Opuscula Math. 42, no. 3 (2022), 439–458
https://doi.org/10.7494/OpMath.2022.42.3.439 Opuscula Mathematica

DISTANCE IRREGULARITY STRENGTH OF GRAPHS WITH PENDANT VERTICES

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Communicated by Mirko Horňák

Abstract. A vertex k-labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$ on a simple graph G is said to be a distance irregular vertex k-labeling of G if the weights of all vertices of G are pairwise distinct, where the weight of a vertex is the sum of labels of all vertices adjacent to that vertex in G. The least integer k for which G has a distance irregular vertex k-labeling is called the distance irregularity strength of G and denoted by $\mathrm{dis}(G)$. In this paper, we introduce a new lower bound of distance irregularity strength of graphs and provide its sharpness for some graphs with pendant vertices. Moreover, some properties on distance irregularity strength for trees are also discussed in this paper.

Keywords: vertex k-labeling, distance irregular vertex k-labeling, distance irregularity strength, pendant vertices.

Mathematics Subject Classification: 05C78, 05C12.

1. INTRODUCTION

All graphs considered here are simple, finite and undirected. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the set of vertices and edges of G, the maximum and minimum degree of G, respectively. For a vertex $v \in V(G)$, we use $N_G(v)$ and $\deg_G(v)$ to denote the neighborhood and the degree of v in G, respectively. When the context is clear, we simply write such notations, respectively, with V, E, Δ, δ, $N(v)$ and $\deg(v)$. The vertex v is called an *isolated vertex* when $\deg(v) = 0$ and is called a *pendant vertex* when $\deg(v) = 1$.

In [14], Miller *et al.* defined a *distance magic labeling* of graphs as a bijection $\psi : V \to \{1, 2, \ldots, |V|\}$ such that all the vertex weights are equal to a *magic constant* k, where the weight of a vertex $v \in V$ is defined as the sum of labels of vertices adjacent to v. A graph that has a distance magic labeling is called *distance magic*.

More general concept was made by Arumugam and Kamatchi [4], where the requirement now is that the vertex weights are not necessarily the same but must form an arithmetic sequence starting from a with common difference d for some fixed integers $a > 0$ and $d \geq 0$. They named such a labeling an (a,d)-*distance antimagic labeling* and a graph that admits an (a,d)-distance antimagic labeling is called an (a,d)-*distance antimagic graph*.

For an edge k-labeling $\varphi : E \to \{1,2,\dots,k\}$ the associated *weight* of a vertex $v \in V$ is defined as

$$wt_{\varphi}(v) = \sum_{uv \in E} \varphi(uv),$$

where the sum is taken over all vertices u adjacent to v. In [10], Chartrand *et al.* defined an edge k-labeling φ of a graph G such that for every two distinct vertices $u, v \in V$ then $wt_{\varphi}(u) \neq wt_{\varphi}(v)$. Such labelings are called *irregular assignments* and the *irregularity strength*, $s(G)$, of a graph G is known as the least integer k such that G has an irregular assignment using labels at most k. This parameter was studied extensively in numerous papers, see [2, 3, 11, 13, 16]. Fascinating modifications on irregular assignments were also developed by some authors, see [1, 5, 6, 12].

In [17], Slamin introduced distance irregular vertex labelings as a unification of distance-based labelings and irregular labelings of graphs. A vertex k-labeling $\phi : V \to \{1,2,\dots,k\}$ is said to be a *distance irregular vertex k-labeling* of G if for every two distinct vertices $u, v \in V$ there is $wt_{\phi}(u) \neq wt_{\phi}(v)$, where the *weight* of a vertex v is

$$wt_{\phi}(v) = \sum_{u \in N(v)} \phi(u).$$

The *distance irregularity strength*, $\mathrm{dis}(G)$, of G is the smallest integer k for which G has a distance irregular vertex k-labeling. Some results on distance irregularity strength for families of graphs have been found, including, for example, complete graphs, paths, cycles and wheels [8, 17], ladders and triangular ladders [15], and some classes of disconnected graphs [18]. In the literature, there are also investigated a variation and generalizations of this concept, see [7, 9].

In [17], it was given a general lower bound for the distance irregularity strength of graphs.

Theorem 1.1 ([17]). *Let G be a graph with minimum degree δ and maximum degree Δ containing no isolated vertex and $N(u) \neq N(v)$ for $u, v \in V$, $u \neq v$. Then*

$$\mathrm{dis}(G) \geq \left\lceil \frac{|V| + \delta - 1}{\Delta} \right\rceil.$$

Susanto *et al.* [18] improved that lower bound for the case if a graph has pendant vertices. They proved the following.

Theorem 1.2 ([18]). *Let G be a graph with maximum degree Δ containing no isolated vertex and $N(u) \neq N(v)$ for $u, v \in V$, $u \neq v$. If G has t pendant vertices then*

$$\mathrm{dis}(G) \geq \max\left\{ t, \left\lceil \frac{|V|}{\Delta} \right\rceil \right\}.$$

In this paper, we introduce a new lower bound of the parameter dis(G) and determine the precise values of the distance irregularity strength for some graphs with pendant vertices. In addition, we study some properties on this invariant for trees. Notice that our lower bound improves the existing bounds in Theorems 1.1 and 1.2.

2. MAIN RESULTS

We begin this section with the following result which presents a new lower bound for the distance irregularity strength.

Theorem 2.1. *Let G be a graph with minimum degree δ and maximum degree Δ containing no isolated vertex and $N(u) \neq N(v)$ for $u, v \in V$, $u \neq v$. Let n_i be the number of vertices of degree i in G for every $i = \delta, \delta+1, \ldots, \Delta$. Then*

$$\mathrm{dis}(G) \geq \max_{\delta \leq i \leq \Delta} \left\{ \left\lceil \frac{\delta + \sum_{j=\delta}^{i} n_j - 1}{i} \right\rceil \right\}.$$

Proof. Let

$$\left\lceil \frac{\delta + \sum_{j=\delta}^{t} n_j - 1}{t} \right\rceil = \max_{\delta \leq i \leq \Delta} \left\{ \left\lceil \frac{\delta + \sum_{j=\delta}^{i} n_j - 1}{i} \right\rceil \right\}$$

for some t. In any distance irregular vertex labeling ϕ of a graph G, the smallest weight of vertices of degrees $\delta, \delta+1, \ldots, t$ is at least δ, and the largest among them must be at least $\delta + \sum_{j=\delta}^{t} n_j - 1$. Such largest weight is obtained from the sum of at most t labels. Therefore

$$\mathrm{dis}(G) \geq \left\lceil \frac{\delta + \sum_{j=\delta}^{t} n_j - 1}{t} \right\rceil = \max_{\delta \leq i \leq \Delta} \left\{ \left\lceil \frac{\delta + \sum_{j=\delta}^{i} n_j - 1}{i} \right\rceil \right\}.$$

□

The lower bound in Theorem 2.1 is tight as can be seen from Theorems 2.2, 2.6 and 2.7.

