

OPERATORS INDUCED BY CERTAIN HYPERCOMPLEX SYSTEMS

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Abstract. In this paper, we consider a family $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$ of rings of hypercomplex numbers, indexed by the real numbers, which contain both the quaternions and the split-quaternions. We consider natural Hilbert-space representations $\{(\mathbb{C}^2, \pi_t)\}_{t \in \mathbb{R}}$ of the hypercomplex system $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$, and study the realizations $\pi_t(h)$ of hypercomplex numbers $h \in \mathbb{H}_t$, as (2×2) -matrices acting on \mathbb{C}^2 , for an arbitrarily fixed scale $t \in \mathbb{R}$. Algebraic, operator-theoretic, spectral-analytic, and free-probabilistic properties of them are considered.

Keywords: scaled hypercomplex ring, scaled hypercomplex monoids, representations, scaled-spectral forms, scaled-spectralization.

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1. INTRODUCTION

In this paper, we study representations of the hypercomplex numbers (a, b) of complex numbers a and b , constructing a ring,

$$\mathbb{H}_t = (\mathbb{C}^2, +, \cdot_t),$$

scaled by a real number $t \in \mathbb{R}$, where $(+)$ is the usual vector addition on the 2-dimensional vector space \mathbb{C}^2 , and (\cdot_t) is the t -scaled vector-multiplication on \mathbb{C}^2 , defined by

$$(a_1, b_1) \cdot_t (a_2, b_2) = (a_1 a_2 + t b_1 \bar{b}_2, a_1 b_2 + b_1 \bar{a}_2),$$

where \bar{z} are the conjugates of z in \mathbb{C} .

Motivated by the canonical Hilbert-space representation (\mathbb{C}^2, π) of the quaternions \mathbb{H} , introduced in [2, 3] and [20], we consider the canonical representation,

$$\Pi_t = (\mathbb{C}^2, \pi_t),$$

of the ring \mathbb{H}_t , and understand each element $h = (a, b)$ of \mathbb{H}_t as its realization,

$$\pi_t(h) \stackrel{\text{denote}}{=} [h]_t \stackrel{\text{def}}{=} \begin{pmatrix} a & tb \\ b & \bar{a} \end{pmatrix} \text{ in } M_2(\mathbb{C}),$$

where $M_2(\mathbb{C}) = B(\mathbb{C}^2)$ is the matricial algebra (or, the operator algebra acting on \mathbb{C}^2) of all (2×2) -matrices over \mathbb{C} (respectively, all bounded linear transformations, or simply operators on \mathbb{C}^2), for each $t \in \mathbb{R}$. Under our setting, one can check that the ring \mathbb{H}_{-1} is nothing but the noncommutative field \mathbb{H} of all quaternions (e.g., [2, 3] and [20]) and the ring \mathbb{H}_1 is the ring of all bicomplex numbers (e.g., [1]).

The spectral-analytic, operator-theoretic (or, matrix-theoretic), and free-probabilistic properties of \mathbb{H}_t are considered and characterized under the canonical representation Π_t . In particular, certain decompositional properties on \mathbb{H}_t are studied algebraically, and spectral-theoretically. And then, it is considered how those properties affect the spectral-analytic, operator-theoretic, and free-probabilistic properties of hypercomplex numbers of \mathbb{H}_t , for $t \in \mathbb{R}$.

1.1. MOTIVATION

The quaternions \mathbb{H} is an interesting object not only in pure mathematics (e.g., [5, 10–12, 15, 16, 18, 20, 23], but also in applied mathematics (e.g., [4, 7, 13, 14, 17] and [21]). Independently, spectral analysis on \mathbb{H} is considered in [2] and [3], under representation, “over \mathbb{C} ”, different from the usual quaternion-eigenvalue problems of quaternion-matrices studied in [13, 15] and [17].

Motivated by the generalized setting of the quaternions so-called the split-quaternions of [1], and by the main results of [2] and [3], we study a new type of hypercomplex numbers induced by the pairs of \mathbb{C}^2 . Especially, we construct a system of the scaled hypercomplex rings $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$, and study how the hypercomplex numbers act as (2×2) -matrices over \mathbb{C} for given scales $t \in \mathbb{R}$, under our canonical Hilbert-space representations $\{\Pi_t = (\mathbb{C}^2, \pi_t)\}_{t \in \mathbb{R}}$. We are interested in algebraic, operator-theoretic, spectral-theoretic, free-probabilistic properties of \mathbb{H}_t under Π_t , for $t \in \mathbb{R}$. Are they similar to those of the quaternions $\mathbb{H}_{-1} = \mathbb{H}$, shown in [2] and [3]? The answers are determined differently case-by-case, up to scales (see below).

1.2. OVERVIEW

In Section 2, we define our main objects, the scaled hypercomplex rings $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$, and their canonical Hilbert-space representations $\{\Pi_t\}_{t \in \mathbb{R}}$. We understand each hypercomplex number of \mathbb{H}_t as an operator, a (2×2) -matrix over \mathbb{C} . We concentrate on studying the invertibility on \mathbb{H}_t , for an arbitrarily fixed scale t . It is shown that if $t < 0$, then \mathbb{H}_t forms a noncommutative field like the quaternions $\mathbb{H} = \mathbb{H}_{-1}$, however, if $t \geq 0$, then it becomes a ring with unity, which is not a noncommutative field.

In Section 3, the spectral theory on (the realizations of) \mathbb{H}_t is studied over \mathbb{C} . After finding the spectra of hypercomplex numbers, we define so-called the t -spectral forms whose main diagonal entries are from the spectra, and off-diagonal entries are 0's. As we have seen in [2] and [3], such spectral forms are similar to the realizations of

quaternions of \mathbb{H}_{-1} . However, if a scale $t \in \mathbb{R} \setminus \{-1\}$ is arbitrary, then such a similarity does not hold in general. We focus on studying such a similarity in detail.

In Section 4, we briefly discuss about how the usual adjoint on $M_2(\mathbb{C})$ acts on the sub-structure \mathcal{H}_2^t of $M_2(\mathbb{C})$, consisting of all realizations of \mathbb{H}_t , for a scale $t \in \mathbb{R}$. Different from the quaternionic case of [2] and [3], in general, the adjoints (conjugate-transposes) of many matrices of \mathcal{H}_2^t are not contained in \mathcal{H}_2^t , especially, if $t \neq -1$. It shows that a bigger, operator-algebraically-better $*$ -algebraic structure generated by \mathcal{H}_2^t is needed in $M_2(\mathbb{C})$, to consider operator-theoretic, and free-probabilistic properties on \mathcal{H}_2^t .

In the final Section 5, on the C^* -algebraic structure of Section 4, we study operator-theoretic, and free-probabilistic properties up to the usual trace, and the normalized trace.

2. THE SCALED HYPERCOMPLEX SYSTEMS $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$

In this section, we define a ring \mathbb{H}_t of hypercomplex numbers, and establish the corresponding canonical Hilbert-space representations Π_t , for an arbitrary fixed scale $t \in \mathbb{R}$. Throughout this section, we let

$$\mathbb{C}^2 = \{(a, b) : a, b \in \mathbb{C}\}$$

be the Cartesian product of two copies of the complex field \mathbb{C} . One may understand \mathbb{C}^2 as the usual 2-dimensional Hilbert space equipped with its canonical orthonormal basis, $\{(1, 0), (0, 1)\}$.

2.1. A t -SCALED HYPERCOMPLEX RING \mathbb{H}_t

In this section, we fix an arbitrary real number t in the real field \mathbb{R} . On the vector space \mathbb{C}^2 (over \mathbb{C}), define the t -scaled vector-multiplication $(\cdot)_t$ by

$$(a_1, b_1) \cdot_t (a_2, b_2) \stackrel{\text{def}}{=} (a_1 a_2 + t b_1 \bar{b}_2, a_1 b_2 + b_1 \bar{a}_2), \tag{2.1}$$

for $(a_l, b_l) \in \mathbb{C}^2$, for all $l = 1, 2$, where \bar{z} are the conjugates of z in \mathbb{C} . It is not difficult to check that such an operation $(\cdot)_t$ is closed on \mathbb{C}^2 . Moreover, it satisfies that

$$\begin{aligned} & ((a_1, b_1) \cdot_t (a_2, b_2)) \cdot_t (a_3, b_3) \\ &= (a_1 a_2 + t b_1 \bar{b}_2, a_1 b_2 + b_1 \bar{a}_2) \cdot_t (a_3, b_3) \\ &= (a_1 a_2 a_3 + t (b_1 \bar{b}_2 a_3 + a_1 b_2 \bar{b}_3 + b_1 \bar{a}_2 \bar{b}_3), a_1 a_2 b_3 + a_1 b_2 \bar{a}_3 + b_1 \bar{a}_2 a_3 + t b_1 \bar{b}_2 b_3), \end{aligned}$$

and

$$\begin{aligned} & (a_1, b_1) \cdot_t ((a_2, b_2) \cdot_t (a_3, b_3)) \\ &= (a_1, b_1) \cdot_t (a_2 a_3 + t b_2 \bar{b}_3, a_2 b_3 + b_2 \bar{a}_3) \\ &= (a_1 (a_2 a_3 + t b_2 \bar{b}_3) + t b_1 (\bar{a}_2 \bar{b}_3 + \bar{b}_2 a_3), a_1 (a_2 b_3 + b_2 \bar{a}_3) + b_1 (\bar{a}_2 a_3 + t \bar{b}_2 b_3)), \end{aligned}$$

implying the equality

$$((a_1, b_1) \cdot_t (a_2, b_2)) \cdot_t (a_3, b_3) = (a_1, b_1) \cdot_t ((a_2, b_2) \cdot_t (a_3, b_3)), \quad (2.2)$$

in \mathbb{C}^2 , for $(a_l, b_l) \in \mathbb{C}^2$, for all $l = 1, 2, 3$.

Furthermore, if $\vartheta = (1, 0) \in \mathbb{C}^2$, then

$$\vartheta \cdot_t (a, b) = (a, b) = (a, b) \cdot_t \vartheta \quad (2.3)$$

by (2.1), for all $(a, b) \in \mathbb{C}^2$.

By (2.2) and (2.3), if

$$\mathbb{C}^{2\times} = \mathbb{C}^2 \setminus \{(0, 0)\},$$

then the pair $(\mathbb{C}^{2\times}, \cdot_t)$ forms a monoid (i.e., semigroup with its identity $(1, 0)$).

Lemma 2.1. *Let $\mathbb{C}^{2\times} = \mathbb{C}^2 \setminus \{(0, 0)\}$, and (\cdot_t) be the closed operation (2.1) on \mathbb{C}^2 . Then the algebraic structure $(\mathbb{C}^{2\times}, \cdot_t)$ forms a monoid with its identity $(1, 0)$.*

Proof. The proof is done by (2.2) and (2.3). \square

Therefore, one can obtain the following ring structure.

Proposition 2.2. *The algebraic triple $(\mathbb{C}^2, +, \cdot_t)$ forms a unital ring with its unity (or the multiplication-identity) $(1, 0)$, where $(+)$ is the usual vector addition on \mathbb{C}^2 , and (\cdot_t) is the vector multiplication (2.1).*

Proof. Clearly, the algebraic pair $(\mathbb{C}^2, +)$ is an Abelian group under the usual addition $(+)$ with its $(+)$ -identity $(0, 0)$. While, by Lemma 2.1, the pair $(\mathbb{C}^{2\times}, \cdot_t)$ forms a monoid (and hence, a semigroup). Observe now that

$$\begin{aligned} & (a_1, b_1) \cdot_t ((a_2, b_2) + (a_3, b_3)) \\ &= (a_1, b_1) \cdot_t (a_2 + a_3, b_2 + b_3) \\ &= (a_1(a_2 + a_3) + tb_1(\overline{b_2} + \overline{b_3}), a_1(b_2 + b_3) + b_1(\overline{a_2} + \overline{a_3})) \\ &= (a_1a_2 + a_1a_3 + tb_1\overline{b_2} + tb_1\overline{b_3}, a_1b_2 + a_1b_3 + b_1\overline{a_2} + b_1\overline{a_3}) \\ &= (a_1a_2 + tb_1\overline{b_2}, a_1b_2 + b_1\overline{a_2}) + (a_1a_3 + tb_1\overline{b_3}, a_1b_3 + b_1\overline{a_3}) \\ &= (a_1, b_1) \cdot_t (a_2, b_2) + (a_1, b_1) \cdot_t (a_3, b_3), \end{aligned}$$

and, similarly,

$$((a_1, b_1) + (a_2, b_2)) \cdot_t (a_3, b_3) = (a_1, b_1) \cdot_t (a_3, b_3) + (a_2, b_2) \cdot_t (a_3, b_3), \quad (2.4)$$

in \mathbb{C}^2 . So, the operations $(+)$ and (\cdot_t) are left-and-right distributive by (2.4).

Therefore, the algebraic triple $(\mathbb{C}^2, +, \cdot_t)$ forms a unital ring with its unity $(1, 0)$. \square

The above proposition characterizes the algebraic structure of $(\mathbb{C}^2, +, \cdot_t)$ as a well-defined unital ring for a fixed $t \in \mathbb{R}$. Remark here that, since a scale t is arbitrary in \mathbb{R} , in fact, we obtain the unital rings $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$.

Definition 2.3. For a fixed $t \in \mathbb{R}$, the ring $(\mathbb{C}^2, +, \cdot_t)$ is called the hypercomplex ring with its scale t (in short, the t -scaled hypercomplex ring). By \mathbb{H}_t , we denote the t -scaled hypercomplex ring.

2.2. THE CANONICAL REPRESENTATION $\Pi_t = (\mathbb{C}^2, \pi_t)$ OF \mathbb{H}_t

In this section, we fix $t \in \mathbb{R}$, and the corresponding t -scaled hypercomplex ring,

$$\mathbb{H}_t = (\mathbb{C}^2, +, \cdot_t),$$

where (\cdot_t) is the vector-multiplication (2.1). We consider a natural finite-dimensional-Hilbert-space representation Π_t of \mathbb{H}_t , and understand each hypercomplex number $h \in \mathbb{H}_t$ as an operator acting on a Hilbert space determined by Π_t . In particular, as in the quaternionic case of [2,3] and [20], a 2-dimensional-Hilbert-space representation of the hypercomplex ring \mathbb{H}_t is established naturally.

Define now a morphism,

$$\pi_t : \mathbb{H}_t \rightarrow B(\mathbb{C}^2) = M_2(\mathbb{C}),$$

by

$$\pi_t((a, b)) = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix}, \quad \forall (a, b) \in \mathbb{H}_t, \tag{2.5}$$

where $B(H)$ is the operator algebra consisting of all bounded (or, continuous linear) operators on a Hilbert space H , and $M_k(\mathbb{C})$ is the matricial algebra of all $(k \times k)$ -matrices over \mathbb{C} , isomorphic to $B(\mathbb{C}^k)$, for all $k \in \mathbb{N}$ (e.g., [9] and [8]).

By definition, the function π_t of (2.5) is an injective map from \mathbb{H}_t into $M_2(\mathbb{C})$. Indeed, if

$$(a_1, b_1) \neq (a_2, b_2) \text{ in } \mathbb{H}_t,$$

then

$$\pi_t((a_1, b_1)) = \begin{pmatrix} a_1 & tb_1 \\ \bar{b}_1 & \bar{a}_1 \end{pmatrix} \neq \begin{pmatrix} a_2 & tb_2 \\ \bar{b}_2 & \bar{a}_2 \end{pmatrix} = \pi_t((a_2, b_2)), \tag{2.6}$$

in $M_2(\mathbb{C})$. Furthermore, it satisfies that

$$\begin{aligned} \pi_t((a_1, b_1) + (a_2, b_2)) &= \begin{pmatrix} a_1 + a_2 & t(b_1 + b_2) \\ \bar{b}_1 + \bar{b}_2 & \bar{a}_1 + \bar{a}_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & tb_1 \\ \bar{b}_1 & \bar{a}_1 \end{pmatrix} + \begin{pmatrix} a_2 & tb_2 \\ \bar{b}_2 & \bar{a}_2 \end{pmatrix} \\ &= \pi_t((a_1, b_1)) + \pi_t((a_2, b_2)). \end{aligned} \tag{2.7}$$

Also, one has

$$\pi_t((a_1, b_1) \cdot_t (a_2, b_2)) = \pi_t((a_1 a_2 + t b_1 \bar{b}_2, a_1 b_2 + b_1 \bar{a}_2))$$

by (2.1)

$$\begin{aligned} &= \begin{pmatrix} a_1 a_2 + t b_1 \bar{b}_2 & t(a_1 b_2 + b_1 \bar{a}_2) \\ \overline{a_1 b_2 + b_1 \bar{a}_2} & \overline{a_1 a_2 + t b_1 \bar{b}_2} \end{pmatrix} \\ &= \begin{pmatrix} a_1 & t b_1 \\ \bar{b}_1 & \bar{a}_1 \end{pmatrix} \begin{pmatrix} a_2 & t b_2 \\ \bar{b}_2 & \bar{a}_2 \end{pmatrix} = \pi_t((a_1, b_1)) \pi_t((a_2, b_2)), \end{aligned} \tag{2.8}$$

where the multiplication (\cdot) in the far-right-hand side of (2.8) is the usual matricial multiplication on $M_2(\mathbb{C})$.

Since our t -scaled hypercomplex ring $\mathbb{H}_t = (\mathbb{C}^2, +, \cdot_t)$ is identified with the 2-dimensional space \mathbb{C}^2 (set-theoretically), one may / can understand this ring \mathbb{H}_t as a topological ring equipped with the usual topology for \mathbb{C}^2 , for any $t \in \mathbb{R}$. From below, we regard the ring \mathbb{H}_t as a topological unital ring under the usual topology for \mathbb{C}^2 .

Lemma 2.4. *The pair (\mathbb{C}^2, π_t) is an injective Hilbert-space representation of the t -scaled hypercomplex ring \mathbb{H}_t , where π_t is an action (2.5).*

Proof. The morphism $\pi_t : \mathbb{H}_t \rightarrow M_2(\mathbb{C})$ of (2.5) is a well-defined injective function by (2.6). Moreover, this map π_t satisfies the relations (2.7) and (2.8), and hence, it is a(n algebraic) ring-action of \mathbb{H}_t , acting on the 2-dimensional vector space \mathbb{C}^2 . So, the pair (\mathbb{C}^2, π_t) forms an algebraic representation of \mathbb{H}_t . By regarding \mathbb{H}_t and $M_2(\mathbb{C})$ as topological spaces equipped with their usual topologies, then it is not difficult to check that the ring-action π_t is continuous from \mathbb{H}_t (which is homeomorphic to \mathbb{C}^2 as a topological space) into $M_2(\mathbb{C})$ (which is $*$ -isomorphic to the C^* -algebra $B(\mathbb{C}^2)$). Thus, the algebraic representation (\mathbb{C}^2, π_t) forms a Hilbert-space representation of \mathbb{H}_t acting on \mathbb{C}^2 via π_t . \square

The above lemma shows that the t -scaled hypercomplex ring \mathbb{H}_t is realized in the matricial algebra $M_2(\mathbb{C})$ as

$$\pi_t(\mathbb{H}_t) = \left\{ \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbb{C}) : (a, b) \in \mathbb{H}_t \right\},$$

as an embedded topological ring in $M_2(\mathbb{C})$.

Definition 2.5. The realization $\pi_t(\mathbb{H}_t)$ of the t -scaled hypercomplex ring \mathbb{H}_t is called the t -scaled (hypercomplex-)realization of \mathbb{H}_t (in $M_2(\mathbb{C})$), for a scale $t \in \mathbb{R}$. And we denote $\pi_t(\mathbb{H}_t)$ by \mathcal{H}_2^t , i.e.,

$$\mathcal{H}_2^t \stackrel{\text{denote}}{=} \pi_t(\mathbb{H}_t) = \left\{ \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} : (a, b) \in \mathbb{H}_t \right\}.$$

Also, by $[\xi]_t$, we denote $\pi_t(\xi) \in \mathcal{H}_2^t$, for all $\xi \in \mathbb{H}_t$.

By the above lemma and definition, we obtain the following result.

Theorem 2.6. *For $t \in \mathbb{R}$, the corresponding t -scaled hypercomplex ring \mathbb{H}_t is topological-ring-isomorphic to the t -scaled realization \mathcal{H}_2^t in $M_2(\mathbb{C})$, i.e.,*

$$\mathbb{H}_t \stackrel{T.R}{=} \mathcal{H}_2^t \text{ in } M_2(\mathbb{C}), \tag{2.9}$$

where “ $\stackrel{T.R}{=}$ ” means “being topological-ring-isomorphic to”.

Proof. The relation (2.9) is proven by Lemma 2.4 and the injectivity (2.6) of π_t . \square

By the above theorem, one can realize that \mathbb{H}_t and \mathcal{H}_2^t as an identical topological ring, for a fixed $t \in \mathbb{R}$. Recall that the relation (2.9) is independently shown in [2] and [3], only for the quaternionic case where $t = -1$.

2.3. SCALED HYPERCOMPLEX MONOIDS

Throughout this section, we fix a scale $t \in \mathbb{R}$, and the corresponding t -scaled hypercomplex ring,

$$\mathbb{H}_t = (\mathbb{C}^2, +, \cdot_t),$$

which is isomorphic to the t -scaled realization,

$$\mathcal{H}_2^t = \left\{ \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbb{C}) : (a, b) \in \mathbb{H}_t \right\},$$

in $M_2(\mathbb{C})$. Let

$$\mathbb{H}_t^\times \stackrel{\text{denote}}{=} \mathbb{H}_t \setminus \{(0, 0)\},$$

set-theoretically, where $(0, 0) \in \mathbb{H}_t$ is the $(+)$ -identity of the Abelian group $(\mathbb{C}^2, +)$. Thus, by Proposition 2.2, this set forms a well-defined semigroup,

$$\mathbb{H}_t^\times \stackrel{\text{denote}}{=} (\mathbb{H}_t^\times, \cdot_t),$$

equipped with its (\cdot_t) -identity $(1, 0)$, and hence, the pair \mathbb{H}_t^\times is the maximal monoid embedded in \mathbb{H}_2^t up to the operation (\cdot_t) .

Definition 2.7. The maximal monoid $\mathbb{H}_t^\times = (\mathbb{H}_t^\times, \cdot_t)$, embedded in the t -scaled hypercomplex ring \mathbb{H}_t , is called the t -scaled hypercomplex monoid.

By (2.9), the following corollary is trivial.

Corollary 2.8. *The t -scaled hypercomplex monoid \mathbb{H}_t^\times is monoid-isomorphic to the monoid $\mathcal{H}_2^{t \times} \stackrel{\text{denote}}{=} (\mathcal{H}_2^{t \times}, \cdot)$, equipped with its identity,*

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \cdot 0 \\ 0 & 1 \end{pmatrix} = [(1, 0)]_t,$$

the (2×2) -identity matrix of $M_2(\mathbb{C})$, where (\cdot) is the usual matricial multiplication inherited from that on $M_2(\mathbb{C})$, i.e.,

$$\mathbb{H}_t^\times = (\mathbb{H}_t^\times, \cdot_t) \stackrel{\text{Monoid}}{=} (\mathcal{H}_2^{t \times}, \cdot) = \mathcal{H}_2^{t \times}, \tag{2.10}$$

where “ $\stackrel{\text{Monoid}}{=}$ ” means “being monoid-isomorphic”.

