Opuscula Math. 42, no. 3 (2022), 361–391
https://doi.org/10.7494/OpMath.2022.42.3.361 Opuscula Mathematica

MONODROMY INVARIANT HERMITIAN FORMS FOR SECOND ORDER FUCHSIAN DIFFERENTIAL EQUATIONS WITH FOUR SINGULARITIES

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Communicated by P.A. Cojuhari

Abstract. We study the monodromy invariant Hermitian forms for second order Fuchsian differential equations with four singularities. The moduli space of our monodromy representations can be realized by certain affine cubic surface. In this paper we characterize the irreducible monodromies having the non-degenerate invariant Hermitian forms in terms of that cubic surface. The explicit forms of invariant Hermitian forms are also given. Our result may bring a new insight into the study of the Painlevé differential equations.

Keywords: Fuchsian differential equations, monodromy representation, monodromy invariant Hermitian form.

Mathematics Subject Classification: 34M35, 34M15.

1. INTRODUCTION

In the theory of Fuchsian differential equations, monodromy invariant Hermitian forms play an important role in studying the geometric nature of solutions. The criterion for the finiteness of monodromies is a typical example. In the famous result of Beukers–Heckman [3], where the generalized hypergeometric equations with finite monodromies are classified, the monodromy invariant Hermitian forms play a basic role. Haraoka [8] studied the finite monodromies for the Pochhammer equation, where the monodromy invariant Hermitian forms play also a basic role. He showed the existence of monodromy invariant Hermitian forms for other classes of Fuchsian system (called Yokoyama's list) in [9].

On the other hand, monodromy invariant Hermitian forms are deeply connected with the theory of integral representations of solutions; the monodromy invariant Hermitian form appears as the inverse of the intersection matrix of twisted cycles associated with the integral representation of the solutions of Euler type [15]. Hence if the differential equation has an integral representation of solutions of Euler type, we may obtain the monodromy invariant Hermitian form. Haraoka–Hamaguchi [10] proved that any irreducible Fuchsian system free from accessory parameters (i.e. rigid) has an integral representation of solutions of Euler type. Hence we see that any rigid Fuchsian differential equation has the monodromy invariant Hermitian form. Incidentally, the differential equations treated in [3, 8, 9] are rigid ones. For non-rigid equations, the existence of invariant Hermitian forms is known only the case when the equation has an integral representation of solutions, as in the Dotsenko–Fateev equation [6].

By looking at these results, one may feel that the existence of invariant Hermitian forms would implies the existence of integral representations of solutions. But recently, we showed the existence of the invariant Hermitian forms for certain family of the third order non-rigid Fuchsian differential equation without assuming the existence of integral representation of solutions [2]. This result suggests the possibility of introducing a new class into the Fuchsian equations. Then it is natural to ask whether there exist other non-rigid equations which belong to this class, that is, the existence of monodromy invariant Hermitian forms for other non-rigid equations.

In this paper, we consider the existence of invariant Hermitian forms for non-rigid second order Fuchsian differential equations of SL type with four singularities $t_1, t_2, t_3, t_4 \in \mathbb{P}^1$. For a fundamental system of solutions at a base point $b \in \mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}$, the monodromy representation of solution is determined as anti-homomorphism of the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}, b)$ to $\mathrm{SL}(2, \mathbb{C})$, describing the analytic continuations of the fundamental system of solutions. The monodromy representation $\rho : \pi_1(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}, b) \to \mathrm{SL}(2, \mathbb{C})$ has an invariant Hermitian form if there exists a Hermitian matrix H satisfying

$$\overline{\rho(\gamma)}^T H \rho(\gamma) = H \quad (\gamma \in \pi_1(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}, b)), \tag{1.1}$$

where $\overline{\rho(\gamma)}^T$ denotes the complex conjugate of transpose of $\rho(\gamma)$.

The monodromy representation depends on the choice of the fundamental system of solutions and the change of the fundamental system of solutions leads to the equivalence relation for the monodromy representations. If a monodromy representation has an invariant Hermitian form, then any other equivalent representations also have invariant Hermitian forms. Therefore, in studying invariant Hermitian forms, it is natural to consider the moduli space of monodromy representations. Note that the moduli space of monodromy representations can be identify with the moduli space of Fuchsian differential equations via Riemann–Hilbert correspondence.

Let $\mathcal{M}(a)$ be the moduli space of monodromy representations with prescribed local monodromy data $a \in \mathbb{C}^4$. Jimbo [14] pointed out that the moduli space $\mathcal{M}(a)$ can be realized by certain affine cubic surface $\mathcal{S}(a) \subset \mathbb{C}^3$. After that, Iwasaki [13] introduced Zariski open subsets $\mathcal{S}^\circ(a) \subset \mathcal{S}(a)$ and $\mathcal{M}^\circ(a) \subset \mathcal{M}(a)$ which are called big opens, and constructed a homeomorphism $\varphi : \mathcal{S}^\circ(a) \xrightarrow{\sim} \mathcal{M}^\circ(a)$ explicitly. Since the big opens $\mathcal{S}^\circ(a)$ and $\mathcal{M}^\circ(a)$ are identical to the entire surface $\mathcal{S}(a)$ and entire space $\mathcal{M}(a)$ respectively for a generic $a \in \mathbb{C}^4$, Iwasaki's result gives an useful correspondence between the moduli space $\mathcal{M}(a)$ and the cubic surface $\mathcal{S}(a)$.

In this paper, we first show that the subspace of $\mathcal{M}^\circ(a)$ consisting of the irreducible elements which have invariant Hermitian forms can be identified with $\mathcal{S}^\circ(a) \cap \mathbb{R}^3$ through the above homeomorphism. Next we consider the case for non-generic $a \in \mathbb{C}^4$. In this case the complement $\mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$ is non-empty. We show that any irreducible element in $\mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$ having an invariant Hermitian form corresponds to the point in $\mathcal{S}(a) \cap \mathbb{R}^3$. Thus we characterized the set of irreducible elements in $\mathcal{M}(a)$ having the invariant Hermitian form in terms of the cubic surface $\mathcal{S}(a)$. As stated above, the irreducible elements in $\mathcal{M}(a)$ can be identified with the irreducible Fuchsian differential equations. Therefore our result implies the existence of the family of the irreducible Fuchsian differential equations having invariant Hermitian forms.

Our result may bring a new insight into the study of the Painlevé differential equations. The sixth Painlevé equation is characterized by the isomonodromic deformation of the Fuchsian systems of rank two with four singularities. Isomonodromic nature leads to an identification of the moduli space $\mathcal{M}(a)$ with the set of solution germs (or the initial value space) at a base point of the sixth Painlevé equation. By looking at this fact, our result implies that we implicitly find out a new class of the solutions of the sixth Painlevé equation, which we will call "Hermitian-class". Iwasaki's pioneering works [12, 13] intended to relate the dynamics on the cubic surface $\mathcal{S}(a)$ to the study of nonlinear monodromy to the solutions of sixth Painlevé equation. Since the relation between the solutions belonging to Hermitian-class and the point of $\mathcal{S}(a)$ is very clear, the study of nonlinear monodromy of the solutions in Hermitian-class through the dynamics on $\mathcal{S}(a)$ is an interesting future problem.

On the other hand, in the study of conformal field theory, the Fourier expansion of tau function of the sixth Painlevé equation is expressed by conformal blocks, which are monodromy invariant Hermitian forms [7, 11]. Hence it is fascinate to investigate the relation between Hermitian-class and conformal blocks.

This paper is organized as follows. In Section 2, we fix some notations and construct the moduli space $\mathcal{M}(a)$. Following Iwasaki's work [12,13], we review the correspondence between $\mathcal{M}(a)$ and $\mathcal{S}(a)$ in Section 3. In Section 4, we give some remarks about monodromy invariant Hermitian forms. Our main results are Theorems 5.1 and 6.12. Their statements and proofs are given in Sections 5 and 6. We note that our proof is constructive, that is, the explicit forms of Hermitian matrices associate to the monodromy invariant Hermitian forms will be given.

2. MODULI SPACE OF MONODROMY REPRESENTATIONS

In keeping with the previous section, we consider the Fuchsian differential equation (resp. Fuchsian system) of order two (resp. rank two) of SL type with four regular singular points $t_1, t_2, t_3, t_4 \in \mathbb{P}^1$. We fix a base point $b \in \mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}$ and take $(+1)$-loops $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as illustrated in Figure 1. Then the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}, b)$ is generated by $\gamma_1, \gamma_2, \gamma_3$ and γ_4 with a relation

$$\gamma_1\gamma_2\gamma_3\gamma_4 = 1.$$

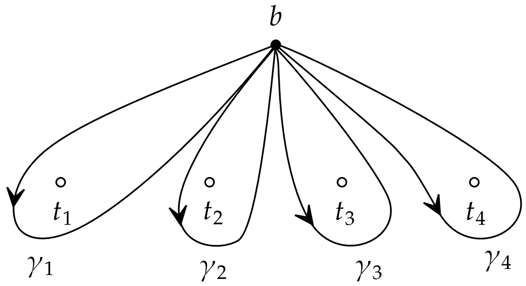

Fig. 1. Singular points and (+1)-loops

By taking a fundamental system of solutions in a neighborhood of b, we obtain the monodromy representation

$$\rho\,:\,\pi_1(\mathbb{P}^1\setminus\{t_1,t_2,t_3,t_4\},b)\to \mathrm{SL}(2,\mathbb{C}).$$

We set

$$\rho(\gamma_i)=M_i \quad (i=1,2,3,4). \tag{2.1}$$

Then the monodromy group is generated by M_1,M_2,M_3,M_4 and the relation

$$M_4M_3M_2M_1=I. \tag{2.2}$$

Note that the each matrix M_i determines the local monodromy at t_i $(i=1,2,3,4)$. Thanks to (2.2), we may identify the representation ρ with the triple (M_1,M_2,M_3), and hence sometimes write $\rho=(M_1,M_2,M_3)$.