Let $G \odot H$ denotes the *Corona product* of two given graphs G and H. It is a graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and joining the ith vertex of G with every vertex of the ith copy of H.

Theorem 2.2. *Let G be a graph on n vertices. Then*

$$\mathrm{dis}(G \odot K_1) = n + r,$$

where $r \geq 0$ is the number of isolated vertices of G.

Proof. Let G be a graph of order n with r vertices of degree 0. Let n_i^H be the number of vertices of degree i in $H \cong G \odot K_1$ for each $i = 1, 2, \ldots, \Delta(H)$. Since $n_1^H = n + r$ and $\delta(H) = 1$ we have

$$\begin{aligned} \operatorname{dis}(H) &\geq \max_{\delta(H) \leq i \leq \Delta(H)} \left\{ \left\lceil \frac{\delta(H) + \sum_{j=\delta(H)}^{i} n_j^H - 1}{i} \right\rceil \right\} \\ &\geq \left\lceil \frac{\delta(H) + \sum_{j=1}^{1} n_j^H - 1}{1} \right\rceil = n + r. \end{aligned}$$

To prove that $n + r$ is also the upper bound for $\operatorname{dis}(H)$ we define a corresponding vertex labeling of H.

Let ϕ be a labeling on vertices of a graph H defined by using the following algorithm.

1. Let $x_1, x_2, \ldots, x_r$ be the isolated vertices of G. Let $y_1, y_2, \ldots, y_r$ be the vertices of H, where y_i is adjacent to x_i. Notice that x_i and y_i are pendant vertices in H. We define $\phi(x_i) = i$ and $\phi(y_i) = r + i$ for $i = 1, 2, \ldots, r$.
2. Denote all non-pendant vertices of H by $v_1, v_2, \ldots, v_{n-r}$ such that $\deg(v_i) \leq \deg(v_j)$ for $1 \leq i < j \leq n - r$. We denote by u_i, $i = 1, 2, \ldots, n - r$, the pendant vertex in $V(H)$ adjacent to v_i. Define $\phi(v_i) = 2r + i$ for $i = 1, 2, \ldots, n - r$.
3. Define $\omega(v_i) = \sum_{z \in N(v_i)} \phi(z)$ as the temporary weight of v_i, $i = 1, 2, \ldots, n - r$.
4. Define a set $W = \{\omega(v_1), \omega(v_2), \ldots, \omega(v_{n-r})\}$.
5. Set $K = n + r$.
6. While $W \neq \emptyset$ do
   a. $i = i + 1$.
   b. If (u_i has not been labeled) and ($\omega(v_i)$ is the smallest element of W) then
      (1) If $\omega(v_i) \leq K$ then
         (a) $K = K + 1$.
         (b) $\phi(u_i) = K - \omega(v_i)$.
         (c) $wt_\phi(v_i) = K$.
         (d) $W = W \setminus \{\omega(v_i)\}$.
      (2) Else
         (a) $K = \omega(v_i)$.
         (b) $K = K + 1$.
         (c) $\phi(u_i) = K - \omega(v_i) = 1$.
         (d) $wt_\phi(v_i) = K$.
         (e) $W = W \setminus \{\omega(v_i)\}$.
   c. If $i = n - r$ then
      (1) $i = 0$.

From the algorithm above, we observe that the labels used in the labeling ϕ are at most $n + r$. For the weights of all pendant vertices of H, we have

$$\begin{aligned} wt_\phi(x_i) &= r + i && \text{for } i = 1, 2, \ldots, r, \\ wt_\phi(y_i) &= i && \text{for } i = 1, 2, \ldots, r, \\ wt_\phi(u_i) &= 2r + i && \text{for } i = 1, 2, \ldots, n - r. \end{aligned}$$

Thus the weights of all pendant vertices lie on the set $\{1, 2, \dots, n + r\}$. Furthermore, one can verify that the weights of all non-pendant vertices of H are distinct and $wt_\phi(v_i) > n + r$ for each $i = 1, 2, \dots, n - r$. It means that ϕ is an optimal distance irregular vertex $(n + r)$-labeling of H. Hence $\mathrm{dis}(H) = n + r$. □

Let G be a graph on n vertices and m edges. The *subdivision* of G, denoted by $S(G)$, is a graph obtained from G by replacing each edge $uv \in E(G)$ with a path uwv of length two. We call the vertex w the *subdivision vertex* of the edge uv.

We define a fern graph with respect to a graph G as follows. Let us denote the vertices of G arbitrarily by the symbols $v_1, v_2, \dots, v_n$. For positive integers s_i, $1 \le i \le n$, the *fern* of a graph G, denoted by $\mathrm{Fern}(G; n; s_1, s_2, \dots, s_n)$, is a graph obtained from G by attaching exactly s_i pendant vertices to the vertex v_i of the graph G. A *monotonous fern* of G, $\mathrm{MoFern}(G; n; s_1, s_2, \dots, s_n)$, is the fern graph with property that for every two distinct vertices $v_i, v_j \in V(G)$ there is $s_i \le s_j$ if and only if $\deg_G(v_i) \le \deg_G(v_j)$. If $s_i = s_j = s$ for every $i \ne j$ then $\mathrm{Fern}(G; n; s, s, \dots, s) \cong \mathrm{Fern}(G; n; s)$ (respectively $\mathrm{MoFern}(G; n; s, s, \dots, s) \cong \mathrm{MoFern}(G; n; s)$). Note that

$$G \odot sK_1 \cong \mathrm{Fern}(G; n; s) \cong \mathrm{MoFern}(G; n; s).$$

Let F be a forest on n vertices and m edges. Let

$$H \cong S(\mathrm{MoFern}(F; n; s_1, s_2, \dots, s_n)), \quad 2 \le s_1 \le s_2 \le \ldots \le s_n,$$

be the subdivision of a monotonous fern of the forest F with

$$V(H) = V(F) \cup V(S_1) \cup V(S_2) \cup V(S_3),$$

where $V(F)$, $V(S_1)$, $V(S_2)$, $V(S_3)$ stand for the set of vertices of the base forest, the set of pendant vertices of H, the set of the subdivision vertices of all edges in F, and the set of the subdivision vertices of all pendant edges in H, respectively. We suppose that $V(F) = \{v_i : 1 \le i \le n\}$, where $\deg_F(v_i) \le \deg_F(v_j)$ for $i < j$. Furthermore, we may split $V(S_1)$ and $V(S_3)$ in such a way that

$$V(S_1) = \bigcup_{i=1}^{n} V(S_1^{v_i}) \quad \text{and} \quad V(S_3) = \bigcup_{i=1}^{n} V(S_3^{v_i}),$$

where $V(S_1^{v_i})$, $i = 1, 2, \dots, n$, is the set consisting of all pendant vertices that have distance 2 to v_i in H and

$$V(S_3^{v_i}) = \{z : z \text{ is adjacent to some vertex } x \in V(S_1^{v_i}) \text{ in } H\}.$$