Proof. The isomorphic relation (2.10) is proven by the proof of Proposition 2.2, and that of Theorem 2.6. □

2.4. INVERTIBILITY ON \mathbb{H}_t

In this section, by identifying our t -scaled hypercomplex ring \mathbb{H}_t as its isomorphic realization \mathcal{H}_2^t , we consider invertibility of elements of \mathbb{H}_t , for an arbitrarily fixed $t \in \mathbb{R}$.

Observe first that, for any $(a, b) \in \mathbb{H}_t$ realized to be $[(a, b)]_t \in \mathcal{H}_2^t$, one can get that

$$\det([(a, b)]_t) = \det \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} = |a|^2 - t|b|^2,$$

i.e.,

$$\det([(a, b)]_t) = |a|^2 - t|b|^2, \tag{2.11}$$

where $\det : M_2(\mathbb{C}) \rightarrow \mathbb{C}$ is the determinant, and $|\cdot|$ is the modulus on \mathbb{C} .

Theorem 2.9. *Let $(a, b) \in \mathbb{H}_t$, realized to be $[(a, b)]_t \in \mathcal{H}_2^t$. Then the following assertions hold.*

- (i) $\det([(a, b)]_t) = |a|^2 - t|b|^2$.
- (ii) *If either $|a|^2 > t|b|^2$, or $|a|^2 < t|b|^2$, then $[(a, b)]_t$ is invertible “in $M_2(\mathbb{C})$ ”, with its inverse matrix,*

$$[(a, b)]_t^{-1} = \frac{1}{|a|^2 - t|b|^2} \begin{pmatrix} \bar{a} & t(-b) \\ (-b) & a \end{pmatrix}.$$

- (iii) *If $|a|^2 - t|b|^2 \neq 0$, then $(a, b) \in \mathbb{H}_t$ is invertible in the sense that there exists a unique $(c, d) \in \mathbb{H}_t$, such that*

$$(a, b) \cdot_t (c, d) = (1, 0) = (c, d) \cdot_t (a, b).$$

In particular, one has that

$$(c, d) = \left(\frac{\bar{a}}{|a|^2 - t|b|^2}, \frac{-b}{|a|^2 - t|b|^2} \right) \in \mathbb{C}^2$$

- (iv) *Assume that (a, b) is invertible in \mathbb{H}_t in the sense of (iii). Then the inverse is also contained “in \mathbb{H}_t ”.*

Proof. The statement (i) is shown by (2.11).

Note-and-recall that a matrix $A \in M_n(\mathbb{C})$ is invertible in $M_n(\mathbb{C})$, if and only if $\det(A) \neq 0$, for all $n \in \mathbb{N}$. Therefore,

$$\det([(a, b)]_t) \neq 0 \iff [(a, b)]_t \text{ is invertible in } M_2(\mathbb{C}).$$

So, by (i),

$$|a|^2 - t|b|^2 \neq 0 \iff [(a, b)]_t \text{ is invertible in } M_2(\mathbb{C}).$$

Moreover, $|a|^2 - t|b|^2 \neq 0$, if and only if

$$[(a, b)]_t^{-1} = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix}^{-1} = \frac{1}{|a|^2 - t|b|^2} \begin{pmatrix} \bar{a} & -tb \\ -\bar{b} & a \end{pmatrix},$$

in $M_2(\mathbb{C})$. Therefore, the statement (ii) holds true in $M_2(\mathbb{C})$.

By (ii), one has $\det([(a, b)]_t) \neq 0$, if and only if

$$[(a, b)]_t^{-1} = \begin{pmatrix} \frac{\bar{a}}{|a|^2 - t|b|^2} & t \left(\frac{-b}{|a|^2 - t|b|^2} \right) \\ \overline{\left(\frac{-b}{|a|^2 - t|b|^2} \right)} & \frac{a}{|a|^2 - t|b|^2} \end{pmatrix} \in M_2(\mathbb{C}),$$

and it is actually contained “in \mathcal{H}_2^t ”, satisfying

$$\pi_t^{-1} \left(\begin{pmatrix} \frac{\bar{a}}{|a|^2 - t|b|^2} & t \left(\frac{-b}{|a|^2 - t|b|^2} \right) \\ \overline{\left(\frac{-b}{|a|^2 - t|b|^2} \right)} & \frac{a}{|a|^2 - t|b|^2} \end{pmatrix} \right) = \left(\frac{\bar{a}}{|a|^2 - t|b|^2}, \frac{-b}{|a|^2 - t|b|^2} \right),$$

in \mathbb{H}_t , by the injectivity of π_t . It shows that $[(a, b)]_t^{-1}$ exists in $M_2(\mathbb{C})$, if and only if it is contained “in \mathcal{H}_2^t ”, i.e., if $[(a, b)]_t$ is invertible, then its inverse is also contained in \mathcal{H}_2^t , too, and vice versa. So, the statements (2.8) and (2.9) hold. \square

The above theorem not only characterizes the invertibility of the monoidal elements of the t -scaled hypercomplex monoid \mathbb{H}_t^\times , but also confirms that the inverses (if exist) are contained in the monoid \mathbb{H}_t^\times , i.e.,

$$(a, b)^{-1} \text{ exists} \iff (a, b)^{-1} = \left(\frac{\bar{a}}{|a|^2 - t|b|^2}, \frac{-b}{|a|^2 - t|b|^2} \right),$$

“in \mathbb{H}_t^\times ”, equivalently,

$$[(a, b)^{-1}]_t = [(a, b)]_t^{-1} \text{ in } \mathcal{H}_2^\times.$$

Corollary 2.10. *Let $(a, b) \in \mathbb{H}_t^\times$. Then it is invertible, if and only if*

$$[(a, b)^{-1}]_t = \left[\left(\frac{\bar{a}}{|a|^2 - t|b|^2}, \frac{-b}{|a|^2 - t|b|^2} \right) \right]_t = [(a, b)]_t^{-1}, \tag{2.12}$$

in \mathcal{H}_2^\times , where $[(a, b)]_t^{-1}$ means the matricial inverse in $M_2(\mathbb{C})$.

Proof. The proof of (2.12) is immediately done by Theorem 2.9(ii)–(iv). \square

The above corollary can be re-stated by that: if $\xi \in \mathbb{H}_t^\times$ is invertible, then

$$\pi_t(\xi^{-1}) = (\pi_t(\xi))^{-1} \text{ in } \mathcal{H}_2^{t \times}.$$

Now consider the cases where

$$|a|^2 - t|b|^2 = 0 \iff |a|^2 = t|b|^2, \tag{2.13}$$

in \mathbb{R} . As we have seen above, the condition (2.13) holds for $(a, b) \in \mathbb{H}_t$, if and only if (a, b) is not invertible in \mathbb{H}_t (and hence, its realization $[(a, b)]_t$ is not invertible in $M_2(\mathbb{C})$, and hence, in \mathcal{H}_2^t). Clearly, we are not interested in the (+)-identity $(0, 0)$

of \mathbb{H}_t automatically satisfying the condition (2.13). So, without loss of generality, we focus on elements (a, b) of the t -scaled hypercomplex monoid \mathbb{H}_t^\times (or, its realizations $[(a, b)]_t$ of $\mathcal{H}_2^{t\times}$), satisfying the condition (2.13).

Recall that an algebraic triple, $(X, +, \cdot)$, is a noncommutative field, if (i) $(X, +)$ is an Abelian group, (ii) (X^\times, \cdot) forms a non-Abelian group, and (iii) the operations $(+)$ and (\cdot) are left-and-right distributive. For instance, the quaternions $\mathbb{H} = \mathbb{H}_{-1}$ is a noncommutative field (e.g., [2] and [3]).

Theorem 2.11. *Suppose the fixed scale $t \in \mathbb{R}$ is negative, i.e., $t < 0$ in \mathbb{R} . Then “all” elements (a, b) of the t -scaled hypercomplex monoid \mathbb{H}_t^\times are invertible in \mathbb{H}_t , with their inverses,*

$$\left(\frac{\bar{a}}{|a|^2 - t|b|^2}, \frac{-b}{|a|^2 - t|b|^2} \right) \in \mathbb{H}_t^\times,$$

i.e.,

$$t < 0 \text{ in } \mathbb{R} \implies \mathbb{H}_t \text{ is a noncommutative field.} \tag{2.14}$$

Proof. Suppose the scale $t \in \mathbb{R}$ is negative. Then, for any $(a, b) \in \mathbb{H}_t^\times$,

$$|a|^2 \neq t|b|^2 \iff |a|^2 - t|b|^2 > 0,$$

since $(a, b) \neq (0, 0)$, i.e., if $t < 0$, then every element $(a, b) \in \mathbb{H}_t^\times$ does “not” satisfy the condition (2.13). It implies that if $t < 0$, then every element $(a, b) \in \mathbb{H}_t^\times$ is invertible in \mathbb{H}_t^\times , by Theorem 2.9(iii)-(iv); and the inverse is determined to be (2.12) in \mathbb{H}_t^\times . Thus, the pair $\mathbb{H}_t^\times = (\mathbb{H}_t^\times, \cdot_t)$ forms a group which is not Abelian by (2.1) and (2.8).

Therefore, if $t < 0$ in \mathbb{R} , then the t -scaled hypercomplex ring \mathbb{H}_t becomes a noncommutative field, proving the statement (2.14). \square

The above theorem characterizes that the algebraic structure of scaled hypercomplex rings $\{\mathbb{H}_t\}_{t < 0}$ as noncommutative fields.

Theorem 2.12. *Suppose $t = 0$ in \mathbb{R} . Then an element (a, b) of the 0-scaled hypercomplex monoid \mathbb{H}_0^\times is invertible in \mathbb{H}_0 , with their inverses,*

$$\left(\frac{\bar{a}}{|a|^2}, \frac{-b}{|a|^2} \right) \in \mathbb{H}_0^\times,$$

if and only if $a \neq 0$ in \mathbb{C} , if and only if only the elements of the subset,

$$\{(a, b) \in \mathbb{H}_0^\times : a \neq 0\} \text{ of } \mathbb{H}_0^\times \tag{2.15}$$

are invertible in \mathbb{H}_0^\times , if and only if $(0, b) \in \mathbb{H}_0^\times$ are not invertible in \mathbb{H}_0^\times , for all $b \in \mathbb{C}$.

Proof. Assume that we have the zero scale, i.e., $t = 0$ in \mathbb{R} . Then, by (2.13),

$$|a|^2 = 0 \cdot |b|^2 \iff |a|^2 = 0 \iff a = 0 \text{ in } \mathbb{C},$$

if and only if $(0, b) \in \mathbb{H}_0^\times$ are not invertible in \mathbb{H}_0^\times , for all $b \in \mathbb{C}$, if and only if all elements (a, b) , contained in the subset (2.15), are invertible in \mathbb{H}_0^\times .

Observe that (a, b) is contained in the subset (2.15) of \mathbb{H}_0^\times , if and only if

$$\begin{aligned} [(a, b)]_0 \left[\left(\frac{\bar{a}}{|a|^2}, \frac{-b}{|a|^2} \right) \right]_0 &= \begin{pmatrix} a & 0 \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \frac{\bar{a}}{|a|^2} & 0 \\ \frac{-b}{|a|^2} & \frac{a}{|a|^2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\bar{a}}{|a|^2} & 0 \\ \frac{-b}{|a|^2} & \frac{a}{|a|^2} \end{pmatrix} \begin{pmatrix} a & 0 \\ \bar{b} & \bar{a} \end{pmatrix} \\ &= \left[\left(\frac{\bar{a}}{|a|^2}, \frac{-b}{|a|^2} \right) \right]_0 [(a, b)]_0, \end{aligned}$$

in \mathbb{H}_0^\times . Therefore, if exists, $(a, b)^{-1} = \left(\frac{\bar{a}}{|a|^2}, \frac{-b}{|a|^2} \right)$ in \mathbb{H}_0^\times . □

The above theorem shows that if we have the zero-scale in \mathbb{R} , then our 0-scaled hypercomplex ring \mathbb{H}_0 cannot be a noncommutative field. It directly illustrates that the algebra on the quaternions $\mathbb{H} = \mathbb{H}_{-1}$, and the algebra on the scaled-hypercomplex rings $\{\mathbb{H}_t\}_{t \in \mathbb{R} \setminus \{-1\}}$ can be different in general, especially, when $t \geq 0$.

Theorem 2.13. *Suppose the scale $t \in \mathbb{R}$ is positive, i.e., $t > 0$ in \mathbb{R} . Then an element $(a, b) \in \mathbb{H}_t^\times$ is invertible in \mathbb{H}_t^\times with its inverse,*

$$\left(\frac{\bar{a}}{|a|^2 - t|b|^2}, \frac{-b}{|a|^2 - t|b|^2} \right) \in \mathbb{H}_t^\times,$$

if and only if $|a|^2 \neq t|b|^2$ in $\mathbb{R}_0^+ = \{r \in \mathbb{R} : r \geq 0\}$, if and only if (a, b) is contained in the subset,

$$\{(a, b) : |a|^2 \neq t|b|^2 \text{ in } \mathbb{R}_0^+\}, \tag{2.16}$$

of \mathbb{H}_t^\times . As application, if $t > 0$ in \mathbb{R} , then the all elements of

$$\{(a, 0) \in \mathbb{H}_t : a \in \mathbb{C}^\times\} \cup \{(0, b) \in \mathbb{H}_t : b \in \mathbb{C}^\times\}, \tag{2.17}$$

are invertible in \mathbb{H}_t , where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Proof. Assume that $t > 0$ in \mathbb{R} , and \mathbb{H}_t^\times , the corresponding t -scaled hypercomplex monoid. Then $(a, b) \in \mathbb{H}_t^\times$ is invertible in \mathbb{H}_t^\times , if and only if the condition (2.13) does not hold, if and only if

$$|a|^2 \neq t|b|^2 \iff \text{either } |a|^2 > t|b|^2, \text{ or } |a|^2 < t|b|^2,$$

in \mathbb{R}_0^+ , since $t > 0$. Therefore, if $t > 0$ in \mathbb{R} , then an element (a, b) is invertible in \mathbb{H}_t^\times , if and only if

$$\text{either } |a|^2 > t|b|^2, \text{ or } |a|^2 < t|b|^2 \text{ in } \mathbb{R}_0^+,$$

if and only if (a, b) is contained in the subset (2.16) in \mathbb{H}_t^\times .

In particular, for $t > 0$ in \mathbb{R} , (i) if $(a, 0) \in \mathbb{H}_t^\times$ with $a \in \mathbb{C}^\times$, then $|a|^2 > 0$; and (ii) if $(0, b) \in \mathbb{H}_t^\times$ with $b \in \mathbb{C}^\times$, then $0 < t|b|^2$. Therefore, the subset (2.17) is properly contained in the subset (2.16) in \mathbb{H}_t^\times , whenever $t > 0$. So, all elements, formed by $(a, 0)$, or by $(0, b)$ with $a, b \in \mathbb{C}^\times$, are invertible in \mathbb{H}_t^\times . □

The above theorem characterizes the invertibility on the t -scaled hypercomplex monoid \mathbb{H}_t^\times , where the scale t is positive in \mathbb{R} . Theorems 2.11, 2.12 and 2.13 refine Theorem 2.9, case-by-case. We again summarize the main results.

Corollary 2.14. *Let \mathbb{H}_t^\times be the t -scaled hypercomplex monoid. If $t < 0$, then all nonzero elements of \mathbb{H}_t^\times are invertible; and if $t = 0$, then*

$$\{(a, b) \in \mathbb{H}_0^\times : a \neq 0\}$$

is the invertible proper subset of \mathbb{H}_0^\times ; and if $t > 0$, then

$$\{(a, b) : |a|^2 \neq t|b|^2 \text{ in } \mathbb{R}_0^+\}$$

is the invertible proper subset of \mathbb{H}_t^\times , where “invertible subset of \mathbb{H}_t^\times ” means “a subset of \mathbb{H}_t^\times containing of all invertible elements”.

Proof. This corollary is nothing but a summary of Theorems 2.11, 2.12 and 2.13. \square

2.5. DECOMPOSITIONS OF THE NONNEGATIVELY-SCALED HYPERCOMPLEX RINGS

In this section, we consider a certain decomposition of the t -scaled hypercomplex ring \mathbb{H}_t , for an arbitrary fixed “positive” scale $t > 0$ in \mathbb{R} . Let $t \geq 0$ and \mathbb{H}_t , the corresponding t -scaled hypercomplex ring. Partition \mathbb{H}_t by

$$\mathbb{H}_t = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{sing}$$

with

$$\mathbb{H}_t^{inv} = \{(a, b) : |a|^2 \neq t|b|^2\}, \tag{2.18}$$

and

$$\mathbb{H}_t^{sing} = \{(a, b) : |a|^2 = t|b|^2\},$$

where \sqcup is the disjoint union. By (2.15) and (2.16), $(a, b) \in \mathbb{H}_t^{inv}$, if and only if it is invertible, equivalently, $(a, b) \in \mathbb{H}_t^{sing}$, if and only if it is not invertible, in \mathbb{H}_t .

Recall-and-note that the determinant is a multiplicative map on $M_n(\mathbb{C})$, for all $n \in \mathbb{N}$, in the sense that:

$$\det(AB) = \det(A)\det(B), \quad \forall A, B \in M_n(\mathbb{C}). \tag{2.19}$$

Thus, by (2.19), one has

$$\xi, \eta \in \mathbb{H}_t^{inv} \Rightarrow \det([\xi \cdot_t \eta]_t) = \det([\xi]_t [\eta]_t) \neq 0. \tag{2.20}$$

Lemma 2.15. *Let $t \geq 0$ in \mathbb{R} . Then the subset $\mathbb{H}_t^{inv} \stackrel{\text{denote}}{=} (\mathbb{H}_t^{inv}, \cdot_t)$ of the t -scaled hypercomplex monoid \mathbb{H}_t^\times forms a non-Abelian group, i.e., \mathbb{H}_t^{inv} is not only a sub-monoid, but also an embedded group in \mathbb{H}_t^\times .*

Proof. By (2.19), if $\xi, \eta \in \mathbb{H}_t^{inv}$, then $\xi \cdot_t \eta \in \mathbb{H}_t^{inv}$, too, i.e., the operation (\cdot_t) is closed, and associative on \mathbb{H}_t^{inv} . Also, the (\cdot_t) -identity $(1, 0)$ is contained in \mathbb{H}_t^{inv} by (2.18). Therefore, the sub-structure $(\mathbb{H}_t^{inv}, \cdot_t)$ forms a sub-monoid of \mathbb{H}_t^\times . But, by (2.14) and (2.20), each element $\xi \in \mathbb{H}_t^{inv}$ has its (\cdot_t) -inverse ξ^{-1} contained in \mathbb{H}_t^{inv} . It shows that \mathbb{H}_t^{inv} forms a non-Abelian group in the monoid \mathbb{H}_t^\times . \square

By the partition (2.18) and the multiplicativity (2.20), one can obtain the following equivalent result of the above theorem.

Lemma 2.16. *Let $t \geq 0$ in \mathbb{R} . Then the pair*

$$\mathbb{H}_t^{\times sing} \stackrel{\text{denote}}{=} \left(\mathbb{H}_t^{sing} \cap \mathbb{H}_t^\times, \cdot_t \right) = \left(\mathbb{H}_t^{sing} \setminus \{(0, 0)\}, \cdot_t \right)$$

forms a semigroup without identity in the t -scaled hypercomplex monoid \mathbb{H}_t^\times .

Proof. By (2.19) and (2.20), the operation (\cdot_t) is closed and associative on the set,

$$\mathbb{H}_t^{\times sing} \stackrel{\text{def}}{=} \mathbb{H}_t^\times \cap \mathbb{H}_t^{sing} = \mathbb{H}_t^{sing} \setminus \{(0, 0)\}.$$

However, the (\cdot_t) -identity $(1, 0)$ is not contained in $\mathbb{H}_t^{\times sing}$, since $I_2 = [(1, 0)]_t$ is in \mathbb{H}_t^{inv} . So, in the monoid \mathbb{H}_t^\times , the sub-structure $(\mathbb{H}_t^{\times sing}, \cdot_t)$ forms a semigroup (without identity). \square

The above lemma definitely includes the fact that: $(\mathbb{H}_t^{sing}, \cdot_t)$ is just a semigroup (without identity), which is not a sub-monoid of \mathbb{H}_t^\times (and hence, not a group).

The above two algebraic characterizations show that the set-theoretical decomposition (2.18) induces an algebraic decomposition of the t -scaled hypercomplex monoid \mathbb{H}_t^\times ,

$$\mathbb{H}_t^\times = (\mathbb{H}_t^{inv}, \cdot_t) \sqcup (\mathbb{H}_t^{\times sing}, \cdot_t),$$

where

$$\mathbb{H}_t^{inv} = \{(a, b) \in \mathbb{H}_t^\times : |a|^2 \neq t|b|^2\}, \tag{2.21}$$

and

$$\mathbb{H}_t^{\times sing} = \{(a, b) \in \mathbb{H}_t^\times : |a|^2 = t|b|^2\},$$

whenever $t \geq 0$ in \mathbb{R} .

Theorem 2.17. *For $t \geq 0$ in \mathbb{R} , the t -scaled hypercomplex monoid \mathbb{H}_t^\times is algebraically decomposed to be*

$$\mathbb{H}_t^\times = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

where \mathbb{H}_t^{inv} is the group, and $\mathbb{H}_t^{\times sing}$ is the semigroup without identity in (2.21).

Proof. The algebraic decomposition,

$$\mathbb{H}_t^\times = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

of the t -scaled hypercomplex monoid \mathbb{H}_t^\times is obtained by the set-theoretic decomposition (2.18) of \mathbb{H}_t^\times , the above two lemmas, and (2.21). \square

By the above theorem, one can have the following concepts whenever a given scale t is nonnegative in \mathbb{R} .

Definition 2.18. Let $t \geq 0$ in \mathbb{R} , and \mathbb{H}_t^\times , the t -scaled hypercomplex monoid. The algebraic block,

$$\mathbb{H}_t^{inv} = (\{(a, b) \in \mathbb{H}_t^\times : |a|^2 \neq t|b|^2\}, \cdot_t),$$

is called the group-part of \mathbb{H}_t^\times (or, of \mathbb{H}_t), and the other algebraic block,

$$\mathbb{H}_t^{\times sing} = (\{(a, b) \in \mathbb{H}_t^\times : |a|^2 = t|b|^2\}, \cdot_t),$$

is called the semigroup-part of \mathbb{H}_t^\times (or, of \mathbb{H}_t).

By the above definition, Theorem 2.17 can be re-stated that: if a scale t is non-negative in \mathbb{R} , then the t -scaled hypercomplex monoid \mathbb{H}_t^\times is decomposed to be the group-part \mathbb{H}_t^{inv} and the semigroup-part $\mathbb{H}_t^{\times sing}$.

One may say that if $t < 0$ in \mathbb{R} , then the semigroup-part $\mathbb{H}_t^{\times sing}$ is empty in \mathbb{H}_t^\times . Indeed, for any scale $t \in \mathbb{R}$, the t -scaled hypercomplex monoid \mathbb{H}_t is decomposed to be (2.21). As we have seen in this section, if $t \geq 0$, then the semigroup-part $\mathbb{H}_t^{\times sing}$ is nonempty, meanwhile, as we considered in Section 2.4, if $t < 0$, then the semigroup-part $\mathbb{H}_t^{\times sing}$ is empty, equivalently, the t -scaled hypercomplex monoid \mathbb{H}_t^\times is identified with its group-part \mathbb{H}_t^{inv} , i.e., $\mathbb{H}_t^\times = \mathbb{H}_t^{inv}$ in \mathbb{H}_t , whenever $t < 0$.