To show that the monodromy representation $\rho=(M_1,M_2,M_3)$ has an invariant Hermitian form, it is sufficient to show the existence of a Hermitian matrix such that

$$\bar{M}_i^T HM_i=H \quad (i=1,2,3), \tag{2.3}$$

where $\bar{M}^T$ denotes the complex conjugate of transpose of M. Indeed, if the equations (2.3) holds, then (1.1) holds from the relations (2.1) and (2.2).

As stated in the previous section, the monodromy representation depends on the choice of the fundamental system of solutions. Indeed, if we change the fundamental system of solutions, the monodromy representation ρ is transformed to $\mathrm{Ad}(P)(\rho)$ defined by

$$\left(\mathrm{Ad}(P)(\rho)\right)(\gamma)=P\rho(\gamma)P^{-1} \quad (\gamma\in\pi_1(\mathbb{P}^1\setminus\{t_1,t_2,t_3,t_4\},b))$$

with some $P\in\mathrm{SL}(2,\mathbb{C})$. The monodromy of the differential equation is defined as the equivalence class $[\rho]$ by the equivalence relation

$$\rho\sim\mathrm{Ad}(P)(\rho) \quad (P\in\mathrm{SL}(2,\mathbb{C})).$$

For the monodromy representation $\rho = (M_1, M_2, M_3)$, the monodromy $[\rho]$ is given by the equivalence class $[(M_1, M_2, M_3)]$ by the equivalence relation

$$(M_1, M_2, M_3) \sim (PM_1P^{-1}, PM_2P^{-1}, PM_3P^{-1}) \quad (P \in \mathrm{SL}(2, \mathbb{C})). \tag{2.4}$$

If $\rho = (M_1, M_2, M_3)$ has an invariant Hermitian form with a Hermitian matrix H, then any equivalent representation

$$\mathrm{Ad}(P)(\rho) = (PM_1P^{-1}, PM_2P^{-1}, PM_3P^{-1})$$

also has an invariant Hermitian form by the Hermitian matrix $(\bar{P}^{-1})^T HP^{-1}$. Therefore in studying our problem, it is natural to consider the moduli space of monodromy representations.

We shall consider the moduli space of monodromy representations prescribed local monodromies. Let $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4)$ be a quartet of conjugacy classes of $\mathrm{SL}(2, \mathbb{C})$, which represents the quartet of the local monodromies at (t_1, t_2, t_3, t_4). Note that taking the local monodromy $\mathcal{C}_i$ is nothing but fixing the spectral type (Jordan canonical form) at the singularity t_i. Then we set the moduli space of monodromy representations prescribed local monodromies as

$$\mathcal{M}_{\mathcal{C}} = \{(M_1, M_2, M_3) \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3\,;\, (M_3M_2M_1)^{-1} \in \mathcal{C}_4\}/\sim.$$

This space is easy to see the correspondence with differential equations, but it does not have good topological properties in general. In fact, this space is nether Hausdorff, nor an algebraic variety. Therefore we use more lough equivalence relation and quotient space, that is, GIT quotient. We set the parameter $a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ by

$$a_i = \operatorname{tr} \mathcal{C}_i \quad (i = 1, 2, 3, 4)$$

and define

$$\mathcal{O}(a_i) = \{M \in \mathrm{SL}(2, \mathbb{C})\,;\, \operatorname{tr} M = a_i\}.$$

Now we define the moduli space of monodromy representations as

$$\begin{aligned} \mathcal{M}(a) = \{(M_1, M_2, M_3) \in \mathcal{O}(a_1) \times \mathcal{O}(a_2) \times \mathcal{O}(a_3)\,;\\ (M_3M_2M_1)^{-1} \in \mathcal{O}(a_4)\}/\!/\,\mathrm{SL}(2, \mathbb{C}). \end{aligned} \tag{2.5}$$

Here the quotient // denotes GIT quotient by the diagonal adjoint action of $\mathrm{SL}(2, \mathbb{C})$. We remark that the condition $(M_3M_2M_1)^{-1} \in \mathcal{O}(a_4)$ is equivalent to $M_3M_2M_1 \in \mathcal{O}(a_4)$.

We sometimes write $(M_1, M_2, M_3) \in \mathcal{M}(a)$ to represent $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ if there is no confuse to write as like this.

The space $\mathcal{M}(a)$ is an algebraic variety and complex manifold of rank two. Moreover, $\mathcal{M}(a)$ is homeomorphic to certain affine cubic surface for general parameters $a \in \mathbb{C}^4$. We shall explain this in the following section.

Finally, we introduce the notion of irreducibility. A triple $(M_1, M_2, M_3) \in \mathrm{SL}(2, \mathbb{C})^3$ or an element $[(M_1, M_2, M_3)] \in \mathcal{M}_{\mathcal{C}}$ is called irreducible if M_1, M_2 and M_3 have no common invariant subspace except trivial subspaces $\{0\}$ and $\mathbb{C}^2$. An element $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ is called irreducible if the any triple of matrices belonging to $[(M_1, M_2, M_3)]$ is irreducible. It is known that any triple in the irreducible element $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ is equivalent in the sense of (2.4). In other words, any irreducible element $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ is also the equivalence classes in $\mathcal{M}_{\mathcal{C}}$.

3. AFFINE CUBIC SURFACE AND BIG OPENS

We explain the relation between the moduli space $\mathcal{M}(a)$ and affine cubic surface in $\mathbb{C}^3$, referring Iwasaki's work [12, 13]. Hereafter, we denote by (i, j, k) any cyclic permutation of $(1, 2, 3)$. For $a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$, we set

$$\begin{aligned} &\theta_i(a) = a_j a_k + a_i a_4 \quad (i = 1, 2, 3), \\ &\theta_4(a) = a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4. \end{aligned} \tag{3.1}$$

and define the affine cubic polynomial in $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ by

$$f_a(x) = x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(a) x_1 - \theta_2(a) x_2 - \theta_3(a) x_3 + \theta_4(a). \tag{3.2}$$

Using this, we define the cubic surface

$$\mathcal{S}(a) = \{x \in \mathbb{C}^3 \,;\, f_a(x) = 0\}. \tag{3.3}$$

In [12], the geometric structure of the surface $\mathcal{S}(a)$ was investigated.

Theorem 3.1 ([12, Theorem 1])**.** *Let $w(a)$ be a polynomial of $a = (a_1, a_2, a_3, a_4)$ defined by*

$$w(a) = \prod_{\varepsilon_1 \varepsilon_2 \varepsilon_3 = 1} (\varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 a_3 + a_4) - \prod_{i=1}^{3} (a_i a_4 - a_j a_k), \tag{3.4}$$

where the first product on the right-hand side is taken over all triples $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3$ satisfying $\varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$. Then the affine cubic surface $\mathcal{S}(a)$ has singular points if and only if

$$w(a) \prod_{i=1}^{4} (a_i^2 - 4) = 0. \tag{3.5}$$

We shall explain the relation between the moduli space $\mathcal{M}(a)$ and the cubic surface $\mathcal{S}(a)$. For the element $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$, we set $x = (x_1, x_2, x_3)$ by

$$x_i = \mathrm{tr}(M_j M_k) \quad (i = 1, 2, 3). \tag{3.6}$$

Note that the value of $x = (x_1, x_2, x_3)$ is invariant for the choice of the representative of $[(M_1, M_2, M_3)]$. We see that that $f_a(x) = 0$ holds, which implies that the mapping $(3.6) : \mathcal{M}(a) \to \mathcal{S}(a)$ is well-defined (Jimbo [14], Boalch [4]). Conversely, the conjugacy classes of $\mathcal{M}(a)$ are parametrized by the coordinate of $\mathcal{S}(a)$ for a generic $a \in \mathbb{C}^4$. To explain this, we introduce some important notions defined in [13].

