u, y, w, x\}$, $R(w, v) = \{x, w, y, v\}$ and in all other cases $R(a, b) = \{a, b\}$. Then R satisfies $(J0)$ and $(J2)$. Next we show that R satisfies $(b2)$. Since $R(u, v) = V$, we can see that $R(u, a) \subseteq R(u, v)$ for all $a \in R(u, v)$ so that for this pair R satisfies $(b2)$. For $R(u, x)$, we can see that $a \in R(u, x) \setminus \{u, x\}$, we have $R(u, a) = \{u, a\}$ and $R(x, a) = \{x, a\}$ which implies that R satisfies $(b2)$ for this pair too. The case is similar for $R(w, v)$. All other pairs correspond to edges. Hence, we can see that R satisfies $(b2)$ axiom. Now $x \in R(u, v)$, $y \in R(u, x)$ but $x \notin R(y, v) = \{y, v\}$, and R violates $(b3)$ axiom.

3. AXIOMATIC CHARACTERIZATION OF THE INTERVAL FUNCTION OF PTOLEMAIC GRAPHS

For the axiomatic characterization of I_G of a Ptolemaic graph G, the essential axiom is $(J0)$. The two forbidden induced subgraphs (3-fan and $C_{n+4}, n \geq 0$) of a Ptolemaic graph is depicted in Figure 1. In [6], Changat *et al.* characterized the graphs for which the interval function satisfies $(J0)$ as follows.

Theorem 3.1 ([6])**.** *Let G be a graph. The interval function I_G satisfies the axiom $(J0)$ if and only if G is a Ptolemaic graph.*

Now, we have the following Theorem on an arbitrary transit function R stating the necessary conditions to have its underlying graph G_R a Ptolemaic graph and R as the interval function of G_R.

Theorem 3.2. *If R is a transit function satisfying $(b3),(J0)$ and $(J2)$, then G_R is Ptolemaic and $R(u, v) = I(u, v)$.*

Proof. Since R satisfies $(b3),(J0)$ and $(J2)$, we have that G_R is a chordal graph by Theorem 2.1. To prove that G_R is Ptolemaic, we have to show that G_R is 3-fan-free. Suppose that G_R contains an induced 3-fan with vertices u, x, y, v, z as shown on Figure 1. Since ux and xy are edges and uy is not an edge, by $(J2)$, $x \in R(u, y)$. Similarly, $y \in R(x, v)$. Since R is a transit function, by $(t2)$, $y \in R(v, x)$ and $x \in R(y, u)$ and hence by $(J0)$, $y \in R(u, v)$. Again, since uz and zy are edges and uy is not an edge, $z \in R(u, y)$. That is, $y \in R(u, v)$ and $z \in R(u, y)$, by $(b3)$, we have $y \in R(z, v)$, which is not true as zv is an edge. That is, we have proved that G_R is a chordal graph which is 3-fan-free and hence G_R is a Ptolemaic graph. By Lemma 2.3, R satisfies axiom $(b2)$ and G_R is connected, moreover (b3) implies (b1).

Now we prove that $R(u, v) = I(u, v)$ for all $u, v \in V$. We prove by induction on the distance between u and v. Clearly $R(u, v) = \{u, v\} = I(u, v)$ when $uv \in E(G_R)$.