Corollary 2.19. For every $t \in \mathbb{R}$, the t -scaled hypercomplex monoid \mathbb{H}_t^\times is partitioned by

$$\mathbb{H}_t^\times = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

where the group-part \mathbb{H}_t^{inv} and the semigroup-part $\mathbb{H}_t^{\times sing}$ are in the sense of (2.21). In particular, if $t < 0$, then

$$\mathbb{H}_t^{\times sing} = \emptyset \iff \mathbb{H}_t^\times = \mathbb{H}_t^{inv};$$

meanwhile, if $t \geq 0$, then $\mathbb{H}_t^{\times sing}$ is a non-empty proper subset of \mathbb{H}_t^\times .

Proof. It is shown conceptually by the discussion of the very above paragraph. Also, see Theorems 2.11 and 2.17. □

3. SPECTRAL ANALYSIS ON $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$ UNDER $\{(\mathbb{C}^2, \pi_t)\}_{t \in \mathbb{R}}$

Throughout this section, we fix an arbitrary scale $t \in \mathbb{R}$, and the corresponding t -scaled hypercomplex ring,

$$\mathbb{H}_t = (\mathbb{C}^2, +, \cdot_t),$$

containing its hypercomplex monoid $\mathbb{H}_t^\times = (\mathbb{H}_t^\times, \cdot_t)$. In Section 2, we showed that for a scale $t \in \mathbb{R}$, the monoid \mathbb{H}_t^\times is partitioned by

$$\mathbb{H}_t^\times = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

where \mathbb{H}_t^{inv} is the group-part, and $\mathbb{H}_t^{\times sing}$ is the semigroup-part of \mathbb{H}_t . In particular, if $t < 0$, then the semigroup-part $\mathbb{H}_t^{\times sing}$ is empty in \mathbb{H}_t^\times , equivalently, $\mathbb{H}_t^\times = \mathbb{H}_t^{inv}$ in \mathbb{H}_t , meanwhile, if $t \geq 0$, then $\mathbb{H}_t^{\times sing}$ is a non-empty proper subset of \mathbb{H}_t^\times .

Motivated by such an analysis of invertibility on \mathbb{H}_t , we here consider spectral analysis on \mathbb{H}_t .

3.1. HYPERCOMPLEX-SPECTRAL FORMS ON \mathbb{H}_t

For $t \in \mathbb{R}$, let \mathbb{H}_t be the t -scaled hypercomplex ring realized to be

$$\mathcal{H}_2^t = \pi_t(\mathbb{H}_t) = \left\{ \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbb{C}) : (a, b) \in \mathbb{H}_t \right\},$$

in $M_2(\mathbb{C})$ under the Hilbert-space representation $\Pi_t = (\mathbb{C}^2, \pi_t)$ of \mathbb{H}_t .

Let $(a, b) \in \mathbb{H}_t$ be an arbitrary element with

$$\pi_t(a, b) = [(a, b)]_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathcal{H}_2^t.$$

Then, in a variable z on \mathbb{C} ,

$$\begin{aligned} \det([(a, b)]_t - z[(1, 0)]_t) &= \det \begin{pmatrix} a - z & tb \\ \bar{b} & \bar{a} - z \end{pmatrix} \\ &= (a - z)(\bar{a} - z) - t|b|^2 \\ &= |a|^2 - az - \bar{a}z + z^2 - t|b|^2 \\ &= z^2 - (a + \bar{a})z + (|a|^2 - t|b|^2) \\ &= z^2 - 2\operatorname{Re}(a)z + \det([(a, b)]_t), \end{aligned}$$

where $\operatorname{Re}(a)$ is the real part of a in \mathbb{C} , and

$$\det([(a, b)]_t) = |a|^2 - t|b|^2,$$

by Theorem 2.9(i). Thus, the equation,

$$\det([(a, b)]_t - z[(1, 0)]_t) = 0,$$

in a variable z on \mathbb{C} , has its solutions,

$$z = \frac{2\operatorname{Re}(a) \pm \sqrt{4\operatorname{Re}(a)^2 - 4\det([(a, b)]_t)}}{2},$$

if and only if

$$z = \operatorname{Re}(a) \pm \sqrt{\operatorname{Re}(a)^2 - \det([(a, b)]_t)}. \tag{3.1}$$

Recall that a matrix $A \in M_n(\mathbb{C})$, for any $n \in \mathbb{N}$, has its spectrum

$$\operatorname{spec}(A) = \{\lambda \in \mathbb{C} : \det(A - \lambda I_n) = 0\},$$

equivalently,

$$\text{spec}(A) = \{\lambda \in \mathbb{C} : \text{there exists } \eta \in \mathbb{C}^n \text{ such that } A\eta = \lambda\eta\}, \tag{3.2}$$

if and only if

$$\text{spec}(A) = \{\lambda \in \mathbb{C} : A - \lambda I_n \text{ is not invertible in } M_n(\mathbb{C})\},$$

as a nonempty discrete (compact) subset of \mathbb{C} , where I_n is the identity matrix of $M_n(\mathbb{C})$ (e.g., [9]). More generally, if $T \in B(H)$ is an operator on a Hilbert space H , then the spectrum $\sigma(T)$ of T is defined to be a nonempty compact subset,

$$\sigma(T) = \{z \in \mathbb{C} : T - zI_H \text{ is not invertible on } H\},$$

where I_H is the identity operator of $B(H)$. Remark that if H is infinite-dimensional, then $\sigma(T)$ is not a discrete subset of \mathbb{C} as in (3.2), in general (e.g., [8]).

Theorem 3.1. *Let $(a, b) \in \mathbb{H}_t$ realized to be $[(a, b)]_t \in \mathcal{H}_2^t$. Then*

$$\text{spec}([(a, b)]_t) = \left\{ \text{Re}(a) \pm \sqrt{\text{Re}(a)^2 - \det([(a, b)]_t)} \right\},$$

in \mathbb{C} . More precisely, if

$$a = x + yi, \quad b = u + vi \in \mathbb{C},$$

with $x, y, u, v \in \mathbb{R}$ and $i = \sqrt{-1}$ in \mathbb{C} , then

$$\text{spec}([(a, b)]_t) = \left\{ x \pm i\sqrt{y^2 - tu^2 - tv^2} \right\} \text{ in } \mathbb{C}. \tag{3.3}$$

Proof. The realization $[(a, b)]_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathcal{H}_2^t$ of a hypercomplex number $(a, b) \in \mathbb{H}_t$ has its spectrum,

$$\text{spec}([(a, b)]_t) = \left\{ \text{Re}(a) \pm \sqrt{\text{Re}(a)^2 - (|a|^2 - t|b|^2)} \right\},$$

in \mathbb{C} , by (3.1) and (3.2). If

$$a = x + yi, \text{ and } b = u + vi \text{ in } \mathbb{C},$$

with $x, y, u, v \in \mathbb{R}$ and $i = \sqrt{-1}$ in \mathbb{C} , then

$$\text{Re}(a) = x,$$

and

$$|a|^2 - t|b|^2 = (x^2 + y^2) - t(u^2 + v^2),$$

in \mathbb{R} , and hence,

$$\text{spec}([(a, b)]_t) = \left\{ x \pm \sqrt{-y^2 + tu^2 + tv^2} \right\},$$

if and only if

$$\text{spec}([(a, b)]_t) = \left\{ x \pm i\sqrt{y^2 - tu^2 - tv^2} \right\},$$

in \mathbb{C} . Therefore, the set-equality (3.3) holds. □

From below, for our purposes, we let

$$a = x + yi \text{ and } b = u + vi \text{ in } \mathbb{C} \tag{3.4}$$

with

$$x, y, u, v \in \mathbb{R}, \text{ and } i = \sqrt{-1}.$$

The above theorem can be refined by the following result.

Corollary 3.2. *Let $(a, b) \in \mathbb{H}_t$, realized to be $[(a, b)]_t \in \mathcal{H}_2^t$, satisfy (3.4). Then the following assertions hold.*

(i) *If $\text{Im}(a)^2 = t|b|^2$ in \mathbb{R} , where $\text{Im}(a)$ is the imaginary part of a in \mathbb{C} , then*

$$\text{spec}([(a, b)]_t) = \{x\} = \{\text{Re}(a)\} \text{ in } \mathbb{R}.$$

(ii) *If $\text{Im}(a)^2 < t|b|^2$ in \mathbb{R} , then*

$$\text{spec}([(a, b)]_t) = \left\{ x \pm \sqrt{tu^2 + tv^2 - y^2} \right\} \text{ in } \mathbb{R}.$$

(iii) *If $\text{Im}(a)^2 > t|b|^2$ in \mathbb{R} , then*

$$\text{spec}([(a, b)]_t) = \left\{ x \pm i\sqrt{y^2 - tu^2 - tv^2} \right\} \text{ in } \mathbb{C} \setminus \mathbb{R}.$$

Proof. For $(a, b) \in \mathbb{H}_t$, satisfying (3.4), one has

$$\text{spec}([(a, b)]_t) = \left\{ x \pm i\sqrt{y^2 - tu^2 - tv^2} \right\},$$

by (3.3). So, one can verify that: (a) if $y^2 - tu^2 - tv^2 = 0$, equivalently, if

$$\text{Im}(a)^2 = t|b|^2 \text{ in } \mathbb{R},$$

then $\text{spec}([(a, b)]_t) = \{x \pm i\sqrt{0}\} = \{x\}$ in \mathbb{R} ; (b) if $y^2 - tu^2 - tv^2 < 0$, equivalently, if

$$\text{Im}(a)^2 < t|b|^2 \text{ in } \mathbb{R},$$

then

$$x \pm i\sqrt{y^2 - tu^2 - tv^2} = x \pm i\sqrt{-|y^2 - tu^2 - tv^2|},$$

implying that

$$x \pm i\sqrt{y^2 - tu^2 - tv^2} = x \pm i^2\sqrt{tu^2 + tv^2 - y^2},$$

and hence,

$$\text{spec}([(a, b)]_t) = \left\{ x \mp \sqrt{tu^2 + tv^2 - y^2} \right\} \text{ in } \mathbb{R};$$

and, finally, (c) if $y^2 - tu^2 - tv^2 > 0$, equivalently, if

$$\text{Im}(a)^2 > t|b|^2 \text{ in } \mathbb{R},$$

then

$$\text{spec}([(a, b)]_t) = \left\{ x \pm i\sqrt{y^2 - tu^2 - tv^2} \right\},$$

contained in $\mathbb{C} \setminus \mathbb{R}$.

Therefore, the refined statements (i), (ii) and (iii) of the spectrum (3.3) of $[(a, b)]_t$ hold true. □

By the above corollary, one immediately obtains the following result.

Corollary 3.3. *Suppose $(a, b) \in \mathbb{H}_t$. If $\operatorname{Im}(a)^2 \leq t|b|^2$, then*

$$\operatorname{spec}([(a, b)]_t) \subset \mathbb{R};$$

meanwhile, if $\operatorname{Im}(b)^2 > t|b|^2$, then

$$\operatorname{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}), \text{ in } \mathbb{C}.$$

Proof. It is shown by (i)–(iii) of Corollary 3.2. \square

Also, we have the following result.

Theorem 3.4. *Assume that the fixed scale $t \in \mathbb{R}$ is negative, i.e., $t < 0$ in \mathbb{R} . If*

$$(a, b) \in \mathbb{H}_t, \text{ with } b \neq 0 \text{ in } \mathbb{C},$$

then

$$\operatorname{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}) \text{ in } \mathbb{C}. \quad (3.5)$$

Meanwhile, if $b = 0$ in \mathbb{C} for $(a, b) \in \mathbb{H}_t$, then

$$a \in \mathbb{R} \implies \operatorname{spec}([(a, 0)]_t) = \{a\} \text{ in } \mathbb{R},$$

and

$$a \in \mathbb{C} \setminus \mathbb{R} \implies \operatorname{spec}([(a, 0)]_t) = \{a, \bar{a}\} \text{ in } \mathbb{C} \setminus \mathbb{R}. \quad (3.6)$$

Proof. Assume that the scale t is given to be negative in \mathbb{R} . Then, for any $(a, b) \in \mathbb{H}_t$, one immediately obtains that

$$\operatorname{Im}(a)^2 \geq t|b|^2,$$

because the left-hand side, $\operatorname{Im}(a)^2$, is nonnegative, but the right-hand side, $t|b|^2$ is either negative or zero in \mathbb{R} by the negativity of t .

Suppose $b \neq 0$ in \mathbb{C} , equivalently, $|b|^2 > 0$, implying $t|b|^2 < 0$ in \mathbb{R} . Then

$$\operatorname{Im}(a)^2 > t|b|^2 \text{ in } \mathbb{R}.$$

Thus, by Corollary 3.2(iii), the spectra, $\operatorname{spec}([(a, b)]_t)$, of the realizations $[(a, b)]_t$ of $(a, b) \in \mathbb{H}_t$, with $b \neq 0$, is contained in $\mathbb{C} \setminus \mathbb{R}$. It proves the relation (3.5).

Meanwhile, if $a = \operatorname{Re}(a)$, and $b = 0$ in \mathbb{C} , then

$$0 = \operatorname{Im}(a)^2 \leq 0 = t \cdot 0 \text{ in } \mathbb{R},$$

implying that

$$\operatorname{spec}([(a, 0)]_t) \subset \mathbb{R} \text{ in } \mathbb{C},$$

by Corollary 3.2(i). However, if $\operatorname{Im}(a) \neq 0$, and $b = 0$, then

$$\operatorname{Im}(a)^2 > 0 = t \cdot 0 \text{ in } \mathbb{R},$$

and hence,

$$\operatorname{spec}([(a, 0)]_t) \subset (\mathbb{C} \setminus \mathbb{R}) \text{ in } \mathbb{C}.$$

So, the relation (3.6) is proven. \square

The above theorem specifies Theorem 3.1 for the case where $t < 0$ in \mathbb{R} , by (3.5) and (3.6).

Theorem 3.5. *Assume that $t = 0$ in \mathbb{R} . If $(a, b) \in \mathbb{H}_0$ with $\text{Im}(a) \neq 0$ in \mathbb{C} , then*

$$\text{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}) \text{ in } \mathbb{C}. \tag{3.7}$$

Meanwhile, if $\text{Im}(a) = 0$, then

$$\text{spec}([(a, b)]_t) \subset \mathbb{R} \text{ in } \mathbb{C}. \tag{3.8}$$

Proof. Suppose the fixed scale t is zero in \mathbb{R} . Then, for any hypercomplex number $(a, b) \in \mathbb{H}_0$, one has

$$[(a, b)]_0 = \begin{pmatrix} a & 0 \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathcal{H}_2^0,$$

and hence,

$$\text{Im}(a)^2 \geq 0 = 0 \cdot |b|^2 \text{ in } \mathbb{R}.$$

In particular, if $\text{Im}(a) \neq 0$ in \mathbb{C} , then the above inequality becomes

$$\text{Im}(a)^2 > 0 \text{ in } \mathbb{R},$$

implying that

$$\text{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}) \text{ in } \mathbb{C},$$

by Corollary 3.2(iii), i.e., for all $(a, b) \in \mathbb{H}_0$, with $a \in \mathbb{C}$ with $\text{Im}(a) \neq 0$, and $b \in \mathbb{C}$ arbitrary, the spectra of the realizations of such (a, b) are contained in $\mathbb{C} \setminus \mathbb{R}$. It shows that the relation (3.7) holds.

Meanwhile, if $\text{Im}(a) = 0$ in \mathbb{C} , then one has

$$\text{Im}(a)^2 = 0 \geq 0 = 0 \cdot |b|^2 \text{ in } \mathbb{R}.$$

So, by Corollary 3.2(i), we have

$$\text{spec}([(a, b)]_t) \subset \mathbb{R} \text{ in } \mathbb{C}.$$

Therefore, the relation (3.8) holds true, too. □

The above theorem specifies Theorem 3.1 for the case where a scale t is zero in \mathbb{R} , by (3.7) and (3.8).

Theorem 3.6. *Assume that the fixed scale t is positive in \mathbb{R} . Then the t -scaled hypercomplex ring \mathbb{H}_t is decomposed to be*

$$\mathbb{H}_t = \mathbb{H}_t^+ \sqcup \mathbb{H}_t^{-0},$$

with

$$\mathbb{H}_t^+ = \{(a, b) \in \mathbb{H}_t : \text{Im}(a)^2 > t|b|^2\}, \tag{3.9}$$

and

$$\mathbb{H}_t^{-0} = \{(a, b) \in \mathbb{H}_t : \text{Im}(a)^2 \leq t|b|^2\},$$

where \sqcup is the disjoint union. Moreover, if $(a, b) \in \mathbb{H}_t^+$, then

$$\text{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}). \quad (3.10)$$

Meanwhile, if $(a, b) \in \mathbb{H}_t^{-0}$, then

$$\text{spec}([(a, b)]_t) \subset \mathbb{R} \text{ in } \mathbb{C}. \quad (3.11)$$

Proof. Suppose that $t > 0$ in \mathbb{R} . Then one can decompose the t -scaled hypercomplex ring \mathbb{H}_t by

$$\mathbb{H}_t = \mathbb{H}_t^+ \sqcup \mathbb{H}_t^{-0},$$

with

$$\begin{aligned} \mathbb{H}_t^+ &= \{(a, b) \in \mathbb{H}_t : \text{Im}(a)^2 > t|b|^2\}, \\ \mathbb{H}_t^{-0} &= \{(a, b) \in \mathbb{H}_t : \text{Im}(a)^2 \leq t|b|^2\}, \end{aligned} \quad (3.12)$$

set-theoretically. Thus, the partition (3.9) holds by (3.12).

If $(a, b) \in \mathbb{H}_t^+$, then

$$\text{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}),$$

meanwhile, if $(a, b) \in \mathbb{H}_t^{-0}$, then

$$\text{spec}([(a, b)]_t) \subset \mathbb{R}, \text{ in } \mathbb{C}.$$

So, the relations (3.10) and (3.11) are proven. \square

The above theorem specifies Theorem 3.1 for the cases where a fixed scale t is positive in \mathbb{R} , by (3.10) and (3.11), up to the decomposition (3.9).

In fact, one can realize that, for “all” $t \in \mathbb{R}$, the corresponding t -scaled hypercomplex ring \mathbb{H}_t is partitioned to be

$$\mathbb{H}_t = \mathbb{H}_t^+ \sqcup \mathbb{H}_t^{-0},$$

where \mathbb{H}_t^+ and \mathbb{H}_t^{-0} are in the sense of (3.9). Especially, Theorems 3.4, 3.5 and 3.6 characterize the above decomposition case-by-case, based on Theorem 3.1 and Corollary 3.2. So, we obtain the following universal spectral properties on \mathbb{H}_t .

Corollary 3.7. *Let $t \in \mathbb{R}$ be an arbitrarily fixed scale for \mathbb{H}_t . Then*

$$\mathbb{H}_t = \mathbb{H}_t^+ \sqcup \mathbb{H}_t^{-0}, \text{ set-theoretically,}$$

where $\{\mathbb{H}_t^+, \mathbb{H}_t^{-0}\}$ is a partition in the sense of (3.9) for t . Moreover, if $(a, b) \in \mathbb{H}_t^+$, then

$$\text{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}),$$

meanwhile, if $(a, b) \in \mathbb{H}_t^{-0}$, then

$$\text{spec}([(a, b)]_t) \subset \mathbb{R} \text{ in } \mathbb{C}.$$

Especially, if $t < 0$, then $\mathbb{H}_t^{-0} = \{(0, 0)\}$, equivalently, $\mathbb{H}_t^\times = \mathbb{H}_t^+$.

Proof. This corollary is nothing but a summary of Theorems 3.4, 3.5 and 3.6. \square

It is not hard to check the converses of the statements of Corollary 3.7 hold true, too.

Theorem 3.8. *Let $\mathbb{H}_t = \mathbb{H}_t^+ \sqcup \mathbb{H}_t^{-0}$ be the fixed t -scaled hypercomplex ring for $t \in \mathbb{R}$. Then the following assertions hold.*

- (i) $(a, b) \in \mathbb{H}_t^+$, if and only if $\text{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R})$.
- (ii) $(a, b) \in \mathbb{H}_t^{-0}$, if and only if $\text{spec}([(a, b)]_t) \subset \mathbb{R}$.

Proof. First, assume that $(a, b) \in \mathbb{H}_t^+$ in \mathbb{H}_t . Then, by Corollary 3.7,

$$\text{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}).$$

Now, suppose that

$$\text{spec}([(a, b)]_t) \subset \mathbb{R} \text{ in } \mathbb{C},$$

and assume that $(a, b) \in \mathbb{H}_t^+$. Then, (a, b) is contained in \mathbb{H}_t^{-0} , equivalently, it cannot be an element of \mathbb{H}_t^+ , by Corollary 3.2(i)–(ii), (3.6), (3.8) and (3.11). It contradicts our assumption. Therefore,

$$(a, b) \in \mathbb{H}_t^+ \iff \text{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}).$$

Thus, the statement (i) holds.

By the decomposition (3.9), the statement (ii) holds true, by (i). \square

By the above theorem, we obtain the following result.

Corollary 3.9. *Let \mathbb{H}_t be the t -scaled hypercomplex ring for an arbitrary $t \in \mathbb{R}$, and suppose it is decomposed to be*

$$\mathbb{H}_t = \mathbb{H}_t^+ \sqcup \mathbb{H}_t^{-0},$$

as in (3.9). Assume that a given element (a, b) satisfies the condition (3.4). Then the following assertions hold.

- (i) $(a, b) \in \mathbb{H}_t^+$, if and only if

$$\text{spec}([(a, b)]_t) = \left\{ x \pm i\sqrt{y^2 - tu^2 - tv^2} \right\} \subset (\mathbb{C} \setminus \mathbb{R}).$$

- (ii) $(a, b) \in \mathbb{H}_t^{-0}$, if and only if either

$$\text{spec}([(a, b)]_t) = \begin{cases} \{x\} & \text{if } \text{Im}(a)^2 = t|b|^2, \\ \left\{ x \pm \sqrt{tu^2 + tv^2 - y^2} \right\} & \text{if } \text{Im}(a)^2 < t|b|^2, \end{cases}$$

in \mathbb{R} .

Proof. The statement (i) holds by (3.5) and Theorem 3.8(i). Meanwhile, the statement (ii) holds by (3.6) and Theorem 3.8(ii). \square

Recall that a Hilbert-space operator $T \in B(H)$ is self-adjoint, if $T^* = T$ in $B(H)$, where T^* is the adjoint of T (see Section 5 below). It is well-known that T is self-adjoint, if and only if its spectrum is contained in \mathbb{R} in \mathbb{C} . So, one obtains the following result.

