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SINGULAR ELLIPTIC PROBLEMS WITH DIRICHLET OR MIXED DIRICHLET–NEUMANN NON-HOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. Let Ω be a C^2 bounded domain in $\mathbb{R}^n$ such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are disjoint closed subsets of $\partial\Omega$, and consider the problem $-\Delta u = g(\cdot, u)$ in Ω, $u = \tau$ on Γ_1, $\frac{\partial u}{\partial \nu} = \eta$ on Γ_2, where $0 \leq \tau \in W^{\frac{1}{2},2}(\Gamma_1)$, $\eta \in (H^1_{0,\Gamma_1}(\Omega))'$, and $g : \Omega \times (0, \infty) \to \mathbb{R}$ is a nonnegative Carathéodory function. Under suitable assumptions on g and η we prove the existence and uniqueness of a positive weak solution of this problem. Our assumptions allow g to be singular at $s = 0$ and also at $x \in S$ for some suitable subsets $S \subset \overline{\Omega}$. The Dirichlet problem $-\Delta u = g(\cdot, u)$ in Ω, $u = \sigma$ on $\partial\Omega$ is also studied in the case when $0 \leq \sigma \in W^{\frac{1}{2},2}(\Omega)$.

Keywords: singular elliptic problems, mixed boundary conditions, weak solutions.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let Ω be a C^2 and bounded domain in $\mathbb{R}^n$ such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, with Γ_1 and Γ_2 disjoint closed subsets of $\partial\Omega$. Our aim in this paper is to state existence and uniqueness results for weak solutions $u \in H^1(\Omega)$ of possibly singular elliptic Dirichlet problems of the form

$$\begin{cases} -\Delta u = g(\cdot, u) & \text{in } \Omega, \\ u = \sigma & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

as well as of possibly singular problems with mixed boundary conditions of the form

$$\begin{cases} -\Delta u = g(\cdot, u) & \text{in } \Omega, \\ u = \tau & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = \eta & \text{on } \Gamma_2, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $g : \Omega \times (0, \infty) \to [0, \infty)$ is a suitable nonnegative Carathéodory function $g(x, s)$ which may be singular at $s = 0$ and at $x \in S$ for some suitable subsets $S \subset \overline{\Omega}$, and, in problem (1.1), $\sigma \in H^{\frac{1}{2}}(\partial\Omega)$, whereas in problem (1.2), $\tau \in H^{\frac{1}{2}}(\Gamma_1)$ and η is a suitable function defined on Γ_2.

Singular elliptic problems appear in the study of many nonlinear physical phenomena: thin films of viscous fluids, chemical catalysis, non-Newtonian fluids, temperature of some electrical conductors, response of a membrane cap under heavy loads, Van der Waal forces, as well as in the study of micro electro-mechanical devices (see, e.g., [8, 12, 18, 20, 21, 30, 34] and the references therein).

In [13], problem (1.1) was studied in the case $\sigma = 0$ (i.e., with homogeneous Dirichlet boundary condition) and there it was proved that if $g \in C^1\big(\overline{\Omega} \times (0, \infty)\big)$ satisfies that $g(x, \cdot)$ is nonincreasing on $(0, \infty)$ for any $x \in \overline{\Omega}$ and $\lim_{s \to 0^+} g(x, s) = \infty$ uniformly on $\overline{\Omega}$, then (1.1) has a unique classical solutions $u \in C^2(\Omega) \cap C\big(\overline{\Omega}\big)$.

In [21, 46], and [45], problem (1.1) was addressed when $\sigma \neq 0$ (non homogeneous Dirichlet boundary condition) obtaining, again in this case, existence and uniqueness of classical solutions when σ is regular enough.

In [11], existence and nonexistence results were obtained for classical solutions of singular bifurcation problems whose model problem is $-\Delta u = u^{-\alpha} + \lambda u^p$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where $\alpha > 0$, $\lambda > 0$, and $p > 1$, and there it was proved that there exists $\lambda^* \in (0, \infty)$ such that for $\lambda < \lambda^*$ there exists at least a solution and for $\lambda > \lambda^*$ no such a solution exists. In [18] it was studied the problem with a parameter $-\Delta u = \lambda f - u^{-\alpha}$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, $u^{-\alpha} \in L^1(\Omega)$, where $\lambda > 0$, $0 < \alpha < 1$, and $0 \leq f \in L^1(\Omega)$. It turns out that the situation is the opposite of that in [11]: there exists $\lambda^* \in (0, \infty)$ such that for $\lambda > \lambda^*$ there exists at least a solution and for $\lambda < \lambda^*$ no such a solution exists.

In [31] it was studied the model problem

$$-\Delta u = k(x)u^{-\alpha} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad u > 0 \text{ in } \Omega. \tag{1.3}$$

There it was proved that if Ω is a $C^{2+\beta}$ bounded domain for some $\beta \in (0, 1)$, and $k \in C^\beta(\overline{\Omega})$ satisfies $\min_{\overline{\Omega}} k > 0$ then, for any $\alpha > 0$, problem (1.3) has a unique classical solution $u \in C^{2+\beta}(\Omega) \cap C(\overline{\Omega})$ which belongs to $C^1(\overline{\Omega})$ if $\alpha < 1$, and belongs to $H_0^1(\Omega)$ if and only if $\alpha < 3$. Moreover, if $\alpha > 1$ then $\frac{1}{c}\varphi_1^{\frac{2}{1+\alpha}} \leq u \leq c\varphi_1^{\frac{2}{1+\alpha}}$ in Ω, where c is a positive constant and φ_1 is a positive eigenfunction corresponding to the first eigenvalue for $-\Delta$ on Ω with homogeneous Dirichlet boundary condition.

After [31], several works studied problem (1.3) under weaker regularity assumptions on k and, in some of them, for more general differential operators than the Laplacian, as well as for more general nonlinearities.

In [15] it was stated the existence and uniqueness of a weak solution $u \in H_0^1(\Omega)$ of problem (1.3) in the case when k is a nonnegative and nonidentically zero function in $L^\infty(\Omega)$, and, for such a u, a global bound for ∇u was obtained. Let us mention some of them.

In [47] it was proved, among other results, that if $\alpha > 1$, $k \in L^1(\Omega)$ and $k > 0$ a.e. in Ω, then (1.3) has a weak solution $u \in H_0^1(\Omega)$ if and only if there exists $u_0 \in H_0^1(\Omega)$

such that $\int_\Omega k u_0^{1-\alpha} < \infty$. These results were extended in [32] to the case where the Laplacian is replaced by the p-Laplacian operator.

Singular problems for differential operators (including the p-Laplacian) more general than the Laplacian and/or with more general nonlinearities were also studied in [2, 22, 32, 37, 39, 48] and [40].

Singular problems on punctured domains were studied in [3]. The paper [9] addressed problem (1.1) in the case where $\alpha = \alpha(x)$ (variable exponent). In [5], [28, 33] and [17] it was studied the existence of solutions (either classical or weak or very weak) of (1.3) in the case where k behaves like $(\operatorname{dist}(\cdot, \partial\Omega))^{-\beta}$ for some $\beta > 0$, and in [36] it was considered the case where k is either a nonnegative function in $L^1(\Omega)$ or a bounded Radon measure on Ω.

In [44] existence and nonexistence results were given for the problem with a parameter $-\Delta u = k(x)u^{-\alpha} + \lambda u^p$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω in the case where $\alpha, p \in (0,1)$, and k may change sign.

Existence results for classical solutions of Lane–Emden–Fowler equations with convection and singular potential were obtained in [19], and related problems were studied in [10, 25] and [4].

Let us mention also that in [30] it was studied the existence of positive classical solutions of the one-dimensional singular problem

$$-u''(t) = f(t)u^{-\beta}(t) + h(t) \quad \text{on } (0,1), \tag{1.4}$$

where $\beta > 0$, f and h belong to $C(0,1)$, $f > 0$ in $(0,1)$, and

$$\int_0^1 t(1-t)(f(t) + |h(t)|) < \infty,$$

and with u such that one of the following boundary conditions holds:

$$u(0) = a, \quad u(1) = b, \tag{1.5}$$

$$u(0) = a, \quad u'(1) = c. \tag{1.6}$$

In [30, Theorem 1.1] it was proved that, if $a \geq 0$ and $b \geq 0$, then problem (1.4), with boundary conditions (1.5), has a unique classical solution; and in [30, Theorem 1.2] it was proved that problem (1.4), with boundary conditions (1.6), has a unique positive solution if $c > c_0 := \inf\{u'_\xi(1) : \xi > 0\}$, and has no positive solution if $c < c_0$, where, for $\xi > 0$, u_ξ is the solution, provided by [30, Theorem 1.1], of problem (1.4) with boundary conditions $u(0) = a$, $u(1) = \xi$.

The interested reader will find an updated account, concerning the topic of singular elliptic Dirichlet problems, as well as additional references, in the research books [23, 24, 41]. See also [16].

As said before, we are interested in the existence and uniqueness of weak solutions of problems (1.1) and (1.2). More specifically, our interest is to obtain a sort of n-dimensional analogous of the above quoted (Theorems 1.1 and 1.2 of [30]).

For a function $u \in H^1(\Omega)$ the value of u on $\partial\Omega$ (or on Γ_1, or on Γ_2) will be always understood in the sense of the trace. Let us present the notion of weak solutions of Dirichlet problems we use.

Definition 1.1. Let $f : \Omega \to \mathbb{R}$ be such that $f\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, and let $\sigma : \partial\Omega \to \mathbb{R}$. We say that $u : \Omega \to \mathbb{R}$ is a weak solution of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = \sigma & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

if $u \in H^1(\Omega)$, $u = \sigma$ on $\partial\Omega$, and

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega f\varphi \quad \text{for any } \varphi \in H_0^1(\Omega). \tag{1.8}$$

For a function $f : \Omega \to \mathbb{R}$, we will write $f \in (H_0^1(\Omega))'$ to mean that $f\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, and that there exists a positive constant c such that $\left|\int_\Omega f\varphi\right| \le c \|\varphi\|_{H_0^1(\Omega)}$ for any $\varphi \in H_0^1(\Omega)$.

Remark 1.2. If $f \in (H_0^1(\Omega))'$ and $\sigma \in H^{\frac{1}{2}}(\partial\Omega)$, then problem (1.7) has a unique weak solution $u \in H^1(\Omega)$, and there exists a positive constant c, independent of f and σ, such that

$$\|u\|_{H^1(\Omega)} \le c\Big(\|f\|_{(H_0^1(\Omega))'} + \|\sigma\|_{H^{\frac{1}{2}}(\partial\Omega)} \Big),$$

for a proof of this fact see, e.g., [43, Section 8.4.1] (there it is assumed that $f \in L^2(\Omega)$, but the arguments given there works also when $f \in (H_0^1(\Omega))'$).

For $S \subset \overline{\Omega}$ we will denote by ρ_S the distance function defined by

$$\rho_S(x) := \operatorname{dist}(x, S) \quad \text{for } x \in \Omega,$$

and, for a Lebesgue measurable subset E of Ω, $|E|$ will denote the Lebesgue measure of E.

We recall that a function $g : \Omega \times (0,\infty) \to \mathbb{R}$ is called a Carathéodory function if $g(\cdot, s)$ is Lebesgue measurable for any $s \in (0,\infty)$ and $g(x,\cdot)$ is continuous on $(0,\infty)$ for a.e. $x \in \Omega$. Our first result, concerning problem (1.1), reads as follows:

Theorem 1.3. *Let Ω be a C^2 and bounded domain in $\mathbb{R}^n$. Let $g : \Omega \times (0,\infty) \to [0,\infty)$ be a function satisfy the following three conditions:*

(H1) $g : \Omega \times (0,\infty) \to \mathbb{R}$ *is a nonnegative Carathéodory function such that, for each* $x \in \Omega$, $g(x,\cdot)$ *is nonincreasing on* $(0,\infty)$.
(H2) *There exists a Lebesgue measurable subset E of Ω such that $|E| > 0$ and $g(x,s) > 0$ for any $s > 0$ and almost all $x \in E$.*
(H3) $\rho_{\partial\Omega} g(\cdot, c\rho_{\partial\Omega}) \in L^2(\Omega)$ *for any* $c \in (0,\infty)$.