Any polynomial $p = p(x)$ of x may be thought as a function on $\mathcal{S}(a)$ if x is regarded as the coordinates of $\mathcal{S}(a)$, and as a function on $\mathcal{M}(a)$ if x is regarded as the invariants in $\mathcal{M}(a)$. Hence we can define the open subset of $\mathcal{S}(a)$ and $\mathcal{M}(a)$ by the same polynomial $p = p(x)$;

$$\mathcal{S}(a)[p] = \mathcal{S}(a) \cap \{p \neq 0\}, \quad \mathcal{M}(a)[p] = \mathcal{M}(a) \cap \{p \neq 0\}.$$

Next we introduce the polynomial

$$p_{i\nu}(x) = \begin{cases} (x_i^2 - 4)\psi(x_i, a_i, a_4) & (\nu = 1), \\ (x_i^2 - 4)\psi(x_i, a_j, a_k) & (\nu = 2), \end{cases} \tag{3.7}$$

where the polynomial $\psi(s, t, u)$ is defined by

$$\psi(s, t, u) = s^2 + t^2 + u^2 - stu - 4. \tag{3.8}$$

Using this we define the open subsets (charts)

$$\mathcal{S}_{i\nu}(a) = \mathcal{S}(a)[p_{i\nu}], \quad \mathcal{M}_{i\nu}(a) = \mathcal{M}(a)[p_{i\nu}] \quad (i = 1, 2, 3,\ \nu = 1, 2). \tag{3.9}$$

We fix a square root of $x_i^2 - 4$ and put

$$r_i = \sqrt{x_i^2 - 4}, \quad \lambda_i^{\pm} = \frac{x_i \pm r_i}{2}. \tag{3.10}$$

Under these preparation, a good parametrization of the space $\mathcal{M}(a)$ is given.

Proposition 3.2 ([13, Definition 3.3 and Lemma 3.4])**.** *We fix* $a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$*. For* $i = 1, 2, 3$ *and* $\nu = 1, 2$*, let*

$$\varphi_{i\nu} : \mathcal{S}_{i\nu}(a) \to \mathcal{M}_{i\nu}(a) \tag{3.11}$$

be the map associating to each $x \in \mathcal{S}_{i\nu}(a)$ *the conjugacy class of the triple* $(M_1, M_2, M_3) \in \mathcal{M}_{i\nu}(a)$ *defined as in Tables 1 (for* $\nu = 1$*) and 2 (for* $\nu = 2$*), where* (i, j, k) *denotes any cyclic permutation of* $(1, 2, 3)$ *and*

$$y_i = \frac{\partial f_a}{\partial x_i} = 2x_i + x_j x_k - \theta_i(a) \quad (i = 1, 2, 3). \tag{3.12}$$

Then the map $\varphi_{i\nu}$ *is well-defined, that is, it does not depend on the choice of the branch in* (3.10)*.*

We would like to obtain a global parametrization of $\mathcal{M}(a)$. Let us set

$$\mathcal{S}^\circ(a) = \bigcup_{i=1}^{3} \bigcup_{\nu=1}^{2} \mathcal{S}_{i\nu}(a), \quad \mathcal{M}^\circ(a) = \bigcup_{i=1}^{3} \bigcup_{\nu=1}^{2} \mathcal{M}_{i\nu}(a) \tag{3.13}$$

and call them big opens. The reason of such naming will be cleared later. Now we give the global parametrization theorem.

Table 1
$(M_1, M_2, M_3) \in \mathcal{M}_{i\nu}(a)$ with $\nu = 1$

$M_i = \begin{pmatrix} \dfrac{a_4 - a_i\lambda_i^-}{r_i} & -\dfrac{\psi(x_i, a_i, a_4)}{x_i^2 - 4} \\ 1 & -\dfrac{a_4 - a_i\lambda_i^+}{r_i} \end{pmatrix}$
$M_j = \begin{pmatrix} -\dfrac{a_k - a_j\lambda_i^+}{r_i} & -\dfrac{y_k - y_j\lambda_i^-}{x_i^2 - 4} \\ \dfrac{y_k - y_j\lambda_i^+}{\psi(x_i, a_i, a_4)} & \dfrac{a_k - a_j\lambda_i^-}{r_i} \end{pmatrix}$
$M_k = \begin{pmatrix} -\dfrac{a_j - a_k\lambda_i^+}{r_i} & -\dfrac{y_j - y_k\lambda_i^+}{x_i^2 - 4} \\ \dfrac{y_j - y_k\lambda_i^-}{\psi(x_i, a_i, a_4)} & \dfrac{a_j - a_k\lambda_i^-}{r_i} \end{pmatrix}$

Table 2
$(M_1, M_2, M_3) \in \mathcal{M}_{i\nu}(a)$ with $\nu = 2$

$M_i = \begin{pmatrix} \dfrac{a_4 - a_i\lambda_i^-}{r_i} & -\dfrac{y_k - y_j\lambda_i^+}{x_i^2 - 4} \\ \dfrac{y_k - y_j\lambda_i^-}{\psi(x_i, a_j, a_k)} & -\dfrac{a_4 - a_i\lambda_i^+}{r_i} \end{pmatrix}$
$M_j = \begin{pmatrix} -\dfrac{a_k - a_j\lambda_i^+}{r_i} & -\dfrac{\psi(x_i, a_j, a_k)}{x_i^2 - 4} \\ 1 & \dfrac{a_k - a_j\lambda_i^-}{r_i} \end{pmatrix}$
$M_k = \begin{pmatrix} -\dfrac{a_j - a_k\lambda_i^+}{r_i} & \dfrac{\lambda_i^+\psi(x_i, a_j, a_k)}{x_i^2 - 4} \\ -\lambda_i^- & \dfrac{a_j - a_k\lambda_i^-}{r_i} \end{pmatrix}$

Theorem 3.3 ([13, Theorem 3.6])**.** *We take a parameter $a \in \mathbb{C}^4$. For each $i = 1, 2, 3$, $\nu = 1, 2$, the map $\varphi_{i\nu} : \mathcal{S}_{i\nu}(a) \to \mathcal{M}_{i\nu}(a)$ in* (3.9) *is a homeomorphism. These six local homeomorphism are patched together to yield a global homeomorphism between the big opens,*

$$\varphi : \mathcal{S}^\circ(a) \to \mathcal{M}^\circ(a). \tag{3.14}$$

For generic $a \in \mathbb{C}^4$, the big open $\mathcal{S}^\circ(a)$ is nothing but the entire surface $\mathcal{S}(a)$. Namely, the following theorem holds.

Theorem 3.4 ([13, Theorem 4.1])**.** *For any $a \in \mathbb{C}^4$, we set*

$$v(a) = \prod_{\varepsilon \in \{\pm 1\}^3} f_a(2\varepsilon),$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and the polynomial $f_a(x)$ is defined in (3.2) *. Then we have $\mathcal{S}^\circ(a) = \mathcal{S}(a)$ if and only if*

$$v(a)w(a) \neq 0, \tag{3.15}$$

where $w(a)$ is the polynomial defined (3.4)*.*

Remark 3.5. Iwasaki pointed out that, the set $\mathcal{S}(a) \setminus \mathcal{S}^\circ(a)$ contains at most 64 points for any $a \in \mathbb{C}$ in [13, Lemma 4.2].

The open set $\mathcal{M}^\circ(a)$ is also big open set of $\mathcal{M}(a)$. In fact, for generic $a \in \mathbb{C}^4$ it holds that $\mathcal{M}(a) = \mathcal{M}^\circ(a)$, and the generality is given by (3.15).

Proposition 3.6. *If the parameter $a \in \mathbb{C}^4$ satisfies the condition* (3.15)*, it holds that $\mathcal{M}^\circ(a) = \mathcal{M}(a)$.*

Thanks to this proposition, any element in $\mathcal{M}^\circ(a)$ can be identified with the monodromy in $\mathcal{M}_{\mathcal{C}}$. Proposition 3.6 can be shown in a same manner as the proof of Theorem 3.4. But the statement itself of this proposition was not stated in [13]. Hence we shall prove this proposition without being afraid of duplication.

Proof. We assume $\mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$ is non-empty and then show $v(a)w(a) = 0$. Let us take $(M_1, M_2, M_3) \in \mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$ and set $x = (x_1, x_2, x_3)$ by $x_i = \mathrm{tr}(M_j M_k)$. Then we have $p_{i\nu}(x) = 0$ for all (i, ν).

If $(x_1, x_2, x_3) = (2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3)$ for some $\varepsilon_i \in \{\pm 1\}$, then we have $v(a) = 0$ immediately.

Next we consider the case that $x_i \neq \pm 2$ for some $i = 1, 2, 3$. Then the condition $p_{i\nu}(x) = 0$ implies

$$\begin{cases} \psi(x_i, a_i, a_4) = x_i^2 + a_i^2 + a_4^2 - x_i a_i a_4 - 4 = 0, \\ \psi(x_i, a_j, a_k) = x_i^2 + a_j^2 + a_k^2 - x_i a_j a_k - 4 = 0. \end{cases} \tag{3.16}$$

We consider these equations divided into two cases.