Proposition 3.10. *A hypercomplex number $(a, b) \in \mathbb{H}_t^{-0}$ in \mathbb{H}_t , if and only if the realization $[(a, b)]_t \in \mathcal{H}_2^t$ is self-adjoint “in $M_2(\mathbb{C})$ ”.*

Proof. (\Rightarrow) Suppose $(a, b) \in \mathbb{H}_t^{-0}$ in \mathbb{H}_t . Then $\text{spec}([(a, b)]_t) \subset \mathbb{R}$ in \mathbb{C} , implying that $[(a, b)]_t$ is self-adjoint in $M_2(\mathbb{C})$.

(\Leftarrow) Suppose $[(a, b)]_t \in \mathcal{H}_2^t$ is self-adjoint in $M_2(\mathbb{C})$, and assume that $(a, b) \notin \mathbb{H}_t^{-0}$, equivalently, $(a, b) \in \mathbb{H}_t^+$ in \mathbb{H}_t . Then,

$$\text{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R}) \text{ in } \mathbb{C},$$

and hence, $[(a, b)]_t$ is not self-adjoint in $M_2(\mathbb{C})$. It contradicts our assumption that it is self-adjoint. \square

Equivalent to the above proposition, one can conclude that $(a, b) \in \mathbb{H}_t^+$ in \mathbb{H}_t , if and only if $[(a, b)]_t$ is not be self-adjoint in $M_2(\mathbb{C})$. The self-adjointness of realizations of hypercomplex numbers would be considered more in detail in Section 5.

3.2. THE SCALED-SPECTRALIZATIONS $\{\sigma_t\}_{t \in \mathbb{R}}$

In this section, we fix an arbitrary scale $t \in \mathbb{R}$, and the corresponding hypercomplex ring \mathbb{H}_t , containing the t -scaled hypercomplex monoid

$$\mathbb{H}_t^\times = (\mathbb{H}_t \setminus \{(0, 0)\}, \cdot_t).$$

Recall that \mathbb{H}_t^\times is algebraically decomposed to be

$$\mathbb{H}_t^\times = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing},$$

with

$$\mathbb{H}_t^{inv} = \{(a, b) : |a|^2 \neq t|b|^2\}, \text{ the group-part,} \tag{3.13}$$

and

$$\mathbb{H}_t^{\times sing} = \{(a, b) : |a|^2 = t|b|^2\}, \text{ the semigroup-part,}$$

as in (2.21). Therefore, the t -scaled hypercomplex ring is set-theoretically decomposed to be

$$\mathbb{H}_t = \mathbb{H}_t^{inv} \sqcup \{(0, 0)\} \sqcup \mathbb{H}_t^{\times sing} = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{sing}, \tag{3.14}$$

by (3.13), where

$$\mathbb{H}_t^{sing} \stackrel{\text{denote}}{=} \{(0, 0)\} \sqcup \mathbb{H}_t^{\times sing} \text{ in (3.2.2).}$$

Also, the ring \mathbb{H}_t is spectrally decomposed to be

$$\mathbb{H}_t = \mathbb{H}_t^+ \sqcup \mathbb{H}_t^{-0},$$

with

$$\mathbb{H}_t^+ = \{(a, b) : \text{Im}(a)^2 > t|b|^2\}, \tag{3.15}$$

and

$$\mathbb{H}_t^{-0} = \{(a, b) : \text{Im}(a)^2 \leq t|b|^2\},$$

satisfying that: $(a, b) \in \mathbb{H}_t^+$ if and only if $\text{spec}([(a, b)]_t) \subset (\mathbb{C} \setminus \mathbb{R})$; meanwhile, $(a, b) \in \mathbb{H}_t^{-0}$ if and only if $\text{spec}([(a, b)]_t) \subset \mathbb{R}$, by Corollary 3.9(i)–(ii).

Corollary 3.11. *Let \mathbb{H}_t be the t -scaled hypercomplex ring for $t \in \mathbb{R}$. Then it is decomposed to be*

$$\begin{aligned} \mathbb{H}_t &= (\mathbb{H}_t^{inv} \cap \mathbb{H}_t^+) \sqcup (\mathbb{H}_t^{inv} \cap \mathbb{H}_t^{-0}) \\ &= (\mathbb{H}_t^{sing} \cap \mathbb{H}_t^+) \sqcup (\mathbb{H}_t^{sing} \cap \mathbb{H}_t^{-0}), \end{aligned} \tag{3.16}$$

set-theoretically.

Proof. It is proven by (3.14) and (3.15). □

Observe now that if $(a, 0) \in \mathbb{H}_t$, then

$$[(a, 0)]_t = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \text{ in } \mathcal{H}_2^t,$$

satisfying

$$\text{spec}([(a, 0)]_t) = \{a, \bar{a}\} \text{ in } \mathbb{C}. \tag{3.17}$$

Indeed, by (3.3), if $(a, 0) \in \mathbb{H}_t$ satisfying $a = x + yi \in \mathbb{C}$ with $x, y \in \mathbb{R}$, then

$$\text{spec}([(a, b)]_t) = \{x \pm i\sqrt{y^2}\} = \{x \pm |y|i\} = \{x \pm yi\},$$

implying (3.17), where $|y|$ is the absolute value of y in \mathbb{R} .

Motivated by (3.15), (3.16) and (3.17), we define a certain \mathbb{C} -valued function σ_t from \mathbb{H}_t . Define a function,

$$\sigma_t : \mathbb{H}_t \rightarrow \mathbb{C},$$

by

$$\sigma_t((a, b)) \stackrel{\text{def}}{=} \begin{cases} a = x + yi & \text{if } b = 0 \text{ in } \mathbb{C}, \\ x + i\sqrt{y^2 - tu^2 - tv^2} & \text{if } b \neq 0 \text{ in } \mathbb{C}, \end{cases} \tag{3.18}$$

for all $(a, b) \in \mathbb{H}_t$ satisfying the condition (3.4):

$$a = x + yi \text{ and } b = u + vi \text{ in } \mathbb{C},$$

with $x, y, u, v \in \mathbb{R}$ and $i = \sqrt{-1}$.

Remark that such a morphism σ_t is indeed a well-defined function assigning all hypercomplex numbers of \mathbb{H}_t to complex numbers of \mathbb{C} . Moreover, by (3.18), it is surjective. But it is definitely not injective. For instance, even though

$$\xi = (1 + 3i, -1 + i) \text{ and } \eta = (1 - 3i, 1 - i)$$

are distinct in \mathbb{H}_t , one has

$$\sigma_t(\xi) = 1 + i\sqrt{9 - 2t} = \sigma_t(\eta),$$

by (3.18).

Definition 3.12. The surjection $\sigma_t : \mathbb{H}_t \rightarrow \mathbb{C}$ of (3.18) is called the t (-scaled)-spectralization on \mathbb{H}_t . The images $\{\sigma_t(\xi)\}_{\xi \in \mathbb{H}_t}$ are said to be t (-scaled)-spectral values. From below, we also understand each t -spectral value $\sigma_t(\xi) \in \mathbb{C}$ of a hypercomplex number $\xi \in \mathbb{H}_t$ as a hypercomplex number $(\sigma_t(\xi), 0)$ in \mathbb{H}_t , i.e., such an assigned hypercomplex number $(\sigma_t(\xi), 0)$ from the t -spectral value $\sigma_t(\xi)$ of ξ is also called the t -spectral value of ξ .

By definition, all t -spectral values are not only \mathbb{C} -quantities for many hypercomplex numbers of \mathbb{H}_t whose realizations of \mathcal{H}_2^t share the same eigenvalues, but also hypercomplex numbers of \mathbb{H}_t , whose first coordinates are the value and the second coordinates are 0.

Definition 3.13. Let $\xi \in \mathbb{H}_t$ be a hypercomplex number inducing its t -spectral value $w \stackrel{\text{denote}}{=} \sigma_t(\xi) \in \mathbb{C}$, also understood to be $\eta = (w, 0) \in \mathbb{H}_t$. The corresponding realization,

$$[\eta]_t = \begin{pmatrix} w & t \cdot 0 \\ 0 & \bar{w} \end{pmatrix} = \begin{pmatrix} \sigma_t(\xi) & 0 \\ 0 & \overline{\sigma_t(\xi)} \end{pmatrix} \in \mathcal{H}_2^t$$

is called the t (-scaled)-spectral form of ξ . By $\Sigma_t(\xi)$, we denote the t -spectral form of $\xi \in \mathbb{H}_t$.

Note that the conjugate-notation in Definition 3.13 is symbolic in the sense that: if $t > 0$, and

$$\sigma_t(\xi) = 1 + i\sqrt{1 - 5t} = 1 - \sqrt{5t - 1},$$

(and hence, $\sigma_t(\xi) \in \mathbb{R}$), then the symbol,

$$\overline{\sigma_t(\xi)} \stackrel{\text{means}}{=} 1 - i\sqrt{1 - 5t} = 1 + \sqrt{5t - 1},$$

in \mathbb{R} , i.e., the conjugate-notation in Definition 3.13 has a symbolic meaning containing not only the usual conjugate on \mathbb{C} , but also the above computational meaning on \mathbb{R} .

Remark 3.14. The conjugate-notation in Definition 3.13 is symbolic case-by-case. If the t -spectral value $\sigma_t(\xi)$ is in \mathbb{C} , then $\overline{\sigma_t(\xi)}$ means the usual conjugate. Meanwhile, if t -spectral value

$$\sigma_t(\xi) = x + \sqrt{tu^2 + tv^2 - y^2},$$

with

$$tu^2 + tv^2 - y^2 \geq 0, \text{ in } \mathbb{R},$$

then

$$\overline{\sigma_t(\xi)} = x - \sqrt{tu^2 + tv^2 - y^2} \text{ in } \mathbb{R},$$

where $\xi \in \mathbb{H}_t$ satisfies the condition (3.4).

For instance, if $\xi_1 = (-2 - i, 0) \in \mathbb{H}_t$, then the t -spectral value is

$$\sigma_t(\xi_1) = -2 - i \text{ in } \mathbb{C},$$

inducing the t -spectral form,

$$\Sigma_t(\xi_1) = \begin{pmatrix} -2 - i & 0 \\ 0 & -2 + i \end{pmatrix} \text{ in } \mathcal{H}_2^t;$$

meanwhile, if $\xi_2 = (-2 - i, 1 + 3i) \in \mathbb{H}_t$, then the t -spectral value is

$$w \stackrel{\text{denote}}{=} \sigma_t(\xi_2) = -2 + i\sqrt{1 - 10t},$$

inducing the t -spectral form,

$$\Sigma_t(\xi_2) = \begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix} = \begin{pmatrix} -2 + i\sqrt{1 - 10t} & 0 \\ 0 & -2 - i\sqrt{1 - 10t} \end{pmatrix},$$

where \bar{w} is symbolic in the sense of Remark 3.14; if $t \leq 0$, then

$$\Sigma_t(\xi_2) = \begin{pmatrix} -2 + i\sqrt{1 - 10t} & 0 \\ 0 & -2 - i\sqrt{1 - 10t} \end{pmatrix},$$

meanwhile, if $t > 0$, then

$$\Sigma_t(\xi_2) = \begin{pmatrix} -2 + \sqrt{10t - 1} & 0 \\ 0 & -2 - \sqrt{10t - 1} \end{pmatrix},$$

in \mathcal{H}_2^t .

Definition 3.15. Two hypercomplex numbers $\xi, \eta \in \mathbb{H}_t$ are said to be t (-scaled)-spectral-related, if

$$\sigma_t(\xi) = \sigma_t(\eta) \text{ in } \mathbb{C},$$

equivalently,

$$\Sigma_t(\xi) = \Sigma_t(\eta) \text{ in } \mathcal{H}_2^t.$$

On the t -hypercomplex ring \mathbb{H}_t , the t -spectral relation of Definition 3.15 is an equivalent relation. Indeed,

$$\sigma_t(\xi) = \sigma_t(\xi), \quad \forall \xi \in \mathbb{H}_t;$$

and if ξ and η are t -spectral related in \mathbb{H}_t , then

$$\sigma_t(\xi) = \sigma_t(\eta) \iff \sigma_t(\eta) = \sigma_t(\xi),$$

and hence, η and ξ are t -spectral related in \mathbb{H}_t ; and if ξ_1 and ξ_2 are t -spectral related, and if ξ_2 and ξ_3 are t -spectral related, then

$$\sigma_t(\xi_1) = \sigma_t(\xi_2) = \sigma_t(\xi_3) \text{ in } \mathbb{C},$$

and hence, ξ_1 and ξ_3 are t -spectral related.

Proposition 3.16. *The t -spectral relation on \mathbb{H}_t is an equivalence relation.*

Proof. The t -spectral relation is reflexive, symmetric and transitive on \mathbb{H}_t , by the discussion of the very above paragraph. \square

Since the t -spectral relation is an equivalence relation, each element ξ of \mathbb{H}_t has its equivalence class,

$$\widetilde{\xi} \stackrel{\text{def}}{=} \{\eta \in \mathbb{H}_t : \eta \text{ is } t\text{-related to } \xi\},$$

and hence, the corresponding quotient set,

$$\widetilde{\mathbb{H}}_t \stackrel{\text{def}}{=} \{\widetilde{\xi} : \xi \in \mathbb{H}_t\}, \quad (3.19)$$

is well-defined to be the set of all equivalence classes.

Theorem 3.17. *Let $\widetilde{\mathbb{H}}_t$ be the quotient set (3.19) induced by the t -spectral relation on \mathbb{H}_t . Then*

$$\widetilde{\mathbb{H}}_t \text{ and } \mathbb{C} \text{ are equipotent.} \quad (3.20)$$

Proof. It is not difficult to check that, for any $z \in \mathbb{C}$, there exist $\xi \in \mathbb{H}_t$, such that $z = \sigma_t(\xi)$ by the surjectivity of the t -spectralization σ_t . It implies that there exists $(z, 0) \in \mathbb{H}_t$, such that

$$\widetilde{(z, 0)} = \widetilde{\xi} \text{ in } \widetilde{\mathbb{H}}_t, \text{ whenever } z = \sigma_t(\xi).$$

Thus, set-theoretically, we have

$$\widetilde{\mathbb{H}}_t = \left\{ \widetilde{(z, 0)} : z \in \mathbb{C} \right\} \stackrel{\text{equip}}{=} \mathbb{C},$$

where “ $\stackrel{\text{equip}}{=}$ ” means “being equipotent (or, bijective) to”. Therefore, the relation (3.20) holds. \square

The above equipotence (3.20) of the quotient set $\widetilde{\mathbb{H}}_t$ of (3.19) with the complex numbers \mathbb{C} shows that the set \mathbb{C} classifies \mathbb{H}_t , for “every” $t \in \mathbb{R}$, up to the t -spectral relation.

3.3. SIMILARITY ON $M_2(\mathbb{C})$ AND THE t -SCALED-SPECTRAL RELATION ON \mathbb{H}_t

In Section 3.2, we defined the t -spectralization σ_t on the t -scaled hypercomplex ring \mathbb{H}_t , for a fixed scale $t \in \mathbb{R}$, and it induces the t -spectral forms $\{\Sigma_t(\xi)\}_{\xi \in \mathbb{H}_t}$ in \mathcal{H}_2^t as complex diagonal matrices whose main diagonals are the eigenvalues of the realizations $\{[\xi]_t\}_{\xi \in \mathbb{H}_t}$, under the symbolic understanding of Remark 3.14. Moreover, σ_t lets the set \mathbb{C} classify \mathbb{H}_t by (3.20) under the t -spectral relation.

Independently, we showed in [2] and [3] that: on the quaternions $\mathbb{H} = \mathbb{H}_{-1}$, the (-1) -spectral relation, called the quaternion-spectral relation in [2] and [3], is equivalent to the similarity “on \mathcal{H}_2^{-1} ”, as equivalence relations. Here, the similarity “on \mathcal{H}_2^{-1} ”

means that the realizations $[q_1]_{-1}$ and $[q_2]_{-1}$ of two quaternions $q_1, q_2 \in \mathbb{H}_{-1}$ are similar “in \mathcal{H}_2^{-1} ”, if there exists invertible element U “in \mathcal{H}_2^{-1} ”, such that

$$[q_2]_{-1} = U^{-1} [q_1]_{-1} U \text{ in } \mathcal{H}_2^{-1}.$$

Here, we consider such property for an arbitrary scale $t \in \mathbb{R}$. Recall that, we showed in [2] and [3] that: the (-1) -spectral form $\Sigma_{-1}(\eta)$ and the realization $[\eta]_{-1}$ are similar “in \mathcal{H}_2^{-1} ”, for “all” quaternions which are the (-1) -scaled hypercomplex numbers $\eta \in \mathbb{H}_{-1} = \mathbb{H}$. Are the t -spectral relation on \mathbb{H}_t and the similarity on \mathcal{H}_2^t same as equivalence relations? In conclusion, the answer is negative in general.

Two matrices A and B of $M_n(\mathbb{C})$, for any $n \in \mathbb{N}$, are said to be similar, if there exists an invertible matrix $U \in M_n(\mathbb{C})$, such that

$$B = U^{-1}AU \text{ in } M_n(\mathbb{C}).$$

Remember that if two matrices A and B are similar, then (i) they share the same eigenvalues, (ii) they have the same traces, and (iii) their determinants are same (e.g., [9] and [8]). We here focus on the fact (iii): the similarity of matrices implies their identical determinants, equivalently, if

$$\det(A) \neq \det(B),$$

then A and B are not similar in $M_n(\mathbb{C})$.

Definition 3.18. Let $A, B \in \mathcal{H}_2^t$ be realizations of certain hypercomplex numbers of \mathbb{H}_t , for $t \in \mathbb{R}$. They are said to be similar “in \mathcal{H}_2^t ”, if there exists an invertible $U \in \mathcal{H}_2^t$, such that

$$B = U^{-1}AU \text{ in } \mathcal{H}_2^t.$$

By abusing notation, we say that two hypercomplex numbers ξ and η are similar in \mathbb{H}_t , if their realizations $[\xi]_t$ and $[\eta]_t$ are similar in \mathcal{H}_2^t .

Let $(a, b) \in \mathbb{H}_t$ be a hypercomplex number satisfying the condition (3.4) and $(a, b) \neq (0, 0)$. Then it has

$$[(a, b)]_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathcal{H}_2^t,$$

$$\sigma_t((a, b)) = x + i\sqrt{y^2 - tu^2 - tv^2} \stackrel{\text{let}}{=} w \in \mathbb{C},$$

and

$$\Sigma_t((a, b)) = \begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix} \in \mathcal{H}_2^t, \tag{3.21}$$

where \bar{w} is symbolic in the sense of Remark 3.14. Observe that

$$\det([(a, b)]_t) = |a|^2 - t|b|^2 = (x^2 + y^2) - t(u^2 + v^2),$$

and

$$\det(\Sigma_t((a, b))) = |w|^2 = x^2 + |y^2 - tu^2 - tv^2|, \tag{3.22}$$

by (3.21). These computations in (3.22) show that, in general, $[(a, b)]_t$ and $\Sigma_t((a, b))$ are “not” similar “as matrices of $M_2(\mathbb{C})$ ”, and hence, not similar in \mathcal{H}_2^t . Indeed, for instance, if

$$t > 0, \text{ and } |a|^2 < t|b|^2,$$

then $\det([(a, b)]_t) < 0$, but $\det(\Sigma_t((a, b))) > 0$ in \mathbb{R} , by (3.22), implying that

$$\det([(a, b)]_t) \neq \det(\Sigma_t((a, b))) \text{ in general,}$$

showing that $[(a, b)]_t$ and $\Sigma_t((a, b))$ are not similar in $M_2(\mathbb{C})$, and hence, they are not similar in \mathcal{H}_2^t , in general.

Proposition 3.19. *Let $(a, b) \in \mathbb{H}_t$ be “nonzero” hypercomplex number satisfying $|a|^2 < t|b|^2$ in \mathbb{R} . Then the realization $[(a, b)]_t$ and the t -spectral form $\Sigma_t((a, b))$ are not similar “in \mathcal{H}_2^t ”.*

Proof. Suppose $(a, b) \in \mathbb{H}_t$ satisfies $(a, b) \neq (0, 0)$ and $|a|^2 < t|b|^2$, for $t > 0$. And assume that $[(a, b)]_t$ and $\Sigma_t((a, b))$ are similar in \mathcal{H}_2^t . Since they are assumed to be similar, their determinants are identically same. However,

$$\det([(a, b)]_t) < 0 \text{ and } \det(\Sigma_t((a, b))) > 0,$$

by (3.22). It contradicts our assumption that they are similar in \mathcal{H}_2^t . □

The above proposition confirms that the realizations and the corresponding t -spectral forms of a t -scaled hypercomplex number are not similar in \mathcal{H}_2^t , in general.

Consider that, in the quaternions $\mathbb{H} = \mathbb{H}_{-1}$, since the scale is $t = -1 < 0$ in \mathbb{R} ,

$$\det([\xi]_{-1}) = \det(\Sigma_{-1}(\xi)) \geq 0, \quad \forall \xi \in \mathbb{H}_{-1},$$

and it is proven that $[\xi]_{-1}$ and $\Sigma_{-1}(\xi)$ are indeed similar in \mathcal{H}_2^{-1} , for “all” $\xi \in \mathbb{H}_{-1}$ in [2] and [3], which motivates a question: if a scale $t < 0$ in \mathbb{R} , then

$$\det([\eta]_t) = \det(\Sigma_t(\eta)) \geq 0, \quad \forall \eta \in \mathbb{H}_t,$$

by (3.22); so, are the realizations $[\eta]_t$ and the corresponding t -spectral forms $\Sigma_t(\eta)$ similar in \mathcal{H}_2^t as in the case of $t = -1$?

First of all, we need to recall that if $t < 0$, then the t -scaled hypercomplex ring \mathbb{H}_t forms a noncommutative field, since the t -scaled hypercomplex monoid \mathbb{H}_t^\times is a non-Abelian group, by (2.14). It allows us to use similar techniques of [2] and [3].

In the rest part of this section, a given scale $t \in \mathbb{R}$ is automatically assumed to be negative in \mathbb{R} .