Let next $d(u, v) = 2$. Let $x \in I(u, v)$. Hence, we can see that $ux, xv \in E(G_R)$. That is, $R(u, x) = \{u, x\}$, $R(x, v) = \{x, v\}$ and $R(u, v) \neq \{u, v\}$, since R satisfies $(J2)$, $x \in R(u, v)$. Therefore, $I(u, v) \subseteq R(u, v)$. Conversely, suppose $x \in R(u, v)$. Suppose $x \notin I(u, v)$. Since $d(u, v) = 2$ there exists at least one element $y \in I(u, v)$ such that uy, yv are edges in G_R. By assumption, x is not adjacent to both u and v. Assume that xu is not an edge. Since $x \in R(u,v)$ and R satisfies $(b2)$ and $(b1)$, $R(u,x) \subset R(u,v)$ with $|R(u,x)| < |R(u,v)|$. By applying axioms $(b2)$ and $(b1)$ continuously on $R(u,x)$, we get vertices $x_i, x_{i+1}, \ldots, x_k, x_{k+1} = x \in R(u,x)$ such that $R(x_i,u) \subset R(x_{i+1},u) \subset R(u,x) \subset R(u,v)$ and $|R(x_i,u)| < |R(x_{i+1},u)|$, for $i \in \{1,\ldots,k\}$ and since V is finite, $R(u,x_i) = \{u,x_i\}$, for some i, say $i = 1$. That is, we have vertices $x_1, x_2, \ldots, x_k, x_{k+1} = x \in R(u,x)$ with $R(x_1,u) = \{x_1,u\}$. If $y \in R(u,x)$ and since $x \in R(u,v)$ then $x \in R(y,v)$ by $(b3)$, a contradiction to $R(y,v) = \{y,v\}$. Therefore, $x_i \neq y$ for all $i \in \{1,\ldots,k\}$. Next, we have to prove that $R(x_1,y) = \{x_1,y\}$. If not let us assume that $R(x_1,y) \neq \{x_1,y\}$. That is $x_1y \notin E(G_R)$. Consider the vertices x_1,u,y,v. By $(J2)$, $u \in R(x_1,y)$ and since $y \in R(u,v)$, by $(J0)$, $u \in R(x_1,v)$. Therefore, $x_1 \in R(v,u)$, $u \in R(v,x_1)$ and hence by $(b3)$, $x_1 \in R(u,u)$, a contradiction. Therefore, $R(x_1,y) = \{x_1,y\}$. This implies that $y \in R(x_1,v)$ by axiom $(J2)$, provided $R(x_1,v) \neq \{x_1,v\}$. That is $x_1 \in R(u,v)$ and $y \in R(x_1,v)$ implies that $x_1 \in R(u,y)$ by $(b3)$, a contradiction since $R(u,y) = \{u,y\}$. Therefore, $R(x_1,v) = \{x_1,v\}$. That is, we have $x \in R(u,v)$, $x_1 \in R(u,x)$ and hence by $(b3)$, $x \in R(x_1,v)$, a final contradiction. Therefore, $R(u,x) = \{u,x\}$. Similarly, we can prove that $R(v,x) = \{v,x\}$. So $x \in I(u,v)$ and hence $R(u,v) \subseteq I(u,v)$, which completes the proof when $d(u,v) = 2$.

Let us assume that the result holds for all distances less than $k > 2$ and let u,v be two vertices such that $d(u,v) = k > 2$. We first prove $I(u,v) \subseteq R(u,v)$. Let $x \in I(u,v)$. Since $d(u,v) > 2$, we can find another vertex y in the shortest u,v-path containing x. Now since I satisfies $(b1)$ and $(b2)$, $I(u,x) \subset I(u,v), I(x,v) \subset I(u,v)$. We may assume that $x \in I(u,y)$. So by induction we have $I(u,x) = R(u,x)$ and $I(x,v) = R(x,v)$. Also, by $(b3)$ axiom $x \in I(u,y) = R(u,y)$, $y \in I(x,v) = R(x,v)$. Then by $(J0)$ axiom $x \in R(u,v)$. Hence, $I(u,v) \subseteq R(u,v)$. Let $x \in R(u,v)$. If possible, let $x \notin I(u,v)$. Since $x \in R(u,v)$, by applying axioms $(b1)$ and $(b2)$ similarly as in the case of $d(u,v) = 2$, we get vertices $x_1, x_2, \ldots, x_k, x_{k+1} = x \in R(u,x)$ with $R(x_1,u) = \{x_1,u\}$ such that $R(x_i,u) \subset R(x_{i+1},u)$ and $|R(x_i,u)| < |R(x_{i+1},u)|$, for $i \in \{1,\ldots,k\}$ and $R(x_1,u) = \{x_1,u\}$. Let y be a vertex such that $R(u,y) = \{u,y\}$ and $y \in I_{G_R}(u,v)$. Similar to the case of $d(u,v) = 2$, we can prove that $R(x_1,y) = \{x_1,y\}$. That is u, x_1, y form a C_3 in G_R. Here there are two possibilities for $d(x_1,v)$.

Case (i)*:* $d(x_1,v) = k$. In this case, since $d(u,v) = k$ and y is on the shortest u,v-path in G_R with $d(y,v) = k-1$, we have that y is on the shortest x_1,v-path in G_R, that is, $y \in I_{G_R}(x_1,v) \subseteq R(x_1,v)$. Therefore, we have $x_1 \in R(u,v)$, $y \in R(x_1,v)$ and hence by $(b3)$, $x_1 \in R(y,u)$, a contradiction as $R(y,u) = \{y,u\}$.