Case 1. $a_i a_4 - a_j a_k \neq 0$. Subtracting above two equations we find that the common root x_i must be

$$z_i = \frac{a_i^2 - a_j^2 - a_k^2 + a_4^2}{a_i a_4 - a_j a_k}.$$

By putting this into $\psi(x_i, a_i, a_4)$ and $\psi(x_i, a_j, a_k)$, we obtain

$$\psi(z_i, a_i, a_4) = \psi(z_i, a_j, a_k) = \frac{w(a)}{(a_i a_4 - a_j a_k)^2}.$$

Since (3.16) holds now, we have $w(a) = 0$.

Case 2. $a_i a_4 - a_j a_k = 0$. In this case the two equations in (3.16) has a common root x_i if and only if the two equations are identical. This means that

$$a_i a_4 = a_j a_k, \quad a_i^2 + a_4^2 = a_j^2 + a_k^2.$$

This is the case if and only if either

$$\begin{cases} a_j = \varepsilon a_i, \\ a_k = \varepsilon a_4 \end{cases} \quad \text{or} \quad \begin{cases} a_j = \varepsilon a_4, \\ a_k = \varepsilon a_i \end{cases}$$

holds for some $\varepsilon \in \{\pm 1\}$. In either case, there exists $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$ such that $\varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$ and

$$\varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 a_3 + a_4 = 0.$$

Combining this and $a_i a_4 - a_j a_k = 0$, we have $w(a) = 0$. □

If the monodromy of the second order differential equation is reducible, then the corresponding differential equation can be reduced into the first order differential equation and solved elementary. Hence we are interested in irreducible monodromies. We find that the big open $\mathcal{M}^\circ(a)$ which introduced in the previous section gives a criterion for the irreducibility, that is, the following lemma holds.

Lemma 3.7. *Let us fix a parameter* $a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$. *Then any element in* $\mathcal{M}^\circ(a)$ *is irreducible.*

Proof. Assume that $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ is reducible. Then by considering the conjugation by some $P \in \mathrm{SL}(2, \mathbb{C})$, we may take a representative of the form

$$M_1 = \begin{pmatrix} \xi_1 & p_1 \\ 0 & \xi_1^{-1} \end{pmatrix}, \quad M_2 = \begin{pmatrix} \xi_2 & p_2 \\ 0 & \xi_2^{-1} \end{pmatrix}, \quad M_3 = \begin{pmatrix} \xi_3 & p_3 \\ 0 & \xi_3^{-1} \end{pmatrix}, \tag{3.17}$$

where $p_i \in \mathbb{C}$ and ξ_i are the non-zero complex numbers such that

$$\xi_i + \xi_i^{-1} = a_i, \quad a_4 = \xi_1\xi_2\xi_3 + \frac{1}{\xi_1\xi_2\xi_3}.$$

Regarding x as an invariant of $\mathcal{M}(a)$ by (3.6), we find that $\psi(x_i, a_i, a_4) = \psi(x_i, a_j, a_k) = 0$ for all $i = 1, 2, 3$, which implies $p_{i\nu}(x) = 0$ for all (i, ν). This means that $[(M_1, M_2, M_3)] \notin \mathcal{M}^\circ(a)$ for all (i, ν), and complete the proof. □

As mentioned in proposition 3.6, it holds that $\mathcal{M}(a) = \mathcal{M}^\circ(a)$ for generic $a \in \mathbb{C}$. As a consequence of this fact and Lemma 3.7, we obtain the following.

Proposition 3.8. *If the parameter* $a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ *satisfies* (3.15), *then any element in* $\mathcal{M}(a)$ *is irreducible.*

4. MONODROMY INVARIANT HERMITIAN FORMS

We give some basic facts about monodromy invariant Hermitian forms. The following lemma states that the monodromy invariant Hermitian forms give a simple criterion for the irreducible monodromies.

Lemma 4.1. *Let* (M_1, M_2, M_3) *be a triple of matrices in* $\mathrm{SL}(n, \mathbb{C})$ *and assume that there is a non-zero Hermitian matrix* H *satisfying*

$$\bar{M}_1^T H M_1 = H, \quad \bar{M}_2^T H M_2 = H, \quad \bar{M}_3^T H M_3 = H.$$

If $\det H = 0$, *then the triple* (M_1, M_2, M_3) *is reducible.*

Since this lemma is proved in a more general case in [2], we omit the proof here. Thanks to this lemma, we see that any invariant Hermitian form of irreducible monodromy is non-degenerate. Therefore we consider non-degenerate Hermitian forms mainly. Next we find the existence of non-degenerate invariant Hermitian form imposes certain constraints on the eigenvalues of monodromies.

Lemma 4.2. *Let* M *be a matrix in* $\mathrm{SL}(2, \mathbb{C})$. *We denote by* $\{\xi, 1/\xi\}$ *the eigenvalues of* M. *If there is a non-degenerate Hermitian matrix* H *satisfying*

$$\bar{M}^T H M = H, \tag{4.1}$$

we have $\{\bar{\xi}, 1/\bar{\xi}\} = \{\xi, 1/\xi\}$.

Proof. We denote by $\sigma(X)$ the set of eigenvalues of X. From the assumption $\sigma(M) = \{\xi, 1/\xi\}$, we have

$$\sigma(M^{-1}) = \{\xi, 1/\xi\}, \quad \sigma(\bar{M}^T) = \sigma(\bar{M}) = \{\bar{\xi}, 1/\bar{\xi}\}.$$

On the other hand, from (4.1) we have

$$\bar{M}^T = H(HM)^{-1} = HM^{-1}H^{-1}.$$

This implies that $\sigma(\bar{M}^T) = \sigma(M^{-1})$ and completes the proof. □

We remark that Lemma 4.2 can be generalized to the case of $M \in \mathrm{GL}(n, \mathbb{C})$ (see [2, Lemma 3.2]). The following corollary is very important for the characterization of the existence of monodromy invariant non-degenerate Hermitian forms.

Corollary 4.3. *Let M be a matrix in* $\mathrm{SL}(2, \mathbb{C})$. *If there is a non-degenerate Hermitian matrix H satisfying* (4.1), *we have* $\operatorname{tr} M \in \mathbb{R}$.

Proof. We set $\sigma(M) = \{\xi, 1/\xi\}$. Then we have

$$\operatorname{tr} M = \xi + \frac{1}{\xi}.$$

On the other hand, from Lemma 4.2 we have

$$\{\bar{\xi}, 1/\bar{\xi}\} = \{\xi, 1/\xi\}.$$

Hence we obtain

$$\xi + \frac{1}{\xi} = \bar{\xi} + \frac{1}{\bar{\xi}} = \overline{\xi + \frac{1}{\xi}}$$

which implies $\operatorname{tr} M \in \mathbb{R}$. □

By looking at this result, for the irreducible elements $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$, we see that the assumption

$$a_i = \operatorname{tr} M_i \in \mathbb{R} \quad (i = 1, 2, 3, 4) \tag{4.2}$$

is needed for the existence of invariant non-degenerate Hermitian forms. Here M_4 is determined by the relation (2.2). Therefore we assume (4.2) in the following sections.

5. INVARIANT HERMITIAN FORMS FOR MONODROMIES IN BIG OPEN

In this section, we characterize the existence condition of the invariant Hermitian forms for the monodromies in big open.

Theorem 5.1. *We take $a \in \mathbb{R}^4$ and fix the homeomorphism $\varphi : \mathcal{S}^\circ(a) \to \mathcal{M}^\circ(a)$ by* (3.14). *Then the element $\varphi(x) = [(M_1, M_2, M_3)] \in \mathcal{M}^\circ(a)$ has an invariant non-degenerate Hermitian forms if and only if $x \in \mathcal{S}^\circ(a) \cap \mathbb{R}^3$.*

Remark 5.2. As seen in Section 3, if the parameter $a \in \mathbb{R}^4$ satisfies the condition (3.15), the open sets $\mathcal{S}^\circ(a)$ and $\mathcal{M}^\circ(a)$ can be replaced to the entire surface $\mathcal{S}(a)$ and the moduli space $\mathcal{M}(a)$, respectively. On the other hand, when the condition (3.15) does not hold, the complements $\mathcal{S}(a) \setminus \mathcal{S}^\circ(a)$ and $\mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$ are non-empty. As seen in (3.17), we see that the several elements in $\mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$ correspond to the one point of $\mathcal{S}(a) \setminus \mathcal{S}^\circ(a)$, hence it is impossible to construct the homeomorphism as like $\varphi(x)$. The treatment of such case will be given in Section 6.

From the point of view of Fuchsian differential equations, Theorem 5.1 shows the existence of the second order irreducible non-rigid Fuchsian differential equations of SL type with four singularities having the monodromy invariant Hermitian forms.