Assume that $(a, 0) \in \mathbb{H}_t$, where $t < 0$. Then

$$[(a, 0)]_t = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} = \Sigma_t((a, 0)),$$

in \mathcal{H}_2^t , since $\sigma_t((a, 0)) = a$ in \mathbb{C} . So, clearly, $[(a, 0)]_t$ and $\Sigma_t((a, 0))$ are similar in \mathcal{H}_2^t , because they are equal in \mathcal{H}_2^t . Indeed, there exist diagonal matrices with nonzero real entries,

$$X = [(x, 0)]_t \in \mathcal{H}_2^t, \text{ with } x = x + 0i \in \mathbb{C}, x \neq 0,$$

such that

$$[(a, 0)]_t = X^{-1}(\Sigma_t(a, 0))X \text{ in } \mathcal{H}_2^t.$$

Thus, we are interested in the cases where $(a, b) \in \mathbb{H}_t$ with $b \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Lemma 3.20. *Let $t < 0$ in \mathbb{R} , and $(a, 0) \in \mathbb{H}_t$, a hypercomplex number. Then the realization $[(a, 0)]_t$ and the t -spectral form $\Sigma_t((a, 0))$ are identically same in \mathcal{H}_2^t , and hence, they are similar in \mathcal{H}_2^t . (Remark that, in fact, the scale t is not necessarily negative in \mathbb{R} here.)*

Proof. It is proven by the discussion of the very above paragraph. Indeed, one has

$$[(a, 0)]_t = \Sigma_t((a, 0)) \text{ in } \mathcal{H}_2^t,$$

since $\sigma_t((a, 0)) = a$ in \mathbb{C} . □

Let $h = (a, b) \in \mathbb{H}_t$ with $b \in \mathbb{C}^\times$, satisfying the condition (3.4), where $t < 0$, having its realization,

$$[h]_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} x + yi & t(u + vi) \\ u - vi & x - yi \end{pmatrix},$$

and its t -spectral form,

$$\Sigma_t(h) = \begin{pmatrix} x + i\sqrt{y^2 - tu^2 - tv^2} & 0 \\ 0 & x - i\sqrt{y^2 - tu^2 - tv^2} \end{pmatrix} \stackrel{\text{let}}{=} \begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix},$$

in \mathcal{H}_2^t . Since $t < 0$ and $b \neq 0$ (by assumption), the t -spectral value $w = \sigma_t(h)$ is a \mathbb{C} -quantity with its conjugate \bar{w} . Define now a matrix,

$$Q_h \stackrel{\text{def}}{=} \begin{pmatrix} 1 & t\left(\frac{w-a}{tb}\right) \\ \frac{w-a}{tb} & 1 \end{pmatrix} \text{ in } M_2(\mathbb{C}).$$

Remark that, by the assumption that $t < 0$ and $b \neq 0$, this matrix is well-defined. Furthermore, one can immediately recognize that $Q_h \in \mathcal{H}_2^t$, i.e.,

$$Q_h = \left[\left(1, \overline{\left(\frac{w-a}{tb} \right)} \right) \right]_t \in \mathcal{H}_2^t. \tag{3.23}$$

One can find that the element $Q_h \in \mathcal{H}_2^t$ of (3.23) is indeed invertible by our negative-scale assumption, since

$$\det(Q_h) = 1 - t \left| \frac{w-a}{tb} \right|^2 \geq 1, \text{ since } t < 0,$$

implying that

$$\det(Q_h) \neq 0 \iff Q_h \text{ is invertible in } \mathcal{H}_2^t.$$

Observe now that

$$Q_h \Sigma_t(h) = \begin{pmatrix} w & t\left(\frac{w^2-aw}{tb}\right) \\ \frac{w^2-aw}{tb} & \bar{w} \end{pmatrix}$$

and

$$[h]_t Q_h = \begin{pmatrix} w & t \left(a \left(\frac{w-a}{tb} \right) + b \right) \\ a \left(\frac{w-a}{tb} \right) + b & \bar{w} \end{pmatrix}, \tag{3.24}$$

in \mathcal{H}_2^t . Now, let us compare the (1,2)-entries of resulted matrices in (3.24). The (1,2)-entry of the element $Q_h \Sigma_t(h)$ is

$$\begin{aligned} t \left(\frac{w^2 - aw}{tb} \right) &= \frac{w(w-a)}{b} \\ &= \frac{(x + i\sqrt{y^2 - tu^2 - tv^2})(i\sqrt{y^2 - tu^2 - tv^2} - yi)}{u + vi} \\ &= \frac{ix\sqrt{R} - xyi - R + y\sqrt{R}}{u + vi}, \end{aligned}$$

where

$$R \stackrel{\text{denote}}{=} y^2 - tu^2 - tv^2 \text{ in } \mathbb{R}, \tag{3.25}$$

and the (1,2)-entry of the matrix $[h]_t Q_h$ is

$$\begin{aligned} &t \left(a \left(\frac{w-a}{tb} \right) + b \right) \\ &= t \left(\bar{a} \left(\frac{w-a}{tb} \right) + \bar{b} \right) = t \left(\frac{\bar{a}w - |a|^2 + t|b|^2}{tb} \right) = \frac{\bar{a}w - |a|^2 + t|b|^2}{b} \\ &= \frac{(x-yi)(x + i\sqrt{y^2 - tu^2 - tv^2}) - (x^2 + y^2) - t(u^2 + v^2)}{u + vi} \\ &= \frac{x^2 + ix\sqrt{R} - xyi + y\sqrt{R} - x^2 - y^2 - tu^2 - tv^2}{u + vi} \\ &= \frac{x^2 + ix\sqrt{R} - xyi + y\sqrt{R} - x^2 - R}{u + vi} \\ &= \frac{ix\sqrt{R} - xyi - R + y\sqrt{R}}{u + vi}, \end{aligned} \tag{3.26}$$

where the \mathbb{R} -quantity R is in the sense of (3.25). As one can see in (3.25) and (3.26), the (1,2)-entries of $[h]_t Q_h$ and $Q_h \Sigma_t(h)$ are identically same, i.e.,

$$Q_h \Sigma_t(h) = [h]_t Q_h \text{ in } \mathcal{H}_2^t, \tag{3.27}$$

where the matrix $Q_h \in \mathcal{H}_2^t$ is in the sense of (3.23).

Lemma 3.21. *Let $t < 0$ in \mathbb{R} , and let $h = (a, b) \in \mathbb{H}_t$ with $b \in \mathbb{C}^\times$. Then the realization $[h]_t$ and the t -spectral form $\Sigma_t(h)$ are similar in \mathcal{H}_2^t . In particular, there exists*

$$q_h = \left(1, t \left(\frac{w-a}{tb} \right) \right) \in \mathbb{H}_t,$$

having its realization,

$$Q_h = [q_h]_t = \begin{pmatrix} 1 & t \left(\frac{\overline{w-a}}{tb} \right) \\ \frac{w-a}{tb} & 1 \end{pmatrix} \in \mathcal{H}_2^t,$$

such that

$$\Sigma_t(h) = Q_h^{-1} [h]_t Q_h \text{ in } \mathcal{H}_2^t. \tag{3.28}$$

Proof. Under the hypothesis, one obtains that

$$Q_h \Sigma_t(h) = [h]_t Q_h \text{ in } \mathcal{H}_2^t,$$

by (3.27). By the invertibility of Q_h , we have

$$\Sigma_t(h) = Q_h^{-1} [h]_t Q_h \text{ in } \mathcal{H}_2^t,$$

implying the relation (3.28). □

The above lemma shows that if a scale t is negative in \mathbb{R} , then the realization $[h]_t$ and the t -spectral form $\Sigma_t(h)$ are similar in \mathcal{H}_2^t , whenever $h = (a, b) \in \mathbb{H}_t$ satisfies $b \neq 0$ in \mathbb{C} .

Theorem 3.22. *If $t < 0$ in \mathbb{R} , then every hypercomplex number $h \in \mathbb{H}_t$ is similar to its t -spectral value $(\sigma_t(h), 0) \in \mathbb{H}_t$, in the sense that:*

$$[h]_t \text{ and } \Sigma_t(h) \text{ are similar in } \mathcal{H}_2^t. \tag{3.29}$$

Proof. Let $h = (a, b) \in \mathbb{H}_t$, for $t < 0$. If $b = 0$ in \mathbb{C} , then $[(a, 0)]_t$ and $\Sigma_t((a, 0))$ are similar in \mathcal{H}_2^t , by the above lemma. Indeed, if $b = 0$, then these matrices are identically same in \mathcal{H}_2^t . Meanwhile, if $b \neq 0$ in \mathbb{C} , then $[h]_t$ and $\Sigma_t(h)$ are similar in \mathcal{H}_2^t by Lemma 3.20. In particular, if $b \neq 0$, then there exists

$$q_h = \left(1, \frac{\overline{w-a}}{tb} \right) \in \mathbb{H}_t,$$

such that

$$\Sigma_t(h) = [q_h]_t^{-1} [h]_t [q_h]_t,$$

in \mathcal{H}_2^t , by (3.28). Therefore, if $t < 0$, then $[h]_t$ and $\Sigma_t(h)$ are similar in \mathcal{H}_2^t , equivalently, two hypercomplex numbers h and $(\sigma_t(h), 0)$ are similar in \mathbb{H}_t , for all $h \in \mathbb{H}_t$. □

The above theorem guarantees that the negative-scale condition on hypercomplex numbers implies the similarity of the realizations and the scaled-spectral forms of them, just like the quaternionic case (whose scale is -1), shown in [2] and [3].

Theorem 3.23. *If $t < 0$ in \mathbb{R} , then the t -spectral relation on \mathbb{H}_t and the similarity on \mathbb{H}_t are same as equivalence relations on \mathbb{H}_t , i.e.,*

$$t < 0 \implies t\text{-spectral relation} \stackrel{\text{equi}}{=} \text{similarity on } \mathbb{H}_t, \tag{3.30}$$

where “ $\stackrel{\text{equi}}{=}$ ” means “being equivalent to, as equivalence relations”.

Proof. Suppose a negative scale $t < 0$ is fixed, and let \mathbb{H}_t be the corresponding t -scaled hypercomplex ring. Assume that two hypercomplex numbers h_1 and h_2 are t -spectral related. Then their t -spectral values are identical in \mathbb{C} , i.e.,

$$\sigma_t(h_1) = \sigma_t(h_2) \stackrel{\text{let}}{=} w \text{ in } \mathbb{C}.$$

Thus the realizations $[h_1]_t$ and $[h_2]_t$ are similar to

$$\Sigma_t(h_1) = \begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix} = \Sigma_t(h_2) \stackrel{\text{let}}{=} W,$$

in \mathcal{H}_2^t , by (3.29), i.e., there exist $q_1, q_2 \in \mathbb{H}_t$ such that

$$[q_1]_t^{-1} [h_1]_t [q_1]_t = W = [q_2]_t^{-1} [h_2]_t [q_2]_t,$$

in \mathcal{H}_2^t . So, one obtains that

$$[h_1]_t = \left([q_1]_t [q_2]_t^{-1} \right) [h_2]_t \left([q_2]_t [q_1]_t^{-1} \right)$$

if and only if

$$[h_1]_t = \left([q_2]_t [q_1]_t^{-1} \right)^{-1} [h_2]_t \left([q_2]_t [q_1]_t^{-1} \right),$$

in \mathcal{H}_2^t , implying that $[h_1]_t$ and $[h_2]_t$ are similar in \mathcal{H}_2^t . Thus, if h_1 and h_2 are t -spectral related, then they are similar in \mathbb{H}_t .

Conversely, suppose $T_1 \stackrel{\text{denote}}{=} [h_1]_t$ and $T_2 \stackrel{\text{denote}}{=} [h_2]_t$ are similar in \mathcal{H}_2^t . Then there exists $U \in \mathcal{H}_2^t$, such that

$$T_1 = U^{-1} T_2 U \text{ in } \mathcal{H}_2^t.$$

Since the realizations T_l and the corresponding t -spectral forms $S_l \stackrel{\text{denote}}{=} \Sigma_t(h_l)$ are similar by (3.29), say,

$$T_l = V_l^{-1} S_l V_l \text{ in } \mathcal{H}_2^t, \text{ for some } V_l \in \mathcal{H}_2^t,$$

for all $l = 1, 2$. Thus,

$$\begin{aligned} T_1 &= U^{-1} T_2 U = U^{-1} (V_2^{-1} S_2 V_2) U \\ &\iff V_1 S_1 V_1^{-1} = T_1 = (V_2 U)^{-1} S_2 (V_2 U) \\ &\iff S_1 = V_1^{-1} (V_2 U)^{-1} S_2 (V_2 U) V_1 \\ &\iff S_1 = (V_2 U V_1)^{-1} S_2 (V_2 U V_1), \end{aligned}$$

and hence, two matrices S_1 and S_2 are similar in \mathcal{H}_2^t . It means that S_1 and S_2 share the same eigenvalues. So, it is either

$$S_1 = \begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix} = S_2,$$

for some $w \in \mathbb{C}$, or

$$S_1 = \begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix}, \text{ and } S_2 = \begin{pmatrix} \bar{w} & 0 \\ 0 & w \end{pmatrix},$$

in \mathcal{H}_2^t . However, by the assumption that $t < 0$, we have

$$S_1 = S_2 \text{ in } \mathcal{H}_2^t,$$

by Corollary 3.2(iii). It shows that, if the realizations T_1 and T_2 are similar, then the t -spectral forms S_1 and S_2 are identically same in \mathcal{H}_2^t , implying that

$$\sigma_t(h_1) = \sigma_t(h_2) \text{ in } \mathbb{C},$$

thus h_1 and h_2 are t -spectral related in \mathbb{H}_t .

Therefore, the equivalence (3.30) between the t -spectral relation and the similarity on \mathbb{H}_t holds, whenever $t < 0$ in \mathbb{R} . \square

The above theorem generalizes the equivalence between the quaternion-spectral relation, which is the (-1) -spectral relation, and the similarity on the quaternions $\mathbb{H}_{-1} = \mathbb{H}$ (e.g., [2] and [3]).

How about the cases where given scale t are nonnegative in \mathbb{R} , i.e., $t \geq 0$? One may need to consider the decomposition (3.16),

$$\begin{aligned} \mathbb{H}_t &= (\mathbb{H}_t^{inv} \cap \mathbb{H}_t^+) \sqcup (\mathbb{H}_t^{inv} \cap \mathbb{H}_t^{-0}) \\ &= (\mathbb{H}_t^{sing} \cap \mathbb{H}_t^+) \sqcup (\mathbb{H}_t^{sing} \cap \mathbb{H}_t^{-0}), \end{aligned}$$

of \mathbb{H}_t , for $t \geq 0$, where

$$\begin{aligned} \mathbb{H}_t^{inv} &= \{(a, b) : |a|^2 \neq t|b|^2\}, \\ \mathbb{H}_t^{sing} &= \{(a, b) : |a|^2 = t|b|^2\}, \\ \mathbb{H}_t^+ &= \{(a, b) : \text{Im}(a)^2 > t|b|^2\}, \end{aligned}$$

and

$$\mathbb{H}_t^{-0} = \{(a, b) : \text{Im}(a)^2 \leq t|b|^2\},$$

block-by-block. In particular, if

$$h \in \mathbb{H}_t^{inv} \cap \mathbb{H}_t^+,$$

then it “seems” that the realization $[h]_t$ and the t -spectral form $\Sigma_t(h)$ are similar in \mathcal{H}_2^t . The proof “may” be similar to the above proofs for negative scales. We leave this problem for a future project.

3.4. THE t -SPECTRAL MAPPING THEOREM

In this section, we let a scale t be arbitrary in \mathbb{R} , and let \mathbb{H}_t be the t -scaled hypercomplex ring. Let $h = (a, b) \in \mathbb{H}_t$ satisfy the condition (3.4), and suppose it has its t -spectral value,

$$\sigma_t(h) = x + i\sqrt{y^2 - tu^2 - tv^2} \stackrel{\text{let}}{=} w,$$

and hence, its t -spectral form

$$\Sigma_t(h) = \begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix} \text{ in } \mathcal{H}_2^t$$

(see Remark 3.14).

Now recall that if $n \in \mathbb{N}$, and $A \in M_n(\mathbb{C})$, and if

$$f \in \mathbb{C}[z] \stackrel{\text{def}}{=} \left\{ g : g = \sum_{k=0}^m z_k z^k, \text{ with } z_1, \dots, z_m \in \mathbb{C}, \text{ for } m \in \mathbb{N} \right\},$$

then

$$\text{spec}(f(A)) = \{f(w) : w \in \text{spec}(A)\}, \quad (3.31)$$

in \mathbb{C} , where $\mathbb{C}[z]$ is the polynomial ring in a variable z over \mathbb{C} , consisting of all polynomials in z whose coefficients are in \mathbb{C} , and

$$f(A) = \sum_{k=0}^N s_k A^k, \text{ with } A^0 = I_n,$$

whenever

$$f(z) = \sum_{k=0}^N s_k z^k \in \mathbb{C}[z], \text{ with } s_1, \dots, s_N \in \mathbb{C},$$

where I_n is the identity matrix of $M_n(\mathbb{C})$, by the spectral mapping theorem (e.g., [9] and [8]). By (3.31), if $\mathbb{R}[x]$ is the polynomial ring in a variable x over the real field \mathbb{R} , then

$$\text{spec}(g(A)) = \{g(w) : w \in \text{spec}(A)\} \text{ in } \mathbb{C}, \quad (3.32)$$

for all $g \in \mathbb{R}[x]$, because $\mathbb{R}[z]$ is a subring of $\mathbb{C}[z]$ if we identify x to z .

It is shown in [2] and [3] that, for $f \in \mathbb{C}[z]$,

$$\text{spec}(f([\xi]_{-1})) = \left\{ f(\sigma_{-1}(\xi)), f(\overline{\sigma_{-1}(\xi)}) \right\}$$

in \mathbb{C} , by (3.31), but

$$f(\overline{\sigma_{-1}(\xi)}) \neq \overline{f(\sigma_{-1}(\xi))}, \text{ in general,}$$

and hence, even though the relation (3.31) holds “on $M_2(\mathbb{C})$, for $[\xi]_{-1} \in \mathcal{H}_2^{-1}$ ”, it does not hold “on \mathcal{H}_2^{-1} ”, in general. It demonstrates that, in a similar manner, the spectral mapping theorem (3.31) holds “on $M_2(\mathbb{C})$,” but it does not hold “on the t -scaled realization \mathcal{H}_2^t of \mathbb{H}_t ”, for $t \in \mathbb{R}$, because the spectra of hypercomplex numbers satisfy

$$\text{spec}([\eta]_t) = \{w, \bar{w}\}, \text{ with } w = \sigma_t(\eta),$$

by (3.3), for all $\eta \in \mathbb{H}_t$ in the sense of Remark 3.14, just like the quaternionic case of [2] and [3].

However, in [2] and [3], it is proven that, for all $g \in \mathbb{R}[x]$, one has

$$\text{spec} (g ([\xi]_{-1})) = \left\{ g (\sigma_t (\xi)), \overline{g (\sigma_t (\xi))} \right\},$$

in \mathbb{C} , by (3.32), since

$$g \in \mathbb{R}[x] \implies g (\bar{w}) = \overline{g (w)}, \forall w \in \mathbb{C}.$$

It means that the “restricted” spectral mapping theorem of (3.32) holds “on the realization \mathcal{H}_2^{-1} of the quaternions \mathbb{H}_{-1} ”. Similarly, we obtain the following result.

Theorem 3.24. *Let $\xi \in \mathbb{H}_t$, realized to be $[\xi]_t \in \mathcal{H}_2^t$. Then, for any $g \in \mathbb{R}[x]$,*

$$\text{spec} (g ([\xi]_t)) = \left\{ g (\sigma_t (\xi)), \overline{g (\sigma_t (\xi))} \right\},$$

i.e.,

$$\text{spec} (g ([\xi]_t)) = \{g (w) : w \in \text{spec} ([\xi]_t)\} \text{ in } \mathbb{C}, \forall t \in \mathbb{R}. \tag{3.33}$$

Proof. By (3.3) and (3.18), if $\xi \in \mathbb{H}_t$, then

$$\text{spec} ([\xi]_t) = \{w, \bar{w}\}, \text{ with } w = \sigma_t (\xi),$$

in \mathbb{C} (under the symbolic understanding of Remark 3.14). For any $g = \sum_{k=1}^N s_k x^k \in \mathbb{R}[x]$, with $s_1, \dots, s_N \in \mathbb{R}$, and $N \in \mathbb{N}$, one has that

$$g (\bar{w}) = \sum_{k=1}^N s_k \bar{w}^k = \sum_{k=1}^N \overline{s_k w^k} = \overline{\sum_{k=1}^N s_k w^k} = \overline{g (w)}, \tag{3.34}$$

in \mathbb{C} . It implies that

$$\text{spec} (g ([\xi]_t)) = \{g (w), g (\bar{w})\} = \left\{ g (w), \overline{g (w)} \right\},$$

in \mathbb{C} , by (3.32) and (3.34). Therefore, the relation (3.33) holds true. \square

One may call the relation (3.33), the hypercomplex-spectral mapping theorem, since it holds for all scales $t \in \mathbb{R}$.

4. THE USUAL ADJOINT ON \mathcal{H}_2^t IN $M_2 (\mathbb{C})$

In this section, we consider how the usual adjoint on $M_2 (\mathbb{C}) = B (\mathbb{C}^2)$ acts on the t -scaled realization \mathcal{H}_2^t of the t -scaled hypercomplex numbers. Throughout this section, we fix an arbitrary scale $t \in \mathbb{R}$, and the corresponding t -scaled hypercomplex ring \mathbb{H}_t realized to be \mathcal{H}_2^t in $M_2 (\mathbb{C})$ under the representation $\Pi_t = (\mathbb{C}^2, \pi_t)$. Recall that every Hilbert-space operator T acting on a Hilbert space H has its unique adjoint T^* on H .

Especially, if $T \in M_n(\mathbb{C}) = B(\mathbb{C}^n)$, for $n \in \mathbb{N}$, is a matrix which is an operator on \mathbb{C}^n , then its adjoint T^* is determined to be the conjugate-transpose of T in $M_n(\mathbb{C})$. For instance,

$$T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C}) \iff T^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix} \in M_2(\mathbb{C}).$$

It says that, if we understand our t -scaled realization \mathcal{H}_2^t as a sub-structure of $M_2(\mathbb{C})$, then each hypercomplex number $(a, b) \in \mathbb{H}_t$ assigns a unique adjoint $[(a, b)]_t^*$ of the realization $[(a, b)]_t$ “in $M_2(\mathbb{C})$ ”.

Let $(a, b) \in \mathbb{H}_t$ realized to be

$$[(a, b)]_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathcal{H}_2^t.$$

Then, as a matrix of $M_2(\mathbb{C})$, this realization has its adjoint,

$$[(a, b)]_t^* = \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} \text{ in } M_2(\mathbb{C}).$$

It shows that the usual adjoint (conjugate-transpose) of $[(a, b)]_t$ is not contained “in \mathcal{H}_2^t ”, in general. In particular, if

$$t^2 \neq 1 \iff \text{either } t \neq 1 \text{ or } t \neq -1, \text{ in } \mathbb{R},$$

then

$$[(a, b)]_t \notin \mathcal{H}_2^t \text{ in general.}$$

Theorem 4.1. *The scale $t \in \mathbb{R}$ satisfies that $t^2 = 1$ in \mathbb{R} , if and only if the adjoint of every realization of a hypercomplex number \mathbb{H}_t is contained in \mathcal{H}_2^t , i.e.,*

$$\text{either } t = 1, \text{ or } t = -1 \iff [\xi]_t^* \in \mathcal{H}_2^t, \quad \forall \xi \in \mathbb{H}_t. \quad (4.1)$$

Proof. For an arbitrary scale $t \in \mathbb{R}$, if $(a, b) \in \mathbb{H}_t$, then

$$[(a, b)]_t^* = \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} \text{ in } M_2(\mathbb{C}).$$

(\Rightarrow) Assume that either $t = 1$, or $t = -1$, equivalently, suppose $t^2 = 1$ in \mathbb{R} . Then

$$[(a, b)]_t^* = \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} = \begin{pmatrix} \bar{a} & t\left(\frac{b}{t}\right) \\ t^2\left(\frac{\bar{b}}{t}\right) & a \end{pmatrix} = \begin{pmatrix} \bar{a} & t\left(\frac{b}{t}\right) \\ \left(\frac{\bar{b}}{t}\right) & a \end{pmatrix},$$

contained in \mathcal{H}_2^t . So, if either $t = 1$, or $t = -1$, then $[(a, b)]_t^* \in \mathcal{H}_2^t$, for all $(a, b) \in \mathbb{H}_t$. Moreover, in such a case,

$$[(a, b)]_t^* = \left[\begin{pmatrix} \bar{a} & b \\ \bar{b} & a \end{pmatrix} \right]_t \text{ in } \mathcal{H}_2^t. \quad (4.2)$$

(\Leftarrow) Assume now that $t^2 \neq 1$ in \mathbb{R} . Then the adjoint $[(a, b)]_t^*$ of $[(a, b)]_t$ is identical to the matrix,

$$[(a, b)]_t^* = \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} \text{ in } M_2(\mathbb{C}),$$

which “can” be

$$\begin{pmatrix} \bar{a} & t\left(\frac{b}{t}\right) \\ t^2\left(\frac{\bar{b}}{t}\right) & a \end{pmatrix} \text{ in } \mathcal{H}_2^t.$$

However, by the assumption that $t^2 \neq 1$, the adjoint $[(a, b)]_t^*$ is not contained in \mathcal{H}_2^t , in general. In particular, if $b \neq 0$ in \mathbb{C} , then the adjoint $[(a, b)]_t^* \notin \mathcal{H}_2^t$ in $M_2(\mathbb{C})$, i.e.,

$$t^2 \neq 1 \text{ and } b \neq 0 \text{ in } \mathbb{C} \implies [(a, b)]_t^* \in (M_2(\mathbb{C}) \setminus \mathcal{H}_2^t). \quad (4.3)$$

Therefore, the characterization (4.1) holds by (4.2) and (4.3). \square

Note that, if $t = -1$, then \mathbb{H}_{-1} is the quaternions; and if $t = 1$, then \mathbb{H}_1 is the bicomplex numbers. The above theorem shows that, only when the scaled hypercomplex ring \mathbb{H}_t is either the quaternions \mathbb{H}_{-1} , or the bicomplex numbers \mathbb{H}_1 , the usual adjoint $(*)$ is closed on \mathcal{H}_2^t , as a well-defined unary operation, by (4.1).