Then for any nonnegative $\sigma \in H^{\frac{1}{2}}(\partial\Omega)$ problem (1.1) has a unique weak solution $u \in H^1(\Omega)$ and there exists a positive constant c such that $u \ge c\rho_{\partial\Omega}$ a.e in Ω.

Let us introduce the space

$$H^1_{0,\Gamma_1}(\Omega) := \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1 \right\},$$

which endowed with the inner product of $H^1(\Omega)$ is a Hilbert space. Let $(H^1_{0,\Gamma_1}(\Omega))'$ denote its topological dual. If f is a function defined on Ω, we will write $f \in (H^1_{0,\Gamma_1}(\Omega))'$ to mean that $f\varphi \in L^1(\Omega)$ and $\left| \int_{\Gamma_2} \eta\varphi \right| \leq c \left\| \varphi \right\|_{H^1(\Omega)}$ for any $\varphi \in H^1_{0,\Gamma_1}(\Omega)$, with c a positive constant independent of φ. Similarly, if η is a function defined on Γ_2 we will say that $\eta \in (H^1_{0,\Gamma_1}(\Omega))'$ to mean that $\eta\varphi \in L^1(\Gamma_2)$ and that $\left| \int_{\Gamma_2} \eta\varphi \right| \leq c \left\| \varphi \right\|_{H^1(\Omega)}$ for any $\varphi \in H^1_{0,\Gamma_1}(\Omega)$, with a positive constant c independent of φ. In both cases, the maps $\varphi \to \int_\Omega f\varphi$ and $\varphi \to \int_{\Gamma_2} \eta\varphi$ will still be denoted by f and η, respectively.

Weak solutions of problems with mixed nonhomogeneous Dirichlet–Neumann boundary conditions are defined as follows:

Definition 1.4. Let $f : \Omega \to \mathbb{R}$ be such that $f\varphi \in L^1(\Omega)$ for any $\varphi \in H^1_{0,\Gamma_1}(\Omega)$, let $\tau \in H^{\frac{1}{2}}(\Gamma_1)$, and let $\eta : \Gamma_2 \to \mathbb{R}$ be a measurable function such that $\eta\varphi \in L^1(\Gamma_2)$ for any $\varphi \in H^1_{0,\Gamma_1}(\Omega)$. We say that $u : \Omega \to \mathbb{R}$ is a weak solution of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = \tau & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = \eta & \text{on } \Gamma_2. \end{cases} \tag{1.9}$$

if $u \in H^1(\Omega)$, $u = \tau$ on Γ_1, and

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega f\varphi + \int_{\Gamma_2} \eta\varphi \quad \text{for any } \varphi \in H^1_{0,\Gamma_1}(\Omega). \tag{1.10}$$

Let $f : \Omega \to \mathbb{R}$ be such that $f\varphi \in L^1(\Omega)$ for any $\varphi \in H^1_{0,\Gamma_1}(\Omega)$, let $\tau : \Gamma_1 \to \mathbb{R}$, and suppose that u is a weak solution of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = \tau & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2. \end{cases} \tag{1.11}$$

If $\varphi \in H^1_{0,\Gamma_1}(\Omega)$ and if φ and u are regular enough on $\overline{\Omega}$, we have

$$-\operatorname{div}(\varphi \nabla u) + \langle \nabla u, \nabla \varphi \rangle = f\varphi$$

and then, from the divergence theorem and the fact that $\varphi = 0$ on Γ_1, we get

$$-\int_{\Gamma_2} \frac{\partial u}{\partial \nu} \varphi + \int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega f\varphi.$$

Therefore,

$$\int_{\Gamma_2} \frac{\partial u}{\partial \nu} \varphi = \int_\Omega \langle \nabla u, \nabla \varphi \rangle - \int_\Omega f\varphi.$$

This suggests the following definition.

Definition 1.5. Let $f : \Omega \to \mathbb{R}$ be such that $f \in (H^1_{0,\Gamma_1}(\Omega))'$, and let $\tau \in H^{\frac{1}{2}}(\Gamma_1)$. If $u \in H^1(\Omega)$ is the weak solution of problem (1.11), we define the (distributional) normal derivative of u on Γ_2, as the linear functional $\frac{\partial u}{\partial \nu}_{\Gamma_2} : H^1_{0,\Gamma_1}(\Omega) \to \mathbb{R}$ defined by

$$\frac{\partial u}{\partial \nu}_{\Gamma_2}(\varphi) := \int_\Omega \langle \nabla u, \nabla \varphi \rangle - \int_\Omega f\varphi \quad \text{for any } \varphi \in H^1_{0,\Gamma_1}(\Omega). \tag{1.12}$$

For η and $\widetilde{\eta}$ in $(H^1_{0,\Gamma_1}(\Omega))'$ we will write $\eta \geq \widetilde{\eta}$ (respectively $\eta \leq \widetilde{\eta}$) to mean that $\eta(\varphi) \geq \widetilde{\eta}(\varphi)$ (resp. $\eta(\varphi) \leq \widetilde{\eta}(\varphi)$) for any nonnegative $\varphi \in H^1_{0,\Gamma_1}(\Omega)$. We will write also $\eta > \widetilde{\eta}$ (respectively $\eta < \widetilde{\eta}$) to mean that $\eta \neq \widetilde{\eta}$ and $\eta \geq \widetilde{\eta}$ (resp. and $\eta \leq \widetilde{\eta}$).

Concerning problem (1.2) we have the following:

Theorem 1.6. *Let Ω be a C^2 and bounded domain in $\mathbb{R}^n$ such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are disjoint closed sets in $\partial\Omega$. Let $g : \Omega \times (0,\infty) \to [0,\infty)$. Assume the conditions* (H1)–(H2) *of Theorem 1.3 and the following:*

(H3') $\rho_{\Gamma_1} g(\cdot, c\rho_{\partial\Omega}) \in L^2(\Omega)$ *for any* $c \in (0,\infty)$.

Let τ be a nonnegative function in $H^{\frac{1}{2}}(\Gamma_1)$, let u_τ be the weak solution of the problem

$$\begin{cases} -\Delta u_\tau = g(\cdot, u_\tau) & \text{in } \Omega, \\ u_\tau = \tau & \text{on } \Gamma_1, \\ u_\tau = 0 & \text{on } \Gamma_2. \end{cases} \tag{1.13}$$

given by Theorem 1.3, and let $\eta : \Gamma_2 \to \mathbb{R}$ be such that $\eta \in (H^1_{0,\Gamma_1}(\Omega))'$. Then:

(i) *if $\eta \geq \frac{\partial u_\tau}{\partial \nu}_{\Gamma_2}$, then* (1.2) *has a unique weak solution $u \in H^1(\Omega)$, and there exists a positive constant c such that $u \geq c\rho_{\partial\Omega}$ in Ω,*
(ii) *if $\eta < \frac{\partial u_\tau}{\partial \nu}_{\Gamma_2}$, then* (1.2) *has no weak solutions.*

As a consequence of Theorem 1.6 and of a weak form of the Hopf boundary lemma given in Lemma 4.4, we will get the following:

Corollary 1.7. *Let $g : \Omega \times (0,\infty) \to [0,\infty)$ satisfy the conditions* (H1)–(H2) *of Theorem 1.3 and the condition* (H3') *of Theorem 1.6, let τ be a nonnegative function in $H^{\frac{1}{2}}(\Gamma_1)$ and let $\eta : \Gamma_2 \to \mathbb{R}$ be such that $\eta \in (H^1_{0,\Gamma_1}(\Omega))'$. If $\eta \geq 0$, then problem* (1.2) *has a unique weak solution $u \in H^1(\Omega)$, and there exists a positive constant c such that $u \geq c\rho_{\partial\Omega}$ in Ω.*

The paper is organized as follows. In Section 2 we recall some general facts we need, and in Section 3 we study problem (1.1) via an approximation approach, which is adapted from [27], where the existence and uniqueness of strong solutions of (1.1) were investigated. We consider, for $\varepsilon \in (0,1]$ and for any nonnegative $\sigma \in H^{\frac{1}{2}}(\partial\Omega)$, the problem of finding $v_\varepsilon \in H^1_0(\Omega)$ such that $-\Delta v_\varepsilon = g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma})$ in Ω, $v_\varepsilon = 0$ on $\partial\Omega$, where $\widetilde{\sigma}$ is the solution of the problem $-\Delta\widetilde{\sigma} = 0$ in Ω, $\widetilde{\sigma} = \sigma$ on $\partial\Omega$, and with $g_\varepsilon : \Omega \times (0,\infty) \to \mathbb{R}$ defined by $g_\varepsilon(x,s) := \min\left\{\varepsilon^{-1},\ g(x, s+\varepsilon)\right\}$. By writing the above problem for v_ε as a fixed point problem and using the Schauder fixed point theorem we prove in Lemma 3.2 the existence and uniqueness of such a v_ε, and that, in addition, the map $\varepsilon \to v_\varepsilon$ is nonincreasing. Lemma 3.3 shows that if $v(x) := \lim_{\varepsilon\to 0^+} v_\varepsilon(x)$, then $v \in H_0^1(\Omega)$,

$$\lim_{\varepsilon\to 0^+} \|v_\varepsilon - v\|_{H_0^1(\Omega)} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \|g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma}) - g(\cdot, v + \widetilde{\sigma})\|_{L^2(\Omega, \rho_{\partial\Omega}^2(x)dx)} = 0.$$

From these facts and from other additional considerations, Theorem 1.3 is proved at the end of Section 3, by showing that $u := v + \widetilde{\sigma}$ is the unique solution of problem (1.1) and that it satisfies $u \geq c\rho_{\partial\Omega}$ for some constant $c > 0$.

In Section 4 we prove Theorem 1.6. The existence assertion of 1.6 is obtained by adapting, to our setting, ideas from the proof of Theorem 1.1 in [35] (which is a sub-supersolution theorem for problems of the form $-\Delta u = f(x, u)$ in Ω, $u = 0$ on $\partial\Omega$. Lemma 4.4 gives a weak form of the Hopf boundary lemma, and Corollary 1.7 is proved as a direct consequence of Theorem 1.6 and of Lemma 4.4.

2. PRELIMINARIES

Let us recall some well known facts.

Remark 2.1.

(i) (Poincaré's inequality for functions in $H_0^1(\Omega)$, see, e.g., [38, Theorem 1.8.1]) There exists a positive constant c such that

$$\|u\|_2 \leq c\, \|\nabla u\|_2 \ \text{ for any } u \in H_0^1(\Omega).$$

(ii) (Poincaré's inequality for functions in $H_{0,\Gamma_1}^1(\Omega)$, see, e.g., [43, Theorem 7.16]) There exists a positive constant c such that

$$\|u\|_2 \leq c\, \|\nabla u\|_2 \ \text{ for any } u \in H_{0,\Gamma_1}^1(\Omega).$$

(iii) The inclusion $H_{0,\Gamma_1}^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Indeed, the inclusion $H_{0,\Gamma_1}^1(\Omega) \hookrightarrow H^1(\Omega)$ is continuous and (see, e.g., [38, Theorem 1.9.15]) $H^1(\Omega)$ has compact inclusion into $L^2(\Omega)$.

(iv) (Hardy's inequality, see, e.g., [6, p. 313], see also [38, Theorem 1.10.15]) There exists a positive constant c such that $\left\| \frac{u}{\rho_{\partial\Omega}} \right\|_2 \leq c\, \|\nabla u\|_2$ for any $u \in H_0^1(\Omega)$.