5.1. PROOF OF NECESSITY

We show the necessity of Theorem 5.1, which is followed from the correspondence (3.6) and Corollary 4.3. We take $x \in \mathcal{S}^\circ(a)$ and assume that

$$\varphi(x) = [(M_1, M_2, M_3)] \in \mathcal{M}^\circ(a)$$

has an invariant non-degenerate Hermitian form. Then for any representative (M_1, M_2, M_3) of $\varphi(x)$, there exists a non-degenerate Hermitian matrix H such that

$$\bar{M}_i^T H M_i = H \quad (i = 1, 2, 3). \tag{5.1}$$

Let (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$. Since $(M_1, M_2, M_3) \in \mathrm{SL}(2, \mathbb{C})^3$, we have $M_j M_k \in \mathrm{SL}(2, \mathbb{C})$ for $i = 1, 2, 3$. Moreover, from (5.1) we obtain

$$(\overline{M_j M_k})^T H (M_j M_k) = H.$$

Hence from Corollary 4.3, we see $\mathrm{tr}(M_j M_k) = x_i \in \mathbb{R}$ and then $x \in \mathbb{R}^3$. This completes the proof of necessity.

5.2. PROOF OF SUFFICIENCY

To show the sufficiency of Theorem 5.1, it is sufficient to prove the following proposition.

Proposition 5.3. *Let $a = (a_1, a_2, a_3, a_4)$ be an element in $\mathbb{R}^4$ and fix the homeomorphism $\varphi_{i\nu} : \mathcal{S}_{i\nu}(a) \to \mathcal{M}_{i\nu}(a)$ by (3.14) for each $i = 1, 2, 3$, $\nu = 1, 2$. For $x \in \mathcal{S}_{i\nu}(a) \cap \mathbb{R}^3$, we take a representative (M_i, M_j, M_k) of the element $\varphi(x) = [(M_1, M_2, M_3)] \in \mathcal{M}_{i\nu}(a)$ by Tables 1 or 2. Then the representative (M_i, M_j, M_k)*

has an invariant non-degenerate Hermitian form and associate Hermitian matrix H is given as follows.

$$H = \begin{cases} h\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} & (x_i^2 > 4) \\ \\ h\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{\psi(x_i, a_i, a_4)}{x_i^2 - 4} \end{pmatrix} & (x_i^2 < 4) \end{cases} \qquad (\nu = 1),$$

$$H = \begin{cases} h\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} & (x_i^2 > 4) \\ \\ h\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{\psi(x_i, a_j, a_k)}{x_i^2 - 4} \end{pmatrix} & (x_i^2 < 4) \end{cases} \qquad (\nu = 2), \tag{5.2}$$

where h is an arbitrary real number.

Proof. We shall only proof the case $\nu = 1$; the other case $\nu = 2$ can be shown in a same manner. Let us take the matrices (M_i, M_j, M_k) as in Table 1, and set the matrix H as

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

We shall determine the entries h_{ij} by solving the algebraic equations

$$\begin{aligned} R_i &:= \bar{M}_i^T H M_i - H = O, \\ R_j &:= \bar{M}_j^T H M_j - H = O, \\ R_k &:= \bar{M}_k^T H M_k - H = O. \end{aligned} \tag{5.3}$$

Now we look at the entries of the matrices M_i, M_j, M_k. Under the assumption $a \in \mathbb{R}^4$ and $x \in \mathbb{R}^3$, we see that

$$\psi(x_i, a_i, a_4),\ y_i,\ y_j,\ y_k \in \mathbb{R}$$

always hold. On the other hand, $r_i = \sqrt{x_i^2 - 4}$ is a real number only and only if $x_i^2 > 4$. Therefore, in the following, we consider the equations (5.3) dividing into two cases.

First we consider the case of $x_i^2 > 4$. In this case we have $\bar{M}_i^T = M_i^T$, $\bar{M}_j^T = M_j^T$, and $\bar{M}_k^T = M_k^T$. We denote $R_*[s,t]$ by the (s,t)-entry of R_*. Let us consider the equation $R_i = O$. By solving

$$\begin{cases} R_i[1,1] = 0, \\ R_i[1,2] = 0 \end{cases}$$

with respect to h_{21} and h_{22}, we obtain

$$h_{21} = -\frac{h_{11}(2a_4 - a_i x_i) + h_{12} r_i}{r_i}, \quad h_{22} = -\frac{h_{11}\psi(x_i, a_i, a_4)}{x_i^2 - 4}, \tag{5.4}$$

which implies $R_i = O$.

Next we solve $R_k = O$. Now $R_k[1,2]$ is given by

$$R_k[1,2] = -\frac{h_{11}(-a_k x_i x_j + a_i x_i + a_j x_j - a_k x_k + a_4 a_k^2 - 2a_4)}{r_i}.$$

By solving $R_k[1,2] = 0$ with respect to h_{11}, we have

$$h_{11} = 0, \tag{5.5}$$

and then we find that $R_k = R_j = O$ hold. Substituting (5.5) into (5.4), we obtain $h_{21} = -h_{12}$ and $h_{22} = 0$. Therefore we obtain

$$H = \begin{pmatrix} 0 & h_{12} \\ -h_{12} & 0 \end{pmatrix}.$$

Since h_{12} can be taken arbitrary, we set $h_{12} = h\sqrt{-1}$ and obtain the assertion for $x_i^2 > 4$.

Next we consider the case of $x_i^2 < 4$. Noting that $\bar{r}_i = -r_i$ and hence $\overline{\lambda_i^{\pm}} = \lambda_i^{\mp}$ in this case, we have

$$\bar{M}_i^T = \begin{pmatrix} -\dfrac{a_4 - a_i\lambda_i^+}{r_i} & 1 \\ -\dfrac{\psi(x_i, a_i, a_4)}{x_i^2 - 4} & \dfrac{a_4 - a_i\lambda_i^-}{r_i} \end{pmatrix},$$

$$\bar{M}_j^T = \begin{pmatrix} \dfrac{a_k - a_j\lambda_i^-}{r_i} & \dfrac{y_k - y_j\lambda_i^-}{\psi(x_i, a_i, a_4)} \\ -\dfrac{y_k - y_j\lambda_i^+}{x_i^2 - 4} & -\dfrac{a_k - a_j\lambda_i^+}{r_i} \end{pmatrix},$$

$$\bar{M}_k^T = \begin{pmatrix} \dfrac{a_j - a_k\lambda_i^-}{r_i} & \dfrac{y_j - y_k\lambda_i^+}{\psi(x_i, a_i, a_4)} \\ -\dfrac{y_j - y_k\lambda_i^-}{x_i^2 - 4} & -\dfrac{a_j - a_k\lambda_i^+}{r_i} \end{pmatrix}.$$

Let us consider the equations

$$\begin{cases} R_i[1,1] = 0, \\ R_i[1,2] = 0. \end{cases}$$

By solving these equations with respect to h_{21}, h_{22}, we obtain

$$\begin{aligned} h_{21} &= -h_{12}, \\ h_{22} &= \frac{h_{11} r_i \psi(x_i, a_i, a_4) + h_{12}\left(x_i^2 - 4\right)(2a_4 - a_i x_i)}{r_i(x_i^2 - 4)} \end{aligned} \tag{5.6}$$

which implies $R_i = O$. Next we solve $R_k = O$. Now $R_k[2,2]$ is given by

$$R_k[2,2] = -\frac{h_{12}(a_k x_i x_j + a_k x_k - a_i x_i - a_j x_j - a_4 a_k^2 + 2a_4)}{r_i}.$$

We solve $R_k[2,2] = 0$ with respect to h_{12} to obtain

$$h_{12} = 0, \tag{5.7}$$

and then we find that $R_k = R_j = O$ hold. Substituting (5.7) into (5.6), we obtain $h_{21} = 0$ and $h_{22} = h_{11}\psi(x_i, a_i, a_4)/(x_i^2 - 4)$. Therefore we obtain

$$H = h_{11} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{\psi(x_i, a_i, a_4)}{x_i^2 - 4} \end{pmatrix}.$$

Since h_{11} can be taken arbitrary, we set $h_{11} = h$ and obtain the assertion for $x_i^2 < 4$. $\square$

Remark 5.4. From the way of the proof, we see that invariant Hermitian matrix is only the form of (5.2). That is, the space of the invariant Hermitian forms for the each monodromy is real one-dimensional. This is the same phenomena with the case of rigid Fuchsian differential equations (see Haraoka [8,9]) and some non-rigid Fuchsian differential equations [2].

6. OUTSIDE OF BIG OPEN

When the condition (3.15) does not hold, there exists an irreducible element $[(M_1, M_2, M_3)] \in \mathcal{M} \setminus \mathcal{M}^\circ(a)$. For such elements, from the definition of big open, we have $p_{i\nu}(x) = 0$ for all (i, ν), where $x = (x_1, x_2, x_3)$ is given by (3.6) and the polynomial $p_{i\nu}(x)$ is given by (3.7) and (3.8). Now we consider the elements dividing into the following two cases:

(I) $x_i^2 = 4$ for all $i = 1, 2, 3$,
(II) at least one of $x_i^2 \neq 4$.

In the following, we first consider a parametrization for the irreducible elements in $\mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$. After that, we show the existence of invariant non-degenerate Hermitian forms and give the explicit forms of associate Hermitian matrices. Here we remark that the strategy of our parametrization is inspired by Iwasaki [13] and Calligaris–Mazzocco [5].