Note that $|V(S_1^{v_i})| = |V(S_3^{v_i})| = s_i$.

Let us consider the smallest positive integer a satisfying

$$\frac{s_1}{2}(2a + (s_1 - 1)n) \ge \sum_{i=1}^{n} s_i + \left\lceil \frac{m}{2} \right\rceil + 1,$$

that is,

$$a = \max\left\{1, \left\lceil \frac{2\sum_{i=1}^{n} s_i - s_1 n(s_1 - 1) + m + 2}{2s_1} \right\rceil\right\}. \tag{2.1}$$

Lemma 2.3. *Let F be a forest on $n \geq 2$ vertices and $m \geq 1$ edges without an isolated vertex and let $H \cong S(MoFern(F; n; s_1, s_2, \ldots, s_n))$ for $2 \leq s_1 \leq s_2 \leq \ldots \leq s_n$. Let a be an integer defined in* (2.1). *Then*

$$a = 1 \quad \textit{if and only if} \quad \sum_{i=1}^{n} s_i < \left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1.$$

Proof. Clearly,

$$a = 1 \quad \text{if and only if} \quad \left\lceil \frac{2\sum_{i=1}^{n} s_i - s_1 n(s_1 - 1) + m + 2}{2s_1} \right\rceil \leq 1,$$

which is equivalent to

$$2\sum_{i=1}^{n} s_i - s_1 n(s_1 - 1) + m + 2 \leq 2s_1.$$

Thus

$$\sum_{i=1}^{n} s_i < \left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1. \qquad \square$$

From Lemma 2.3 we immediately get the following.

Lemma 2.4. *Let F be a forest on $n \geq 2$ vertices and $m \geq 1$ edges without an isolated vertex and let $H \cong S(MoFern(F; n; s_1, s_2, \ldots, s_n))$ for $2 \leq s_1 \leq s_2 \leq \ldots \leq s_n$. Let a be an integer defined in* (2.1). *Then*

$$a = \left\lceil \frac{2\sum_{i=1}^{n} s_i - s_1 n(s_1 - 1) + m + 2}{2s_1} \right\rceil$$

if and only if

$$\sum_{i=1}^{n} s_i \geq \left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1.$$

Lemma 2.5. *Let F be a forest on $n \geq 2$ vertices and $m \geq 1$ edges without an isolated vertex and let $H \cong S(MoFern(F; n; s_1, s_2, \ldots, s_n))$ for $2 \leq s_1 \leq s_2 \leq \ldots \leq s_n$. Let a be an integer defined in* (2.1). *Then*

$$a + n_1^F - 1 + (s_1 - 1)n \leq \sum_{i=1}^{n} s_i$$

if and only if one of the following statements is satisfied:

(i)

$$\sum_{i=1}^{n} s_i < \left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1$$

or

(ii)

$$\sum_{i=1}^{n} s_i \geq \max\left\{ \left\lceil \frac{2s_1(n_1^F - 1 + (s_1-1)n) - s_1 n(s_1-1) + m + 2}{2(s_1-1)} \right\rceil, \left\lfloor \frac{s_1 n(s_1-1) - m}{2} \right\rfloor + s_1 \right\},$$

where n_1^F is the number of vertices of degree 1 in F.

Proof. We first show the necessity. Let

$$a + n_1^F - 1 + (s_1 - 1)n \leq \sum_{i=1}^{n} s_i.$$

If $a = 1$ then

$$a + n_1^F - 1 + (s_1-1)n = n_1^F - n + s_1 n \leq \sum_{i=1}^{n} s_i$$

and by Lemma 2.3, we have

$$\sum_{i=1}^{n} s_i < \left\lfloor \frac{s_1 n(s_1-1) - m}{2} \right\rfloor + s_1.$$

However, the condition $n_1^F - n + s_1 n \leq \sum_{i=1}^{n} s_i$ is trivial since $n_1^F - n \leq 0$ and $s_1 n \leq \sum_{i=1}^{n} s_i$. So

$$\sum_{i=1}^{n} s_i < \left\lfloor \frac{s_1 n(s_1-1) - m}{2} \right\rfloor + s_1.$$

If

$$a = \left\lceil \frac{2\sum_{i=1}^{n} s_i - s_1 n(s_1-1) + m + 2}{2s_1} \right\rceil$$

then

$$a + n_1^F - 1 + (s_1-1)n = \left\lceil \frac{2\sum_{i=1}^{n} s_i - s_1 n(s_1-1) + m + 2}{2s_1} \right\rceil + n_1^F - 1 + (s_1-1)n \leq \sum_{i=1}^{n} s_i,$$

which is equivalent to

$$\sum_{i=1}^{n} s_i \geq \left\lceil \frac{2s_1\left(n_1^F - 1 + (s_1-1)n\right) - s_1 n(s_1-1) + m + 2}{2(s_1-1)} \right\rceil. \tag{2.2}$$

Moreover, from Lemma 2.4 we get

$$\sum_{i=1}^{n} s_i \geq \left\lfloor \frac{s_1 n(s_1-1) - m}{2} \right\rfloor + s_1. \tag{2.3}$$

Combining (2.2) and (2.3) then

$$\sum_{i=1}^{n} s_i \geq \max\left\{ \left\lceil \frac{2s_1\big(n_1^F - 1 + (s_1 - 1)n\big) - s_1 n(s_1 - 1) + m + 2}{2(s_1 - 1)} \right\rceil, \left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1 \right\}.$$