$H_{0,\Gamma_1}^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ and thus, provided with the norm of $H^1(\Omega)$, it is a Hilbert space, and the Poincaré inequality of Remark 2.1(ii) gives that $u \to \|\nabla u\|_2$ is a norm on $H_{0,\Gamma_1}^1(\Omega)$, equivalent to the norm $\|.\|_{H^1(\Omega)}$. From now on, we will consider $H_{0,\Gamma_1}^1(\Omega)$ as a Hilbert space provided with the norm $\|u\|_{H_{0,\Gamma_1}^1(\Omega)} := \|\nabla u\|_2$. Similarly, $H_0^1(\Omega)$ will be considered as a Hilbert space with the same norm.

For $\delta > 0$, let

$$\Omega_\delta := \{x \in \Omega : \rho_{\partial\Omega}(x) > \delta\} \quad \text{and} \quad A_\delta := \{x \in \Omega : \rho_{\partial\Omega}(x) \leq \delta\}.$$

Similarly, for $i = 1, 2$, we set

$$\Omega_{\Gamma_i,\delta} := \{x \in \Omega : \rho_{\Gamma_i}(x) > \delta\} \quad \text{and} \quad A_{\Gamma_i,\delta} := \{x \in \Omega : \rho_{\Gamma_i}(x) \le \delta\}. \tag{2.1}$$

The following lemma provides an analogous of the Hardy inequality for functions in $H^1_{0,\Gamma_1}(\Omega)$.

Lemma 2.2 (Hardy's inequality for functions in $H^1_{0,\Gamma_1}(\Omega)$)**.** *There exists a positive constant c such that*

$$\left\| \frac{u}{\rho_{\Gamma_1}} \right\|_2 \le c \|\nabla u\|_2$$

for any $u \in H^1_{0,\Gamma_1}(\Omega)$.

Proof. Along the proof, c, c', c'' etc., will denote positive constants independent of u. Let δ_1, δ_2 be such that such that $0 < \delta_1 < \delta_2$ and $\Omega_{\Gamma_1,\delta_2} \neq \varnothing$. Let $\psi \in C^\infty(\overline{\Omega})$ be such that $0 \le \psi \le 1$ in Ω, $\psi = 1$ in A_{Γ_1,δ_1} and $\psi = 0$ in $\Omega_{\Gamma_1,\delta_2}$. Then, for $u \in H^1_{0,\Gamma_1}(\Omega)$,

$$\left\| \frac{u}{\rho_{\Gamma_1}} \right\|_2^2 = \int_\Omega \frac{u^2}{\rho^2_{\Gamma_1}} = \int_{A_{\Gamma_1,\delta_1}} \frac{u^2}{\rho^2_{\Gamma_1}} + \int_{\Omega\setminus A_{\Gamma_1,\delta_1}} \frac{u^2}{\rho^2_{\Gamma_1}}. \tag{2.2}$$

Now, $u\psi \in H^1_0(\Omega)$ and so, taking into account the Hardy inequality in $H^1_0(\Omega)$,

$$\int_{A_{\Gamma_1,\delta_1}} \frac{u^2}{\rho^2_{\Gamma_1}} = \int_{A_{\Gamma_1,\delta_1}} \frac{u^2\psi^2}{\rho^2_{\Gamma_1}} \le \int_\Omega \frac{u^2\psi^2}{\rho^2_{\partial\Omega}} \le c \int_\Omega |\nabla(u\psi)|^2 = c \int_\Omega |\psi\nabla u + u\nabla\psi|^2 .$$

Thus

$$\begin{aligned} \left\| \frac{u}{\rho_{\Gamma_1}} \right\|_{L^2(A_{\Gamma_1,\delta_1})} &\le c \|\psi\nabla u + u\nabla\psi\|_{L^2(\Omega)} \\ &\le c \|\psi\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} + c \|\nabla\psi\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} . \end{aligned} \tag{2.3}$$

and, by the Poincaré inequality of Remark 2.1(ii), $\|u\|_{L^2(\Omega)} \le c' \|\nabla u\|_{L^2(\Omega)}$. Thus,

$$\left\| \frac{u}{\rho_{\Gamma_1}} \right\|_{L^2(A_{\Gamma_1,\delta_1})} \le c''(\|\psi\|_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^\infty(\Omega)}) \|\nabla u\|_{L^2(\Omega)} .$$

On the other hand,

$$\int_{\Omega\setminus A_{\Gamma_1,\delta_1}} \frac{u^2}{\rho^2_{\Gamma_1}} \le \frac{1}{\delta_1^2} \int_{\Omega\setminus A_{\Gamma_1,\delta_1}} u^2 \le \frac{1}{\delta_1^2} \int_\Omega u^2 \le c''' \|\nabla u\|^2_{L^2(\Omega)} , \tag{2.4}$$

the last inequality by the Poincaré inequality of Remark 2.1(ii), and the lemma follows from (2.2), (2.3), and (2.4). □

Corollary 2.3.

(i) *If* $f:\Omega\to\mathbb{R}$ *and* $f\in L^2(\Omega,\rho^2_{\partial\Omega}(x)dx)$*, then* $f\in(H^1_0(\Omega))'$ *and*

$$\|f\|_{(H^1_0(\Omega))'}\le c\,\|f\|_{L^2(\Omega,\rho^2_{\partial\Omega}(x)dx)}$$

with c *a positive constant independent of* f*.*

(ii) *If* $f:\Omega\to\mathbb{R}$ *and* $f\in L^2(\Omega,\rho^2_{\Gamma_1}(x)dx)$*, then* $f\in(H^1_{0,\Gamma_1}(\Omega))'$ *and it holds that*

$$\|f\|_{(H^1_{0,\Gamma_1}(\Omega))'}\le c\,\|f\|_{L^2(\Omega,\rho^2_{\Gamma_1}(x)dx)},$$

where c *is a positive constant independent of* f*.*

Proof. Suppose that $\rho_{\partial\Omega}f\in L^2(\Omega)$ and let $\varphi\in H^1_0(\Omega)$. Then, for some positive constant c independent of φ,

$$\int\limits_\Omega |f\varphi| = \int\limits_\Omega \left|\rho_{\partial\Omega}f\frac{\varphi}{\rho_{\partial\Omega}}\right| \le \|\rho_{\partial\Omega}f\|_2\left\|\frac{\varphi}{\rho_{\partial\Omega}}\right\|_2 \le c\,\|\rho_{\partial\Omega}f\|_2\,\|\varphi\|_{H^1_0(\Omega)},$$

the last inequality by Remark 2.1(iii). Thus (i) holds. The proof of (ii) is similar, using Lemma 2.2 instead of Remark 2.1(iii). □

Remark 2.4 (see, e.g., [43, Theorem 8.9])**.** If $0\le f\in(H^1_0(\Omega))'$, $0\le\sigma\in H^{\frac12}(\partial\Omega)$, and if u is the weak solution of problem (1.7), then $u\ge0$ in Ω.

Remark 2.5.

(i) (see [7, Lemma 3.2]) Suppose $0\le f\in L^\infty(\Omega)$, and let ζ be the solution of the problem

$$\begin{cases}-\Delta\zeta=f & \text{in }\Omega,\\ \zeta=0 & \text{on }\partial\Omega.\end{cases}\tag{2.5}$$

Then $\zeta\ge c\rho_{\partial\Omega}\int_\Omega f\rho_{\partial\Omega}$ in Ω, with c a positive constant independent of f.

(ii) If $0\le f\in L^\infty(\Omega)$, $0\le\sigma\in H^{\frac12}(\partial\Omega)$ and if $u\in H^1(\Omega)$ is a weak solution of the problem

$$\begin{cases}-\Delta u=f & \text{in }\Omega,\\ u=\sigma & \text{on }\partial\Omega,\end{cases}\tag{2.6}$$

then $u\ge c\rho_{\partial\Omega}\int_\Omega f\rho_{\partial\Omega}$ in Ω with c a positive constant independent of f. Indeed, let ζ be as in (i), then $u-\zeta$ satisfies, in a weak sense,

$$\begin{cases}-\Delta(u-\zeta)=f & \text{in }\Omega,\\ u-\zeta=\sigma & \text{on }\partial\Omega,\end{cases}$$

and then, by Remark 2.4, $u\ge\zeta$. Thus, by (i), $u\ge c\rho_{\partial\Omega}\int_\Omega f\rho_{\partial\Omega}$ in Ω, with c as in (i).

(iii) Let $f:\Omega\to\mathbb{R}$ be a nonnegative and measurable function such that $f\in(H^1_0(\Omega))'$ and $|\{x\in\Omega:f(x)>0\}|>0$. If $0\le\sigma\in H^{\frac12}(\partial\Omega)$ and if $u\in H^1(\Omega)$ is a weak solution of the problem (2.6), then there exists a positive constant c such that $u \geq c\rho_{\partial\Omega}$ in Ω. In fact, in such a case there exist a measurable subset $F \subset \Omega$ with $|F| > 0$ and $\lambda \in (0,\infty)$ such that $f \geq \lambda\chi_F$ in Ω. Let w be the solution of $-\Delta w = \lambda\chi_F$ in Ω, $w = 0$ on $\partial\Omega$. Then, by (i), there exists a positive constant c such that $w \geq c\rho_{\partial\Omega}$ in Ω. Also, $-\Delta(u-w) = f - \lambda\chi_F \geq 0$ in Ω and $u - w = \sigma \geq 0$ on $\partial\Omega$. Thus, by Remark 2.4, $u - w \geq 0$, and then $u \geq c\rho_{\partial\Omega}$ in Ω.

Remark 2.6. Suppose $0 \leq f \in (H^1_{0,\Gamma_1}(\Omega))'$, $0 \leq \tau \in H^{\frac{1}{2}}(\Gamma_1)$ and let $\eta : \Gamma_2 \to \mathbb{R}$ be such that $0 \leq \eta \in (H^1_{0,\Gamma_1}(\Omega))'$. If u is the weak solution of problem (1.9), then $u \geq 0$ in Ω. Indeed, since $\tau \geq 0$ we have $u^- = 0$ on Γ_1 and thus $u^- \in H^1_{0,\Gamma_1}(\Omega)$. Taking $\varphi = -u^-$ in (1.10) we get

$$-\int_\Omega \langle \nabla u, \nabla u^- \rangle + \int_\Omega f u^- + \int_{\Gamma_2} \eta u^- = 0,$$

and so

$$\int_\Omega \left|\nabla u^-\right|^2 = -\int_\Omega f u^- - \int_{\Gamma_2} \eta u^- \leq 0.$$

Thus $\int_\Omega \left|\nabla u^-\right|^2 = 0$. Therefore, by the Poincaré inequality of Remark 2.1(ii), $u^- = 0$ in Ω. Then $u \geq 0$ in Ω. Moreover, from Remark 2.5 (iii) used with $\sigma := u_{|\partial\Omega} \geq 0$ on $\partial\Omega$ (the restriction in the sense of the trace), it follows that, if in addition, $|\{x \in \Omega : f(x) > 0\}| > 0$, then there exists a positive constant c such that $u \geq c\rho_{\partial\Omega}$ in Ω.

