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A NOTE ON HAUSDORFF CONVERGENCE OF PSEUDOSPECTRA

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Abstract. For a bounded linear operator on a Banach space, we study approximation of the spectrum and pseudospectra in the Hausdorff distance. We give sufficient and necessary conditions in terms of pointwise convergence of appropriate spectral quantities.

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1. INTRODUCTION

Given a bounded linear operator A on a Banach space X, we denote its *spectrum* and *pseudospectra* [14], respectively, by

$$\operatorname{spec} A := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$$

and

$$\operatorname{spec}_\varepsilon A := \{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| > \tfrac{1}{\varepsilon}\}, \quad \varepsilon > 0, \tag{1.1}$$

where we identify $\|B^{-1}\| := \infty > \frac{1}{\varepsilon}$ if B is not invertible, so that $\operatorname{spec} A \subseteq \operatorname{spec}_\varepsilon A$ for all $\varepsilon > 0$.

Fairly convenient access to the norm of the inverse is given by the so-called *lower norm*, the number

$$\nu(A) := \inf_{\|x\|=1} \|Ax\|. \tag{1.2}$$

Indeed, putting $\mu(A) := \min\{\nu(A), \nu(A^*)\}$, we have

$$\|A^{-1}\| = 1/\mu(A), \tag{1.3}$$

where A^* is the adjoint on the dual space X^* and equation (1.3) takes the form $\infty = 1/0$ if and only if A is not invertible.

One big advantage of this approach is that, in case $X = \ell^p(\mathbb{Z}^d, Y)$ with $p \in [1,\infty]$, $d \in \mathbb{N}$ and a Banach space Y, $\nu(A)$ can be approximated by the same infimum (1.2) with $x \in X$ restricted to elements with finite support of given diameter D. We can even quantify the approximation error against D, see [1] and [10] (as well as [8] for a corresponding result on the norm).

By means of (1.3), we can rewrite spectrum and pseudospectra as follows:

$$\operatorname{spec} A = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\}$$

and

$$\operatorname{spec}_\varepsilon A = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) < \varepsilon\}, \quad \varepsilon > 0.$$

In other words, $\operatorname{spec} A$ is the level set of the function $f : \mathbb{C} \to [0,\infty)$ with

$$f(\lambda) := \mu(A - \lambda I) \tag{1.4}$$

for the level zero, and $\operatorname{spec}_\varepsilon A$ is the sublevel set of f for the level $\varepsilon > 0$.

For a function $g : \mathbb{C} \to [0,\infty)$ and $\varepsilon > 0$, let

$$\operatorname{sub}_\varepsilon(g) := \{\lambda \in \mathbb{C} : g(\lambda) < \varepsilon\}$$

denote the *sublevel set of g for the level ε.*

In general, pointwise convergence $g_n \to g$ of functions $\mathbb{C} \to [0,\infty)$ need not coincide with Hausdorff convergence of their sublevel sets:

Example 1.1. Suppose we have g and g_n such that a) $g_n \to g$ as well as b) $\operatorname{sub}_\varepsilon(g_n) \overset{\mathrm{H}}{\to} \operatorname{sub}_\varepsilon(g)$ hold for all $\varepsilon > 0$. Increasing $g(\lambda)$ to a certain level $\varepsilon > 0$ in a point λ, where g was continuous and below ε before, changes the state of a), while it does not affect $\operatorname{clos}(\operatorname{sub}_\varepsilon(g))$ and hence b).

So let us look at continuous examples from here on.

Example 1.2. For $g_n(\lambda) := \frac{|\lambda|}{n} \to 0 =: g(\lambda)$, the Hausdorff distance of the sublevel sets

$$\operatorname{sub}_\varepsilon(g_n) = n\varepsilon\mathbb{D} \quad \text{and} \quad \operatorname{sub}_\varepsilon(g) = \mathbb{C}$$

remains infinite, where $\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$.

Of course, this problem was due to the unboundedness of $\operatorname{sub}_\varepsilon(g)$. So let us further focus on functions that go to infinity at infinity, so that all sublevel sets are bounded.