Next we prove the sufficiency. If

$$\sum_{i=1}^{n} s_i < \left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1,$$

by Lemma 2.3, we have $a = 1$. Therefore,

$$a + n_1^F - 1 + (s_1 - 1)n = n_1^F + (s_1 - 1)n \leq n + (s_1 - 1)n = s_1 n \leq \sum_{i=1}^{n} s_i.$$

Consider

$$\sum_{i=1}^{n} s_i \geq \max\left\{ \left\lceil \frac{2s_1\left(n_1^F - 1 + (s_1 - 1)n\right) - s_1 n(s_1 - 1) + m + 2}{2(s_1 - 1)} \right\rceil, \left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1 \right\}.$$

As

$$\sum_{i=1}^{n} s_i \geq \left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1,$$

from Lemma 2.4, we have

$$a = \left\lceil \frac{2\sum_{i=1}^{n} s_i - s_1 n(s_1 - 1) + m + 2}{2s_1} \right\rceil. \tag{2.4}$$

Moreover, as

$$\sum_{i=1}^{n} s_i \geq \left\lceil \frac{2s_1\left(n_1^F - 1 + (s_1 - 1)n\right) - s_1 n(s_1 - 1) + m + 2}{2(s_1 - 1)} \right\rceil,$$

we get

$$2s_1\big(n_1^F - 1 + (s_1 - 1)n\big) - s_1 n(s_1 - 1) + m + 2 \leq 2(s_1 - 1)\sum_{i=1}^{n} s_i. \tag{2.5}$$

Using (2.4) and (2.5), we obtain

$$\begin{aligned}
&a + n_1^F - 1 + (s_1 - 1)n \\
&= \left\lceil \frac{2\sum_{i=1}^n s_i - s_1 n(s_1 - 1) + m + 2}{2s_1} \right\rceil + n_1^F - 1 + (s_1 - 1)n \\
&= \left\lceil \frac{2\sum_{i=1}^n s_i + 2s_1\left(n_1^F - 1 + (s_1 - 1)n\right) - s_1 n(s_1 - 1) + m + 2}{2s_1} \right\rceil \\
&\le \left\lceil \frac{2\sum_{i=1}^n s_i + 2(s_1 - 1)\sum_{i=1}^n s_i}{2s_1} \right\rceil = \sum_{i=1}^n s_i.
\end{aligned}$$

□

The next theorem gives the exact value of the distance irregularity strength for a subdivision of the monotonous fern of forests.

Theorem 2.6. *Let F be a forest on $n \ge 2$ vertices and $m \ge 1$ edges without an isolated vertex. Let n_1^F be the number of vertices of degree 1 in F and let $2 \le s_1 \le s_2 \le \ldots \le s_n$. If either*

(i)

$$\sum_{i=1}^n s_i < \left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1$$

or

(ii)

$$\begin{aligned}
\sum_{i=1}^n s_i \ge \max\Bigg\{ &\left\lceil \frac{2s_1\left(n_1^F - 1 + (s_1 - 1)n\right) - s_1 n(s_1 - 1) + m + 2}{2(s_1 - 1)} \right\rceil, \\
&\left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1 \Bigg\},
\end{aligned}$$

then

$$\operatorname{dis}(S(\mathit{MoFern}(F; n; s_1, s_2, \ldots, s_n))) = \sum_{i=1}^n s_i + \left\lceil \frac{m}{2} \right\rceil.$$

Proof. Let $H \cong S(\mathrm{MoFern}(F; n; s_1, s_2, \ldots, s_n))$. Let n_i be the number of vertices in H having degree i. We first show the lower bound. Evidently, $n_1 = \sum_{i=1}^n s_i$ and $n_2 = \sum_{i=1}^n s_i + m$. From Theorem 2.1, we have

$$\begin{aligned}
\operatorname{dis}(H) &\ge \max\left\{ \sum_{i=1}^n s_i, \sum_{i=1}^n s_i + \left\lceil \frac{m}{2} \right\rceil, \ldots, \left\lceil \frac{2\sum_{i=1}^n s_i + m + \sum_{i=3}^{\Delta(H)} n_i}{\Delta(H)} \right\rceil \right\} \\
&\ge \sum_{i=1}^n s_i + \left\lceil \frac{m}{2} \right\rceil.
\end{aligned}$$

Next we show the upper bound. Put $k = \sum_{i=1}^{n} s_i + \lceil \frac{m}{2} \rceil$. The construction of the labeling of vertices is as follows. Label all vertices in $V(S_2)$ with k. Next we label the vertices in $V(S_3)$ with integers from the set $U = \{1, 2, \dots, \sum_{i=1}^{n} s_i\}$. For each $i = 1, 2, \dots, n$, label $(s_1 - 1)$ vertices in $V(S_3^{v_i})$ with integers $a + i - 1, a + i - 1 + n, \dots, a + i - 1 + (s_1 - 2)n$ and for $i = 1, 2, \dots, n_1^F$, label 1 vertex (unlabeled vertex) in $V(S_3^{v_i})$ with $a + i - 1 + (s_1 - 1)n$, where a is an integer defined in (2.1). Order values in U which have not previously used on vertices in $V(S_3)$ ascendingly. Let us name this ordered set by U'. Thus

$$\begin{aligned} U' &= U \setminus \{a, a+1, \dots, a + n_1^F - 1 + (s_1 - 1)n\} \\ &= \left\{1, 2, \dots, a - 1, a + n_1^F + (s_1 - 1)n, a + n_1^F + (s_1 - 1)n + 1, \dots, \sum_{i=1}^{n} s_i \right\}. \end{aligned}$$

The conditions (i) and (ii) of the statement along with Lemma 2.5 guarantee that the set $\{a, a + 1, \dots, a + n_1^F - 1 + (s_1 - 1)n\}$ belongs to U. Note that if $a + n_1^F - 1 + (s_1 - 1)n = \sum_i^n s_i$ and $a = 1$ then U' becomes an empty set.

Label all the $(s_i - s_1)$ remaining vertices in $V(S_3^{v_i})$, $i = 1, 2, \dots, n_1^F$, and the $(s_i - s_1 + 1)$ remaining vertices in $V(S_3^{v_i})$, $i = n_1^F + 1, n_1^F + 2, \dots, n$, using integers from U' with requirement that vertices in $V(S_3^{v_i})$ receive smaller label than vertices in $V(S_3^{v_j})$ for each $1 \le i < j \le n$. In other words, all vertices (unlabeled vertices) in $V(S_3^{v_1}), V(S_3^{v_2}), \dots, V(S_3^{v_n})$ are labeled successively with integers from U'. Observe that

$$\sum_{i=1}^{n_1^F} (s_i - s_1) + \sum_{i=n_1^F+1}^{n} (s_i - s_1 + 1) = \sum_{i=1}^{n} s_i - \left(n_1^F + (s_1 - 1)n\right) = |U'|,$$

so this is possible.