5. FREE PROBABILITY ON \mathbb{H}_t

In this section, we establish a universal free-probabilistic model on our t -scaled hypercomplex ring \mathbb{H}_t , for “every” scale $t \in \mathbb{R}$. First, recall that, on $M_2(\mathbb{C})$, we have the usual trace tr , defined by

$$tr \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) = a_{11} + a_{22},$$

for all $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$; and the normalized trace τ ,

$$\tau = \frac{1}{2} tr \text{ on } M_2(\mathbb{C}).$$

i.e., we have two typical free-probabilistic models,

$$(M_2(\mathbb{C}), tr) \text{ and } (M_2(\mathbb{C}), \tau).$$

5.1. FREE PROBABILITY

For more about free probability theory, see e.g., [20] and [22]. Let A be a noncommutative algebra over \mathbb{C} , and $\varphi : A \rightarrow \mathbb{C}$, a linear functional on A . Then the pair (A, φ) is called a (noncommutative) free probability space. By definition, free probability spaces are the noncommutative version of classic measure spaces (X, μ) consisting of a set X and a measure μ on the σ -algebra of X . As in measure theory, the (noncommutative)

free probability on (A, φ) is dictated by the linear functional φ . Meanwhile, if (A, φ) is unital in the sense that (i) the unity 1_A of A exists, and (ii) $\varphi(1_A) = 1$, then it is called a unital free probability space. These unital free probability spaces are the noncommutative analogue of classical probability spaces (Y, ρ) where the given measures ρ are the probability measures satisfying $\rho(Y) = 1$.

If A is a topological algebra, and if φ is bounded (and hence, continuous under linearity), then the corresponding free probability space (A, φ) is said to be a topological free probability space. Similarly, if A is a topological $*$ -algebra equipped with the adjoint $(*)$, then the topological free probability space (A, φ) is said to be a topological (free) $*$ -probability space. More in detail, if A is a C^* -algebra, or a von Neumann algebra, or a Banach $*$ -algebra, we call (A, φ) , a C^* -probability space, respectively, a W^* -probability space, respectively, a Banach $*$ -probability space, etc. For our main purposes, we focus on C^* -probability spaces from below.

If (A, φ) is a C^* -probability space, and $a \in A$, then the algebra-element a is said to be a free random variable of (A, φ) . For any arbitrarily fixed free random variables $a_1, \dots, a_s \in (A, \varphi)$ for $s \in \mathbb{N}$, one can get the corresponding free distribution of a_1, \dots, a_s , characterized by the joint free moments,

$$\varphi \left(\prod_{l=1}^n a_{i_l}^{r_l} \right) = \varphi (a_{i_1}^{r_1} a_{i_2}^{r_2} \dots a_{i_n}^{r_n}),$$

for all $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$ and $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where a_l^* are the adjoints of a_l , for all $l = 1, \dots, s$. For instance, if $a \in (A, \varphi)$ is a free random variable, then the free distribution of a is fully characterized by the joint free moments of $\{a, a^*\}$,

$$\varphi \left(\prod_{l=1}^n a^{r_l} \right) = \varphi (a^{r_1} a^{r_2} \dots a^{r_n}),$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$ (e.g., [20] and [22]). So, if a free random variable $a \in (A, \varphi)$ is self-adjoint in the sense that: $a^* = a$ in A , then the free distribution of a is determined by the free-moment sequence,

$$(\varphi(a^n))_{n=1}^\infty = (\varphi(a), \varphi(a^2), \varphi(a^3), \dots)$$

(e.g., [20] and [22]).

5.2. FREE-PROBABILISTIC MODELS INDUCED BY \mathbb{H}_t

By identifying the t -scaled hypercomplex ring \mathbb{H}_t and its realization \mathcal{H}_2^t as the same ring, we identify the t -scaled hypercomplex monoid \mathbb{H}_t^\times and its realization $\mathcal{H}_2^{t \times}$ as the same monoid. As a subset in $M_2(\mathbb{C})$, we define a subset,

$$\mathcal{H}_2^{t \times} (*) \stackrel{\text{def}}{=} \{[\xi]_t^* \in M_2(\mathbb{C}) : \xi \in \mathbb{H}_t^\times\},$$

i.e.,

$$\mathcal{H}_2^{t \times} (*) = \left\{ \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} \in M_2(\mathbb{C}) : (a, b) \in \mathbb{H}_t^\times \right\}, \quad (5.1)$$

by the subset of all adjoints of realizations in $\mathcal{H}_2^{\times t}$. Indeed,

$$[(a, b)]_t^* = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix}^* = \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} \text{ in } M_2(\mathbb{C}).$$

As we have seen in Section 4, the adjoint is not closed on \mathcal{H}_2^t in general, and hence,

$$\mathcal{H}_2^{t \times} (*) \neq \mathcal{H}_2^{t \times} \text{ in } M_2(\mathbb{C}),$$

in general. In particular, the scale t satisfies $t^2 \neq 1$ in \mathbb{R} , if and only if the above non-equality holds in $M_2(\mathbb{C})$, by (4.1). Now, let

$$\mathcal{H}_2^{t \times} (1, *) \stackrel{\text{denote}}{=} \mathcal{H}_2^{t \times} \cup \mathcal{H}_2^{t \times} (*),$$

i.e.,

$$\mathcal{H}_2^{t \times} (1, *) = \left\{ \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix}, \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} : (a, b) \in \mathbb{H}_t^{\times} \right\}, \tag{5.2}$$

in $M_2(\mathbb{C})$, set-theoretically. By (4.1), (5.1) and (5.2),

$$\mathcal{H}_2^{t \times} (1, *) \supsetneq \mathcal{H}_2^{t \times} \text{ in } M_2(\mathbb{C}), \text{ in general.}$$

Define now the C^* -algebra \mathfrak{H}_2^t by the C^* -subalgebra of $M_2(\mathbb{C})$ generated by the set $\mathcal{H}_2^{t \times} (1, *)$ of (5.2), i.e.,

$$\mathfrak{H}_2^t \stackrel{\text{denote}}{=} C^* (\mathcal{H}_2^{t \times}) \stackrel{\text{def}}{=} \overline{\mathbb{C} [\mathcal{H}_2^{t \times} (1, *)]}, \tag{5.3}$$

in $M_2(\mathbb{C})$, where $C^*(Z)$ means the C^* -subalgebra of $B(\mathbb{C}^2)$ generated by the subset Z and their adjoints, and $\mathbb{C}[X]$ is the (pure-algebraic) algebra (over \mathbb{C}) generated by a subset X of $M_2(\mathbb{C})$, and \bar{Y} means the operator-norm-topology closure of a subset Y of the operator algebra $M_2(\mathbb{C}) = B(\mathbb{C}^2)$, which is a C^* -algebra over \mathbb{C} .

Definition 5.1. The C^* -algebra \mathfrak{H}_2^t of (5.3), generated by the t -scaled hypercomplex monoid $\mathbb{H}_t^{\times} \stackrel{\text{monoid}}{=} \mathcal{H}_2^{t \times}$, is called the t -scaled(-hypercomplex)-monoidal C^* -algebra of \mathbb{H}_t^{\times} (or, of \mathbb{H}_t).

Clearly, by the definition (5.3), the t -scaled-monoidal C^* -algebra \mathfrak{H}_2^t is well-determined in $M_2(\mathbb{C})$. So, the usual trace tr and the normalized trace τ on $M_2(\mathbb{C})$ are well-defined on \mathfrak{H}_2^t , i.e., we have two trivial free-probabilistic models of \mathfrak{H}_2^t ,

$$(\mathfrak{H}_2^t, \text{tr}) \text{ and } (\mathfrak{H}_2^t, \tau),$$

as C^* -probability spaces (e.g., see Section 5.1). Note that such free-probabilistic structures are independent from the choice of the scales $t \in \mathbb{R}$.

Observe that, if $\begin{pmatrix} \bar{a}_l & b_l \\ t\bar{b}_l & a_l \end{pmatrix} \in \mathcal{H}_2^{t \times} (*)$ in \mathfrak{H}_2^t , for $l = 1, 2$, then

$$\begin{pmatrix} \bar{a}_1 & b_1 \\ t\bar{b}_1 & a_1 \end{pmatrix} \begin{pmatrix} \bar{a}_2 & b_2 \\ t\bar{b}_2 & a_2 \end{pmatrix} = \begin{pmatrix} \bar{a}_1 \bar{a}_2 + t b_1 \bar{b}_2 & \bar{a}_1 b_2 + b_1 a_2 \\ t (\bar{b}_1 \bar{a}_2 + a_1 \bar{b}_2) & t \bar{b}_1 b_2 + a_1 a_2 \end{pmatrix},$$

identifying to be

$$\begin{pmatrix} \overline{a_1 a_2 + t \overline{b_1} b_2} & b_1 a_2 + \overline{a_1} b_2 \\ t \overline{(b_1 a_2 + \overline{a_1} b_2)} & a_1 a_2 + t \overline{b_1} b_2 \end{pmatrix} \text{ in } \mathfrak{H}_2^t. \tag{5.4}$$

Therefore,

$$\begin{pmatrix} \overline{a_1} & b_1 \\ t \overline{b_1} & a_1 \end{pmatrix} \begin{pmatrix} \overline{a_2} & b_2 \\ t \overline{b_2} & a_2 \end{pmatrix} \in \mathcal{H}_2^{t \times}(*), \text{ too.}$$

i.e., the matricial multiplication is closed on the set $\mathcal{H}_2^{t \times}(*)$ of (5.2), by (5.4). In fact, under the closed-ness (5.4), the algebraic pair,

$$\mathcal{H}_2^{t \times}(*)^{\text{denote}} = (\mathcal{H}_2^{t \times}(*), \cdot),$$

forms a monoid with its identity I_2 . So, the generating set $\mathcal{H}_2^{t \times}(1, *)$ of the t -scaled-monoidal C^* -algebra \mathfrak{H}_2^t is the set-theoretical union of two monoids $\mathcal{H}_2^{t \times}$ and $\mathcal{H}_2^{t \times}(*)$, under the matricial multiplication. Note, however, that the matricial multiplication is not closed on the generating set $\mathcal{H}_2^{t \times}(1, *)$ of (5.2). Indeed, if

$$\begin{pmatrix} a_1 & t b_1 \\ b_1 & \overline{a_1} \end{pmatrix} \in \mathcal{H}_2^{t \times}, \quad \begin{pmatrix} \overline{a_2} & b_2 \\ t \overline{b_2} & a_2 \end{pmatrix} \in \mathcal{H}_2^{t \times}(*)$$

in \mathfrak{H}_2^t , then

$$\begin{aligned} \begin{pmatrix} a_1 & t b_1 \\ b_1 & \overline{a_1} \end{pmatrix} \begin{pmatrix} \overline{a_2} & b_2 \\ t \overline{b_2} & a_2 \end{pmatrix} &= \begin{pmatrix} a_1 \overline{a_2} + t^2 b_1 \overline{b_2} & a_1 b_2 + t a_2 b_1 \\ a_2 b_1 + t a_1 b_2 & \overline{b_1} b_2 + \overline{a_1} a_2 \end{pmatrix}, \\ \begin{pmatrix} \overline{a_2} & b_2 \\ t \overline{b_2} & a_2 \end{pmatrix} \begin{pmatrix} a_1 & t b_1 \\ b_1 & \overline{a_1} \end{pmatrix} &= \begin{pmatrix} a_1 \overline{a_2} + \overline{b_1} b_2 & t b_1 \overline{a_2} + \overline{a_1} b_2 \\ t a_1 \overline{b_2} + \overline{b_1} a_2 & t^2 b_1 \overline{b_2} + \overline{a_1} a_2 \end{pmatrix}, \end{aligned} \tag{5.5}$$

in \mathfrak{H}_2^t . However, the resulted products of (5.5), contained in \mathfrak{H}_2^t , are not contained in $\mathcal{H}_2^{t \times}(1, *)$, in general.

Observation 5.2. *By (5.4) and (5.5), one can realize that:*

- (i) *if $A, B \in \mathcal{H}_2^{t \times}$, then $AB \in \mathcal{H}_2^{t \times}$,*
- (ii) *if $C, D \in \mathcal{H}_2^{t \times}(*)$, then $CD \in \mathcal{H}_2^{t \times}(*)$,*
- (iii) *if $T, S \in \mathcal{H}_2^{t \times}(1, *)$, then $TS \notin \mathcal{H}_2^{t \times}(1, *)$, in general, as elements of the t -scaled-monoidal C^* -algebra \mathfrak{H}_2^t .*

*Even though the non-closed rule (iii) is satisfied “on $\mathcal{H}_2^t(1, *)$ ”, at least, we have a multiplication rule (5.5) “in the C^* -algebra \mathfrak{H}_2^t ”.*

Assume that $[(a, b)]_t \in \mathcal{H}_2^{t \times}$ in \mathfrak{H}_2^t . Then

$$\text{tr}([(a, b)]_t) = a + \overline{a} = 2\text{Re}(a),$$

and

$$\tau([(a, b)]_t) = \frac{1}{2} \text{tr}([(a, b)]_t) = \text{Re}(a), \tag{5.6}$$

where $\operatorname{Re}(a)$ is the real part of a in \mathbb{C} . Similarly, if $[(a, b)]_t^* \in \mathcal{H}_2^{t \times} (*)$ in \mathfrak{H}_2^t , then we have

$$\operatorname{tr}([(a, b)]_t^*) = \operatorname{tr} \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} = \bar{a} + a = 2\operatorname{Re}(a),$$

and

$$\tau([(a, b)]_t^*) = \frac{1}{2}(2\operatorname{Re}(a)) = \operatorname{Re}(a). \tag{5.7}$$

Remark that, since tr and τ are well-defined linear functional on the C^* -algebra \mathfrak{H}_2^t , they satisfy

$$\operatorname{tr}(T^*) = \overline{\operatorname{tr}(T)}, \text{ and } \tau(T^*) = \overline{\tau(T)},$$

for all $T \in \mathfrak{H}_2^t$. So, the relation (5.7) is well-verified, too.

Also, if $[(a_1, b_1)]_t, [(a_2, b_2)]_t^* \in \mathcal{H}_2^{t \times}(1, *)$ in \mathfrak{H}_2^t , then

$$\operatorname{tr}([(a_1, b_1)]_t [(a_2, b_2)]_t^*) = \operatorname{tr} \left(\begin{pmatrix} a_1\bar{a}_2 + t^2\bar{b}_1\bar{b}_2 & a_1b_2 + ta_2b_1 \\ a_2\bar{b}_1 + ta_1\bar{b}_2 & \bar{b}_1b_2 + \bar{a}_1a_2 \end{pmatrix} \right)$$

by (5.5)

$$\begin{aligned} &= a_1\bar{a}_2 + t^2\bar{b}_1\bar{b}_2 + \bar{b}_1b_2 + \bar{a}_1a_2 \\ &= 2\operatorname{Re}(a_1\bar{a}_2) + t^2\bar{b}_1\bar{b}_2 + \bar{b}_1b_2, \end{aligned}$$

and similarly,

$$\operatorname{tr}([(a_1, b_1)]_t^* [(a_2, b_2)]_t) = 2\operatorname{Re}(\bar{a}_1a_2) + t^2\bar{b}_1b_2 + b_1\bar{b}_2, \tag{5.8}$$

and hence,

$$\tau([(a_1, b_1)]_t [(a_2, b_2)]_t^*) = \operatorname{Re}(a_1\bar{a}_2) + \frac{t^2\bar{b}_1\bar{b}_2 + \bar{b}_1b_2}{2},$$

and

$$\tau([(a_1, b_1)]_t^* [(a_2, b_2)]_t) = \operatorname{Re}(\bar{a}_1a_2) + \frac{t^2\bar{b}_1b_2 + b_1\bar{b}_2}{2}, \tag{5.9}$$

by (5.8).

Proposition 5.3. *Let $(a, b), (a_l, b_l) \in \mathbb{H}_t$, for $l = 1, 2$, and let $A = [(a, b)]_t$ and $A_l = [(a_l, b_l)]_t$ be the corresponding realizations of \mathcal{H}_2^t , regarded as elements of the t -scaled-monoidal C^* -algebra \mathfrak{H}_2^t . Then*

$$\tau(A) = \frac{1}{2}\operatorname{tr}(A) = \operatorname{Re}(a) = \frac{1}{2}\operatorname{tr}(A^*) = \tau(A^*),$$

and

$$\tau(A_1A_2^*) = \frac{1}{2}\operatorname{tr}(A_1A_2^*) = \operatorname{Re}(a_1\bar{a}_2) + \frac{t^2\bar{b}_1\bar{b}_2 + \bar{b}_1b_2}{2}, \tag{5.10}$$

and

$$\tau(A_1^*A_2) = \frac{1}{2}\operatorname{tr}(A_1^*A_2) = \operatorname{Re}(\bar{a}_1a_2) + \frac{t^2\bar{b}_1b_2 + b_1\bar{b}_2}{2}.$$

Proof. The joint free moments in (5.10) are proven by (5.6), (5.7), (5.8) and (5.9). \square

The above computations in (5.10) provide a general way to compute free-distributional data, in particular, the joint free moments of matrices in the t -scaled-monoidal C^* -algebra \mathfrak{H}_2^t , up to the trace tr , and up to the normalized trace τ . And, they demonstrate that computing such free-distributional data is not easy. So, we will restrict our interests to a certain specific case.

5.3. FREE PROBABILITY ON $(\mathfrak{H}_2^t, \text{tr})$

In this section, we fix a scale $t \in \mathbb{R}$, and the corresponding t -scaled-monoidal C^* -algebra \mathfrak{H}_2^t generated by the t -scaled hypercomplex monoid \mathbb{H}_t^\times . Let $(\mathfrak{H}_2^t, \text{tr})$ be the C^* -probability space with respect to the usual trace tr on \mathfrak{H}_2^t .

Recall that if a scale t is negative, then the realization $[\xi]_t$ and the t -spectral form $\Sigma_t(\xi)$ are similar “in \mathcal{H}_2^t ” by (3.29), for all $\xi \in \mathbb{H}_t$. It implies that the similarity “on \mathcal{H}_2^t ” is equivalent to the t -spectral relation on \mathbb{H}_t by (3.30). Also, recall that if two matrices A and B are similar in $M_n(\mathbb{C})$, for any $n \in \mathbb{N}$,

$$\text{tr}(A) = \text{tr}(B).$$

So, if the realization $[\xi]_t$ and the t -spectral form $\Sigma_t(\xi)$ are similar in \mathcal{H}_2^t , then the free-moment computations would be much simpler than the computations of (5.10). Note again that if $(a, b) \in \mathbb{H}_t$ satisfies the condition (3.4), then

$$\text{tr}([(a, b)]_t) = 2\text{Re}(a) = 2x = (x + i\sqrt{R}) + (x - i\sqrt{R}) = \text{tr}(\Sigma_t(a, b)),$$

where

$$R = y^2 - tu^2 - tv^2 \text{ in } \mathbb{R}, \tag{5.11}$$

in the sense of Remark 3.14. Even though the identical results hold in (5.11) (without similarity), if $[(a, b)]_t$ and $\Sigma_t(a, b)$ are not similar in \mathcal{H}_2^t , then

$$\text{tr}([(a, b)]_t^n) \neq \text{tr}((\Sigma_t(a, b))^n),$$

for some $n \in \mathbb{N}$, by (5.5). It implies that some (joint) free-moments of $[(a, b)]_t$ and those of $\Sigma_t(a, b)$ are not identical, and hence, the free distributions of them are distinct.

Lemma 5.4. *Suppose the realization $[(a, b)]_t$ and the t -spectral form $\Sigma_t(a, b)$ are similar in \mathcal{H}_2^t for $(a, b) \in \mathbb{H}_t$. Then*

$$\text{tr}([(a, b)]_t^n) = 2\text{Re}(\sigma_t(a, b)^n) = \text{tr}([(a, b)]_t^{*n}) \tag{5.12}$$

for all $n \in \mathbb{N}$, where $\sigma_t(a, b)$ is the t -spectral value of (a, b) .

Proof. Suppose $(a, b) \in \mathbb{H}_t$ satisfies the condition (3.4). Then

$$[(a, b)]_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \quad \text{and} \quad \Sigma_t((a, b)) = \begin{pmatrix} \sigma_t(a, b) & 0 \\ 0 & \sigma_t(a, b) \end{pmatrix},$$

in \mathcal{H}_2^t , where

$$\sigma_t(a, b) = x + i\sqrt{y^2 - tu^2 - tv^2},$$

in the sense of Remark 3.14. Assume that $[(a, b)]_t$ and $\Sigma_t((a, b))$ are similar in \mathcal{H}_2^t . Then the matrices $[(a, b)]_t^n$ and $\Sigma_t((a, b))^n$ are similar in \mathcal{H}_2^t , for all $n \in \mathbb{N}$. Indeed, if two elements A and B are similar in \mathcal{H}_2^t , satisfying $B = U^{-1}AU$ in \mathcal{H}_2^t , for an invertible element $U \in \mathcal{H}_2^t$, then

$$B^n = (U^{-1}AU)^n = U^{-1}A^nU \text{ in } \mathcal{H}_2^t,$$

implying the similarity of A^n and B^n , for $n \in \mathbb{N}$. Thus,

$$\text{tr}([(a, b)]_t^n) = \text{tr}(\Sigma_t((a, b))^n),$$

and

$$\text{tr}(\Sigma_t((a, b))^n) = \text{tr}\left(\begin{pmatrix} \sigma_t(a, b)^n & 0 \\ 0 & \overline{\sigma_t(a, b)^n} \end{pmatrix}\right),$$

implying that

$$\text{tr}([(a, b)]_t^n) = \text{tr}(\Sigma_t((a, b))^n) = 2\text{Re}(\sigma_t(a, b)^n),$$

for all $n \in \mathbb{N}$. Therefore, the first equality in (5.12) holds.