Example 1.3 (locally constant)**.** Let $g(\lambda) := h(|\lambda|)$ and $g_n(\lambda) := h_n(|\lambda|)$ for $n \in \mathbb{N}$, where

$$h(x) := \max\{\min\{|x|, 1\}, |x| - 1\}, \quad h_1(x) := \tfrac{1}{4}x^2, \quad h_n := h + \tfrac{1}{n}(h_1 - h) \to h$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, unlike any g_n, g is locally constant in $2\mathbb{D} \setminus \mathbb{D}$. Consequently,

$$\operatorname{sub}_1(g_n) \equiv 2\mathbb{D} \overset{\mathrm{H}}{\not\to} \mathbb{D} = \operatorname{sub}_1(g) \quad \text{but} \quad g_n \to g.$$

Example 1.4 (increasingly oscillating)**.** Let $g_n(\lambda) := h_n(|\lambda|)$ for $n \in \mathbb{N}$, where

$$h_n(x) := \begin{cases} |\sin(n\pi x)|, & x \in [0,1], \\ x-1, & x > 1. \end{cases}$$

Then $h_n(x) < \varepsilon$ for all $\varepsilon > 0$ and

$$x \in \tfrac{1}{n}\mathbb{Z} \cap [0,1] \overset{\mathrm{H}}{\to} [0,1] \quad \text{as} \quad n \to \infty.$$

It follows that $\mathrm{sub}_\varepsilon(g_n) \overset{\mathrm{H}}{\to} (1+\varepsilon)\mathbb{D}$ for all $\varepsilon > 0$, while g_n does not converge pointwise at all.

For a sequence of bounded operators A_n on X and their corresponding functions $f_n : \mathbb{C} \to [0,\infty)$ with

$$f_n(\lambda) := \mu(A_n - \lambda I), \quad n \in \mathbb{N}, \tag{1.5}$$

we show equivalence of pointwise convergence $f_n \to f$ and Hausdorff convergence of their sublevel sets, i.e. of the corresponding pseudospectra,

$$f_n \to f \quad \iff \quad \forall \varepsilon > 0 : \mathrm{spec}_\varepsilon A_n \overset{\mathrm{H}}{\to} \mathrm{spec}_\varepsilon A.$$

This result is not surprising (and similar arguments have been used e.g. in [2] in a more specific situation) but there are some little details that deserve to be written down as this separate note.

In [5], the approximation of the lower norm of $H(b) - \lambda I$ for a (generalized) discrete Schrödinger operator $H(b)$ and $\lambda \in \mathbb{C}$, is established via successive exhaustion of the set of finite subwords of the potential $b \in \ell^\infty(\mathbb{Z})$. Together with our paper here, this yields Hausdorff approximation of the pseudospectrum of $H(b)$, also see [9].

2. LIPSCHITZ CONTINUITY AND NON-CONSTANCY OF μ

Our functions ν and μ, and hence f and f_n, have two properties that rule out effects as in Examples 1.1–1.4: Lipschitz continuity and the fact that their level sets have no interior points, i.e. μ is not constant on any open set.

The first property is straightforward, but the latter is a very nontrivial subject [3, 6, 11–13], and it actually limits the choice of our Banach space X as shown in Lemma 2.2.

Lemma 2.1. *For all bounded operators B, C on X, one has*

$$|\nu(B) - \nu(C)| \leq \|B - C\|,$$

so that also $\mu(B) = \min\{\nu(B), \nu(B^*)\}$ is Lipschitz continuous with Lipschitz constant 1*. The same follows for the functions f from* (1.4) *and f_n with $n \in \mathbb{N}$ from* (1.5)*.*

This result is absolutely standard but we give the (short) proof, for the reader's convenience.

Proof of Lemma 2.1. For all $x \in X$ with $\|x\| = 1$, one has

$$\|B - C\| \geq \|Bx - Cx\| \geq \|Bx\| - \|Cx\| \geq \nu(B) - \|Cx\|.$$

Now pass to the infimum in $\|Cx\|$ to get $\|B - C\| \geq \nu(B) - \nu(C)$. Finally, swap B and C. □

It will become crucial to understand when the resolvent norm of a bounded operator cannot be constant on an open subset. This is a surprisingly rich and deep problem. As it turns out it is connected to a geometric property, the complex uniform convexity, of the underlying Banach space (see [11, Definition 2.4 (ii)]).