So far, we have completely obtained the final weights of vertices in $V(S_1)$ and in $V(F)$. For the weights of vertices in $V(S_1)$, we have

$$\{wt(v) : v \in V(S_1)\} = \left\{1, 2, \dots, \sum_{i=1}^{n} s_i \right\}. \tag{2.6}$$

Let us consider the weights of vertices in $V(F)$. For $i = 1, 2, \dots, n_1^F$, we obtain

$$wt(v_i) = \sum_{j=1}^{s_1 - 1} (a + i - 1 + (j - 1)n) + (a + i - 1 + (s_1 - 1)n) + k + A_i,$$

and for $i = n_1^F + 1, n_1^F + 2, \dots, n$, we get

$$wt(v_i) = \sum_{j=1}^{s_1 - 1} (a + i - 1 + (j - 1)n) + k(\deg_F(v_i)) + A_i,$$

where A_i is the sum of $(s_i - s_1)$ labels in $V(S_3^{v_i})$ for $i = 1, 2, \ldots, n_1^F$ and is the sum of $(s_i - s_1 + 1)$ labels in $V(S_3^{v_i})$ for $i = n_1^F + 1, n_1^F + 2, \ldots, n$; such values are obtained from the process on the preceding paragraph.

We show that the weights of vertices in $V(F)$ are distinct. For every $i = 1, 2, \ldots, n_1^F$,

$$\begin{aligned} wt(v_i) &= \sum_{j=1}^{s_1-1}(a + i - 1 + (j-1)n) + (a + i - 1 + (s_1 - 1)n) + k + A_i \\ &\geq \sum_{j=1}^{s_1-1}(a + (j-1)n) + (a + (s_1 - 1)n) + k + A_i \\ &= \frac{s_1}{2}(2a + (s_1 - 1)n) + k + A_i > 2k + A_i \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} wt(v_i) &= \sum_{j=1}^{s_1-1}(a + i - 1 + (j-1)n) + (a + i - 1 + (s_1 - 1)n) + k + A_i \\ &\leq \sum_{j=1}^{s_1-1}\big(a + n_1^F - 1 + (j-1)n\big) + \big(a + n_1^F - 1 + (s_1 - 1)n\big) + k + A_i \\ &\leq \sum_{j=1}^{s_1-1}\big(a + n_1^F - 1 + (j-1)n\big) + \sum_{i=1}^{n} s_i + k + A_i \\ &< \sum_{j=1}^{s_1-1}\big(a + n_1^F - 1 + (j-1)n\big) + 2k + A_i. \end{aligned} \tag{2.8}$$

On the other hand, for $i = n_1^F + 1, n_1^F + 2, \ldots, n$, we have $\deg_F(v_i) \geq 2$. Thus for each $i = n_1^F + 1, n_1^F + 2, \ldots, n$,

$$\begin{aligned} wt(v_i) &= \sum_{j=1}^{s_1-1}(a + i - 1 + (j-1)n) + k(\deg_F(v_i)) + A_i \\ &\geq \sum_{j=1}^{s_1-1}\big(a + n_1^F + (j-1)n\big) + 2k + A_i. \end{aligned} \tag{2.9}$$

Combining (2.7), (2.8), (2.9) and using the facts that $A_1 = 0$ and $A_i < A_j$ for $1 \leq i < j \leq n$, it is not surprising that

$$2k < wt(v_1) < wt(v_2) < \ldots < wt(v_n). \tag{2.10}$$

Next we label vertices in $V(F)$ with integers $\{k - n + 1, k - n + 2, \ldots, k\}$. For a positive integer t, let $V(F) = \bigcup_{j=1}^{t} V(T_j)$, where $V(T_j)$ is the set of vertices of a tree which is the jth component of F, $j = 1, 2, \ldots, t$. We may assume, without loss of generality, that T_j is a rooted tree (i.e., a tree in which one vertex has been designated as the root). Furthermore, let

$$V(T_j) = \bigcup_{b=0}^{h_j-1} V(T_j^b),$$

where

$$V(T_j^b) = \{v_{(bc)_j} : c = 1, 2, \ldots, |V(T_j^b)|\}$$

is an ordered set of vertices (say from the left most to the right most) in the bth level of T_j and h_j is the height of T_j (the level of a vertex $v \in V(T_j)$ is the length of the unique path from v to the root and the height of T_j is defined by $h_j = \max\{b : b \text{ is the level of vertices in } T_j\}$). Just keep in mind that

$$\{v_{(bc)_j} : j = 1, 2, \ldots, t; b = 0, 1, \ldots, h_j - 1; c = 1, 2, \ldots, |V(T_j^b)|\}$$
$$= \{v_i : i = 1, 2, \ldots, n\}$$

which means that for each triple (b, c, j) there is an integer i such that $v_{(bc)_j} = v_i$, and vice versa. The vertex $v_{(01)_j}$ is always the root of T_j and we can choose arbitrarily one vertex v_i, for some $i \in \{1, 2, \ldots, n\}$, in each component to be the root. For example, in Figure 1, it is shown a forest with two components and with vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9$ (the indices are ordered ascendingly based on its degree). The components of such a forest are then represented with rooted trees T_1 and T_2; say, T_1 is rooted at v_1 and T_2 is rooted at v_8. So all the vertices can now be written: $v_1 = v_{(01)_1}$, $v_2 = v_{(23)_1}$, $v_3 = v_{(21)_1}$, $v_4 = v_{(13)_2}$, $v_5 = v_{(12)_2}$, $v_6 = v_{(11)_2}$, $v_7 = v_{(22)_1}$, $v_8 = v_{(01)_2}$, $v_9 = v_{(11)_1}$ (see Figure 2).