6.1. CASE (I)

We consider the case (I), that is, we assume $v(a)w(a) = 0$ and $(M_1, M_2, M_3) \in \mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$ satisfies

$$x_i^2 = (\mathrm{tr}(M_j M_k))^2 = 4 \quad (i = 1, 2, 3). \tag{6.1}$$

First of all, we consider the case that at least one of the local monodromies is diagonalizable and similar with scalar matrix.

Proposition 6.1. *Take an irreducible element* $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ *satisfying* (6.1) *and set* $x = (x_1, x_2, x_3) = (2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3)$ *by using* $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3$. *If the matrix* M_i *in the representative* (M_1, M_2, M_3) *is identical to the scaler matrix* $\xi_i I$ $(\xi_i \in \{\pm 1\})$, *then we have*

$$(a_i, a_j, a_k, a_4) = (2\xi_i, 2\xi_i\varepsilon_k, 2\xi_i\varepsilon_j, 2\xi_i\varepsilon_i) \tag{6.2}$$

and can take a representative (M_i, M_j, M_k) *as*

$$M_i = \begin{pmatrix} \xi_i & 0 \\ 0 & \xi_i \end{pmatrix}, \quad M_j = \begin{pmatrix} \varepsilon_k\xi_i & 2(\varepsilon_i - \varepsilon_j\varepsilon_k) \\ 0 & \varepsilon_k\xi_i \end{pmatrix}, \quad M_k = \begin{pmatrix} \varepsilon_j\xi_i & 0 \\ 1 & \varepsilon_j\xi_i \end{pmatrix}. \tag{6.3}$$

We see $\varepsilon_i\varepsilon_j\varepsilon_k = 1$ *if and only if* (M_i, M_j, M_k) *is reducible.*

Proof. The last assertion is seen easily from the representative (6.3). Hence we show only the first and second assertions. We take a representative $(M_1, M_2, M_3) \in \mathcal{M}(a)$ such that $M_i = \xi_i I$. Then we see that $a_i = \operatorname{tr} M_i = 2\xi_i$ and

$$2\varepsilon_k = x_k = \operatorname{tr}(M_j M_i) = \xi_i \operatorname{tr} M_j = \xi_i a_j.$$

From this and $\xi_i^2 = 1$, we obtain $a_j = 2\xi_i\varepsilon_k$. Similarly we can get $a_k = 2\xi_i\varepsilon_j$. Lastly,

$$a_4 = \operatorname{tr}(M_k M_j M_i) = \xi_i \operatorname{tr}(M_k M_j) = \xi_i x_i = 2\xi_i\varepsilon_i$$

and we obtain (6.2). Now we note that $a_j = 2\varepsilon_k\xi_i$ and $a_k = 2\varepsilon_j\xi_i$ imply $a_j, a_k \in \{\pm 2\}$, respectively. This means

$$\sigma(M_j) = \{\varepsilon_k\xi_i, \varepsilon_k\xi_i\}, \quad \sigma(M_k) = \{\varepsilon_j\xi_i, \varepsilon_j\xi_i\}.$$

Next we shall show the assertion (6.3). If $M_j \sim \varepsilon_k\xi_i I$ (resp. $M_k \sim \varepsilon_j\xi_i I$) holds, then the triple (M_i, M_j, M_k) is reducible. Indeed, the eigenspace of M_k (resp. M_j) is a non-trivial common invariant subspace. Hence we have

$$M_j \sim \begin{pmatrix} \varepsilon_k\xi_i & 1 \\ 0 & \varepsilon_k\xi_i \end{pmatrix}, \quad M_k \sim \begin{pmatrix} \varepsilon_j\xi_i & 1 \\ 0 & \varepsilon_j\xi_i \end{pmatrix}.$$

In this case, by considering the suitable conjugation, we can set

$$M_j = \begin{pmatrix} \varepsilon_k\xi_i & p \\ 0 & \varepsilon_k\xi_i \end{pmatrix}, \quad M_k = \begin{pmatrix} \varepsilon_j\xi_i & 0 \\ q & \varepsilon_j\xi_i \end{pmatrix}.$$

Now we have $pq \neq 0$ from the irreducibility. Then by considering the further conjugation by the suitable diagonal matrix we can take $q = 1$. The condition $\operatorname{tr}(M_j M_k) = x_i = 2\varepsilon_i$ yield

$$2\varepsilon_j\varepsilon_k + p = 2\varepsilon_i.$$

By solving this equation with respect to p, we obtain $p = 2(\varepsilon_i - \varepsilon_j\varepsilon_k)$. □

In the same way as Proposition 5.3, we can show the following proposition.

Proposition 6.2. *In addition to the assumptions as in Proposition 6.1, we assume* $\varepsilon_i\varepsilon_j\varepsilon_k \neq 1$*. Then the representative* (6.3) *of the element* $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ *has an invariant non-degenerate Hermitian form and associate Hermitian matrix* H *is given by*

$$H = h \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix},$$

where h *is an arbitrary real number.*

Next we consider the case at least one of the local monodromy is diagonalizable and has distinct eigenvalues.

Proposition 6.3. *Take an irreducible element* $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ *satisfying* (6.1) *and set* $x = (x_1, x_2, x_3) = (2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3)$ *by using* $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3$*. If the matrix* M_i *in the representative* (M_1, M_2, M_3) *has distinct eigenvalues, then we can take a representative* (M_i, M_j, M_k) *as follows.*

(i) *If* $a_i \neq \varepsilon_k a_j$*, then*

$$\begin{aligned}
M_i &= \begin{pmatrix} \dfrac{a_i + s_i}{2} & 0 \\ 0 & \dfrac{a_i - s_i}{2} \end{pmatrix}, \\
M_j &= \begin{pmatrix} \dfrac{a_j(s_i - a_i) + 4\varepsilon_k}{2s_i} & -\dfrac{(a_i - \varepsilon_k a_j)^2}{a_i^2 - 4} \\ 1 & \dfrac{a_j(s_i + a_i) - 4\varepsilon_k}{2s_i} \end{pmatrix}, \\
M_k &= \begin{pmatrix} \dfrac{a_k(s_i - a_i) + 4\varepsilon_j}{2s_i} & w_{12} \\ w_{21} & \dfrac{a_k(s_i + a_i) - 4\varepsilon_j}{2s_i} \end{pmatrix},
\end{aligned} \tag{6.4}$$

where the branch of

$$s_i = \sqrt{a_i^2 - 4} \tag{6.5}$$

is fixed and

$$\begin{aligned}
w_{12} &= \frac{4\{a_4 - \varepsilon_i(a_i - s_i)\}s_i - \{4\varepsilon_j - a_k(a_i - s_i)\}\{4\varepsilon_k - a_j(a_i - s_i)\}}{4(a_i^2 - 4)}, \\
w_{21} &= \frac{4\{a_4 - \varepsilon_i(a_i + s_i)\}s_i + \{4\varepsilon_j - a_k(a_i + s_i)\}\{4\varepsilon_k - a_j(a_i + s_i)\}}{4(a_i - \varepsilon_k a_j)^2}.
\end{aligned} \tag{6.6}$$

This representative does not depend on the choice of the branch in (6.5).

(ii) *If* $a_i = \varepsilon_k a_j$ *and* $a_i \neq \varepsilon_j a_k$*, then*

$$M_i = \begin{pmatrix} \frac{a_i + s_i}{2} & 0 \\ 0 & \frac{a_i - s_i}{2} \end{pmatrix},$$

$$M_j = \begin{pmatrix} \frac{\varepsilon_k(a_i - s_i)}{2} & v_{12} \\ v_{21} & \frac{\varepsilon_k(a_i + s_i)}{2} \end{pmatrix}, \tag{6.7}$$

$$M_k = \begin{pmatrix} \frac{a_k(s_i - a_i) + 4\varepsilon_j}{2s_i} & -\frac{(a_i - \varepsilon_j a_k)^2}{a_i^2 - 4} \\ 1 & \frac{a_k(s_i + a_i) - 4\varepsilon_j}{2s_i} \end{pmatrix},$$

where

$$v_{12} = -\frac{4\{a_4 - \varepsilon_i(a_i + s_i)\}s_i + \{4\varepsilon_j - a_k(a_i + s_i)\}\{4\varepsilon_k - a_j(a_i + s_i)\}}{4(a_i^2 - 4)},$$
$$v_{21} = -\frac{4\{a_4 - \varepsilon_i(a_i - s_i)\}s_i - \{4\varepsilon_j - a_k(a_i - s_i)\}\{4\varepsilon_k - a_j(a_i - s_i)\}}{4(a_i - \varepsilon_j a_k)^2}. \tag{6.8}$$