Case (ii)*:* $d(x_1,v) = k-1$. In this case, $x_1 \in I_{G_R}(u,v)$. Since $x \in R(u,v)$ and so by $(b2)$ axiom, $R(x,v) \subseteq R(u,v)$. We have also $x \in R(u,v), x_1 \in R(u,x)$ and hence by axiom $(b3)$, we have $x \in R(x_1,v) = I_{G_R}(x_1,v)$, by induction hypothesis. That is $x \in I_{G_R}(x_1,v) \subseteq I_{G_R}(u,v)$, since $x_1 \in R(u,v)$, which is a contradiction to our assumption.

Therefore, in all cases, we get contradictions to the assumption and hence our assumption is wrong, that is $x \in R(u,v) \subseteq I_{G_R}(u,v)$ and hence the theorem. □

From Theorem 3.2 and Theorem 3.1, we have the following theorem characterizing the interval function of Ptolemaic graphs.

Theorem 3.3. *Let R be a transit function on a non-empty set V. Then R satisfies the axioms $(b3)$, $(J0)$ and $(J2)$ if and only if G_R is a Ptolemaic graph and R coincides the interval function I_{G_R}.*

We now give examples of transit functions R to show that the transit axioms $(t1)$, $(t2)$, $(t3)$ and the axioms $(J0)$, $(J2)$ and $(b3)$ are independent.

Example 3.4 ($(t2), (t3), (J0)$, $(J2), (b3)$ but not $(t1)$)**.** Let $V = \{a, b, c, d\}$ and define a transit function R on V as $R(a, b) = R(b, a) = \{a\}$, $R(a, c) = \{a, c\}$, $R(a, d) = \{a, c, d\}$, $R(b, c) = \{b, c\}$, $R(b, d) = \{b, c, d\}$, $R(c, d) = \{c, d\}$, $R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for all $x, y \in V$. We can see that R satisfies $(t2), (t3), (J0), (J2)$ and $(b3)$. But R does not satisfy axiom $(t1)$.

Example 3.5 ($(t1), (t3), (J0)$, $(J2), (b3)$ but not $(t2)$)**.** Let $V = \{a, b, c, d\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\} = R(b, a)$, $R(a, c) = \{a, b, c\}$, $R(c, a) = \{a, c\}$ $R(a, d) = \{a, b, c, d\} = R(d, a)$, $R(b, c) = \{b, c\} = R(c, b)$, $R(b, d) = \{b, c, d\}$ $R(d, b) = \{d, b\}$, $R(c, d) = \{c, d\} = R(d, c)$, $R(x, x) = \{x\}$. We can see that R satisfies $(t1), (t3), (J0), (J2)$ and $(b3)$. But $R(a, b) \neq R(b, a)$. Therefore, R does not satisfy the $(t2)$ axiom.

Example 3.6 ($(t1), (t2), (J0)$, $(J2), (b3)$ but not $(t3)$)**.** Let $V = \{a, b, c, d\}$ and define a transit function R on V as follows: $R(a, a) = \{a, b\}$, $R(a, b) = \{a, b\}$, $R(a, c) = \{a, b, c\}$, $R(a, d) = \{a, b, c, d\}$, $R(b, c) = \{b, c\}$, $R(b, d) = \{b, c, d\}$, $R(c, d) = \{c, d\}$, $R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for all $x, y \in V$. We can see that R satisfies $(t1), (t2), (J0), (J2)$, and $(b3)$. But $b \in R(a, a)$. Therefore, R does not satisfy the $(t3)$ axiom.

Example 3.7 ($(t1), (t2), (t3), (J0), (J2)$ but not $(b3)$)**.** Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}$, $R(a, c) = \{a, c\}$, $R(a, d) = \{a, b, c, d\}$, $R(a, e) = V$, $R(b, c) = \{b, c\}$, $R(b, d) = \{b, d\}$, $R(b, e) = \{b, e\}$, $R(c, d) = \{c, d\}$, $R(c, e) = \{b, c, d, e\}$, $R(d, e) = \{d, e\}$, $R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for all $x, y \in V$. We can see that R satisfies $(t1)$, $(t2)$, $(t3)$, $(J0)$ and $(J2)$. But $d \in R(a, e)$, $b \in R(a, d)$, and $d \notin R(b, e)$. Therefore, R does not satisfy the $(b3)$ axiom.

Example 3.8 ($(t1), (t2), (t3), (J2), (b3)$ but not $(J0)$)**.** Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}$, $R(a, c) = \{a, c\}$, $R(a, d) = \{a, b, c, d\}$, $R(a, e) = \{a, b, e\}$, $R(b, c) = \{b, c\}$, $R(b, d) = \{b, d\}$, $R(b, e) = \{b, e\}$, $R(c, d) = \{c, d\}$, $R(c, e) = \{b, c, d, e\}$, $R(d, e) = \{d, e\}$, $R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for all $x, y \in V$. Here R satisfies $(t1)$, $(t2)$, $(t3)$, $(J2)$ and $(b3)$. We can see that $c \in R(a, d)$, $d \in R(c, e)$ but $c \notin R(a, e)$. So R does not satisfy $(J0)$.