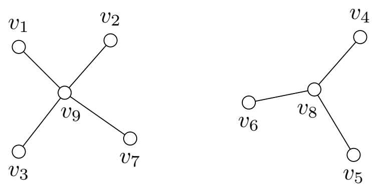

Fig. 1. A forest on 9 vertices with two components

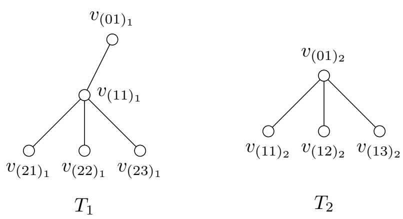

Fig. 2. A forest which its components are represented with rooted trees T_1 and T_2

For each $j = 1, 2, \dots, t$, $b = 0, 1, \dots, h_j - 1$ and $c = 1, 2, \dots, |V(T_j^b)|$ we label vertex $v_{(bc)_j}$ with $k - n + c + \sum_{x=0}^{b-1} |V(T_j^x)| + \sum_{y=1}^{j-1} |V(T_y)|$. Note that $\sum_{x=0}^{-1} |V(T_j^x)| = 0$ when $b = 0$ and $\sum_{y=1}^{0} |V(T_y)| = 0$ when $j = 1$.

It is easy to see that the weight of a subdivision vertex $y \in V(S_2)$ of an edge $v_i v_{i'} \in E(F)$ is equal to the sum of labels of $v_i = v_{(bc)_j}$ and $v_{i'} = v_{(b'c')_{j'}}$ for some $i, i', b, b', c, c', j, j'$. Furthermore, we can easily check that

$$\begin{aligned}(k - n + 1) + (k - n + 2) &= 2k - 2n + 3 \\ &\leq wt(y) \neq wt(y') \leq 2k - 1 = (k - 1) + k \qquad (2.11)\end{aligned}$$

for every two distinct vertices $y, y' \in V(S_2)$.

Next we label vertices in $V(S_1)$. Beforehand, let us consider

$$V(S_1) = \bigcup_{i=1}^{n} V\left(S_1^{v_i}\right) = \bigcup_{j=1}^{t} \bigcup_{b=0}^{h_j-1} \bigcup_{c=1}^{|V(T_j^b)|} V\left(S_1^{v_{(bc)_j}}\right)$$

and

$$V(S_3) = \bigcup_{i=1}^{n} V\left(S_3^{v_i}\right) = \bigcup_{j=1}^{t} \bigcup_{b=0}^{h_j-1} \bigcup_{c=1}^{|V(T_j^b)|} V\left(S_3^{v_{(bc)_j}}\right).$$

Let $(b, c, j) < (b', c', j')$ if either (i) $j < j'$; (ii) $j = j'$ and $b < b'$; or (iii) $j = j'$, $b = b'$ and $c < c'$. Our strategy is that the vertices in $V(S_3)$ are weighted with values from

$$W = \left\{ \sum_{i=1}^{n} s_i + 1, \sum_{i=1}^{n} s_i + 2, \dots, 2k \right\} \setminus \{wt(y) : y \in V(S_2)\} \qquad (2.12)$$

and show that it is possible to label the vertices in $V(S_1)$ using integers from 1 up to k to reach these weights. To do that, we distribute the weights W to the vertices in $V(S_3)$ such that

$$wt(z) < wt(z') \qquad (2.13)$$

for every two distinct vertices $z \in V\left(S_3^{v_{(bc)_j}}\right)$ and $z' \in V\left(S_3^{v_{(b'c')_{j'}}}\right)$, where $(b, c, j) < (b', c', j')$. This is possible since $|V(S_2)| = m$ and

$$|W| = 2k - \sum_{i=1}^{n} s_i - m = 2\Big(\sum_{i=1}^{n} s_i + \left\lceil \frac{m}{2} \right\rceil\Big) - \sum_{i=1}^{n} s_i - m \geq \sum_{i=1}^{n} s_i = |V(S_3)|.$$

We label every vertex $x \in V\left(S_1^{v_{(bc)_j}}\right)$ with $\left(wt(z) - \text{the label of } v_{(bc)_j}\right)$, where z is adjacent to x and $v_{(bc)_j}$. We do this for all (b, c, j). Observe that, from this strategy, for each $x \in V\left(S_1^{v_{(bc)_j}}\right)$ and $x' \in V\left(S_1^{v_{(b'c')_{j'}}}\right)$ with $(b, c, j) < (b', c', j')$, it holds that

$$\text{the label of } x \leq \text{the label of } x'. \qquad (2.14)$$

We show that $1 \leq$ the label of $x \leq k$ for every $x \in V(S_1)$. Let us consider vertices $x_{\min} \in V\left(S_1^{v_{(01)_1}}\right)$ and $x_{\max} \in V\left(S_1^{v_{\left((h_t-1)(|V(T_t^{h_t-1})|)\right)_t}}\right)$ such that

$$\text{the label of } x_{\min} = \min\left\{\text{the label of } x : x \in V\left(S_1^{v_{(01)_1}}\right)\right\},$$
$$\text{the label of } x_{\max} = \max\left\{\text{the label of } x : x \in V\left(S_1^{v_{\left((h_t-1)(|V(T_t^{h_t-1})|)\right)_t}}\right)\right\}.$$

According to (2.14), we only need to show that the label of $x_{\min} \geq 1$ and the label of $x_{\max} \leq k$. From (2.11) and (2.12), it is not hard to show that $wt(z_{\min}) = \sum_{i=1}^{n} s_i + 1$ and $wt(z_{\max}) \leq 2k$, where $z_{\min}$ and $z_{\max}$ are vertices adjacent to $x_{\min}$ and $x_{\max}$, respectively. Since the label of $v_{(01)_1} = k - n + 1$ then the label of

$$x_{\min} = \sum_{i=1}^{n} s_i + 1 - (k - n + 1) = \sum_{i=1}^{n} s_i - \left(\sum_{i=1}^{n} s_i + \left\lceil \frac{m}{2} \right\rceil\right) + n = n - \left\lceil \frac{m}{2} \right\rceil \geq 1$$

and since the label of $v_{\left((h_t-1)(|V(T_t^{h_t-1})|)\right)_t} = k$ then the label of $x_{\max} \leq 2k - k = k$.

Finally, we have to show that the weights of all vertices are distinct. However, this is true as we see from (2.6), (2.10), (2.11), (2.12) and (2.13). It allows us to conclude that our labeling is the desired distance irregular vertex k-labeling and we are done. □

Observe that, in the proof of Theorem 2.6, one member of W in (2.12) was not used when m is odd since $|W| - |V(S_3)| = 1$. This observation leads us to the following result.