Lemma 2.2 (Globevnik [6], Shargorodsky *et al.* [3, 11–13])**.** *Let X be a Banach space which satisfies at least one of the following properties:*

(a) $\dim(X) < \infty$,
(b) *X is complex uniform convex,*
(c) *its dual X^* is complex uniform convex.*

Then, for every bounded operator A on X, the resolvent norm,

$$\lambda \mapsto \|(A - \lambda I)^{-1}\| = 1/\mu(A - \lambda I),$$

cannot be locally constant on any open set in $\mathbb{C}$, and, consequently,

$$\forall \varepsilon > 0 : \operatorname{clos}(\operatorname{spec}_\varepsilon A) = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) \leq \varepsilon\}.$$

For example, every Hilbert space is subject to the condition (b) above, and every space $X = \ell^p(\mathbb{Z}^d, Y)$ with $p \in [1, \infty]$ and $d \in \mathbb{N}$ falls into this category as soon as Y does [4].

3. SET SEQUENCES AND HAUSDORFF CONVERGENCE

Let (S_n) be a sequence of bounded sets in $\mathbb{C}$ and recall the following notations (see e.g. [7, §3.1.2]):

$$\liminf S_n = \text{ the set of all limits of sequences } (s_n) \text{ with } s_n \in S_n,$$
$$\limsup S_n = \text{ the set of all partial limits of sequences } (s_n) \text{ with } s_n \in S_n.$$

Both limiting sets are closed. Moreover, let us write $S_n \to S$ if

$$\liminf S_n = \limsup S_n = S.$$

It holds that $S_n \to S$ if and only if $\operatorname{clos} S_n \to S$, where, as we know, automatically $S = \operatorname{clos} S$ holds.

Here is an apparently different approach to set convergence: For $z \in \mathbb{C}$ and $S \subseteq \mathbb{C}$, set $\operatorname{dist}(z, S) := \inf_{s \in S} |z - s|$. The *Hausdorff distance* of two bounded sets $S, T \subseteq \mathbb{C}$ is defined via

$$d_{\mathrm{H}}(S, T) := \max\left\{\sup_{s \in S} \operatorname{dist}(s, T), \sup_{t \in T} \operatorname{dist}(t, S)\right\}.$$

Although d_H is a metric on the set of all compact subsets of $\mathbb{C}$, it is just a pseudometric on the set of all bounded subsets of $\mathbb{C}$: it enjoys symmetry and triangle inequality, but not definiteness. Indeed, one has $d_H(S,T) = 0$ if and only if $\operatorname{clos} S = \operatorname{clos} T$ since

$$d_H(S,T) = d_H(\operatorname{clos} S, T) = d_H(S, \operatorname{clos} T) = d_H(\operatorname{clos} S, \operatorname{clos} T).$$

Let us still write $S_n \stackrel{H}{\to} S$ if $d_H(S_n, S) \to 0$, also for merely bounded sets S_n, S, knowing that the limit S in $S_n \stackrel{H}{\to} S$ is not unique: one has $S_n \stackrel{H}{\to} S$ and $S_n \stackrel{H}{\to} T$ if and only if $d_H(S,T) = 0$, i.e. $\operatorname{clos} S = \operatorname{clos} T$.

Both notions of set convergence are connected, via the Hausdorff theorem:

$$S_n \stackrel{H}{\to} S \quad \Longleftrightarrow \quad S_n \to \operatorname{clos} S. \tag{3.1}$$

Lemma 3.1. *Let S_n and T_n be bounded subsets of $\mathbb{C}$ with $S_n \to S$ and $T_n \to T$. In addition, suppose $S_n \setminus T_n \neq \varnothing$. Then:*

(a) *in general, it does not follow that*

$$S_n \setminus T_n \to S \setminus T,$$

(b) *however, it always holds that*

$$\liminf (S_n \setminus T_n) \supseteq S \setminus T.$$

Proof. (a) Consider

$$S_n := [0,1] \to [0,1] =: S \quad \text{and} \quad T_n := \frac{1}{n}\mathbb{Z} \cap [0,1] \to [0,1] =: T.$$