3. THE CASE OF DIRICHLET BOUNDARY CONDITION

We assume, for the whole section, that $g : \Omega \times (0,\infty) \to \mathbb{R}$ satisfies the conditions (H1)–(H3) of Theorem 1.3. We first study, for $\varepsilon \in (0,1]$ and for a nonnegative $\sigma \in H^{\frac{1}{2}}(\partial\Omega)$, the approximated problem

$$\begin{cases} -\Delta u = g_\varepsilon(\cdot, u) & \text{in } \Omega, \\ u = \sigma & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $g_\varepsilon : \Omega \times (0,\infty) \to \mathbb{R}$ is defined by

$$g_\varepsilon(x,s) := \min\left\{\varepsilon^{-1},\ g(x, s+\varepsilon)\right\}. \tag{3.2}$$

Observe that, since g satisfies (H1)–(H3), the same conditions hold for each g_ε. Let $\widetilde{\sigma} \in H^1(\Omega)$ be the weak solution of the problem

$$\begin{cases} -\Delta \widetilde{\sigma} = 0 & \text{in } \Omega, \\ \widetilde{\sigma} = \sigma & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Then, by Remark 2.4(i), $\widetilde{\sigma} \geq 0$ in Ω. By writing $u = \widetilde{\sigma} + v$, problem (3.1) becomes equivalent to the problem of finding a weak solution $v \in H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta v = g_\varepsilon(\cdot, v + \widetilde{\sigma}) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Let $(-\Delta)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ be the solution operator of the homogeneous Dirichlet problem defined by $(-\Delta)^{-1} h = u$, where $u \in H_0^1(\Omega)$ is the weak solution of the problem $-\Delta u = h$ in Ω, $u = 0$ on $\partial\Omega$. We recall that $(-\Delta)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ is continuous and that, since $H_0^1(\Omega)$ has compact inclusion into $L^2(\Omega)$, $(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is a compact operator. Let $T_\varepsilon : L^2(\Omega) \to H_0^1(\Omega)$ be defined by

$$T_\varepsilon(v) := (-\Delta)^{-1}(g_\varepsilon(\cdot, v + \widetilde{\sigma})),$$

and let $C_\varepsilon := \left\{ v \in L^2(\Omega) : 0 \leq v \leq \frac{1}{\varepsilon}(-\Delta)^{-1}(\mathbf{1}) \right\}$. We have the following:

Lemma 3.1.

(i) *C_ε is a bounded, closed and convex subset of $L^2(\Omega)$.*
(ii) *$T_\varepsilon(C_\varepsilon) \subset C_\varepsilon$.*
(iii) *$T_\varepsilon : C_\varepsilon \to C_\varepsilon$ is continuous.*
(iv) *$T_\varepsilon : C_\varepsilon \to C_\varepsilon$ is a compact operator.*

Proof. (i) is obvious.

To show (ii) observe that if $v \in C_\varepsilon$ then $0 \leq g_\varepsilon(\cdot, v + \widetilde{\sigma}) \leq \frac{1}{\varepsilon}$ a.e. in Ω and so, by Remark 2.4,

$$0 \leq (-\Delta)^{-1}(g_\varepsilon(x, v + \widetilde{\sigma})) \leq \frac{1}{\varepsilon}(-\Delta)^{-1}(\mathbf{1}).$$

Thus $T_\varepsilon(v) \in C_\varepsilon$.

To prove (iii) it is enough to see that if $v \in C_\varepsilon$, and if $\{v_j\}_{j\in\mathbb{N}}$ is a sequence in C_ε that converges to v in $L^2(\Omega)$, then there exists a subsequence $\{v_{j_k}\}_{k\in\mathbb{N}}$ such that $\{T_\varepsilon(v_{j_k})\}_{k\in\mathbb{N}}$ converges to $T_\varepsilon(v)$ in $L^2(\Omega)$. Let $v \in C_\varepsilon$, and let $\{v_j\}_{j\in\mathbb{N}}$ be a sequence in C_ε which converges to v in $L^2(\Omega)$, then there exists a subsequence $\{v_{j_k}\}_{k\in\mathbb{N}}$ such that $\{v_{j_k}\}_{k\in\mathbb{N}}$ converges to v a.e. in Ω. Thus, since g_ε is a Carathéodory function, $\{g_\varepsilon(\cdot, v_{j_k} + \widetilde{\sigma})\}_{k\in\mathbb{N}}$ converges to $g_\varepsilon(\cdot, v + \widetilde{\sigma})$ a.e. in Ω. Then

$$\lim_{k\to\infty} |g_\varepsilon(\cdot, v_{j_k} + \widetilde{\sigma}) - g_\varepsilon(\cdot, v + \widetilde{\sigma})|^2 = 0$$

a.e. in Ω. Since $|g_\varepsilon(\cdot, v_{j_k} + \widetilde{\sigma}) - g_\varepsilon(\cdot, v + \widetilde{\sigma})|^2 \leq \frac{1}{\varepsilon^2}$, the Lebesgue dominated convergence theorem gives that $\{g_\varepsilon(\cdot, v_{j_k} + \widetilde{\sigma})\}_{k\in\mathbb{N}}$ converges to $g_\varepsilon(\cdot, v + \widetilde{\sigma})$ in $L^2(\Omega)$. Then $\left\{(-\Delta)^{-1}(g_\varepsilon(\cdot, v_{j_k} + \widetilde{\sigma}))\right\}_{k\in\mathbb{N}}$ converges to $(-\Delta)^{-1}(g_\varepsilon(\cdot, v + \widetilde{\sigma}))$ in $L^2(\Omega)$, i.e., $\{T_\varepsilon(v_{j_k})\}_{k\in\mathbb{N}}$ converges to $T_\varepsilon(v)$ in $L^2(\Omega)$. Thus (iii) holds.

To see (iv), note that $\{g_\varepsilon(\cdot, v_j + \widetilde{\sigma})\}_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ for any sequence $\{v_j\}_{j\in\mathbb{N}}$ in C_ε, and so (iv) follows immediately from the compactness of the solution operator $(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$. □

Lemma 3.2.

(i) *For* $\varepsilon \in (0,1]$*, the problem*

$$\begin{cases} -\Delta v_\varepsilon = g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma}) & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

has a unique weak solution $v_\varepsilon \in H_0^1(\Omega)$*.*

(ii) *The map* $\varepsilon \to v_\varepsilon$ *is nonincreasing.*

(iii) *There exists a positive constant* c *such that* $v_\varepsilon \geq c\rho_{\partial\Omega}$ *for any* $\varepsilon \in (0,1]$*.*

(iv) $\{v_\varepsilon\}_{\varepsilon \in (0,1]}$ *is bounded in* $H_0^1(\Omega)$*.*

Proof. From Lemma 3.1 and the Schauder fixed point theorem, T_ε has a fixed point $v_\varepsilon \in C_\varepsilon$, and so v_ε is a weak solution of problem (3.4). Suppose that $w \in H^1(\Omega)$ is another solution of (3.4). Then $v_\varepsilon - w \in H_0^1(\Omega)$ and it satisfies, in weak sense

$$\begin{cases} -\Delta(v_\varepsilon - w) = g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma}) - g_\varepsilon(., w + \widetilde{\sigma}) & \text{in } \Omega, \\ v_\varepsilon - w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Now, $g_\varepsilon(x, \cdot)$ is nonincreasing on $(0, \infty)$ for a.e. $x \in \Omega$, and so

$$g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma}) - g_\varepsilon(\cdot, w + \widetilde{\sigma})(v_\varepsilon - w) \leq 0 \quad \text{a.e in } \Omega.$$

Thus, taking $v_\varepsilon - w$ as a test function in (3.6), we get that $\| |\nabla(v_\varepsilon - w)| \|_2 = 0$, and so, by the Poincaré inequality, $v_\varepsilon = w$ in Ω. Thus (i) holds.

To prove (ii), suppose that $0 < \varepsilon < \theta \leq 1$. Then $g_\varepsilon \geq g_\theta$ on $\Omega \times (0, \infty)$. Thus, in a weak sense,

$$\begin{cases} -\Delta(v_\varepsilon) = g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma}) \geq g_\theta(\cdot, v_\varepsilon + \widetilde{\sigma}) & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Also,

$$\begin{cases} -\Delta(v_\theta) = g_\theta(\cdot, v_\theta + \widetilde{\sigma}) & \text{in } \Omega, \\ v_\theta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

and so, again in a weak sense,

$$\begin{cases} -\Delta(v_\varepsilon - v_\theta) = g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma}) - g_\theta(\cdot, v_\theta + \widetilde{\sigma}) & \\ \geq g_\theta(\cdot, v_\varepsilon + \widetilde{\sigma}) - g_\theta(\cdot, v_\theta + \widetilde{\sigma}) & \text{in } \Omega, \\ v_\varepsilon - v_\theta = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

and so, taking $-(v_\varepsilon - v_\theta)^-$ as a test function in (3.9) we get

$$\int_\Omega \left|\nabla((v_\varepsilon - v_\theta)^-)\right|^2 \leq -\int_{\{v_\varepsilon - v_\theta < 0\}} (g_\theta(\cdot, v_\varepsilon + \widetilde{\sigma}) - g_\theta(\cdot, v_\theta + \widetilde{\sigma}))(v_\varepsilon - v_\theta)^- \leq 0.$$

The last inequality because $g_\varepsilon(x,\cdot)$ is nonincreasing on $(0,\infty)$ for a.e. $x \in \Omega$. Thus $\int_\Omega \left|\nabla((v_\varepsilon - v_\theta)^-)\right|^2 = 0$, and so, by Remark 2.1(i), $(v_\varepsilon - v_\theta)^- = 0$ in Ω, and then $v_\varepsilon \geq v_\theta$ a.e. in Ω. Thus (ii) holds.

To see (iii), observe that for $\varepsilon \in (0,1]$, by (ii), $v_\varepsilon \geq v_1$. Since $-\Delta v_1 = g_1(\cdot, v_1)$ in Ω and $0 \leq g_1(\cdot, v_1) \in L^\infty(\Omega)$, and taking into account that, by (H2), $g_1(\cdot, v_1)$ is not identically zero, Remark 2.5(i) gives that $v_1 \geq c\rho_{\partial\Omega}$ for some positive constant c. Thus $v_\varepsilon \geq c\rho_{\partial\Omega}$ and (iii) holds.

It remains to show (iv). Let c be as in (iii). We take v_ε as a test function in (3.4) to obtain

$$\begin{aligned}\|\nabla v_\varepsilon\|_2^2 &= \int_\Omega |\nabla v_\varepsilon|^2 = \int_\Omega v_\varepsilon g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma}) \\ &\leq \int_\Omega v_\varepsilon g(\cdot, v_\varepsilon + \widetilde{\sigma}) \leq \int_\Omega v_\varepsilon g(\cdot, \widetilde{\sigma} + c\rho_{\partial\Omega}) \\ &= \int_\Omega \frac{v_\varepsilon}{\rho_{\partial\Omega}} \rho_{\partial\Omega} g(\cdot, \widetilde{\sigma} + c\rho_{\partial\Omega}) \leq \int_\Omega \frac{v_\varepsilon}{\rho_{\partial\Omega}} \rho_{\partial\Omega} g(\cdot, c\rho_{\partial\Omega})\end{aligned} \tag{3.10}$$

where we have used (iii), (H1), and that $g_\varepsilon \leq g$, as well as that $g(x,s)$ is nonincreasing in s. Now, by the Hölder inequality and Remark 2.1(iv), we have, for some positive constant c' independent of ε,

$$\int_\Omega \frac{v_\varepsilon}{\rho_{\partial\Omega}} \rho_{\partial\Omega} g(\cdot, c\rho_{\partial\Omega}) \leq \left\| \frac{v_\varepsilon}{\rho_{\partial\Omega}} \right\|_2 \|\rho_{\partial\Omega} g(\cdot, c\rho_{\partial\Omega})\|_2 \leq c' \, \||\nabla v_\varepsilon|\|_2 \, \|\rho_{\partial\Omega} g(\cdot, c\rho_{\partial\Omega})\|_2 \tag{3.11}$$

and, by (H3), $\|\rho_{\partial\Omega} g(\cdot, c\rho_{\partial\Omega})\|_2 < \infty$. Thus, from (3.10) and (3.11), we get

$$\||\nabla v_\varepsilon|\|_2 \leq c' \, \|\rho_{\partial\Omega} g(\cdot, c\rho_{\partial\Omega})\|_2 ,$$

which ends the proof of the lemma. □

Lemma 3.3. *For $\varepsilon \in (0,1]$, let $v_\varepsilon \in H_0^1(\Omega)$ be as given by Lemma 3.2, and let $v := \lim_{\varepsilon \to 0^+} v_\varepsilon$. Then:*

(i) *$v \in H_0^1(\Omega)$ and $\lim_{\varepsilon\to 0^+} v_\varepsilon = v$ with convergence in $H_0^1(\Omega)$,*
(ii) *$\lim_{\varepsilon\to 0^+} g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma}) = g(\cdot, v + \widetilde{\sigma})$ with convergence in $L^2(\Omega, \rho_{\partial\Omega}^2(x)dx)$.*

Proof. Observe that $v \in H_0^1(\Omega)$. Indeed, let $\{\theta_j\}_{j\in\mathbb{N}} \subset (0,1]$ be a sequence such that $\lim_{j\to\infty} \theta_j = 0$, By Lemma 3.2, $\left\{v_{\theta_j}\right\}_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Thus there exist a subsequence $\{v_{\theta_{j_k}}\}_{k\in\mathbb{N}}$ and a function $w \in H_0^1(\Omega)$ such that $\{v_{\theta_{j_k}}\}_{k\in\mathbb{N}}$ converges to w strongly in $L^2(\Omega)$, and $\{\nabla v_{\theta_{j_k}}\}_{k\in\mathbb{N}}$ converges to ∇w weakly in $L^2(\Omega, \mathbb{R}^n)$. After pass to a further subsequence if necessary, we can assume also that $\{v_{\theta_{j_k}}\}_{k\in\mathbb{N}}$ converges to w a.e. in Ω. Since $v := \lim_{\varepsilon\to 0^+} v_\varepsilon$ it follows that $w = v$ and then $v \in H_0^1(\Omega)$.