Theorem 2.7. *Let F be a forest on $n \geq 2$ vertices and $m \geq 1$ edges without an isolated vertex. Let n_1^F be the number of vertices of degree 1 in F and let $2 \leq s_1 \leq s_2 \leq \ldots \leq s_n$. If m is odd and*

$$\sum_{i=1}^{n} s_i = \left\lceil \frac{2s_1\left(n_1^F - 1 + (s_1 - 1)n\right) - s_1 n(s_1 - 1) + m + 2}{2(s_1 - 1)} \right\rceil - 1 \tag{2.15}$$

then

$$\operatorname{dis}(S(\mathit{MoFern}(F; n; s_1, s_2, \ldots, s_n))) = \sum_{i=1}^{n} s_i + \left\lceil \frac{m}{2} \right\rceil = \sum_{i=1}^{n} s_i + \frac{m+1}{2}.$$

Proof. The proof is the same as of Theorem 2.6 with two exceptions, that, in this case, the set

$$U = \left\{1, 2, \ldots, \sum_{i=1}^{n} s_i + 1\right\} \setminus \{j\} \tag{2.16}$$

for any $j \in \{1, 2, \ldots, a - 1\}$ and the set

$$W = \left\{\sum_{i=1}^{n} s_i + 2, \sum_{i=1}^{n} s_i + 3, \ldots, 2k\right\} \setminus \{wt(y) : y \in V(S_2)\} \tag{2.17}$$

are used in the labeling construction instead of the set

$$U = \Big\{1, 2, \ldots, \sum_{i=1}^{n} s_i\Big\}$$

and the set

$$W = \left\{\sum_{i=1}^{n} s_i + 1, \sum_{i=1}^{n} s_i + 2, \ldots, 2k\right\} \setminus \{wt(y) : y \in V(S_2)\},$$

respectively.

Note that due to (2.15) and Lemma 2.4, we have

$$a = \left\lceil \frac{2\sum_{i=1}^{n} s_i - s_1 n(s_1 - 1) + m + 2}{2s_1} \right\rceil > 1. \tag{2.18}$$

By using (2.15) and the fact that $-(x+y) < -y\lceil \frac{x}{y} \rceil \leq -x$ for $xy > 0$ then

$$\begin{aligned}
& s_1 n(s_1 - 1) + m + 2 + 2s_1\big(n_1^F - 1\big) - 2(s_1 - 1)\sum_{i=1}^{n} s_i \\
&= s_1 n(s_1 - 1) + m + 2 + 2s_1\big(n_1^F - 1\big) \\
&\quad - 2(s_1 - 1)\left(\left\lceil \frac{2s_1\big(n_1^F - 1 + (s_1 - 1)n\big) - s_1 n(s_1 - 1) + m + 2}{2(s_1 - 1)} \right\rceil - 1\right) \\
&> s_1 n(s_1 - 1) + m + 2 + 2s_1\big(n_1^F - 1\big) - \Big(2s_1\big(n_1^F - 1 + (s_1 - 1)n\big) \\
&\quad - s_1 n(s_1 - 1) + m + 2 + 2(s_1 - 1)\Big) + 2(s_1 - 1) = 0
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
& s_1 n(s_1 - 1) + m + 2 + 2s_1\big(n_1^F - 1\big) - 2(s_1 - 1)\sum_{i=1}^{n} s_i \\
&= s_1 n(s_1 - 1) + m + 2 + 2s_1\big(n_1^F - 1\big) \\
&\quad - 2(s_1 - 1)\left(\left\lceil \frac{2s_1\big(n_1^F - 1 + (s_1 - 1)n\big) - s_1 n(s_1 - 1) + m + 2}{2(s_1 - 1)} \right\rceil - 1\right) \\
&\leq s_1 n(s_1 - 1) + m + 2 + 2s_1\big(n_1^F - 1\big) - \Big(2s_1\big(n_1^F - 1 + (s_1 - 1)n\big) \\
&\quad - s_1 n(s_1 - 1) + m + 2\Big) + 2(s_1 - 1) = 2(s_1 - 1) < 2s_1,
\end{aligned} \tag{2.20}$$

and so by (2.18), (2.19) and (2.20),

$$\begin{aligned} &a+n_1^F-1+(s_1-1)n-\sum_{i=1}^{n}s_i \\ &=\left\lceil \frac{2\sum_{i=1}^{n}s_i-s_1n(s_1-1)+m+2}{2s_1}\right\rceil+n_1^F-1+(s_1-1)n-\sum_{i=1}^{n}s_i \\ &=\left\lceil \frac{s_1n(s_1-1)+m+2+2s_1\big(n_1^F-1\big)-2(s_1-1)\sum_{i=1}^{n}s_i}{2s_1}\right\rceil=1, \end{aligned}$$

or equivalently,

$$a+n_1^F-1+(s_1-1)n=\sum_{i=1}^{n}s_i+1,$$

which implies that the set $\{a,a+1,\dots,a+n_1^F-1+(s_1-1)n\}$ belongs to (2.16).

Furthermore, due to (2.17), we get that $wt(z_{\min}) = \sum_{i=1}^{n}s_i+2$ and $wt(z_{\max})=2k$. Since m is odd then the label of

$$x_{\min}=\sum_{i=1}^{n}s_i+2-(k-n+1)=\sum_{i=1}^{n}s_i-\left(\sum_{i=1}^{n}s_i+\frac{m+1}{2}\right)+n+1=n+1-\frac{m+1}{2}\geq 2$$

and the label of $x_{\max}=2k-k=k$, meaning that there is no vertex with label greater than k. □

We end this section by discussing some properties of the distance irregularity strength for trees. First, we show that the bound in Theorem 2.1 can be reduced such that it is determined only by n_1 or $\lceil\frac{n_1+n_2}{2}\rceil$ as we state in the following theorem.

Theorem 2.8. *Let T be a tree with maximum degree Δ such that $N(u)\neq N(v)$ for $u,v\in V$, $u\neq v$. Let n_i be the number of vertices of degree i for every $i=1,2,\dots,\Delta$. Then*

$$\operatorname{dis}(T)\geq\max\left\{n_1,\left\lceil\frac{n_1+n_2}{2}\right\rceil\right\}.$$

Proof. With respect to Theorem 2.1, evidently, the theorem holds for $1\leq\Delta\leq 2$. We now suppose that $\Delta\geq 3$. Let $t_i=\left\lceil\frac{\sum_{j=1}^{i}n_j}{i}\right\rceil$ for $i=1,2,\dots,\Delta$.