Example 3.9 ($(t1), (t2), (t3)$, $(J0), (b3)$ but not $(J2)$)**.** Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, e) = \{a, e\}$, $R(b, e) = \{b, e\}$, $R(a, b) = \{a, b, c\}$ and for all other pair $R(x, y) = \{x, y\}$, $R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for all $x, y \in V$. We can see that R satisfies $(t1)$, $(t2)$, $(t3)$, $(J0)$, $(b3)$. But since $e \notin R(a, b)$ we can see that R fails to satisfy $(J2)$.

4. INDUCED PATH FUNCTION OF CHORDAL GRAPHS

In this section we characterize the induced path function of chordal graphs. We prove that even though the induced path transit function of an arbitrary connected graph is not first order definable as shown by Nebeský in [23], the family of chordal graphs possess a characterization in terms of a set of first order axioms. It is proved that for the class of HHD-free graphs [7], HHP-free graphs [6], and distance hereditary graphs [8], the induced path transit function possess a first order axiomatic characterization. We need the following axiom and the theorem from [6] for the characterization of the induced path transit function of a chordal graph.

($J1$) If $w \in R(u,v)$ and $w \neq u,v$, then there exist $u_1 \in R(u,w) \setminus R(v,w)$, $v_1 \in R(v,w) \setminus R(u,w)$, such that $R(u_1,w) = \{u_1,w\}$, $R(v_1,w) = \{v_1,w\}$ and $w \in R(u_1,v_1)$ for all $u,v,w \in V$.

Theorem 4.1 ([6])**.** *Let G be a graph. The induced path transit function J of G satisfies the axiom ($J0$) if and only if G is a chordal graph.*

Now we have the following theorem.

Theorem 4.2. *Let $R : V \times V \to 2^V$ be a function on a non-empty set V. Then R satisfies the axioms $(t1), (t2), (b2), (J0), (J1)$ and $(J2)$ if and only if G_R is a chordal graph and R coincides the induced path function J_{G_R}.*

Proof. First we prove that when R is a function satisfying axioms $(t1), (t2), (b2)$ and $(J1)$, then R satisfies $(b1)$. If possible, assume that R doesn't satisfy $(b1)$. Therefore, there exists u, v, w with $v \neq w$, $w \in R(u,v)$ and $v \in R(u,w)$. Since R satisfies $(b2)$, and $v \in R(u,w)$ we have $R(u,v) \subseteq R(u,w)$. Again since, $w \in R(u,v)$, we have $R(u,w) \subseteq R(u,v)$ which implies that $R(u,w) = R(u,v)$. Now since R satisfies $(J1)$, there exist an element $y \in R(v,w) \setminus R(u,w)$. Since, $R(v,w) \subseteq R(u,v)$, $R(v,w) \setminus R(u,w) = R(v,w) \setminus R(u,v) = \emptyset$, a contradiction to R satisfying axiom $(J1)$ and so R satisfies $(b1)$.

If R is a function satisfy axioms $(t1), (t2), (b1)$ and $(b2)$, then R satisfy axiom $(t3)$. For if not, let $R(u,u) \neq \{u\}$, for some $u \in V$. Let $x (\neq u) \in R(u,u)$. Then by axiom $(b2)$, we have $R(u,x) \subseteq R(u,u)$. By $(t1)$, $u \in R(u,x)$ and by $(b1)$ and $(t2)$, $x \notin R(u,u)$, a contradiction. That is, the function R that satisfy axioms $(t1), (t2), (b1)$ and $(b2)$, is a transit function. Since R satisfy axiom $(J0)$ and $(J2)$, by Theorem 2.1, G_R is a chordal graph.