To prove the lemma it is enough to see that for any sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,1]$ such that $\lim_{j\to\infty} \varepsilon_j = 0$ there exists a subsequence, which we still denoted by $\{\varepsilon_j\}_{j\in\mathbb{N}}$, such that

$$\lim_{j\to\infty}\left\|v_{\varepsilon_j}-v\right\|^2_{H_0^1(\Omega)}=0$$

and

$$\lim_{j\to\infty}\left\|g_{\varepsilon_j}(\cdot,v_{\varepsilon_j}+\widetilde{\sigma})-g(\cdot,v+\widetilde{\sigma})\right\|_{L^2(\Omega,\rho^2_{\partial\Omega}(x)dx)}=0.$$

Now, in a weak sense,

$$\begin{cases}-\Delta(v_{\varepsilon_j}-v)=g_{\varepsilon_j}(\cdot,v_{\varepsilon_j}+\widetilde{\sigma})-g(\cdot,v+\widetilde{\sigma}) & \text{in } \Omega,\\ v_{\varepsilon_j}-v=0 & \text{on } \partial\Omega.\end{cases} \tag{3.12}$$

We take $v_{\varepsilon_j}-v$ as a test function in (3.12) and we use the Hardy inequality of Remark 2.1(iv) to obtain

$$\begin{aligned}\left\|v_{\varepsilon_j}-v\right\|^2_{H_0^1(\Omega)}&=\int_\Omega\left|\nabla(v_{\varepsilon_j}-v)\right|^2=\int_\Omega(g_{\varepsilon_j}(\cdot,v_{\varepsilon_j}+\widetilde{\sigma})-g(\cdot,v+\widetilde{\sigma}))(v_{\varepsilon_j}-v)\\&=\int_\Omega\rho_{\partial\Omega}(g_{\varepsilon_j}(\cdot,v_{\varepsilon_j}+\widetilde{\sigma})-g(\cdot,v+\widetilde{\sigma}))\frac{v_{\varepsilon_j}-v}{\rho_{\partial\Omega}}\\&\le c\left\|\rho_{\partial\Omega}(g_{\varepsilon_j}(\cdot,v_{\varepsilon_j}+\widetilde{\sigma})-g(\cdot,v+\widetilde{\sigma}))\right\|_2\left\|v_{\varepsilon_j}-v\right\|_{H_0^1(\Omega)}.\end{aligned}$$

where c is a positive constant independent of j. Then, in order to prove the lemma, it suffices to show that

$$\lim_{j\to\infty}\left\|\rho_{\partial\Omega}(g_{\varepsilon_j}(\cdot,v_{\varepsilon_j}+\widetilde{\sigma})-g(\cdot,v+\widetilde{\sigma}))\right\|_2=0. \tag{3.13}$$

Now,

$$\begin{aligned}&\left\|\rho_{\partial\Omega}(g_{\varepsilon_j}(\cdot,v_{\varepsilon_j}+\widetilde{\sigma})-g(\cdot,v+\widetilde{\sigma}))\right\|_2^2\\&=\int\limits_{\left\{g(\cdot,v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j)\le\frac{1}{\varepsilon_j}\right\}}\rho^2_{\partial\Omega}(g_{\varepsilon_j}(\cdot,v_{\varepsilon_j}+\widetilde{\sigma})-g(\cdot,v+\widetilde{\sigma}))^2\\&\quad+\int\limits_{\left\{g(\cdot,v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j)>\frac{1}{\varepsilon_j}\right\}}\rho^2_{\partial\Omega}(g_{\varepsilon_j}(\cdot,v_{\varepsilon_j}+\widetilde{\sigma})-g(\cdot,v+\widetilde{\sigma}))^2\\&=\int\limits_{\left\{g(\cdot,v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j)\le\frac{1}{\varepsilon_j}\right\}}\rho^2_{\partial\Omega}(g(\cdot,v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j)-g(\cdot,v+\widetilde{\sigma}))^2\\&\quad+\int\limits_{\left\{g(\cdot,v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j)>\frac{1}{\varepsilon_j}\right\}}\rho^2_{\partial\Omega}\Big(\frac{1}{\varepsilon}-g(\cdot,v+\widetilde{\sigma})\Big)^2\end{aligned}$$

and so

$$
\begin{aligned}
&\left\|\rho_{\partial\Omega}(g_{\varepsilon_j}(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}) - g(\cdot, v+\widetilde{\sigma}))\right\|_2^2 \\
&= \int\limits_{\Omega} \rho_{\partial\Omega}^2 (g(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) - g(\cdot, v+\widetilde{\sigma}))^2 \\
&\quad - \int\limits_{\left\{g(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) > \frac{1}{\varepsilon_j}\right\}} \rho_{\partial\Omega}^2 (g(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) - g(\cdot, v+\widetilde{\sigma}))^2 \\
&\quad + \int\limits_{\left\{g(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) > \frac{1}{\varepsilon_j}\right\}} \rho_{\partial\Omega}^2 \Big(\frac{1}{\varepsilon} - g(\cdot, v+\widetilde{\sigma})\Big)^2 \\
&= I_{1,j} + I_{2,j} + I_{3,j},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{1,j} &:= \int\limits_{\Omega} \rho_{\partial\Omega}^2 (g(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) - g(\cdot, v+\widetilde{\sigma}))^2, \\
I_{2,j} &:= - \int\limits_{\left\{g(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) > \frac{1}{\varepsilon_j}\right\}} \rho_{\partial\Omega}^2 (g(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) - g(\cdot, v+\widetilde{\sigma}))^2, \\
I_{3,j} &:= \int\limits_{\left\{g(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) > \frac{1}{\varepsilon_j}\right\}} \rho_{\partial\Omega}^2 \Big(\frac{1}{\varepsilon} - g(\cdot, v+\widetilde{\sigma})\Big)^2.
\end{aligned}
$$

Now, since g is Carathéodory,

$$
\lim_{j\to\infty} \rho_{\partial\Omega}^2 (g(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) - g(\cdot, v+\widetilde{\sigma}))^2 = 0
$$

a.e. in Ω. Also,

$$
\begin{aligned}
&\rho_{\partial\Omega}^2 (g(., v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) - g(\cdot, v+\widetilde{\sigma}))^2 \\
&\le 2\rho_{\partial\Omega}^2 g^2(\cdot, v_{\varepsilon_j}+\widetilde{\sigma}+\varepsilon_j) + 2\rho_{\partial\Omega}^2 g^2(\cdot, v+\widetilde{\sigma}) \le 4\rho_{\partial\Omega}^2 g^2(\cdot, v+\widetilde{\sigma}),
\end{aligned}
$$

and since $v \ge c\rho_{\partial\Omega}$ and $\widetilde{\sigma} \ge 0$, (H3) gives $\rho_{\partial\Omega}^2 g^2(\cdot, v+\widetilde{\sigma}) \in L^1(\Omega)$. Then, by the Lebesgue dominated convergence theorem,

$$
\lim_{j\to\infty} I_{1,j} = 0.
$$

Let

$$
U_j := \left\{ g(\cdot, v_1+\widetilde{\sigma}) > \frac{1}{\varepsilon_j} \right\}.
$$

Then $U_{j+1} \subset U_j$ for any j, $U_1 \subset \Omega$, and $\bigcap_{j=1}^{\infty} U_j = \{g(\cdot, v_1 + \widetilde{\sigma}) = \infty\}$. Since $\rho_{\partial\Omega}^2 g^2(\cdot, v + \widetilde{\sigma}) \in L^1(\Omega)$ it follows that $\left|\bigcap_{j=1}^{\infty} U_j\right| = 0$. Then $\lim_{j\to\infty} |U_j| = 0$, and thus

$$\lim_{j\to\infty} \int\limits_{U_j} \rho_{\partial\Omega}^2 g^2(\cdot, v + \widetilde{\sigma}) = 0. \tag{3.14}$$

Now,

$$\rho_{\partial\Omega}^2 (g(\cdot, v_{\varepsilon_j} + \widetilde{\sigma} + \varepsilon_j) - g(\cdot, v + \widetilde{\sigma}))^2 \le 2\rho_{\partial\Omega}^2 g^2(\cdot, v + \widetilde{\sigma},)$$

and so

$$\begin{aligned} |I_{2,j}| &\le \int\limits_{\left\{g(\cdot, v_{\varepsilon_j} + \widetilde{\sigma} + \varepsilon_j) > \frac{1}{\varepsilon_j}\right\}} \rho_{\partial\Omega}^2 (g(\cdot, v_{\varepsilon_j} + \widetilde{\sigma} + \varepsilon_j) - g(\cdot, v + \widetilde{\sigma}))^2 \\ &\le \int\limits_{U_j} 2\rho_{\partial\Omega}^2 g^2(\cdot, v + \widetilde{\sigma}). \end{aligned}$$

Then, by (3.14),

$$\lim_{j\to\infty} I_{2,j} = 0.$$

Finally,

$$\rho_{\partial\Omega}^2 \Big(\frac{1}{\varepsilon} - g(\cdot, v + \widetilde{\sigma})\Big)^2 \le 2\rho_{\partial\Omega}^2 \frac{1}{\varepsilon^2} + 2\rho_{\partial\Omega}^2 g^2(\cdot, v + \widetilde{\sigma}),$$

and then

$$\begin{aligned} |I_{3,j}| &\le \int\limits_{\left\{g(\cdot, v_{\varepsilon_j} + \widetilde{\sigma} + \varepsilon_j) > \frac{1}{\varepsilon_j}\right\}} \Big(2\rho_{\partial\Omega}^2 \frac{1}{\varepsilon^2} + 2\rho_{\partial\Omega}^2 g^2(\cdot, v + \widetilde{\sigma})\Big) \\ &\le 2 \int\limits_{\left\{g(\cdot, v_{\varepsilon_j} + \widetilde{\sigma} + \varepsilon_j) > \frac{1}{\varepsilon_j}\right\}} \rho_{\partial\Omega}^2 g^2(\cdot, v_{\varepsilon_j} + \widetilde{\sigma} + \varepsilon_j) \\ &\quad + 2 \int\limits_{\left\{g(\cdot, v_{\varepsilon_j} + \widetilde{\sigma} + \varepsilon_j) > \frac{1}{\varepsilon_j}\right\}} \rho_{\partial\Omega}^2 g^2(\cdot, v + \widetilde{\sigma}) \\ &\le 4 \int\limits_{U_j} \rho_{\partial\Omega}^2 g^2(\cdot, v + \widetilde{\sigma}), \end{aligned}$$

and thus, by (3.14),

$$\lim_{j\to\infty} I_{3,j} = 0,$$

which concludes the proof of the lemma. □

Proof of Theorem 1.3. For $\varepsilon \in (0,1]$, let $v_\varepsilon \in H_0^1(\Omega)$ be as given by Lemma 3.2 and let $v = \lim_{\varepsilon\to 0^+} v_\varepsilon$. Let $u_\varepsilon := \widetilde{\sigma} + v_\varepsilon$ and let $u := \lim_{\varepsilon\to 0^+} u_\varepsilon = \widetilde{\sigma} + v$. By Lemma 3.2, v_ε is a weak solution of the problem

$$\begin{cases} -\Delta v_\varepsilon = g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma}) & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

and thus $\int_\Omega \langle \nabla v_\varepsilon, \nabla\varphi\rangle = \int_\Omega g_\varepsilon(\cdot, v_\varepsilon + \widetilde{\sigma})\varphi$ for any $\varphi \in H_0^1(\Omega)$, and so (since $\int_\Omega \langle \nabla\widetilde{\sigma}, \nabla\varphi\rangle = 0$ for any $\varepsilon \in (0,1]$ and $\varphi \in H_0^1(\Omega)$)

$$\int_\Omega \langle \nabla u_\varepsilon, \nabla\varphi\rangle = \int_\Omega g_\varepsilon(\cdot, u_\varepsilon)\varphi \quad \text{for any } \varphi \in H_0^1(\Omega). \tag{3.15}$$