This representative does not depend on the choice of the branch in (6.5)*.*

(iii) *If* $a_i = a_j \varepsilon_k$ *and* $a_i = \varepsilon_j a_k$*, then*

$$a_4 = a_i \varepsilon_j \varepsilon_k + \frac{(a_i + s_i)\{2\varepsilon_i - (a_i^2 - 2)\varepsilon_j \varepsilon_k\}}{2} \tag{6.9}$$

and

$$M_i = \begin{pmatrix} \frac{a_i + s_i}{2} & 0 \\ 0 & \frac{a_i - s_i}{2} \end{pmatrix},$$

$$M_j = \begin{pmatrix} \frac{(a_i - s_i)\varepsilon_k}{2} & 0 \\ 1 & \frac{(a_i + s_i)\varepsilon_k}{2} \end{pmatrix}, \tag{6.10}$$

$$M_k = \begin{pmatrix} \frac{(a_i - s_i)\varepsilon_j}{2} & 2\varepsilon_i - (a_i^2 - 2)\varepsilon_j \varepsilon_k \\ 0 & \frac{(a_i + s_i)\varepsilon_j}{2} \end{pmatrix}$$

or

$$a_4 = a_i \varepsilon_j \varepsilon_k + \frac{(a_i - s_i)\{2\varepsilon_i - (a_i^2 - 2)\varepsilon_j \varepsilon_k\}}{2} \tag{6.11}$$

and

$$M_i = \begin{pmatrix} \dfrac{a_i + s_i}{2} & 0 \\ 0 & \dfrac{a_i - s_i}{2} \end{pmatrix},$$

$$M_j = \begin{pmatrix} \dfrac{(a_i - s_i)\varepsilon_k}{2} & 1 \\ 0 & \dfrac{(a_i + s_i)\varepsilon_k}{2} \end{pmatrix}, \tag{6.12}$$

$$M_k = \begin{pmatrix} \dfrac{(a_i - s_i)\varepsilon_j}{2} & 0 \\ 2\varepsilon_i - (a_i^2 - 2)\varepsilon_j\varepsilon_k & \dfrac{(a_i + s_i)\varepsilon_j}{2} \end{pmatrix}.$$

Note that the triple (M_i, M_j, M_k) *is reducible only and only if*

$$2\varepsilon_i - (a_i^2 - 2)\varepsilon_j\varepsilon_k = 0.$$

By taking the other branch in (6.5)*, the value* (6.9) *and the representative* (6.10) *changed into the value* (6.11) *and the representative equivalent with* (6.12) *respectively, and vice versa.*

Proof. Take $[(M_1, M_2, M_3)] \in \mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$ satisfying (6.1) and assume that M_i has distinct eigenvalues $\xi_i^\pm$. Note that this condition yields $a_i^2 - 4 \neq 0$. We may assume that M_i is already diagonalized;

$$M_i = \begin{pmatrix} \xi_i^+ & 0 \\ 0 & \xi_i^- \end{pmatrix},$$

where

$$\xi_i^+ + \xi_i^- = a_i, \qquad \xi_i^+\xi_i^- = 1.$$

Hence we can set

$$\xi_i^\pm = \frac{a_i \pm s_i}{2}, \quad s_i = \sqrt{a_i^2 - 4}. \tag{6.13}$$

Then we have

$$M_i = \begin{pmatrix} \dfrac{a_i + s_i}{2} & 0 \\ 0 & \dfrac{a_i - s_i}{2} \end{pmatrix}.$$

To make the expressions simple, we forget the relation (6.13) for a while. We set

$$M_j = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}, \qquad M_k = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}.$$

The conditions $\operatorname{tr}(M_iM_j) = x_k = 2\varepsilon_k$ and $\operatorname{tr} M_j = a_j$ leads to

$$\begin{cases} \xi_i^+ v_{11} + \xi_i^- v_{22} = 2\varepsilon_k, \\ v_{11} + v_{22} = a_j. \end{cases}$$

By solving these equations we have

$$v_{11} = \frac{2\varepsilon_k - a_j\xi_i^-}{s_i}, \qquad v_{22} = -\frac{2\varepsilon_k - a_j\xi_i^+}{s_i}. \tag{6.14}$$

Similarly the conditions $\operatorname{tr}(M_k M_i) = x_j = 2\varepsilon_j$ and $\operatorname{tr} M_k = a_k$ leads to

$$\begin{cases} \xi_i^+ w_{11} + \xi_i^- w_{22} = 2\varepsilon_j, \\ w_{11} + w_{22} = a_k, \end{cases}$$

and we obtain

$$w_{11} = \frac{2\varepsilon_j - a_k\xi_i^-}{s_i}, \qquad w_{22} = -\frac{2\varepsilon_j - a_k\xi_i^+}{s_i}. \tag{6.15}$$

Next we consider the conditions $\operatorname{tr}(M_k M_j) = x_i = 2\varepsilon_i$ and $\operatorname{tr}(M_k M_j M_i) = a_4$. They lead to the equations

$$\begin{cases} w_{11}v_{11} + w_{12}v_{21} + w_{21}v_{12} + w_{22}v_{22} = 2\varepsilon_i, \\ \xi_i^+(w_{11}v_{11} + w_{12}v_{21}) + \xi_i^-(w_{21}v_{12} + w_{22}v_{22}) = a_4. \end{cases}$$

From these equations and (6.14) and (6.15), we have

$$w_{12}v_{21} = \frac{(a_4 - 2\varepsilon_i\xi_i^-)s_i - (2\varepsilon_j - a_k\xi_i^-)(2\varepsilon_k - a_j\xi_i^-)}{a_i^2 - 4}, \tag{6.16}$$

$$w_{21}v_{12} = -\frac{(a_4 - 2\varepsilon_i\xi_i^+)s_i + (2\varepsilon_j - a_k\xi_i^+)(2\varepsilon_k - a_j\xi_i^+)}{a_i^2 - 4}. \tag{6.17}$$

On the other hand, the conditions $\det M_j = \det M_k = 1$ yield

$$\begin{cases} v_{11}v_{22} - v_{12}v_{21} = 1, \\ w_{11}w_{22} - w_{12}w_{21} = 1. \end{cases}$$

Substituting (6.14) and (6.15) into them, we obtain

$$v_{12}v_{21} = -\frac{(a_i - \varepsilon_k a_j)^2}{a_i^2 - 4}, \tag{6.18}$$

$$w_{12}w_{21} = -\frac{(a_i - \varepsilon_j a_k)^2}{a_i^2 - 4}. \tag{6.19}$$

We would like to determine the unknowns $v_{12}, v_{21}, w_{12}, w_{21}$ so that (6.16), (6.17), (6.18) and (6.19) hold. Now we find the following lemma holds.

Lemma 6.4. *We assume* $(x_1, x_2, x_3) = (2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3)$ *for* $\varepsilon_i \in \{\pm 1\}$ *and* $f_a(x) = 0$. *Then it holds that*

$$\begin{aligned} &\{(a_4 - 2\varepsilon_i\xi_i^-)s_i - (2\varepsilon_j - a_k\xi_i^-)(2\varepsilon_k - a_j\xi_i^-)\} \\ &\times \{(a_4 - 2\varepsilon_i\xi_i^+)s_i + (2\varepsilon_j - a_k\xi_i^+)(2\varepsilon_k - a_j\xi_i^+)\} \\ &= -(a_i - \varepsilon_j a_k)^2(a_i - \varepsilon_k a_j)^2. \end{aligned} \tag{6.20}$$

This lemma can be shown by a direct calculation. The relation (6.20) guarantees that if the variables $v_{12}, v_{21}, w_{12}, w_{21}$ satisfy the three of (6.16), (6.17), (6.18) and (6.19), then the remaining one is also satisfied.

In the following we divide the situation into three cases.

Case (i). The case of $a_i \neq \varepsilon_k a_j$.

In this case, the condition (6.18) implies $v_{12}v_{21} \neq 0$. Considering the conjugation by the suitable diagonal matrix, we can assume

$$v_{21} = 1.$$

Then we obtain

$$w_{12} = \frac{(a_4 - 2\varepsilon_i \xi_i^-)s_i - (2\varepsilon_j - a_k \xi_i^-)(2\varepsilon_k - a_j \xi_i^-)}{a_i^2 - 4}, \tag{6.21}$$

$$v_{12} = -\frac{(a_i - \varepsilon_k a_j)^2}{a_i^2 - 4} \tag{6.22}$$

from (6.16) and (6.18) respectively. Substituting (6.21) into (6.19) we have

$$w_{21} = -\frac{(a_i - \varepsilon_j a_k)^2}{(a_4 - 2\varepsilon_i \xi_i^-)s_i - (2\varepsilon_j - a_k \xi_i^-)(2\varepsilon_k - a_j \xi_i^-)}.$$

Using (6.20), this can be rewritten

$$w_{21} = \frac{(a_4 - 2\varepsilon_i \xi_i^+)s_i + (2\varepsilon_j - a_k \xi_i^+)(2\varepsilon_k - a_j \xi_i^+)}{(a_i - \varepsilon_k a_j)^2}. \tag{6.23}$$

Lastly, by substituting (6.13) into (6.21), (6.22) and (6.23), we obtain (6.4) and (6.6).