Since tr is a well-defined linear functional on the C^* -algebra \mathfrak{H}_2^t , one has

$$\text{tr}(A^*) = \overline{\text{tr}(A)}, \text{ for all } A \in \mathfrak{H}_2^t.$$

Since

$$\text{tr}([(a, b)]_t^{*n}) = \text{tr}([(a, b)]_t^n)^* = \overline{\text{tr}([(a, b)]_t^n)},$$

one has

$$\text{tr}([(a, b)]_t^{*n}) = \overline{\text{tr}(\sigma_t(a, b)^n)} = 2\text{Re}(\sigma_t(a, b)^n),$$

for all $n \in \mathbb{N}$. So, the second equality in (5.12) holds, too. \square

Note that the formula (5.12) holds true under the similarity assumption of the realization and the t -spectral form.

Remark that every complex number $w \in \mathbb{C}$ is polar-decomposed to be

$$w = |w|w_o \text{ with } w_o \in \mathbb{T},$$

uniquely, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in \mathbb{C} . So, all our t -spectral values $\sigma_t(\xi)$ are polar-decomposed to be

$$\sigma_t(\xi) = |\sigma_t(\xi)|\sigma_t(\xi)_o \text{ with } \sigma_t(\xi)_o \in \mathbb{T},$$

for all $\xi \in \mathbb{H}_t$. In such a sense, we have that

$$\text{tr}([\xi]_t^n) = 2|\sigma_t(\xi)|^n \text{Re}(\sigma_t(\xi)_o^n),$$

for all $n \in \mathbb{N}$, by (5.12).

Corollary 5.5. *Suppose the realization $[\xi]_t$ and the t -spectral form $\Sigma_t(\xi)$ are similar in \mathcal{H}_2^t for $\xi \in \mathbb{H}_t$. Then*

$$\text{tr}([\xi]_t^n) = 2|\sigma_t(\xi)|^n \text{Re}(\sigma_t(\xi)_o^n) = \text{tr}([\xi]_t^*), \tag{5.13}$$

for all $n \in \mathbb{N}$, where $\sigma_t(\xi) = |\sigma_t(\xi)|\sigma_t(\xi)_o$ is the polar decomposition of $\sigma_t(\xi)$, with $\sigma_t(\xi)_o \in \mathbb{T}$.

Proof. The free-distributional data (5.13) is immediately obtained by (5.12) under the polar decomposition of the t -spectral value $\sigma_t(\xi)$ in \mathbb{C} . \square

Assume again that a hypercomplex number $(a, b) \in \mathbb{H}_t$ satisfies our similarity assumption, i.e., $T \stackrel{\text{denote}}{=} [(a, b)]_t$ and $S \stackrel{\text{denote}}{=} \Sigma_t((a, b))$ are similar in \mathcal{H}_2^t . Then, for any

$$(r_1, \dots, r_n) \in \{1, *\}^n, \text{ for } n \in \mathbb{N},$$

the matrix $\prod_{l=1}^n T^{r_l}$ is similar to $\prod_{l=1}^n S^{r_l}$ in \mathcal{H}_2^t (and hence, in \mathfrak{H}_2^t).

Theorem 5.6. *Let $(a, b) \in \mathbb{H}_t$ satisfy the similarity assumption that: $T \stackrel{\text{denote}}{=} [(a, b)]_t$ and $S \stackrel{\text{denote}}{=} \Sigma_t((a, b))$ are similar in \mathcal{H}_2^t . If*

$$\sigma_t(a, b) = rw_o, \text{ polar decomposition,}$$

with

$$r = |\sigma_t(a, b)| \text{ and } w_o \in \mathbb{T}, \tag{5.14}$$

then

$$\text{tr} \left(\prod_{l=1}^n T^{r_l} \right) = 2r^n \text{Re} \left(w_o^{\sum_{l=1}^n e_l} \right), \tag{5.15}$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where

$$e_l = \begin{cases} 1 & \text{if } r_l = 1, \\ -1 & \text{if } r_l = *, \end{cases}$$

for all $l = 1, \dots, n$.

Proof. Since the realization T and the t -spectral form S are assumed to be similar in \mathcal{H}_2^t , their adjoints T^* and S^* are similar in $\mathcal{H}_2^{t \times}(\ast) \cup \{(0, 0)_t\}$; and hence, the matrix $\prod_{l=1}^n T^{r_l}$ and $\prod_{l=1}^n S^{r_l}$ are similar “in \mathfrak{H}_2^t ”. Consider that

$$S = \begin{pmatrix} \sigma_t(a, b) & 0 \\ 0 & \overline{\sigma_t(a, b)} \end{pmatrix} = \begin{pmatrix} rw_o & 0 \\ 0 & r\overline{w_o} \end{pmatrix} = r \begin{pmatrix} w_o & 0 \\ 0 & w_o^{-1} \end{pmatrix},$$

under hypotheses, because $\bar{z} = \frac{1}{z} = z^{-1}$ in \mathbb{T} , whenever $z \in \mathbb{T}$ in \mathbb{C} . It shows that

$$S^j = r^j \begin{pmatrix} w_o^j & 0 \\ 0 & w_o^{-j} \end{pmatrix}, \text{ for all } j \in \mathbb{N} \cup \{0\},$$

and

$$S^* = \bar{r} \begin{pmatrix} \overline{w_o} & 0 \\ 0 & w_o \end{pmatrix} = r \begin{pmatrix} w_o^{-1} & 0 \\ 0 & w_o \end{pmatrix},$$

satisfying that

$$(S^*)^j = (S^j)^*, \text{ for all } j \in \mathbb{N}.$$

It implies that, for any $(r_1, \dots, r_n) \in \{1, *\}^n$, for $n \in \mathbb{N}$, there exists $(e_1, \dots, e_n) \in \{\pm 1\}^n$, such that

$$e_l = \begin{cases} 1 & \text{if } r_l = 1, \\ -1 & \text{if } r_l = *, \end{cases}$$

for all $l = 1, \dots, n$, and

$$\prod_{l=1}^n S^{r_l} = r^n \begin{pmatrix} \sum_{l=1}^n e_l & 0 \\ w_o^{\sum_{l=1}^n e_l} & -\left(\sum_{l=1}^n e_l\right) \\ 0 & w_o \end{pmatrix}, \tag{5.16}$$

in \mathfrak{H}_2^t . Thus, under our similarity assumption,

$$\text{tr} \left(\prod_{l=1}^n T^{r_l} \right) = \text{tr} \left(\prod_{l=1}^n S^{r_l} \right) = r^n \left(w_o^{\sum_{l=1}^n e_l} + w_o^{-\left(\sum_{l=1}^n e_l\right)} \right),$$

implying that

$$\text{tr} \left(\prod_{l=1}^n T^{r_l} \right) = r^n \left(2\text{Re} \left(w_o^{\sum_{l=1}^n e_l} \right) \right),$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where $(e_1, \dots, e_n) \in \{\pm 1\}^n$ satisfies (5.16).

Therefore, under our similarity assumption and the polar decomposition (5.14), the free-distributional data (5.15) holds. \square

By the above theorem, one immediately obtain the following result.

Corollary 5.7. *Let $(a, b) \in \mathbb{H}_t$ satisfy the similarity assumption that: $T \stackrel{\text{denote}}{=} [(a, b)]_t$ and $S \stackrel{\text{denote}}{=} \Sigma_t((a, b))$ are similar in \mathfrak{H}_2^t . If*

$$\sigma_t(a, b) = r w_o, \text{ polar decomposition,}$$

with

$$r = |\sigma_t(a, b)| \text{ and } w_o \in \mathbb{T}, \tag{5.17}$$

then

$$\tau \left(\prod_{l=1}^n T^{r_l} \right) = r^n \operatorname{Re} \left(w_o^{\sum_{l=1}^n e_l} \right), \tag{5.18}$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where

$$e_l = \begin{cases} 1 & \text{if } r_l = 1, \\ -1 & \text{if } r_l = *, \end{cases}$$

for all $l = 1, \dots, n$.

Proof. By (5.15), the free-distributional data (5.18) holds up to the normalized trace $\tau = \frac{1}{2} \operatorname{tr}$ on \mathfrak{H}_2^t , under (5.17). \square

Under our similarity assumption and the condition (5.17), the free-distributional data (5.18) fully characterizes the free distribution of $[(a, b)]_t \in \mathcal{H}_2^t$ in the C^* -probability space (\mathfrak{H}_2^t, τ) .

Corollary 5.8. *Suppose a given scale t is negative in \mathbb{R} . Let $(a, b) \in \mathbb{H}_t$, and let $T \stackrel{\text{denote}}{=} [(a, b)]_t$ and $S \stackrel{\text{denote}}{=} \Sigma_t((a, b))$ in \mathcal{H}_2^t . If*

$$\sigma_t(a, b) = r w_o, \text{ polar decomposition,}$$

with

$$r = |\sigma_t(a, b)| \text{ and } w_o \in \mathbb{T}, \tag{5.19}$$

then

$$\operatorname{tr} \left(\prod_{l=1}^n T^{r_l} \right) = 2r^n \operatorname{Re} \left(w_o^{\sum_{l=1}^n e_l} \right) = 2\tau \left(\prod_{l=1}^n T^{r_l} \right), \tag{5.20}$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where

$$e_l = \begin{cases} 1 & \text{if } r_l = 1, \\ -1 & \text{if } r_l = *, \end{cases}$$

for all $l = 1, \dots, n$.

Proof. In Theorem 5.6 and Corollary 5.7, we showed that if T and S are similar in \mathcal{H}_2^t , then the free-distributional data (5.20) holds under the condition (5.19), by (5.15) and (5.18), respectively. So, it suffices to show that the realization T and the t -spectral form S are similar in \mathcal{H}_2^t . However, since $t < 0$ in \mathbb{R} , the matrices T and S are similar in \mathcal{H}_2^t by (3.29). \square

The above corollary shows that, if a given scale t is negative in \mathbb{R} , then the free-distributional data (5.20) fully characterizes the free distributions of the realizations $[\xi]_t$ in the t -scaled-monoidal C^* -algebra \mathfrak{H}_2^t up to the usual trace tr , and the

normalized trace τ , for “all” $\xi \in \mathbb{H}_t$. In other words, it illustrates that, if $t < 0$ in \mathbb{R} , then the free-distributional data on the C^* -probability spaces,

$$(\mathfrak{H}_2^t, \text{tr}) \text{ and } (\mathfrak{H}_2^t, \tau),$$

are fully characterized by the spectra of hypercomplex numbers of \mathbb{H}_t , by (5.19) and (5.20).

But, if $t \geq 0$, and hence, there are some hypercomplex numbers η of \mathbb{H}_t whose realization and spectral form are not similar in \mathcal{H}_2^t , then computing joint free moments of $[\eta]_t$ in \mathfrak{H}_2^t would not be easy, e.g., see (5.10).

5.4. MORE FREE-DISTRIBUTIONAL DATA ON (\mathfrak{H}_2^t, τ) FOR $t < 0$

In this section, a fixed scale t is automatically assumed to be negative, i.e., $t < 0$ in \mathbb{R} . At this moment, we emphasize that most main results of this section would hold even though t is not negative in \mathbb{R} . However, we assume a given scale t is negative for convenience (e.g., see (5.20)). Let \mathfrak{H}_2^t be the t -scaled-monoidal C^* -algebra inducing a C^* -probability space (\mathfrak{H}_2^t, τ) , where τ is the normalized trace on \mathfrak{H}_2^t . Since t is assumed to be negative in \mathbb{R} , the realizations $T = [\eta]_t$ and the t -spectral forms $S = \Sigma_t(\eta)$ are similar in \mathcal{H}_2^t by (3.29), and hence,

$$\tau \left(\prod_{l=1}^n T^{r_l} \right) = r^n \text{Re} \left(w_o^{\sum_{l=1}^n e_l} \right) = \tau \left(\prod_{l=1}^n S^{r_l} \right),$$

by (5.15), where

$$\sigma_t(\eta) = r w_o \in \mathbb{C}, \text{ polar decomposition,} \tag{5.21}$$

with $r = |\sigma_t(\eta)|$ and $w_o \in \mathbb{T}$, for all $(r_1, \dots, r_n) \in \{1, *\}^n$, where $(e_1, \dots, e_n) \in \{\pm 1\}^n$ satisfies (5.16), for all $n \in \mathbb{N}$, for “all” $\eta \in \mathbb{H}_t$. And the free-distributional data (5.21) fully characterizes the free distribution of $[\eta]_t \in (\mathfrak{H}_2^t, \tau)$, for all $\eta \in \mathbb{H}_t$.

In this section, we refine (5.21) case-by-case, up to operator-theoretic properties of elements of (\mathfrak{H}_2^t, τ) .

Definition 5.9. Let \mathcal{A} be a unital C^* -algebra with its unity $1_{\mathcal{A}}$, and let $T \in \mathcal{A}$, and $T^* \in \mathcal{A}$, the adjoint of T .

- (1) T is said to be self-adjoint, if $T^* = T$ in \mathcal{A} .
- (2) T is a projection, if $T^* = T = T^2$ in \mathcal{A} .
- (3) T is normal, if $T^*T = TT^*$ in \mathcal{A} .
- (4) T is a unitary, if $T^*T = 1_{\mathcal{A}} = TT^*$ in \mathcal{A} .

Let $(a, b) \in \mathbb{H}_t$, satisfying the condition (3.4), and $T \stackrel{\text{denote}}{=} [(a, b)]_t \in \mathcal{H}_2^t$, as an element of (\mathfrak{H}_2^t, τ) . Then its adjoint,

$$T^* = \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} \in \mathcal{H}_2^t(*),$$

is well-defined in (\mathfrak{H}_2^t, τ) , and the corresponding t -spectral form,

$$S \stackrel{\text{denote}}{=} \Sigma_t((a, b)) = \begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix} \in \mathcal{H}_2^t,$$

is contained in (\mathfrak{H}_2^t, τ) , where \bar{w} is determined by Remark 3.14, and

$$w = \sigma_t(a, b) = x + i\sqrt{y^2 - tu^2 - tv^2}$$

is the t -spectral value, uniquely polar-decomposed to be

$$w = rw_o \text{ with } r = |\sigma_t(a, b)| \text{ and } w_o \in \mathbb{T}.$$

For a given hypercomplex number $(a, b) \in \mathbb{H}_t$, let us assume that

it has its realization denoted by T , its t -spectral form denoted by S , determined by the t -spectral value denoted by w , which is polar-decomposed to be $w = rw_o$, as indicated in the very above paragraph. (5.22)

From now on, if we say that “a given hypercomplex number $(a, b) \in \mathbb{H}_t$ satisfies (5.22)”, we understand that the above properties hold.

Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22). Then the self-adjointness of the realization $T \in \mathcal{H}_2^t$ in \mathfrak{H}_2^t says that

$$T^* = T \iff \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix},$$

if and only if

$$\bar{a} = a \text{ and } tb = b \text{ in } \mathbb{C},$$

if and only if

$$a \in \mathbb{R} \text{ and } b = 0. \tag{5.23}$$

Epecially, the equality $b = 0$ in (5.23) is obtained by our negative-scale assumption: $t < 0$ in \mathbb{R} .

Proposition 5.10. *Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22). Then the realization $T \in \mathcal{H}_2^t$ is self-adjoint in \mathfrak{H}_2^t , if and only if*

$$a \in \mathbb{R} \text{ and } b = 0 \iff (a, b) = (\text{Re}(a), 0) \text{ in } \mathbb{H}_t. \tag{5.24}$$

Proof. The self-adjointness (5.24) is shown by (5.23). □

The self-adjointness (5.24) illustrates that the self-adjoint generating elements $T \in \mathcal{H}_2^t$ of (\mathfrak{H}_2^t, τ) have their forms,

$$T = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in \mathcal{H}_2^t(1, *) \text{ with } x \in \mathbb{R}.$$

Remark 5.11. The above self-adjointness characterization (5.24) is obtained for the case where $t < 0$ in \mathbb{R} . How about the other cases? Generally, one has T is self-adjoint in \mathfrak{H}_2^t , if and only if

$$\bar{a} = a \quad \text{and} \quad tb = b,$$

like (5.23). Thus one can verify that: (i) if $t = 0$, then T is self-adjoint, if and only if $a \in \mathbb{R}$ and $b = 0$, just like (5.24); (ii) if $t > 0$ and $t \neq 1$, then T is self-adjoint, if and only if $a \in \mathbb{R}$ and $b = 0$, just like (5.24); meanwhile, (iii) if $t = 1$ (equivalently, if (a, b) is a bicomplex number of \mathbb{H}_1), then T is self-adjoint in \mathfrak{H}_2^1 , if and only if $a \in \mathbb{R}$, if and only if $(a, b) = (\text{Re}(a), b)$ in \mathbb{H}_1 . In summary,

$$T \text{ is self-adjoint in } \mathfrak{H}_2^t \iff (a, b) = (\text{Re}(a), 0) \text{ in } \mathbb{H}_t,$$

like (5.24), whenever $t \in \mathbb{R} \setminus \{1\}$, meanwhile,

$$T \text{ is self-adjoint in } \mathfrak{H}_2^1 \iff (a, b) = (\text{Re}(a), b) \in \mathbb{H}_1.$$

Now, let $(a, b) \in \mathbb{H}_t$, under (5.22) and our negative-scale assumption, satisfy the self-adjointness (5.24), i.e., it is actually $(a, 0)$ with $a \in \mathbb{R}$. Then

$$T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = S \text{ in } \mathcal{H}_2^t(1, *),$$

as an element of \mathfrak{H}_2^t .

Theorem 5.12. Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22), and assume that the realization T is self-adjoint in (\mathfrak{H}_2^t, τ) . Then

$$\tau \left(\prod_{l=1}^n T^{r_l} \right) = \tau(T^n) = a^n \quad \text{in } \mathbb{R} \tag{5.25}$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. By the self-adjointness (5.24) of the realization T of $(a, b) \in \mathbb{H}_t$, one has $(a, b) = (a, 0)$ in \mathbb{H}_t , with $a \in \mathbb{R}$, and

$$T = S = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = S^* = T^* \text{ in } \mathfrak{H}_2^t.$$

So,

$$\tau \left(\prod_{l=1}^n T^{r_l} \right) = \tau(T^n) = \tau(S^n) = \tau \left(\begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix} \right),$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$. Therefore, the free-distributional data (5.25) holds true. \square

Remark 5.13. Similar to the above theorem, one can verify that: if $t \in \mathbb{R} \setminus \{1\}$, then the free-distributional data (5.25) holds for self-adjoint realizations $T \in (\mathfrak{H}_2^t, \tau)$ of $(a, 0) \in \mathbb{H}_t$ with $a \in \mathbb{R}$.

By (5.24), the realization T of a hypercomplex number $(a, b) \in \mathbb{H}_t$, satisfying (5.22), is self-adjoint, if and only if $(a, b) = (a, 0)$ with $a \in \mathbb{R}$. And, by definition, such a self-adjoint matrix T can be a projection, if and only if it is idempotent in the sense that

$$T^2 = T \text{ in } \mathfrak{H}_2^t.$$

Observe that a self-adjoint realization T satisfies the above idempotence, if and only if

$$T^2 = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = T,$$

if and only if

$$a^2 = a \iff a = 0, \text{ or } a = 1, \text{ in } \mathbb{R}. \quad (5.26)$$

Proposition 5.14. *Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22). Then the realization T is a projection, if and only if*

$$\text{either } T = I_2, \text{ or } T = O_2 \text{ in } \mathcal{H}_2^t, \quad (5.27)$$

where $I_2 = [(1, 0)]_t$ is the identity matrix, and $O_2 = [(0, 0)]_t$ is the zero matrix of \mathfrak{H}_2^t .

Proof. The operator-equality (5.27) holds in \mathcal{H}_2^t (and hence, in \mathfrak{H}_2^t) by (5.26). \square

Remark 5.15. Like in the above proposition, one can conclude that: if $t \in \mathbb{R} \setminus \{1\}$, then the realization T is a projection in \mathfrak{H}_2^t , if and only if it is either the identity matrix I_2 , or the zero matrix O_2 of \mathfrak{H}_2^t . How about the case where $t = 1$? As we discussed above, $T \in \mathfrak{H}_2^1$ is self-adjoint, if and only if $(a, b) = (\operatorname{Re}(a), b)$ in \mathbb{H}_1 , if and only if

$$T = \begin{pmatrix} x & b \\ \bar{b} & x \end{pmatrix} \in \mathcal{H}_2^1, \quad \text{and} \quad S = \begin{pmatrix} x + i\sqrt{-u^2 - v^2} & 0 \\ 0 & x - i\sqrt{-u^2 - v^2} \end{pmatrix},$$

implying that

$$S = \begin{pmatrix} x - |b| & 0 \\ 0 & x + |b| \end{pmatrix} \text{ in } \mathfrak{H}_2^1,$$

under (5.22). Such a self-adjoint T is a projection, if and only if $T^2 = T$ in \mathfrak{H}_2^1 , if and only if

$$x^2 + |b|^2 = x \quad \text{and} \quad 2xb = b.$$

Thus if $b = 0$, then $x \in \{0, 1\}$, meanwhile, if $b \neq 0$, then

$$x^2 + |b|^2 = x \quad \text{and} \quad x = \frac{1}{2},$$

if and only if

$$x = \frac{1}{2} \quad \text{and} \quad \frac{1}{4} + |b|^2 = \frac{1}{2},$$

if and only if

$$x = \frac{1}{2} \quad \text{and} \quad |b|^2 = \frac{1}{4},$$

if and only if

$$(a, b) = \left(\frac{1}{2}, b\right) \text{ with } |b|^2 = \frac{1}{4}.$$

It implies that T is a projection in \mathfrak{H}_2^1 , if and only if

$$(a, b) = (0, 0), \text{ or } (a, b) = (1, 0),$$

or

$$(a, b) = \left(\frac{1}{2}, b\right) \text{ with } |b|^2 = \frac{1}{4},$$

in \mathbb{H}_1 .

The above proposition says that, under our negative-scale assumption, the only projections of \mathfrak{H}_2^t induced by hypercomplex numbers of \mathbb{H}_t are the identity element $I_2 = [(1, 0)]_t$, and the zero element $O_2 = [(0, 0)]_t$ in \mathfrak{H}_2^t . For any unital C^* -probability spaces (\mathcal{A}, φ) , the unity $1_{\mathcal{A}}$ has its free distributions characterized by its free-moment sequence,

$$(\varphi(1_{\mathcal{A}}^n) = \varphi(1_{\mathcal{A}}))_{n=1}^{\infty} = (1, 1, 1, 1, 1, \dots);$$

and the free distribution of the zero element $0_{\mathcal{A}}$ is nothing but the zero-free distribution, characterized by the free-moment sequence,

$$(\varphi(0_{\mathcal{A}}^n) = \varphi(0_{\mathcal{A}}))_{n=1}^{\infty} = (0, 0, 0, 0, \dots).$$

Theorem 5.16. *Let $(a, b) \in \mathbb{H}_t$, satisfying (5.22), have its realization $T \in \mathcal{H}_2^t$, which is a “non-zero” projection in \mathfrak{H}_2^t . Then*

$$\tau(T^n) = 1, \quad \forall n \in \mathbb{N}.$$

(In fact, this result holds true for all $t \in \mathbb{R} \setminus \{1\}$.)