Let $\varphi \in H_0^1(\Omega)$. From Lemma 3.3 it follows that $u \in H^1(\Omega))$ and that $\lim_{\varepsilon\to 0^+} u_\varepsilon = u$ with convergence in $H_0^1(\Omega)$. Then $\lim_{\varepsilon\to 0^+} \int_\Omega \langle \nabla u_\varepsilon, \nabla\varphi\rangle = \int_\Omega \langle \nabla u, \nabla\varphi\rangle$. Again by Lemma 3.3, $\lim_{\varepsilon\to 0^+} g_\varepsilon(\cdot, u_\varepsilon) = g(\cdot, u)$ with convergence in $L^2(\Omega, \rho_{\partial\Omega}^2(x)dx)$ and thus $\lim_{\varepsilon\to 0^+} \int_\Omega g_\varepsilon(\cdot, u_\varepsilon)\varphi = \int_\Omega g(\cdot, u)\varphi$. Then, from (3.15),

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle = \int_\Omega g(\cdot, u)\varphi.$$

Thus u is a weak solution of problem (1.1). Also, Lemma 3.2 gives that $v_\varepsilon \geq c\rho_{\partial\Omega}$ for some positive constant c independent of ε, and then $u \geq c\rho_{\partial\Omega}$ in Ω.

If w is another weak solution of (1.1), then $u - w \in H_0^1(\Omega)$ and

$$\int_\Omega \langle \nabla(u-w), \nabla\varphi\rangle = \int_\Omega (g(\cdot, u) - g(\cdot, w))\varphi$$

for any $\varphi \in H_0^1(\Omega)$. We take $\varphi = u - w$ and, since $g(x,s)$ is nonincreasing in s, we get

$$\int_{N\Omega} |\nabla(u-w)|^2 = \int_\Omega (g(\cdot, u) - g(\cdot, w))(u-w) \leq 0.$$

Thus $\int_\Omega |\nabla(u-w)|^2 = 0$ which, by the Poincaré inequality, gives $u = w$. □

4. THE CASE OF MIXED DIRICHLET–NEUMAN BOUNDARY CONDITIONS

Our aim in this section is to prove Theorems 1.6 and 1.7. We assume, from now on, that $g : \Omega \times (0,\infty) \to \mathbb{R}$ satisfies the conditions (H1) and (H2) of Theorem 1.3 as well as the condition (H3') of Theorem 1.6. Since the condition (H3') implies the condition (H3) of Theorem 1.3 (because $\rho_{\partial\Omega} \leq \rho_{\Gamma_1}$), all the results of the previous section for the Dirichlet problems still hold under our new assumptions.

Remark 4.1.

(i) If $f \in L^2(\Omega, \rho^2_{\Gamma_1}(x)dx)$, $\tau \in H^{\frac{1}{2}}(\Gamma_1)$ and $\eta \in (H^1_{0,\Gamma_1}(\Omega))'$ (notice that we are not assuming that η is a function defined in Γ_2, then the problem of finding $u \in H^1(\Omega)$ such that

$$\begin{cases} \int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega f\varphi + \eta(\varphi) \text{ for any } \varphi \in H^1_{0,\Gamma_1}(\Omega), \\ u = \tau \text{ on } \Gamma_1 \end{cases} \tag{4.1}$$

has a unique solution, and it satisfies

$$\|u\|_{H^1(\Omega)} \le c(\|f\|_{(H^1_0(\Omega))'} + \|\tau\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\eta\|_{(H^1_{0,\Gamma_1}(\Omega))'}). \tag{4.2}$$

for some positive constant c independent of f, τ and η. Indeed, let $\sigma \in H^{\frac{1}{2}}(\partial\Omega)$ be defined by $\sigma = \tau$ on Γ_1 and $\sigma = 0$ on Γ_2, and let $\xi \in H^1(\Omega)$ be such that $\xi = \sigma$ on $\partial\Omega$. By writing $u = z + \xi$, the problem of finding u becomes equivalent to the problem of finding $z \in H^1_{0,\Gamma_1}(\Omega)$ such that

$$\int_\Omega \langle \nabla z, \nabla \varphi \rangle = \int_\Omega f\varphi - \int_\Omega \langle \nabla \xi, \nabla \varphi \rangle + \eta(\varphi) \quad \text{for any } \varphi \in H^1_{0,\Gamma_1}(\Omega), \tag{4.3}$$

i.e., such that

$$B(z, \varphi) = L(\varphi) \text{ for any } \varphi \in H^1_{0,\Gamma_1}(\Omega),$$

where, for $w \in H^1_{0,\Gamma_1}(\Omega)$ and $\varphi \in H^1_{0,\Gamma_1}(\Omega)$,

$$B(w, \varphi) := \int_\Omega \langle \nabla w, \nabla \varphi \rangle \quad \text{and} \quad L(\varphi) := \int_\Omega f\varphi - \int_\Omega \langle \nabla \xi, \nabla \varphi \rangle + \eta(\varphi).$$

Since B is a continuous and coercive bilinear form on $H^1_{0,\Gamma_1}(\Omega) \times H^1_{0,\Gamma_1}(\Omega)$ and $L \in (H^1_{0,\Gamma_1}(\Omega))'$, the Lax Milgram theorem gives the existence and uniqueness of the solution $z \in H^1_{0,\Gamma_1}(\Omega)$ of (4.3), and that it satisfies $\|z\|_{H^1_{0,\Gamma_1}(\Omega)} \le c' \|L\|_{(H^1_{0,\Gamma_1}(\Omega))'}$ for some positive constant c' independent of f, τ, and η. Then problem (4.1) has a unique solution $u \in H^1(\Omega)$ given by $u := z + \xi$. And, since

$$\|L\|_{(H^1_{0,\Gamma_1}(\Omega))'} \le \|f\|_{(H^1_0(\Omega))'} + \|\xi\|_{H^1(\Omega)} + \|\eta\|_{(H^1_{0,\Gamma_1}(\Omega))'}$$

and (see [43, Section 7.9.3, formula (7.48)])

$$\|\tau\|_{H^{\frac{1}{2}}(\Gamma_1)} = \|\sigma\|_{H^{\frac{1}{2}}(\Gamma_1)} = \inf\left\{\|w\|_{H^1(\Omega)} : w \in H^1(\Omega) \text{ and } w = \sigma \text{ on } \partial\Omega\right\},$$

we get (4.2).

(ii) From (i) it follows that if $f \in L^2(\Omega, \rho^2_{\Gamma_1}(x)dx)$, $\tau \in H^{\frac{1}{2}}(\Gamma_1)$ and if $\eta : \Gamma_2 \to \mathbb{R}$ belongs to $(H^1_{0,\Gamma_1}(\Omega))'$, then the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = \tau & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = \eta \end{cases} \tag{4.4}$$

has a unique weak solution $u \in H^1(\Omega)$.

Definition 4.2. For $\tau \in H^{\frac{1}{2}}(\Gamma_1)$, $\eta \in (H^1_{0,\Gamma_1}(\Omega))'$, let

$$S_{\tau,\eta} : L^2(\Omega, \rho^2_{\Gamma_1}(x)dx) \to H^1(\Omega)$$

be the solution operator of the problem

$$\begin{cases} \int\limits_\Omega \langle \nabla u, \nabla \varphi \rangle = \int\limits_\Omega h\varphi + \eta(\varphi) \quad \text{for any } \varphi \in H^1_{0,\Gamma_1}(\Omega), \\ u = \tau \text{ on } \Gamma_1, \end{cases} \tag{4.5}$$

defined by $S_{\tau,\eta}(h) = u$, where u is the weak solution of (4.5). If no confusion arises, we will write S instead of $S_{\tau,\eta}$.

Lemma 4.3. *Let $\tau \in H^{\frac{1}{2}}(\Gamma_1)$, $\eta \in (H^1_{0,\Gamma_1}(\Omega))'$. Then:*

(i) *$S : L^2(\Omega, \rho^2_{\Gamma_1}(x)dx) \to H^1(\Omega)$ is continuous,*
(ii) *$S : L^2(\Omega, \rho^2_{\Gamma_1}(x)dx) \to L^2(\Omega)$ is a continuous and compact operator,*
(iii) *if h_1 and h_2 belong to $L^2(\Omega, \rho^2_{\Gamma_1}(x)dx)$ and $h_1 \le h_2$ then $S(h_1) \le S(h_2)$,*
(iv) *if, in addition, $\tau \ge 0$ and $\eta \ge 0$ then $S(h) \ge 0$ for any nonnegative $h \in L^2(\Omega, \rho^2_{\Gamma_1}(x)dx)$.*

Proof. If h_1 and h_2 belong to $L^2(\Omega, \rho^2_{\Gamma_1}(x)dx)$ and if $u_1 = S(h_1)$ and $u_2 = S(h_2)$ then $u_1 - u_2$ satisfies

$$\begin{cases} \int\limits_\Omega \langle \nabla(u_1 - u_2), \nabla \varphi \rangle = \int\limits_\Omega (h_1 - h_2)\varphi \quad \text{for any } \varphi \in H^1_{0,\Gamma_1}(\Omega), \\ u_1 - u_2 = 0 \text{ on } \Gamma_1, \end{cases} \tag{4.6}$$

and so, by (4.2),

$$\|u_1 - u_2\|_{H^1(\Omega)} \le c\,\|h_1 - h_2\|_{(H^1(\Omega))'} \le c'\,\|h_1 - h_2\|_{L^2(\Omega, \rho^2_{\Gamma_1}(x)dx)}$$

with c and c' positive constants independent of h_1 and h_2. Then (i) holds, and (ii) follows from (i) and from the fact that the inclusion $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous and compact.