Now we prove that $R(u,v) = J(u,v)$ for all $u,v \in V$. Let u, v and x be distinct elements in V. Suppose $x \in R(u,v)$. Then by $(J1)$, there exists $u_1 \in R(u,x) \setminus R(v,x)$, $v_1 \in R(v,x) \setminus R(u,x)$, such that $R(u_1,x) = \{u_1,x\}$, $R(v_1,x) = \{v_1,x\}$ and $x \in R(u_1,v_1)$. Since $u_1 \in R(u,x)$ by $(J1)$, there exists $u_2 \in R(u,u_1) \setminus R(u_1,x)$ such that $u_1 \in R(u_2,x)$. Now applying $(J1)$ successively to $R(u,u_2)$ and so on, we get a sequence of vertices $x = u_0, u_1, u_2, u_3, \ldots, u_k, u_{k+1} = u$ such that:

(i) $R(u_i,u_{i+1}) = \{u_i,u_{i+1}\}$, $i \in \{1,2,\ldots,k\}$,
(ii) $u_i \in R(u_{i-1},u_{i+1})$, $i \in \{1,2,\ldots,k\}$,
(iii) $R(u_{i+1},u) \subset R(u_i,u)$, $i \in \{1,2,\ldots,k\}$.

Also, we have $x = v_0, v_1, v_2, v_3, \ldots, v_m, v_{m+1} = v$ satisfying conditions similar to (i), (ii) and (iii) such that $x \in R(u_1, v_1)$

We claim that $P : uu_k \ldots u_1xv_1 \ldots v_mv$ is an induced path. We need to prove that $u_iu_{i+\ell} \notin E_{G_R}$, for $i \in \{0, 1, 2, \ldots, k - \ell\}$ with $\ell \geq 2$. When $\ell = 2$, the result follows by (ii). In the case $\ell > 3$, assume the contrary that $u_iu_{i+\ell} \in E_{G_R}$. Then it contradicts the fact that G_R is chordal. Similarly, $v_iv_{i+\ell} \notin E_{G_R}$ for $\ell \geq 2$. Now we need to prove that no vertex u_i, with $i \in \{1, 2, 3, \ldots, k\}$, is adjacent to a vertex v_i, with $i \in \{1, 2, 3, \ldots, m\}$. Now $x \in R(u_1, v_1)$. Therefore, $u_1v_1 \notin E_{G_R}$. Let v_r be the first vertex in v_j's adjacent to u_s. Then $u_sv_rv_{r-1} \ldots xu_1u_2 \ldots u_{s-1}u_s$ is an induced cycle of length greater than four, which is a contradiction to G_R is chordal. Hence, $P : uu_k \ldots u_1xv_1 \ldots v_mv$ is an induced u, v-path and x lies on it.

Suppose x belongs to some u, v-induced path say P. We prove that $x \in R(u, v)$ by induction on the length $l(P)$ of P. When $l(P) = 2$, the result follows by $(J2)$. Assume that the result is true for $l(P) < m$. Suppose now that $l(P) = m$ with $m > 2$. Then, either u or v has a neighbor on P different from x. Let u' be the neighbor of u on P. So $u'u$ lies on the induced x, u-subpath of P and x lies on the induced v, u'-subpath of P. By the induction hypothesis we have $x \in R(v, u')$ and $u' \in R(x, u)$, hence by $(J0)$ we have $x \in R(v, u)$. Since R is a transit function it follows that $x \in R(u, v)$. Hence, $R = J_{G_R}$.

The induced path function satisfy transit axioms $(t1)$, $(t2)$ and the axiom $(J2)$ for any graph. Conversely, assume that, the underlying graph G_R of a transit function R is a chordal graph and R is the induced path function J of G_R. By Theorem 4.1, it is clear that the induced path function satisfies axiom $(J0)$ on a chordal graph. Now assume that J does not satisfy axiom $(J1)$. Take the induced u, v-path, say P, in G_R containing w with u_1 and v_1 are neighbors of w in the path P. Since $(J1)$ is not satisfied, we have $u_1 \in J(v, w)$ or $v_1 \in J(u, w)$. We may assume that $u_1 \in J(v, w)$. Then there exists an induced w, v-path Q containing u_1. Evidently Q starts with the edge wu_1. Let v_r be the first vertex on Q which is also a vertex on the path P. Then $v_r \neq v_1$ otherwise wv_1 will act as a chord of Q. Since P is an induced path, $u_1v_1 \notin E(G_R)$. Consider the w, v_r-subpath say Q' of Q and the w, v_r-subpath say P' of P. Q' has length at least three and P' has length at least two. Together they form a cycle of length at least five. To avoid the long cycle, there must exist chord between an internal vertex of P' and Q'. But no vertex of Q' except u_1 is not adjacent with w. Let u_2 be the vertex on Q' adjacent to u_1. Then the vertices v_1, w, u_1, u_2 and some of the vertices in path P' induces a cycle of length at least four, a contradiction to the assumption that G_R is chordal.