Proof. Under hypothesis, the realization $T \in \mathcal{H}_2^t$ is a projection in \mathfrak{H}_2^t , if and only if $(a, b) = (1, 0)$, or $(0, 0)$ in \mathbb{H}_t , by (5.27). Since $T \in \mathcal{H}_2^t$ is assumed to a non-zero projection in \mathfrak{H}_2^t , we have

$$(a, b) = (1, 0) \text{ in } \mathbb{H}_t \iff T = I_2 = S \text{ in } \mathfrak{H}_2^t.$$

Therefore,

$$\tau(T^n) = \tau(I_2^n) = 1, \quad \forall n \in \mathbb{N}.$$

(Note that it holds true for all $t \in \mathbb{R} \setminus \{1\}$.) □

Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22), and let $T \in \mathcal{H}_2^t$ be the realization in \mathfrak{H}_2^t . Observe that

$$T^*T = \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & (t+1)\bar{a}b \\ (t+1)a\bar{b} & t^2|b|^2 + |a|^2 \end{pmatrix},$$

and

$$TT^* = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} = \begin{pmatrix} |a|^2 + t^2|b|^2 & (t+1)ab \\ (t+1)\bar{a}b & |b|^2 + |a|^2 \end{pmatrix}, \tag{5.28}$$

in \mathfrak{H}_2^t . So, the realization T of (a, b) is normal in \mathfrak{H}_2^t , if and only if

$$|a|^2 + t^2|b|^2 = |a|^2 + |b|^2 \text{ and } (t+1)\bar{a}b = (t+1)ab, \quad (5.29)$$

in \mathbb{C} , by (5.28).

Proposition 5.17. *Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22). Then the realization $T \in \mathcal{H}_2^t$ is normal in \mathfrak{H}_2^t , if and only if*

$$t^2|b|^2 = |b|^2 \text{ and } (t+1)\bar{a}b = (t+1)ab, \quad (5.30)$$

in \mathbb{C} . In particular, if $t = -1$ (equivalently, if $(a, b) \in \mathbb{H}_{-1}$ is a quaternion), then T is normal in \mathfrak{H}_2^{-1} ; if $t = 1$, (equivalently, if $(a, b) \in \mathbb{H}_1$ is a bicomplex number), then T is normal in \mathfrak{H}_2^1 , if and only if

$$\text{either } (a, b) = (\operatorname{Re}(a), b) \text{ or } (a, b) = (a, 0) \text{ in } \mathbb{H}_1; \quad (5.31)$$

meanwhile, if $t \in \mathbb{R} \setminus \{\pm 1\}$, then T is normal in \mathfrak{H}_2^t , if and only if

$$b = 0 \text{ in } \mathbb{C} \iff (a, b) = (a, 0) \in \mathbb{H}_t. \quad (5.32)$$

Proof. By (5.29), the normality characterization (5.30) holds.

By (5.30), if $t = -1$ in \mathbb{R} , and hence, if $(a, b) \in \mathbb{H}_{-1}$ is a quaternion, then the condition (5.30) is identified with

$$|b|^2 = |b|^2, \text{ and } 0 = 0,$$

which are the identities on \mathbb{C} . These identities demonstrate that the realization of every quaternion is automatically normal in \mathfrak{H}_2^{-1} .

Suppose $t = 1$ in \mathbb{R} . Then the condition (5.30) is equivalent to

$$|b|^2 = |b|^2 \text{ and } 2\bar{a}b = 2ab,$$

if and only if either

$$\bar{a} = a \text{ in } \mathbb{C} \iff (a, b) = (\operatorname{Re}(a), b) \in \mathbb{H}_1 \text{ (if } b \neq 0),$$

or

$$(a, b) = (a, 0) \in \mathbb{H}_1 \text{ (if } b = 0).$$

Thus, if $t = 1$, then T is normal, if and only if the condition (5.31) holds.

Assume now that both $t \neq 1$ and $t \neq -1$, i.e., suppose $t^2 \neq 1$ in \mathbb{R} . So, the first condition of (5.30) is identified with

$$t^2|b|^2 = |b|^2 \iff b = 0 \text{ in } \mathbb{C}.$$

So, the second condition of (5.30) automatically holds, since

$$(t+1)\bar{a} \cdot 0 = (t+1)a \cdot 0 \iff 0 = 0.$$

Therefore, the realization $T \in \mathcal{H}_2^t$ of $(a, b) \in \mathbb{H}_t$ is normal in \mathfrak{H}_2^t , if and only if $(a, b) = (a, 0)$ in \mathbb{H}_t , whenever $t \in \mathbb{R} \setminus \{\pm 1\}$, i.e., the normality (5.32) holds. \square

The above proposition illustrates that: (i) the realizations of “all” quaternions are normal in \mathfrak{H}_2^{-1} , (ii) the realizations of bicomplex numbers are normal in \mathfrak{H}_2^1 , if and only if either $(a, b) = (\operatorname{Re}(a), b)$, or $(a, b) = (a, 0)$ in \mathbb{H}_1 , by (5.31), and (iii) the only realizations $[(a, 0)]_t$ are normal in \mathfrak{H}_2^t , whenever $t \in \mathbb{R} \setminus \{\pm 1\}$, by (5.32).

Theorem 5.18. *Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22).*

- (i) *Suppose $t = -1$. Then T is normal in \mathfrak{H}_2^{-1} , and its free distribution is characterized by the formula (5.20).*
- (ii) *Let $t \in \mathbb{R} \setminus \{\pm 1\}$. If T is “non-zero” normal in \mathfrak{H}_2^t , then*

$$\tau \left(\prod_{l=1}^n T^{r_l} \right) = R^n \operatorname{Re} \left(W_o^{\sum_{l=1}^n e_l} \right),$$

with

$$R = |a| \text{ and } W_o = \frac{a}{|a|} \in \mathbb{T}, \tag{5.33}$$

where

$$e_l = \begin{cases} 1 & \text{if } r_l = 1, \\ -1 & \text{if } r_l = *, \end{cases}$$

for $l = 1, \dots, n$, for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. The statement (i) holds by (5.20).

Of course, if $t < 0$, and if $T \in \mathcal{H}_2^t$, then the free-distributional data (5.33) holds by (5.20), because T and the t -spectral form S are similar in \mathcal{H}_2^t as elements of (\mathfrak{H}_2^t, τ) . However, in the statement (ii), the normality works for all the scales $t \in \mathbb{R} \setminus \{\pm 1\}$. Assume that the realization T is a “non-zero”, “normal” element of \mathfrak{H}_2^t . Then

$$(a, b) = (a, 0) \in \mathbb{H}_t, \text{ with } a \neq 0,$$

by (5.32). Therefore,

$$T = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} = S,$$

because $\sigma_t(a, 0) = a$ in \mathbb{C} , i.e., the realization T and the t -spectral form S are identical in \mathfrak{H}_2^t , implying the similarity of them. So, under (5.22),

$$a = w \stackrel{\text{denote}}{=} \sigma_t(a, 0),$$

polar-decomposed to be

$$w = a = |a| \begin{pmatrix} a \\ |a| \end{pmatrix} \in \mathbb{C},$$

i.e., $r = |a|$ and $w_o = \frac{a}{|a|}$ under (5.22). Therefore, similar to (5.20), the free-distributional data (5.33) holds. \square

Note that, in the proof of the statement (ii) of Theorem 5.18, we did not use our negative-scale assumption for the cases where $t < 0$, but $t \neq -1$. Indeed, even though $t \geq 0$, but $t \neq 1$, the normality (5.32) shows that the realization T is a diagonal matrix not affected by the scale t . So, whatever scales t are given in $\mathbb{R} \setminus \{\pm 1\}$, the free-distributional data (5.33) holds in (\mathfrak{H}_2^t, τ) , under normality. Then, how about the case where $t = 1$? Recall that if $t = 1$, then the realization T of $(a, b) \in \mathbb{H}_1$ is normal in \mathfrak{H}_2^1 , if and only if either

$$(a, b) = (\operatorname{Re}(a), b), \text{ if } b \neq 0,$$

or

$$(a, b) = (a, 0), \text{ if } b = 0,$$

in \mathbb{H}_1 , by (5.31). So, if $(a, b) = (a, 0)$ in \mathbb{H}_1 , the joint free moments of T are determined similarly by the formula (5.33), by the identity (and hence, the similarity) of T and S (under (5.22)). However, if $(a, b) = (\operatorname{Re}(a), b)$ with $b \neq 0$, then we need a better tool than (5.10) to compute the corresponding free-distributional data, because we cannot use our similarity technique (of Theorem 5.6) here.

By the definition of the unitarity, if an element U of a C^* -algebra \mathcal{A} is a unitary, then it is automatically normal, i.e., the unitarity implies the normality. Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22) with its realization $T \in \mathcal{H}_2^t$ in (\mathfrak{H}_2^t, τ) , and suppose it is a unitary in \mathfrak{H}_2^t . By the assumption that T is a unitary in \mathfrak{H}_2^t , it is normal.

Assume first that $t = -1$ in \mathbb{R} , and hence, $(a, b) \in \mathbb{H}_{-1}$ is a quaternion. Then the realization T is automatically normal in \mathfrak{H}_2^{-1} by Theorem 5.18(i). Indeed, in this case,

$$T = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix} \quad \text{with} \quad T^* = \begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix} = [(\bar{a}, -b)]_{-1},$$

in \mathcal{H}_2^{-1} , as elements of \mathfrak{H}_2^{-1} . So, the normality is guaranteed;

$$T^*T = \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix} = TT^*,$$

in \mathcal{H}_2^{-1} , as elements of \mathfrak{H}_2^{-1} . It shows that T is a unitary in \mathfrak{H}_2^{-1} , if and only if

$$|a|^2 + |b|^2 = 1. \tag{5.34}$$

Meanwhile, if $t \in \mathbb{R} \setminus \{\pm 1\}$ in \mathbb{R} , then T is normal, if and only if $(a, b) = (a, 0)$ in \mathbb{H}_t by (5.32), if and only if

$$T = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \in \mathcal{H}_2^t,$$

which is identical (and hence, similar) to the t -spectral form S of $(a, 0)$ in \mathfrak{H}_2^t . This normal element T is a unitary in \mathfrak{H}_2^t , if and only if

$$T^*T = I_2 = TT^* \iff \begin{pmatrix} |a|^2 & 0 \\ 0 & |a|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

if and only if

$$|a|^2 = 1 \text{ in } \mathbb{C}. \tag{5.35}$$

Proposition 5.19. *Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22).*

- (i) *Let $t = -1$. Then T is a unitary in \mathfrak{H}_2^t , if and only if $|a|^2 + |b|^2 = 1$.*
- (ii) *Let $t \in \mathbb{R} \setminus \{\pm 1\}$. Then T is a unitary in \mathfrak{H}_2^t , if and only if $|a|^2 = 1$ and $b = 0$.*

Proof. The statements (i) and (ii) hold by (5.34) and (5.35), respectively, because a unitary realization T of (a, b) automatically satisfies the normality (5.30). \square

Observation 5.20. *Now, assume that $t = 1$, and let $(a, b) \in \mathbb{H}_1$ be a bicomplex number satisfying (5.22). By (5.31), the realization $T \in \mathcal{H}_2^1$ is normal in \mathfrak{H}_2^1 , if and only if either*

$$(a, b) = (\operatorname{Re}(a), b), \text{ or } (a, b) = (a, 0),$$

in \mathbb{H}_1 . So, if $(a, b) = (a, 0)$ in \mathbb{H}_1 , then one obtains the unitarity that: T is a unitary in \mathfrak{H}_2^1 , if and only if $|a|^2 = 1$, just like Proposition 5.19 (ii). However, if

$$(a, b) = (\operatorname{Re}(a), b) = (x, b) \text{ in } \mathbb{H}_1,$$

with $b \neq 0$ in \mathbb{C} , then T is a unitary in \mathfrak{H}_2^1 , if and only if

$$\begin{pmatrix} x & \bar{b} \\ b & x \end{pmatrix} \begin{pmatrix} x & b \\ \bar{b} & x \end{pmatrix} = \begin{pmatrix} x^2 + \bar{b}^2 & 2x\operatorname{Re}(b) \\ 2x\operatorname{Re}(b) & x^2 + b^2 \end{pmatrix} = I_2,$$

and

$$\begin{pmatrix} x & b \\ \bar{b} & x \end{pmatrix} \begin{pmatrix} x & \bar{b} \\ b & x \end{pmatrix} = \begin{pmatrix} x^2 + b^2 & 2x\operatorname{Re}(b) \\ 2x\operatorname{Re}(b) & x^2 + \bar{b}^2 \end{pmatrix} = I_2,$$

in \mathfrak{H}_2^1 , if and only if

$$x^2 + \bar{b}^2 = x^2 + b^2 = 1 \quad \text{and} \quad 2x\operatorname{Re}(b) = 0,$$

if and only if

$$b^2 = \bar{b}^2 = 1 - x^2 \quad \text{and} \quad 2x\operatorname{Re}(b) = 0,$$

if and only if

$$b^2 = 1 - x^2 \in \mathbb{R} \quad \text{and} \quad x = 0,$$

because b is assumed not to be zero in \mathbb{C} , if and only if

$$x = 0 \quad \text{and} \quad b = \pm 1 \quad \text{in } \mathbb{R},$$

if and only if

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ or } T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \text{ in } \mathcal{H}_2^1,$$

if and only if

$$(a, b) = (0, 1), \text{ or } (a, b) = (0, -1) \text{ in } \mathbb{H}_1.$$

i.e., if $(a, b) = (\operatorname{Re}(a), b)$ in \mathbb{H}_1 , then T is a unitary in \mathfrak{H}_2^1 , if and only if

$$(a, b) = (0, 1), \text{ or } (a, b) = (0, -1),$$

in \mathbb{H}_1 . In summary, the realization $T \in \mathcal{H}_2^1$ of a bicomplex number $(a, b) \in \mathbb{H}_1$ is a unitary in \mathfrak{H}_2^t , if and only if either

$$(a, b) = (a, 0) \text{ with } |a|^2 = 1,$$

or

$$(a, b) = (0, 1), \text{ or } (a, b) = (0, -1),$$

in \mathbb{H}_1 .

By Proposition 5.19, one has the following result.

Theorem 5.21. *Let $(a, b) \in \mathbb{H}_t$ satisfy (5.22).*

- (i) *Suppose $t = -1$. If T is a unitary in \mathfrak{H}_2^t , then its free distribution is characterized by the formula (5.20) with $r = 1$.*
- (ii) *Let $t \in \mathbb{R} \setminus \{\pm 1\}$. If T is a unitary in \mathfrak{H}_2^t , then*

$$\tau \left(\prod_{l=1}^n T^{r_l} \right) = \text{Re} \left(a^{\sum_{l=1}^n e_l} \right), \text{ with } a \in \mathbb{T} \text{ in } \mathbb{C},$$

where

$$e_l = \begin{cases} 1 & \text{if } r_l = 1, \\ -1 & \text{if } r_l = *, \end{cases} \tag{5.36}$$

for $l = 1, \dots, n$, for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. The statement (i) holds by (5.20). In particular, by the unitarity characterization in Proposition 5.19(i), the free-distributional data in (5.20) must have $r = 1$, since

$$|\sigma_t(a, b)| = |w| \stackrel{\text{denote}}{=} r = 1,$$

under the similarity of T and S , by Proposition 5.19(i).

By Theorem 5.18(ii), if $t \neq \pm 1$, then the free-distributional data (5.36) holds by (5.33). Indeed, under the unitarity of T , the formula (5.33) satisfies

$$R = |a| = 1 \text{ and } W_o = a \in \mathbb{T}.$$

Therefore, the joint free moments (5.36) holds. □

The above theorem characterizes the free distributions of unitary elements of (\mathfrak{H}_2^t, τ) induced by \mathbb{H}_t , where $t \in \mathbb{R} \setminus \{1\}$.

Suppose $t = 1$, and $(a, b) \in \mathbb{H}_1$ satisfies (5.22). In the above observation, we showed that the realization $T \in \mathcal{H}_2^1$ of (a, b) is a unitary, if and only if either

$$(a, b) = (a, 0) \text{ with } a \in \mathbb{T},$$

or

$$(a, b) = (0, 1), \text{ or } (a, b) = (0, -1),$$

in \mathbb{H}_1 , equivalently, either

$$T = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \text{ with } a \in \mathbb{T},$$

or

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ or } T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

in \mathcal{H}_2^1 (as an element of \mathfrak{H}_2^1). Thus, if $(a, b) = (a, 0) \in \mathbb{H}_1$ with $|a|^2 = 1$, then the free distribution of T is similarly characterized by the formula (5.36). Meanwhile, if $T = [(0, 1)]_1$, then

$$T^* = T \in \mathcal{H}_2^1 \subset \mathcal{H}_2^1(1, *) \text{ in } \mathfrak{H}_2^1,$$

and

$$T^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

in \mathfrak{H}_2^1 , satisfying that

$$(T^n)_{n=1}^\infty = (T, I_2, T, I_2, T, I_2, \dots); \tag{5.37}$$

and, if $T = [(0, -1)]_1$, then

$$T^* = T \in \mathcal{H}_2^1 \subset \mathcal{H}_2^1(1, *) \text{ in } \mathfrak{H}_2^1,$$

and

$$T^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

in \mathfrak{H}_2^1 , satisfying that

$$(T^n)_{n=1}^\infty = (T, I_2, T, I_2, T, I_2, \dots). \tag{5.38}$$

Therefore, one obtains the following result in addition with Theorem 5.21.

Theorem 5.22. *Let $(a, b) \in \mathbb{H}_1$ be a bicomplex number satisfying (5.22). Then the realization T is a unitary in (\mathfrak{H}_2^1, τ) , if and only if either*

$$(a, b) = (a, 0), \text{ with } |a|^2 = 1,$$

or

$$(a, b) = (0, 1), \text{ or } (a, b) = (0, -1) \text{ in } \mathbb{H}_1. \tag{5.39}$$

- (i) *If $(a, b) = (a, 0)$, with $|a|^2 = 1$, in \mathbb{H}_1 , then the free distribution of T is characterized by the formula (5.36).*
- (ii) *If either $(a, b) = (0, 1)$, or $(a, b) = (0, -1)$ in \mathbb{H}_1 , then the free distribution of the unitary realization T is fully characterized by the free-moment sequence,*

$$(\tau(T^n))_{n=1}^\infty = (0, 1, 0, 1, 0, 1, 0, 1, \dots). \tag{5.40}$$

Proof. By Observation 5.20, it is shown that the realization $T \in \mathcal{H}_2^1$ of a bicomplex number $(a, b) \in \mathbb{H}_1$ is a unitary in \mathfrak{H}_2^1 , if and only if the condition (5.39) holds true.

The statement (i) is shown similarly by the proof of the statement Theorem 5.21(ii). So, the free-distributional data (5.36) holds.

Now, if either $T = [(0, 1)]_1$, or $T = [(0, -1)]_1$ in \mathcal{H}_2^1 , it is not only a unitary, but also a self-adjoint element of (\mathfrak{H}_2^1, τ) , and hence, the free distribution of T is fully characterized by the free-moment sequence $(\tau(T^n))_{n=1}^\infty$. However, by (5.37) and (5.38), one immediately obtain the free-moment sequence (5.40). Therefore, the statement (ii) holds. \square

The above theorem fully characterizes the free distributions of the unitaries of (\mathfrak{H}_2^1, τ) induced by bicomplex numbers of \mathbb{H}_1 .


REFERENCES

- [1] D. Alpay, M.E. Luna-Elizarrarás, M. Shapiro, D. Struppa, *Gleason's problem, rational functions and spaces of left-regular functions: The split-quaternion setting*, Isr. J. Math. **226** (2018), 319–349.
- [2] I. Cho, P.E.T. Jorgensen, *Spectral analysis of equations over quaternions*, Conference Proceeding for International Conference on Stochastic Processes and Algebraic Structures from Theory towards Applications (SPAS 2019), Vastras, Sweden (2021).
- [3] I. Cho, P.E.T. Jorgensen, *Multi-variable quaternionic spectral analysis*, Opuscula Math. **41** (2021), no. 3, 335–379.
- [4] C. Doran, A. Lasenby, *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge, 2003.
- [5] F.O. Farid, Q.-W. Wang, F. Zhang, *On the eigenvalues of quaternion matrices*, Linear Multilinear Algebra **59** (2011), no. 4, 451–473.
- [6] C. Flaut, *Eigenvalues and eigenvectors for the quaternion matrices of degree two*, An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. **10** (2002), no. 2, 39–44.
- [7] P.R. Girard, *Einstein's equations and Clifford algebra*, Adv. Appl. Clifford Algebras **9** (1999), no. 2, 225–230.
- [8] P.R. Halmos, *A Hilbert Space Problem Book*, Graduate Texts in Mathematics, vol. 19, Springer-Verlag, New York-Berlin, 1982.
- [9] P.R. Halmos, *Linear Algebra Problem Book*, The Dolciani Mathematical Expositions, vol. 16, Mathematical Association of America, Washington, DC, 1995.
- [10] W.R. Hamilton, *Lectures on Quaternions*, Cambridge Univ. Press., 1853.
- [11] I.L. Kantor, A.S. Solodnikov, *Hypercomplex Numbers: An Elementary Introduction to Algebras*, Springer, Berlin, 1989.
- [12] V. Kravchenko, *Applied Quaternionic Analysis*, Research and Exposition in Mathematics, vol. 28, Heldermann Verlag, Lemgo, 2003.

- [13] S.D. Leo, G. Sclarici, L. Solombrino, *Quaternionic eigenvalue problem*, J. Math. Phys. **43** (2002), no. 11, 5815–5829.
- [14] T.S. Li, *Eigenvalues and eigenvectors of quaternion matrices*, J. Central China Normal Univ. Natur. Sci. **29** (1995), no. 4, 407–411.
- [15] N. Mackey, *Hamilton and Jacobi meet again: quaternions and the eigenvalue problem*, SIAM J. Matrix Anal. Appl. **16** (1995), no. 2, 421–435.
- [16] S. Qaisar, L. Zou, *Distribution for the standard eigenvalues of quaternion matrices*, Int. Math. Forum **7** (2012), no. 17–20, 831–838.
- [17] L. Rodman, *Topics in Quaternion Linear Algebra*, Princeton University Press, Princeton, NJ, 2014.
- [18] B.A. Rozenfeld, *A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space*, Studies in the History of Mathematics and Physical Sciences, vol. 12, Springer, 1988.
- [19] R. Speicher, *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, Mem. Amer. Math. Soc. **132** (1998), no. 627.
- [20] A. Sudbery, *Quaternionic analysis*, Math. Proc. Cambridge Philos. Soc. **85** (1979), no. 2, 199–224.
- [21] J. Vince, *Geometric Algebra for Computer Graphics*, Springer-Verlag London, Ltd., London, 2008.
- [22] D.V. Voiculescu, K.J. Dykema, A. Nica, *Free Random Variables*, American Mathematical Society, Providence, RI, 1992.
- [23] J. Voight, *Quaternion Algebra*, Graduate Texts in Mathematics, 288. Springer, Cham, 2021.

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