Then

$$S_n \setminus T_n \to [0,1] \neq \varnothing = S \setminus T.$$

(b) Let $x \in S \setminus T$. Since $x \in S$, there is a sequence (x_n) with $x_n \in S_n$ such that $x_n \to x$. We show that $x_n \notin T_n$, eventually. Suppose $x_n \in T_n$ for infinitely many $n \in \mathbb{N}$. Then there is a strictly monotonic sequence (n_k) in $\mathbb{N}$ with $x_{n_k} \in T_{n_k}$. But then

$$x = \lim_n x_n = \lim_k x_{n_k} \in \limsup_n T_n = T,$$

which contradicts $x \in S \setminus T$. Consequently, just finitely many elements of the sequence (x_n) can be in T_n. Replacing these by elements from $S_n \setminus T_n$ does not change the limit x. So $x \in \liminf(S_n \setminus T_n)$. □

4. EQUIVALENCE OF POINTWISE CONVERGENCE $f_n \to f$ AND HAUSDORFF CONVERGENCE OF THE PSEUDOSPECTRA

Here is our main theorem. Note that we do not require any convergence of A_n to A.

Theorem 4.1. *Let X be a Banach space with the properties from Lemma 2.2 and let A and $A_n, n \in \mathbb{N}$, be bounded linear operators on X. Then the following are equivalent for the functions and sets introduced in* (1.1), (1.4) *and* (1.5)*:*

(i) *$f_n \to f$ pointwise,*
(ii) *for all $\varepsilon > 0$, one has $\operatorname{spec}_\varepsilon A_n \xrightarrow{\text{H}} \operatorname{spec}_\varepsilon A$.*

Proof. (i)⇒(ii). Assume (i) and take $\varepsilon > 0$. For $f(\lambda) < \varepsilon$, (i) implies $f_n(\lambda) < \varepsilon$ for sufficiently large n. So it follows that

$$\operatorname{spec}_\varepsilon A \subseteq \liminf \operatorname{spec}_\varepsilon A_n \subseteq \limsup \operatorname{spec}_\varepsilon A_n.$$

Now let $\lambda \in \limsup \operatorname{spec}_\varepsilon A_n$, i.e. $\lambda = \lim \lambda_{n_k}$ with $\lambda_{n_k} \in \operatorname{spec}_\varepsilon A_{n_k}$, so that $f_{n_k}(\lambda_{n_k}) < \varepsilon$. Then

$$|f(\lambda) - f_{n_k}(\lambda_{n_k})| \leq \underbrace{|f(\lambda) - f_{n_k}(\lambda)|}_{\to 0 \text{ by } (i)} + \underbrace{|f_{n_k}(\lambda) - f_{n_k}(\lambda_{n_k})|}_{\leq |\lambda - \lambda_{n_k}| \to 0} \to 0.$$

Consequently, $f(\lambda) \leq \varepsilon$ and hence $\lambda \in \operatorname{clos}(\operatorname{spec}_\varepsilon A)$, by Lemma 2.2. We get

$$\operatorname{spec}_\varepsilon A \subseteq \liminf \operatorname{spec}_\varepsilon A_n \subseteq \limsup \operatorname{spec}_\varepsilon A_n \ \subseteq \operatorname{clos}(\operatorname{spec}_\varepsilon A).$$

Passing to the closure everywhere in this chain of inclusions, just changes $\operatorname{spec}_\varepsilon A$ at the very left into $\operatorname{clos}(\operatorname{spec}_\varepsilon A)$, and we have $\operatorname{spec}_\varepsilon A_n \to \operatorname{clos}(\operatorname{spec}_\varepsilon A)$. Hence, by (3.1), we obtain (ii).

(ii)⇒(i). Take $\lambda \in \mathbb{C}$ and put $\varepsilon := f(\lambda)$.

Case 1. $\varepsilon = 0$. Take an arbitrary $\delta > 0$. Then, by (ii), $\lambda \in \operatorname{spec}_\delta A \xleftarrow{\text{H}} \operatorname{spec}_\delta A_n$. So, by (3.1), there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \in \operatorname{spec}_\delta A_n = f_n^{-1}([0,\delta))$ and $\lambda_n \to \lambda$.