Now, we have to prove that J satisfy axiom $(b2)$ in G_R. We will prove that J satisfy a stronger axiom than $(b2)$, namely the monotone axiom (m), which states that for all $x, y \in J(u, v)$, $J(x, y) \subseteq J(u, v)$, for every $u, v \in V(G_R)$. It follows that axiom $(b2)$ is a special case of (m). Assume that J does not satisfy axiom (m) in G_R. That is, $x, y \in J(u, v), z \in J(x, y)$ but $z \notin J(u, v)$.

Case 1. x and y are in the same induced path.

Let P be an induced u,v-path containing x and y and Q be an induced x,y- path containing z. Since $z \notin J(u,v), Q$ is not a subpath of P. Let a be the vertex closest to x and common to both P and Q. Let a' be vertex closest to y and common to both P and Q. Let $P: u = u_0u_1 \ldots u_r = xu_{r+1} \ldots u_s = y \ldots u_t = v$. Let the a,a'-induced subpath of Q containing z be $Q': av_1 \ldots v_m = zv_{m+1} \ldots v_n = a'$. Since $z \notin J(u,v)$, there exist chords from the u,a-subpath, say P' of P to the a',z-subpath, say Q' of Q or there exists chords from the v,a'-subpath of P to the a,z-subpath of Q. Without loss of generality we may assume that there exists chords from the vertices from P' to the vertices in Q'. Clearly the chords start from a vertex before the vertex a as we traverse along P' and must end before the vertex z as we traverse along Q'. Let bb' be the chord, where b is a vertex in P' closest to a and b' is a vertex in Q' closest to z. The cycle C formed by the union of the chord bb', the b',a subpath of Q containing z and the a,b-subpath of P will be an induced cycle of length at least four (the worst case we can allow is that b is adjacent to a and b' is adjacent to z and the b',a subpath of Q is of length exactly two, so that the cycle C is a four cycle), a contradiction to G_R being a chordal graph.

Case 2. x and y belong to different induced paths.

Let P be an induced u,v-path containing x and Q be an induced u,v-path containing y and R be an induced x,y- path containing z. Let a be the last vertex before x and common to both P and Q and let a' be the first vertex after y and common to both P and Q as we traverse along P from u. Clearly x is in the a,a'-induced subpath of P and y is in the a,a'-induced subpath of Q. So we may replace a by u and a' by v so that u and v are the only common vertex of the paths P and Q. Now let b be the first vertex before z and common to both P and R and b' be the first vertex after z and common to both Q and R as we traverse along R from x. Replace b by x and b' by y so that, we can assume that x is the only vertex common between P and R. Similarly, y is the only vertex common between Q and R. Let C_1 be the cycle formed by $u \xrightarrow{P} x \xrightarrow{R} y \xrightarrow{Q} u$ and C_2 be the cycle formed by $v \xrightarrow{P} x \xrightarrow{R} y \xrightarrow{Q} v$. If the path $x \xrightarrow{P} u \xrightarrow{Q} y$ is not induced path, then consider the chord $u'y'$ from the path $u \xrightarrow{P} x$ to the path $u \xrightarrow{Q} y$, where u' is closest to x and y' is closest to y. Here also we can replace u' by u and again we see that the cycle C_1 is an induced cycle of length at least five. To avoid induced long cycles, there should be chords from $u \xrightarrow{P} x$ to $y \xrightarrow{R} x$. Consider the chord u_1z_1, where u_1 closest to u on $u \xrightarrow{P} x$ and z_1 closest to y on $y \xrightarrow{R} x$, then the cycle formed by the union of the edge uy' and the paths $y' \xrightarrow{Q} y$, $y \xrightarrow{R} z_1$, the edge z_1u_1 and the path $u_1 \xrightarrow{P} u$ is an induced cycle of length at least five. So the only way to avoid the length of the induced cycle thus formed being of length less than four is that the vertex u and u_1 should be adjacent to y. Now replace u_1 by u. Now the cycle formed by $u_1 \xrightarrow{P} x \xrightarrow{R} z_1u_1$ is a cycle of length at least four. If $ux \in E(G_R)$, then clearly the cycle has length exactly four. Then u has a chord with all the vertices in the path $x \xrightarrow{R} y$, in particular uz should also form an edge, since G_R is chordal. If ux is not an edge then the cycle formed by $u \xrightarrow{P} x \xrightarrow{R} z_1u$ is of at least length five. Since G_R is chordal there exists a chord from vertices in u, x-subpath of P to x, y-subpath of R. Then there exists a vertex say u_2 closest to x in the u, x-subpath of P which is adjacent to both z and z_1 (neighbor z in z, y-subpath of R). Otherwise, suppose u_2 is adjacent only to z. Let z_1 is adjacent to a vertex say u'' closest to u_2 in the u, u_2-subpath of P. Then $u'' \xrightarrow{P} u_2 z z_1 u''$ forms an induced cycle of length at least four. That is, either u_2 or u'' is adjacent to both z and z_1 and let it be u_2.