It is enough to show that $t_1-t_l\geq 0$ or $t_2-t_l\geq 0$ for each $l=3,4,\dots,\Delta$. If $t_1-t_l\geq 0$ then we are done. Now assume that $t_1-t_l<0$. Then

$$ln_1-\sum_{j=1}^{l}n_j<0 \quad\Leftrightarrow\quad n_2>(l-1)n_1-\sum_{j=3}^{l}n_j. \tag{2.21}$$

Using (2.21) along with the facts that $\lceil x\rceil - \lceil y\rceil > x-y-1$ and $n_1 = 2+\sum_{i=3}^{\Delta}(i-2)n_i$, we have

$$\begin{aligned}
t_2 - t_l &= \left\lceil \frac{n_1+n_2}{2} \right\rceil - \left\lceil \frac{\sum_{j=1}^{l} n_j}{l} \right\rceil \\
&> \frac{n_1+n_2}{2} - \frac{\sum_{j=1}^{l} n_j}{l} - 1 = \frac{(l-2)n_1 + (l-2)n_2}{2l} - \frac{2\sum_{j=3}^{l} n_j}{2l} - 1 \\
&> \frac{(l-2)n_1 + (l-2)\left((l-1)n_1 - \sum_{j=3}^{l} n_j\right)}{2l} - \frac{2\sum_{j=3}^{l} n_j}{2l} - 1 \\
&= \frac{l(l-2)n_1 - l\sum_{j=3}^{l} n_j}{2l} - 1 \\
&= \frac{(l-2)\left(2+\sum_{j=3}^{\Delta}(j-2)n_j\right) - \sum_{j=3}^{l} n_j}{2} - 1 \\
&= \frac{(l-3)\sum_{j=3}^{l}(j-2)n_j + \sum_{j=3}^{l}(j-3)n_j}{2} \\
&\quad + \frac{(l-2)\left(2+\sum_{j=l+1}^{\Delta}(j-2)n_j\right)}{2} - 1 \geq 0.
\end{aligned}$$

As $l \geq 3$, clearly, the last inequality is true. Therefore $t_2 - t_l > 0$ and so the assertion holds. □

In the next lemmas, we give the necessary condition for which $\operatorname{dis}(T)$ is equal to n_1 or $\lceil \frac{n_1+n_2}{2} \rceil$.

Lemma 2.9. *Let T be a tree with maximum degree Δ such that $N(u) \neq N(v)$ for $u, v \in V$, $u \neq v$. Let n_i be the number of vertices of degree i for every $i = 1, 2, \dots, \Delta$. If $\operatorname{dis}(T) = n_1$ then $n_1 \geq n_2$.*

Proof. Evidently $\operatorname{dis}(T) \geq n_1$. Then by Theorem 2.8,

$$n_1 \geq \left\lceil \frac{n_1+n_2}{2} \right\rceil \Leftrightarrow 2n_1 \geq n_1 + n_2 \Leftrightarrow n_1 \geq n_2. \qquad \square$$

From Lemma 2.9 we immediately have the following.

Lemma 2.10. *Let T be a tree with maximum degree Δ such that $N(u) \neq N(v)$ for $u, v \in V$, $u \neq v$. Let n_i be the number of vertices of degree i for every $i = 1, 2, \dots, \Delta$. If $\operatorname{dis}(T) = \lceil \frac{n_1+n_2}{2} \rceil$ then $n_1 < n_2$.*

3. FINAL REMARKS

In this paper, we introduced a new lower bound for the distance irregularity strength and determined the exact values of this parameter for some graphs with pendant vertices. We also presented some properties on distance irregularity strength for trees.

In particular, we gave the necessary condition for a tree to have distance irregularity strength n_1 or $\lceil \frac{n_1+n_2}{2} \rceil$.

The necessary condition stated in Lemma 2.9 is not sufficient as we can see from a counterexample shown in Figure 3.

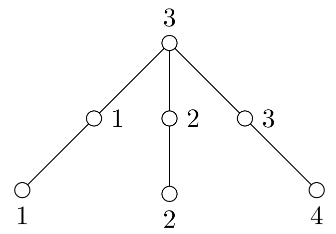

Fig. 3. A tree T with $\mathrm{dis}(T) = 4$.

We believe, however, that this happens only if $n_1 = n_2$ as stated in the following conjecture.

Conjecture 3.1. *Let T be a tree with maximum degree Δ such that $N(u) \neq N(v)$ for $u, v \in V$, $u \neq v$. Let n_i be the number of vertices of degree i for every $i = 1, 2, \dots, \Delta$. If $\mathrm{dis}(T) = n_1 + 1$ then $n_1 = n_2$.*

When trees with $n_1 = n_2$ are ignored, the following conjecture is most likely to be true.

Conjecture 3.2. *Let T be a tree with maximum degree Δ such that $N(u) \neq N(v)$ for $u, v \in V$, $u \neq v$. Let n_i be the number of vertices of degree i for every $i = 1, 2, \dots, \Delta$ and $n_1 \neq n_2$. Then $\mathrm{dis}(T) = n_1$ if and only if $n_1 > n_2$.*

As we have not found its counter example, we conjecture that the necessary condition given in Lemma 2.10 is also sufficient.

Conjecture 3.3. *Let T be a tree with maximum degree Δ such that $N(u) \neq N(v)$ for $u, v \in V$, $u \neq v$. Let n_i be the number of vertices of degree i for every $i = 1, 2, \dots, \Delta$. Then $\mathrm{dis}(T) = \lceil \frac{n_1+n_2}{2} \rceil$ if and only if $n_1 < n_2$.*

A problem below is based on the result from Theorems 2.6 and 2.7.

Problem 3.4. *Let F be a forest on $n \geq 2$ vertices and $m \geq 1$ edges without an isolated vertex. Let n_1^F be the number of vertices of degree 1 in F and let $2 \leq s_1 \leq s_2 \leq \ldots \leq s_n$. Determine the exact value of $\mathrm{dis}(MoFern(F; n; s_1, s_2, \dots, s_n))$ if*

(i)

$$\left\lfloor \frac{s_1 n(s_1 - 1) - m}{2} \right\rfloor + s_1 \leq \sum_{i=1}^{n} s_i < \left\lceil \frac{2s_1\big(n_1^F - 1 + (s_1 - 1)n\big) - s_1 n(s_1 - 1) + m + 2}{2(s_1 - 1)} \right\rceil - 1,$$

(ii) *m is even and*

$$\sum_{i=1}^{n} s_i = \left\lceil \frac{2s_1\big(n_1^F - 1 + (s_1-1)n\big) - s_1 n(s_1-1) + m + 2}{2(s_1-1)} \right\rceil - 1.$$

Acknowledgements

The authors would like to thank the anonymous referee whose comments and suggestions improved the paper.

The research for this article was supported by APVV-19-0153 and by VEGA 1/0233/18.

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Received: November 9, 2020.
Revised: January 25, 2022.
Accepted: January 26, 2022.