To prove (iii) observe that if $u_1 = S(h_1)$ and $u_2 = S(h_2)$, then, from (4.6) used with $\varphi = (u_1 - u_2)^+$, we get $\int_\Omega |\nabla((u_1 - u_2)^+)|^2 \le 0$ and so

$$\int\limits_\Omega \left|\nabla((u_1 - u_2)^+)\right|^2 = 0.$$

Then, by the Poincaré inequality of Remark 2.1(ii), $(u_1 - u_2)^+ = 0$ and thus $u_1 \le u_2$. To see (iv) suppose $\eta \ge 0$ and $0 \le h \in L^2(\Omega, \rho^2_{\Gamma_1}(x)dx)$. Let $u = S(h)$. Then

$$\int\limits_\Omega \langle \nabla u, \nabla \varphi \rangle = \int\limits_\Omega h\varphi + \eta(\varphi) \quad \text{for any } \varphi \in H^1_{0,\Gamma_1}(\Omega).$$

We take $\varphi = -u^-$ to obtain

$$\int_\Omega \left|\nabla u^-\right|^2 = -\int_\Omega \left\langle \nabla u, \nabla u^- \right\rangle = -\int_\Omega hu^- - \int_{\Gamma_2} \eta u^- \leq 0.$$

Then $\int_\Omega \left|\nabla u^-\right|^2 = 0$ and so, by Remark 2.1(ii), $u^- = 0$. □

Proof of Theorem 1.6. Let $u_\tau \in H^1(\Omega)$ be the solution of the problem

$$\begin{cases} -\Delta u_\tau = g(\cdot, u_\tau) & \text{in } \Omega, \\ u_\tau = \tau & \text{on } \Gamma_1, \\ u_\tau = 0 & \text{on } \Gamma_2 \end{cases} \tag{4.7}$$

given by Theorem 1.3. Let $\eta : \Gamma_2 \to \mathbb{R}$ be such that $\eta \in (H^1_{0,\Gamma_1}(\Omega))'$ and $\eta \geq \frac{\partial u_\tau}{\partial \nu}_{\Gamma_2}$ in $(H^1_{0,\Gamma_1}(\Omega))'$, let $\Phi \in H^1(\Omega)$ be the solution of the problem

$$\begin{cases} \int_\Omega \langle \nabla \Phi, \nabla \varphi \rangle = \Big(\eta - \dfrac{\partial u_\tau}{\partial \nu}_{\Gamma_2}\Big)(\varphi) \text{ for any } \varphi \in H^1_{0,\Gamma_1}(\Omega), \\ \Phi = 0 \text{ on } \Gamma_1 \end{cases} \tag{4.8}$$

(by Remark 4.1(i), there exists such a unique Φ), and let $z = \Phi + u_\tau$. Since $\eta - \frac{\partial u_\tau}{\partial \nu}_{\Gamma_2} \geq 0$, Lemma 4.3(iv) gives that $\Phi \geq 0$, thus $u_\tau \leq z$. Note that for any nonnegative $\varphi \in H^1_{0,\Gamma_1}(\Omega)$,

$$\int_\Omega \langle \nabla u_\tau, \nabla \varphi \rangle = \int_\Omega g(\cdot, u_\tau)\varphi + \frac{\partial u_\tau}{\partial \nu}_{\Gamma_2}(\varphi) \leq \int_\Omega g(\cdot, u_\tau)\varphi + \eta(\varphi),$$

and so, for any nonnegative $\varphi \in H^1_{0,\Gamma_1}(\Omega)$,

$$\begin{aligned} \int_\Omega \langle \nabla z, \nabla \varphi \rangle &= \int_\Omega \langle \nabla u_\tau, \nabla \varphi \rangle + \int_\Omega \langle \nabla \Phi, \nabla \varphi \rangle \\ &= \int_\Omega g(\cdot, u_\tau)\varphi + \frac{\partial u_\tau}{\partial \nu}_{\Gamma_2}(\varphi) + \Big(\eta - \frac{\partial u_\tau}{\partial \nu}_{\Gamma_2}\Big)(\varphi) \\ &\geq \int_\Omega g(\cdot, \Phi + u_\tau)\varphi + \eta(\varphi) = \int_\Omega g(\cdot, z)\varphi + \eta(\varphi), \end{aligned}$$

where we have used that $\frac{\partial \Phi}{\partial \nu}_{\Gamma_2}(\varphi) = \int_\Omega \langle \nabla \Phi, \nabla \varphi \rangle$ and that $g = g(x, s)$ is nonincreasing in s.

Then for any nonnegative $\varphi \in H^1_{0,\Gamma_1}(\Omega)$,

$$\int_\Omega \langle \nabla u_\tau, \nabla \varphi \rangle \leq \int_\Omega g(\cdot, u_\tau)\varphi + \eta(\varphi) \tag{4.9}$$

and

$$\int_\Omega \langle \nabla z, \nabla \varphi \rangle \geq \int_\Omega g(\cdot, z)\varphi + \eta(\varphi). \tag{4.10}$$

To prove the existence assertion of the theorem we will show that problem (1.2) has a solution u^* such that $u_\tau \leq u^* \leq z$.

As in the proof of [35, Theorem 1.1] we define $\overline{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{g}(x,s) := \begin{cases} g(x, u_\tau(x)) & \text{if } s \leq u_\tau(x), \\ g(x,s) & \text{if } u_\tau(x) < s < z(x), \\ g(x, z(x)) & \text{if } s \geq z(x). \end{cases}$$

It is easy to check that $\overline{g}$ is a nonnegative Carathéodory function on $\Omega \times \mathbb{R}$ (because g is a Carathéodory function on $\Omega \times (0,\infty)$ and u_τ, z are measurable functions) and that $\overline{g}(x,s)$ is nonincreasing in s. Moreover, if $E \subset \Omega$ is the set given by the condition (H2) then $\overline{g}(x,s) > 0$ for any $x \in E$ and $s > 0$. Also, since $u_\tau \leq z$ and, taking into account that, by Theorem 1.3, $u_\tau \geq c\rho_{\partial\Omega}$ for some $c \in (0,\infty)$ and that $\overline{g}(x,s)$ is nonnegative and nonincreasing in s, we obtain that

$$0 \leq \overline{g}(\cdot, s) \leq \overline{g}(\cdot, u_\tau) = g(\cdot, u_\tau) \leq g(\cdot, c\rho_{\partial\Omega}) \tag{4.11}$$

for any $s \in \mathbb{R}$, and so, for any $v \in L^2(\Omega)$, $0 \leq \rho_{\Gamma_1}\overline{g}(\cdot, v) \leq \rho_{\Gamma_1} g(\cdot, c\rho_{\partial\Omega})$, and, by (H3'), $\rho_{\Gamma_1} g(\cdot, c\rho_{\partial\Omega}) \in L^2(\Omega)$. Therefore, taking into account the Hardy inequality of Lemma 2.2 we have, for any $\varphi \in H^1_{0,\Gamma_1}(\Omega)$,

$$\int_\Omega |\overline{g}(\cdot, v)\varphi| = \int_\Omega \rho_{\Gamma_1}\overline{g}(\cdot, v) \left| \frac{\varphi}{\rho_{\Gamma_1}} \right| \leq \|\rho_{\Gamma_1}\overline{g}(\cdot, v)\|_2 \left\| \frac{\varphi}{\rho_{\Gamma_1}} \right\|_2 \leq c' \|\varphi\|_{H^1(\Omega)}$$

with c' a positive constant independent of v and φ. Thus $\overline{g}(\cdot, v) \in (H^1_{0,\Gamma_1}(\Omega))'$ and

$$\|\overline{g}(\cdot, v)\|_{(H^1_{0,\Gamma_1}(\Omega))'} \leq c' \tag{4.12}$$

with c' independent of v. Following the lines of the proof of [35, Theorem 1.1] we consider the operator $T : L^2(\Omega) \to L^2(\Omega)$ defined by

$$T(v) := S(\overline{g}(\cdot, v)).$$

with S given by Definition 4.2.

We prove that:

(1) T is continuous,
(2) T is a compact operator,
(3) there exists $R > 0$ such that $T(L^2(\Omega)) \subset \overline{B}$, where $\overline{B}$ is the closed ball in $L^2(\Omega)$ centered at 0 and with radius R.

To prove (1) we proceed similarly to the proof of Lemma 3.1(iii). It is enough to see that if $v \in L^2(\Omega)$, and if $\{v_j\}_{j\in\mathbb{N}}$ is a sequence in $L^2(\Omega)$ that converges to v in $L^2(\Omega)$, then there exists a subsequence $\{v_{j_k}\}_{k\in\mathbb{N}}$ such that $\{T(v_{j_k})\}_{k\in\mathbb{N}}$ converges to $T(v)$ in $L^2(\Omega)$. Let $v \in L^2(\Omega)$, and let $\{v_j\}_{j\in\mathbb{N}}$ be a sequence in $L^2(\Omega)$ which converges to v in $L^2(\Omega)$, then there exists a subsequence $\{v_{j_k}\}_{k\in\mathbb{N}}$ such that $\{v_{j_k}\}_{k\in\mathbb{N}}$ converges to v a.e. in Ω. Thus, since $\overline{g}$ is a Carathéodory function, $\{\overline{g}(\cdot, v_{j_k})\}_{k\in\mathbb{N}}$ converges to $\overline{g}(\cdot, v)$ a.e. in Ω. Then $\lim_{k\to\infty} |\overline{g}(\cdot, v_{j_k}) - \overline{g}(\cdot, v)|^2 = 0$ a.e. in Ω. By (4.11), $|\overline{g}(\cdot, v_{j_k}) - \overline{g}(\cdot, v)|^2 \leq 4g^2(., c\rho_{\partial\Omega})$, and, by (H3') we have $\int_\Omega \rho_{\Gamma_1}^2 g^2(\cdot, c\rho_{\partial\Omega}) < \infty$. Then, by the Lebesgue dominated convergence theorem, $\{\overline{g}(\cdot, v_{j_k})\}_{k\in\mathbb{N}}$ converges to $\overline{g}(\cdot, v)$ in $L^2(\Omega, \rho_{\Gamma_1}^2(x)dx)$. Then, by Lemma 4.3(ii), $\{S(\overline{g}(\cdot, v_{j_k}))\}_{k\in\mathbb{N}}$ converges to $S(\overline{g}(\cdot, v))$ in $L^2(\Omega)$, i.e., $\{T(v_{j_k})\}_{k\in\mathbb{N}}$ converges to $T(v)$ in $L^2(\Omega)$. Thus (1) holds.

To see (2) note that, by (4.11), $\{\overline{g}(\cdot, v_j)\}_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega, \rho_{\Gamma_1}^2(x)dx)$ for any sequence $\{v_j\}_{j\in\mathbb{N}}$ in $L^2(\Omega)$, and that $S : L^2(\Omega, \rho_{\Gamma_1}^2(x)dx) \to L^2(\Omega)$ is compact.

To see (3) observe that, by (4.2) and (4.12), we have, for any $v \in L^2(\Omega)$,

$$\begin{aligned}
\|T(v)\|_2 &= \|S(\overline{g}(\cdot, v))\|_2 \\
&\leq c(\|\overline{g}(\cdot, v)\|_{(H_0^1(\Omega))'} + \|\tau\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\eta\|_{(H_{0,\Gamma_1}^1(\Omega))'}) \\
&\leq c(c' + \|\tau\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\eta\|_{(H_{0,\Gamma_1}^1(\Omega))'})
\end{aligned}$$

with c and c' positive constants independent of v.

Now, as in [35, Theorem 1.1], from (1), (2), (3) and the Schauder fixed point theorem, there exists $u^* \in L^2(\Omega)$ such that $T(u^*) = u^*$, i.e., such that

$$\begin{cases}
-\Delta u^* = \overline{g}(\cdot, u^*) & \text{in } \Omega, \\
u^* = \tau & \text{on } \Gamma_1, \\
\frac{\partial u^*}{\partial \nu} = \eta & \text{on } \Gamma_2.
\end{cases} \tag{4.13}$$

To complete the proof of the existence assertion of the theorem it suffices to see that $u_\tau \leq u^* \leq z$ (because in such a case $\overline{g}(\cdot, u^*) = g(\cdot, u^*)$ and, by Theorem 1.3, $u_\tau \geq c\rho_{\partial\Omega}$ for some positive constant c). From (4.10), (4.13), and since $\overline{g}(\cdot, z) = g(\cdot, z)$ we have, for any nonnegative $\varphi \in H^1_{0,\Gamma_1}(\Omega)$,

$$\begin{aligned}\int_\Omega \langle \nabla(z-u^*), \nabla\varphi\rangle &= \int_\Omega \langle \nabla z, \nabla\varphi\rangle - \int_\Omega \langle \nabla u^*, \nabla\varphi\rangle \\ &\geq \int_\Omega g(\cdot,z)\varphi + \eta(\varphi) - \int_\Omega \overline{g}(\cdot,u^*) - \eta(\varphi) \\ &= \int_\Omega (\overline{g}(\cdot,z) - \overline{g}(\cdot,u^*))\varphi\end{aligned}$$

which, by taking $\varphi = (z-u^*)^-$ gives

$$\int_\Omega \left|\nabla((z-u^*)^-)\right|^2 \leq -\int_\Omega (\overline{g}(\cdot,z) - \overline{g}(\cdot,u^*))(z-u^*)^- \leq 0,$$

the last inequality because $\overline{g}(x,s)$ is nonincreasing in s. Then, by Remark 2.1(ii), $(z-u^*)^- = 0$ and so $u^* \leq z$.