Case 2.1. If u_2 is adjacent to z_2, the neighbor of z in the z, x-subpath of R. Replace u_2 by u, z_2 by x and z_1 by y, we get an induced $K_4 \setminus \{e\}$.

Case 2.2. If u_2 is not adjacent to z_2, the neighbor of z in the z, x-subpath of R. Then the neighbor of u_2 say u_3 is adjacent to z_2. If u_3 is not adjacent to z, then $zu_2u_3z_2z$ is a cycle of length four. Since u_2 is not adjacent to z_2, u_3 is adjacent to z. If we replace u_2 by u, z_2 by x and z_1 by y, we get an induced 3-fan as xu_3uy forms the path and z is the common vertex.

Using a similar argument, we can prove that in the cycle C_2, either the vertices x, v, y and z induce $K_4 \setminus \{e\}$ or the vertices x, v, y, v_3 and z induces a 3-fan. Then, we have that the path formed by the union of the edges uz and zv is an induced path containing z, a contradiction to our assumption that $z \notin J(u, v)$ and completes the proof. □

The following examples show that the axioms $(t1), (t2), (b2), (J0), (J1)$ and $(J2)$ are independent.

Example 3.4 forms an example for $(t2), (J0), (J1)$, $(J2), (b2)$ but not $(t1)$ and Example 3.5 form an example for $(t1), (J0), (J1), (J2), (b2)$ but not $(t2)$. If we define R as in Example 3.8, then R satisfy $(t1), (t2), (J1), (J2), (b2)$ but not $(J0)$ and if we define R as in Example 3.9, then R satisfy $(t1), (t2), (J0), (J1), (b2)$ but not $(J2)$. The examples below establish the independence of the remaining sets of axioms.

Example 4.3 (There exists a transit function that satisfies $(t1), (t2), (J0)$, $(J1), (J2)$ but not $(b2)$)**.** Let $V = \{a, b, c, d\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}$, $R(a, c) = \{a, b, c\}$, $R(a, d) = \{a, c, d\}$, $R(b, c) = \{b, c\}$, $R(b, d) = \{b, d\}$, $R(c, d) = \{c, d\}$, $R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for all $x, y \in V$. We can see that R satisfies $(t1)$, $(t2)$, $(J0)$, $(J1)$ and $(J2)$. But $c \in R(a, d)$, $b \in R(a, c)$, and $b \notin R(a, d)$. Therefore, R does not satisfy the $(b2)$ axiom.

Example 4.4 (There exists a transit function that satisfies $(t1), (t2), (J0)$, $(J2), (b2)$ but not $(J1)$)**.** Let $V = \{a, b, c, d\}$ and define a transit function R on V as follows: $R(a, d) = V$, $R(b, d) = \{b, d\}$, $R(a, c) = \{a, b, c\}$ and for all other pair $R(x, y) = \{x, y\}$, $R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for all $x, y \in V$. We can see that R satisfies $(t1)$, $(t2)$, $(J0)$, $(J2)$, $(b2)$. But we can see that R fails to satisfy $(J1)$. For $c \in R(a, d)$ there does not exist a u_1 and v_1 such that $u_1 \in R(a, c) \setminus R(c, d)$, $v_1 \in R(c, d) \setminus R(a, c)$, such that $R(u_1, c) = \{u_1, c\}$, $R(v_1, c) = \{v_1, c\}$ and $c \in R(u_1, v_1)$.

Acknowledgements. *Manoj Changat acknowledges the financial support from SERB, Department of Science & Technology, Govt. of India (research project under MATRICS scheme No. MTR/2017/000238). Lekshmi Kamal K Sheela acknowledges the financial support from CSIR, Government India for providing CSIR Senior Research Fellowship (CSIR-SRF) (No 09/102(0260)/2019-EMR-I). Prasanth G. Narasimha-Shenoi acknowledges the financial support from SERB, Department of Science & Technology, Govt. of India (research project under MATRICS scheme No. MTR/2018/000012).*

REFERENCES

[1] K. Balakrishnan, M. Changat, A.K. Lakshmikuttyamma, J. Mathew, H.M. Mulder, P.G. Narasimha-Shenoi, N. Narayanan, *Axiomatic characterization of the interval function of a block graph*, Discrete Math. **338** (2015), 885–894.