Case 2. $\varepsilon > 0$. Take an arbitrary $\delta \in (0, \varepsilon)$. By (ii), we have

$$S_n := \operatorname{spec}_{\varepsilon+\delta} A_n \xrightarrow{\text{H}} \operatorname{spec}_{\varepsilon+\delta} A =: S, \quad \text{i.e. } S_n \to \operatorname{clos} S, \text{ by (3.1)},$$

and

$$T_n := \operatorname{spec}_{\varepsilon-\delta} A_n \xrightarrow{\text{H}} \operatorname{spec}_{\varepsilon-\delta} A =: T, \quad \text{i.e. } T_n \to \operatorname{clos} T, \text{ by (3.1)}.$$

By Lemma 3.1 (b),

$$\lambda \in \operatorname{clos}(\operatorname{spec}_{\varepsilon+\delta} A) \setminus \operatorname{clos}(\operatorname{spec}_{\varepsilon-\delta} A) \subseteq \liminf \Big(\operatorname{spec}_{\varepsilon+\delta} A_n \setminus \ \operatorname{spec}_{\varepsilon-\delta} A_n\Big),$$

in short:

$$\lambda \in f^{-1}\big((\varepsilon - \delta, \varepsilon + \delta]\big) \subseteq \liminf f_n^{-1}\big([\varepsilon - \delta, \varepsilon + \delta)\big).$$

So there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \in f_n^{-1}\big([\varepsilon - \delta, \varepsilon + \delta)\big)$ and $\lambda_n \to \lambda$.

In both cases, we conclude

$$|f(\lambda) - f_n(\lambda)| \le \underbrace{|\overbrace{f(\lambda)}^{\varepsilon} - f_n(\lambda_n)|}_{\le \delta} + \underbrace{|f_n(\lambda_n) - f_n(\lambda)|}_{\le |\lambda_n - \lambda| \to 0} < 2\delta$$

for all sufficiently large n, and hence $f_n(\lambda) \to f(\lambda)$ as $n \to \infty$, i.e. (i) holds. □

Corollary 4.2. *Let the assumptions of Theorem 4.1 be satisfied. If X is a Hilbert space and the operators A and A_n, $n \in \mathbb{N}$, are normal then* (i) *and* (ii) *are also equivalent to*

(iii) $\operatorname{spec} A_n \xrightarrow{\mathrm{H}} \operatorname{spec} A$.

Proof. For normal operators, the ε-pseudospectrum is exactly the ε-neighborhood of the spectrum, e.g. [14]. But $B_\varepsilon(S_n) \xrightarrow{\mathrm{H}} B_\varepsilon(S)$ for all $\varepsilon > 0$ implies $S_n \xrightarrow{\mathrm{H}} S$. □

Remark 4.3.

(a) The pointwise convergence $f_n \to f$ is uniform on compact subsets of $\mathbb{C}$. (Take an $\frac{\varepsilon}{3}$-net for the compact set and use the uniform Lipschitz continuity of the f_n).

(b) It is well-known [14] that $\operatorname{spec}_\varepsilon A \subseteq r\mathbb{D}$ with $r = \|A\| + \varepsilon$. So if $(A_n)_{n\in\mathbb{N}}$ is a bounded sequence then $\operatorname{spec}_\varepsilon B \subset r\mathbb{D}$ for all $B \in \{A, A_n : n \in \mathbb{N}\}$ with $r = \max\{\|A\|, \sup \|A_n\|\} + \varepsilon$. By (a), the convergence $f_n \to f$ is uniform on $\operatorname{clos}(r\mathbb{D})$.

Remark 4.4. Sometimes (especially in earlier works), pseudospectra are defined in terms of non-strict inequality:

$$\operatorname{Spec}_\varepsilon A := \{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| \ge \tfrac{1}{\varepsilon}\}, \quad \varepsilon > 0.$$

One benefit is to get compact pseudospectra, in which case d_H is a metric and $\xrightarrow{\mathrm{H}}$ has a unique limit. By Lemma 2.2, $\operatorname{Spec}_\varepsilon A = \operatorname{clos}(\operatorname{spec}_\varepsilon A)$ for all $\varepsilon > 0$. But since $S_n \xrightarrow{\mathrm{H}} S$ if and only if $\operatorname{clos}(S_n) \xrightarrow{\mathrm{H}} \operatorname{clos}(S)$, one could add this further equivalent statement to Theorem 4.1:

(iv) $\forall \varepsilon > 0 : \operatorname{Spec}_\varepsilon A_n \xrightarrow{\mathrm{H}} \operatorname{Spec}_\varepsilon A$.

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