Similarly, from (4.7), (4.13) and since $\overline{g}(\cdot,u_\tau) = g(\cdot,u_\tau)$, we have, for any nonnegative $\varphi \in H^1_{0,\Gamma_1}(\Omega)$,

$$\begin{aligned}\int_\Omega \langle \nabla(u^*-u_\tau), \nabla\varphi\rangle &= \int_\Omega (\overline{g}(.,u^*) - \overline{g}(\cdot,u_\tau))\varphi + \eta(\varphi) - \frac{\partial u_\tau}{\partial\nu}_{\Gamma_2}(\varphi) \\ &\leq \int_\Omega (\overline{g}(\cdot,u^*) - \overline{g}(\cdot,u_\tau))\varphi,\end{aligned} \tag{4.14}$$

the last inequality by our assumption that $\eta \geq \frac{\partial u_\tau}{\partial\nu}_{\Gamma_2}$. Observe that $u^* - u_\tau \in H^1_{0,\Gamma_1}(\Omega)$ and that, since $\overline{g}(\cdot,s)$ is nonincreasing in s,

$$(\overline{g}(\cdot,u^*) - \overline{g}(\cdot,u_\tau))(u^*-u_\tau)^- \geq 0.$$

Thus, taking $\varphi = -(u^*-u_\tau)^-$ in (4.14) we obtain $\int_\Omega \left|\nabla((u^*-u_\tau)^-)\right|^2 = 0$, which implies $(u^*-u_\tau)^- = 0$ and so $u_\tau \leq u^*$.

Suppose that $w \in H^1(\Omega)$ is another solution of (1.2). Then $u^* - w \in H^1_{0,\Gamma_1}(\Omega)$ and, in a weak sense,

$$\begin{cases} -\Delta(u^*-w) = g(\cdot,u^*) - g(.,w) & \text{in } \Omega, \\ u^* - w = 0 & \text{on } \Gamma_1, \\ \frac{\partial(u^*-w)}{\partial\nu} = 0 & \text{on } \Gamma_2, \end{cases} \tag{4.15}$$

that is,

$$\int_\Omega \langle \nabla(u^*-w), \nabla\varphi\rangle = \int_\Omega (g(\cdot,u^*) - g(\cdot,w))\varphi \text{ for any } \varphi \in H^1_{0,\Gamma_1}(\Omega), \tag{4.16}$$

Now, $g(x,\cdot)$ is nonincreasing on $(0,\infty)$ for a.e. $x\in\Omega$, and so

$$g(\cdot,u^*)-g(\cdot,w)(u^*-w)\leq 0 \text{ a.e. in } \Omega.$$

Thus, taking $\varphi=u^*-w$ in (4.16), we get

$$\int_\Omega |\nabla(u^*-w)|^2=\int_\Omega (g(\cdot,u^*)-g(\cdot,w))(u^*-w)\leq 0$$

and so $\|\nabla(u^*-w)\|_2=0$, Then, by Remark 2.1(ii), $u^*=w$ in Ω. This concludes the proof of the part (i) of the theorem.

To see (ii), suppose that $\eta<\frac{\partial u_\tau}{\partial\nu}_{\Gamma_2}$ and that u is a weak solution of problem (1.2). Then, for any nonnegative $\varphi\in H^1_{0,\Gamma_1}(\Omega)$,

$$\int_\Omega \langle\nabla u,\nabla\varphi\rangle=\int_\Omega g(\cdot,u)\varphi+\eta(\varphi)\leq\int_\Omega g(\cdot,u)\varphi+\frac{\partial u_\tau}{\partial\nu}_{\Gamma_2}(\varphi),$$

and

$$\int_\Omega \langle\nabla u_\tau,\nabla\varphi\rangle=\int_\Omega g(\cdot,u_\tau)\varphi+\frac{\partial u_\tau}{\partial\nu}_{\Gamma_2}(\varphi).$$

Thus, for any nonnegative $\varphi\in H^1_{0,\Gamma_1}(\Omega)$,

$$\int_\Omega \langle\nabla(u-u_\tau),\nabla\varphi\rangle\leq\int_\Omega (g(\cdot,u)-g(\cdot,u_\tau))\varphi.$$

Now we take $\varphi=(u-u_\tau)^+$ to obtain that

$$\int_\Omega \left|\nabla((u-u_\tau)^+)\right|^2\leq\int_\Omega (g(\cdot,u)-g(\cdot,u_\tau))(u-u_\tau)^+\leq 0,$$

the last inequality because $g(x,s)$ is nonincreasing in s. Thus $(u-u_\tau)^+=0$ and so $u\leq u_\tau$. Since u is nonnegative and $u_\tau=0$ on Γ_2 we conclude that $u=0$ on Γ_2. Then u is a solution of problem (1.13) and, by Theorem 1.3, this problem has a unique solution. Then $u=u_\tau$, and so $\eta=\frac{\partial u_\tau}{\partial\nu}_{\Gamma_2}$, which is a contradiction. Therefore no such a solution u exists. □

Lemma 4.4. *If $0\leq f\in L^2(\Omega,d^2_{\Gamma_1}(x)dx)$, $0\leq\tau\in H^{\frac12}(\Gamma_1)$, and if $u\in H^1(\Omega)$ is the weak solution of the problem*

$$\begin{cases}-\Delta u=f & \textit{in}\,\Omega,\\ u=\tau & \textit{on}\,\Gamma_1,\\ u=0 & \textit{on}\,\Gamma_2,\end{cases}$$

then $\frac{\partial u}{\partial\nu}_{\Gamma_2}\leq 0$.

Proof. Let $\Psi : \partial\Omega \to \mathbb{R}$ be defined by $\Psi = \tau$ on Γ_1 and $\Psi = 0$ on Γ_2. Then $0 \le \Psi \in H^{\frac{1}{2}}(\partial\Omega)$ and thus there exists $\widetilde{\Psi} \in H^1(\Omega)$ such that $\widetilde{\Psi} = \Psi$ on $\partial\Omega$. By replacing $\widetilde{\Psi}$ by $\widetilde{\Psi}^+$ if necessary, we can assume that $\widetilde{\Psi} \ge 0$ in Ω. Now, Ω is a bounded domain with C^2 boundary, and then $C^\infty(\overline{\Omega})$ is dense in $W^{1,2}(\Omega)$ (see [1, Theorem 3.18]). Then there exists a sequence $\{\widetilde{\Psi}_j\}_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that $\{\widetilde{\Psi}_j\}_{j\in\mathbb{N}}$ converges to $\widetilde{\Psi}$ in $H^1(\Omega)$. An inspection of the proof of [1, Theorem 3.18] shows that, since $\widetilde{\Psi}$ is nonnegative, the functions $\widetilde{\Psi}_j$ can be chosen nonnegative. For $\gamma > 0$, let $\Omega_{\Gamma_2,\gamma}$ and $A_{\Gamma_2,\gamma}$ be defined as in (2.1). Let δ be a positive number such that $\Gamma_1 \cap A_{\Gamma_2,4\delta} = \varnothing$, and let $\phi \in C^\infty(\overline{\Omega})$ be such that $0 \le \phi \le 1$, $\phi = 0$ in $A_{\Gamma_2,\delta}$ and $\phi = 1$ in $\Omega_{\Gamma_2,2\delta}$. Then $0 \le \phi\widetilde{\Psi}_j \in C^\infty(\overline{\Omega})$, $\phi\widetilde{\Psi}_j = 0$ on Γ_2, and $\{\phi\widetilde{\Psi}_j\}_{j\in\mathbb{N}}$ converges to $\phi\widetilde{\Psi}$ in $H^1(\Omega)$. For $j \in \mathbb{N}$, let $\Psi_j := \phi\widetilde{\Psi}_{j|\partial\Omega}$ and let $f_j : \Omega \to \mathbb{R}$ be defined by $f_j(x) := \min\{j, f(x)\}$. Then $\Psi_j = 0$ on Γ_2, $\left\{\Psi_{j|\Gamma_1}\right\}_{j\in\mathbb{N}}$ converges to τ in $H^{\frac{1}{2}}(\Gamma_1)$ and $\{f_j\}_{j\in\mathbb{N}}$ converges to f in $L^2(\Omega,\ d^2_{\Gamma_1}(x)dx)$. In particular, $\{f_j\}_{j\in\mathbb{N}}$ converges to f in $(H^1_{0,\Gamma_1}(\Omega))'$. Now, $f_j \in L^\infty(\Omega)$ and Ψ_j is the restriction to $\partial\Omega$ of a function in $C^\infty(\overline{\Omega})$. Then (see, e.g., [29, Theorem 2.4.2.5], see also [26, Theorem 9.15]), the problem

$$\begin{cases} -\Delta u_j = f_j & \text{in } \Omega, \\ u_j = \Psi_j & \text{on } \partial\Omega \end{cases} \tag{4.17}$$

has a unique strong solution $u_j \in \bigcap_{1<p<\infty} W^{2,p}(\Omega) \subset C^1(\overline{\Omega})$. Since $f_j \ge 0$ and $\Psi_j \ge 0$ we have $u_j \ge 0$. Also, $u_j = 0$ on Γ_2, and then the Hopf boundary lemma, as stated in [42, Theorem 1.1], gives that $\frac{\partial u_j}{\partial\nu}(x) < 0$ for any $x \in \Gamma_2$. On the other hand, $\{f_j\}_{j\in\mathbb{N}}$ converges to f in $(H^1_{0,\Gamma_1}(\Omega))'$ and $\{\Psi_j\}_{j\in\mathbb{N}}$ converges to Ψ in $H^{\frac{1}{2}}(\partial\Omega)$, then $\{u_j\}_{j\in\mathbb{N}}$ converges to u in $H^1(\Omega)$. Let φ be an arbitrary nonnegative function in $H^1_{0,\Gamma_1}(\Omega)$. From (4.17), we have $-\operatorname{div}(\varphi\nabla u_j) + \langle\nabla u_j, \nabla\varphi\rangle = f_j\varphi$ in Ω, and so, by the divergence theorem (as stated, for example, in [14, Lemma A.1]),

$$-\int\limits_{\Gamma_2} \varphi\frac{\partial u_j}{\partial\nu} + \int\limits_{\Omega} \langle\nabla u_j, \nabla\varphi\rangle = \int\limits_{\Omega} f_j\varphi.$$

Then $\int_\Omega \langle\nabla u_j, \nabla\varphi\rangle - \int_\Omega f_j\varphi \ge 0$ and thus, taking into account that $\{\nabla u_j\}_{j\in\mathbb{N}}$ converges to ∇u in $L^2(\Omega, \mathbb{R}^n)$ and that $\{f_j\}_{j\in\mathbb{N}}$ converges to f in $(H^1_{0,\Gamma_1}(\Omega))'$, we get that $\int_\Omega \langle\nabla u, \nabla\varphi\rangle - \int_\Omega f\varphi \ge 0$. Then $\frac{\partial u}{\partial\nu}_{\Gamma_2} \le 0$. □

Proof of Corollary 1.7. Let u_τ be the solution (given by Theorem 1.3) of problem (4.7). By Lemma 4.4, we have $\frac{\partial u}{\partial\nu}_{\Gamma_2} \le 0$. Then the corollary follows immediately from Theorem 1.6. □

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