[2] A. Brandstädt, V.B. Le, J.P. Spinrad, *Graph Classes – A Survey*, SIAM Monogr. J. Discrete Math., 1999.

[3] M. Changat, F. Hossein Nezhad, H.M. Mulder, N. Narayanan, *A note on the interval function of a disconnected graph*, Discuss. Math. Graph Theory **38** (2018), no. 1, 39–48.

[4] M. Changat, F. Hossein Nezhad, N. Narayanan, *Axiomatic characterization of claw and paw-free graphs using graph transit functions*, [in:] Conference on Algorithms and Disc. Appl. Math. Springer LNCS (2016), pp. 115–125.

[5] M. Changat, F. Hossein Nezhad, N. Narayanan, *Axiomatic characterization of the interval function of a bipartite graph*, [in:] Conference on Algorithms and Disc. Appl. Math. Springer LNCS (2017), 96–106.

[6] M. Changat, A.K. Lakshmikuttyamma, J. Mathew, I. Peterin, P.G. Narasimha-Shenoi, G. Seethakuttyamma, S. Špacapan, *A forbidden subgraph characterization of some graph classes using betweenness axioms*, Discrete Math. **313** (2013), 951–958.

[7] M. Changat, J. Mathew, H.M. Mulder, *The induced path function, monotonicity and betweenness*, Discrete Appl. Math. **158** (2010), no. 5, 426–433.

[8] M. Changat, L.K. Sheela, P.G. Narasimha-Shenoi, *The axiomatic characterization of the interval function of distance hereditary graphs*, submitted.

[9] V. Chepoi, *Some properties of d-convexity in triangulated graphs*, Math. Res. **87** (1986), 164–177 [in Russian].

[10] V. Chvátal, D. Rautenbach, P.M. Schäfer, *Finite Sholander trees, trees, and their betweenness*, Discrete Math. **311** (2011), 2143–2147.

[11] H.N. de Ridder et al., *Information System on Graph Classes and their Inclusions*, (ISGCI), https://www.graphclasses.org.

[12] E. Howorka, *A characterization of Ptolemaic graphs*, J. Graph Theory **5** (1981), 323–331.

[13] D. Kay, G. Chartrand, *A characterization of certain Ptolemaic graphs*, Canad. J. Math. **17** (1965), 342–346.

[14] H.M. Mulder, *The Interval Function of a Graph*, MC Tract 132, Amsterdam, 1980.

[15] H.M. Mulder, *Transit functions on graphs (and posets)*, Proc. Int. Conf. - Convexity in Discrete Structures **5** (2008), 117–130.

[16] H.M. Mulder, L. Nebeský, *Axiomatic characterization of the interval function of a graph*, European J. Combin. **30** (2009), 1172–1185.

[17] L. Nebeský, *A characterization of the interval function of a connected graph*, Czechoslovak Math. J. **44** (1994), 173–178.

[18] L. Nebeský, *A characterization of the set of all shortest paths in a connected graph*, Math. Bohem. **119** (1994), no. 1, 15–20.

[19] L. Nebeský, *A characterization of geodetic graphs*, Czechoslovak Math. J. **45** (1995), no. 3, 491–493.

[20] L. Nebeský, *Characterizing the interval function of a connected graph*, Math. Bohem. **123** (1998), no. 2, 137–144.

[21] L. Nebeský, *A new proof of a characterization of the set of all geodesics in a connected graph*, Czechoslovak Math. J. **48** (1998), no. 4, 809–813.

[22] L. Nebeský, *Characterization of the interval function of a (finite or infinite) connected graph*, Czechoslovak Math. J. **51** (2001), 635–642.

[23] L. Nebeský, *The induced paths in a connected graph and a ternary relation determined by them*, Math. Bohem. **127** (2002), 397–408.

[24] M. Sholander, *Trees, lattices, order, and betweenness*, Proc. Amer. Math. Soc. **3** (1952), 369–381.

Manoj Changat
mchangat@keralauniversity.ac.in

Department of Futures Studies
University of Kerala
Trivandrum, India – 695 581

Lekshmi Kamal K. Sheela
lekshmisanthoshgr@gmail.com

Department of Futures Studies
University of Kerala
Trivandrum, India – 695 581

Prasanth G. Narasimha-Shenoi (corresponding author)
prasanthgns@gmail.com

Department of Mathematics
Government College Chittur
Palakkad, India – 678 104

Received: June 8, 2022.
Revised: February 27, 2023.
Accepted: March 2, 2023.