Taking the other branch in (6.13) has the effect that $s_i \leftrightarrow -s_i$, which results in a change of the representative (6.4). But this change is canceled by taking conjugation by a matrix

$$\begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \quad \text{such that} \quad \alpha^2 = \frac{(a_i - \varepsilon_k a_j)^2}{a_i^2 - 4}.$$

Case (ii). The case of $a_i = \varepsilon_k a_j$ and $a_i \neq \varepsilon_j a_k$.

In this case we have $v_{12}v_{21} = 0$ and $w_{12}w_{21} \neq 0$ from (6.18) and (6.19). Considering the conjugation by a suitable diagonal matrix, we take

$$w_{21} = 1.$$

Then we have

$$v_{12} = -\frac{(a_4 - 2\varepsilon_i \xi_i^+)s_i + (2\varepsilon_j - a_k \xi_i^+)(2\varepsilon_k - a_j \xi_i^+)}{a_i^2 - 4}, \tag{6.24}$$

$$w_{12} = -\frac{(a_i - \varepsilon_j a_k)^2}{a_i^2 - 4}, \tag{6.25}$$

from (6.17) and (6.19) respectively. Substituting (6.25) into (6.16) we get

$$v_{21} = -\frac{(a_4 - 2\varepsilon_i\xi_i^-)s_i - (2\varepsilon_j - a_k\xi_i^-)(2\varepsilon_k - a_j\xi_i^-)}{(a_i - \varepsilon_j a_k)^2}. \tag{6.26}$$

By substituting $a_j = \varepsilon_k^{-1}a_i$ and (6.13), we have (6.7) and (6.8). Here we note that from the assumption $a_i = \varepsilon_k a_j$ and (6.20), at least one of (6.24) and (6.26) vanishes. The independency on the choice of the branch in (6.13) can be checked in a same manner in Case (i).

Case (iii). The case of $a_i = \varepsilon_k a_j = \varepsilon_j a_k$.

In this case, the conditions (6.18) and (6.19) imply that $v_{12}v_{21} = 0$ and $w_{12}w_{21} = 0$. Note that in this case, if any of the conditions

(a) $v_{12} = w_{12} = 0$,
(b) $v_{21} = w_{21} = 0$,
(c) $v_{12} = v_{21} = 0$,
(d) $w_{12} = w_{21} = 0$

is satisfied, then the triple (M_i, M_j, M_k) is reducible. Hence we assume that the all above four conditions do not hold. Under this assumption, we see that one of v_{12} and v_{21} is non-zero and the other is zero.

Now we consider the case of $v_{21} \neq 0$ and $v_{12} = 0$. Then, we have $w_{12} \neq 0$ from the irreducibility, which implies $w_{21} = 0$. By considering the conjugation by the suitable diagonal matrix, we can send

$$v_{21} = 1.$$

On the other hand, the condition $a_j = \varepsilon_j^{-1}a_k = \varepsilon_j a_k$ yields

$$v_{11} = \frac{2\varepsilon_k - \varepsilon_k a_i \xi_i^-}{s_i}.$$

Since $a_i = \operatorname{tr} M_i = \xi_i^+ + \xi_i^-$, we have

$$v_{11} = \frac{2\varepsilon_k - \varepsilon_k(\xi_i^+ + \xi_i^-)\xi_i^-}{s_i} = \frac{\varepsilon_k(\xi_i^+ - \xi_i^-)\xi_i^-}{s_i} = \frac{\varepsilon_k(\xi_i^+ - \xi_i^-)\xi_i^-}{\xi_i^+ - \xi_i^-} = \varepsilon_k\xi_i^-.$$

We can calculate the other variables v_{22}, w_{11}, w_{22} in the same manner. Eventually we obtain

$$M_j = \begin{pmatrix} \varepsilon_k\xi_i^- & 0 \\ 1 & \varepsilon_k\xi_i^+ \end{pmatrix}, \qquad M_k = \begin{pmatrix} \varepsilon_j\xi_i^- & w_{12} \\ 0 & \varepsilon_j\xi_i^+ \end{pmatrix}. \tag{6.27}$$

The condition $\operatorname{tr}(M_kM_j) = x_i = 2\varepsilon_i$ and $\operatorname{tr}(M_kM_jM_i) = a_4$ leads to

$$\begin{cases} \varepsilon_k\varepsilon_j\{(\xi_i^+)^2 + (\xi_i^-)^2\} + w_{12} = 2\varepsilon_i, \\ \varepsilon_j\varepsilon_k(\xi_i^+ + \xi_i^-) + \xi_i^+ w_{12} = a_4. \end{cases}$$

Since $(\xi_i^+)^2 + (\xi_i^-)^2 = (\xi_i^+ + \xi_i^-)^2 - 2 = a_i^2 - 2$, the first equation yields

$$w_{12} = 2\varepsilon_i - (a_i^2 - 2)\varepsilon_j\varepsilon_k.$$

Substituting this and (6.13) into the second equation, we have (6.9). Lastly, by substituting (6.13) into (6.27), we obtain the representative (6.10).

The other case $v_{21} = 0$ and $v_{12} \neq 0$ can be treated in a similar manner and we have (6.11) and (6.12). □

When $a_1, a_2, a_3, a_4 \in \mathbb{R}$, in the same way as Proposition 5.3, we can show that each irreducible element treated in Proposition 6.3 has invariant non-degenerate Hermitian forms.

Proposition 6.5. *In addition to the assumptions as in Proposition 6.3, we assume $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Then the each representatives* (6.4)*,* (6.7)*,* (6.10) *or* (6.12) *of the element $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ has an invariant non-degenerate Hermitian form and associate Hermitian matrix H is given as follows.*

(i) *If $a_i \neq \varepsilon_k a_j$ and (M_i, M_j, M_k) is given by* (6.4)*, then*

$$H = \begin{cases} h\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} & (a_i^2 > 4), \\ h\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{(a_i - \varepsilon_k a_j)^2}{a_i^2 - 4} \end{pmatrix} & (a_i^2 < 4), \end{cases} \tag{6.28}$$

where h is an arbitrary real number.

(ii) *If $a_i = \varepsilon_k a_j$, $a_i \neq \varepsilon_j a_k$ and (M_i, M_j, M_k) is given by* (6.7)*, then*

$$H = \begin{cases} h\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} & (a_i^2 > 4), \\ h\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{(a_i - \varepsilon_j a_k)^2}{a_i^2 - 4} \end{pmatrix} & (a_i^2 < 4), \end{cases} \tag{6.29}$$

where h is an arbitrary real number.

(iii) *If $a_i = \varepsilon_k a_j = \varepsilon_j a_k$ and (M_i, M_j, M_k) is given by* (6.10) *or* (6.12)*, then*

$$H = h\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \tag{6.30}$$

where h is an arbitrary real number.

Remark 6.6. In case (iii), a_4 is given by (6.9). In order to make it real number while keeping irreducibility, we have to take $s_i \in \mathbb{R}$, that is $a_i^2 > 4$.

Next we consider the case that $(M_1, M_2, M_3) \in \mathcal{M}(a)$ satisfies the assumption (I) and all the local monodromies are not diagonalizable.

Proposition 6.7. *Take an irreducible element* $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ *satisfying* (6.1) *and set* $x = (x_1, x_2, x_3) = (2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3)$ *by using* $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3$. *If none of the matrices* M_1, M_2, M_3 *is diagonalizable, then*

$$a_1^2 = a_2^2 = a_3^2 = 4 \tag{6.31}$$

and there exists $i \in \{1, 2, 3\}$ *such that*

$$a_i a_j - 4\varepsilon_k \neq 0. \tag{6.32}$$

and can take a representative (M_i, M_j, M_k) *of the form*

$$M_i = \begin{pmatrix} \frac{a_i}{2} & 0 \\ 1 & \frac{a_i}{2} \end{pmatrix}, \quad M_j = \begin{pmatrix} \frac{a_j}{2} & -\frac{a_i a_j - 4\varepsilon_k}{2} \\ 0 & \frac{a_j}{2} \end{pmatrix},$$
$$M_k = \begin{pmatrix} \frac{-4a_4 - a_i(a_j a_k - 4\varepsilon_i) + 4a_j\varepsilon_j}{2(a_i a_j - 4\varepsilon_k)} & -\frac{a_k a_i - 4\varepsilon_j}{2} \\ \frac{a_j a_k - 4\varepsilon_i}{a_i a_j - 4\varepsilon_k} & a_k + \frac{4a_4 + a_i(a_j a_k - 4\varepsilon_i) - 4a_j\varepsilon_j}{2(a_i a_j - 4\varepsilon_k)} \end{pmatrix}. \tag{6.33}$$

Proof. The matrices (M_i, M_j, M_k) are all similar with Jordan block of size 2;

$$M_i \sim \begin{pmatrix} \xi_i & 1 \\ & \xi_i \end{pmatrix}, \quad (\xi_i^2 = 1, \ i = 1, 2, 3). \tag{6.34}$$

From the condition $a_i = \mathrm{tr}(M_i)$ yields $a_i = 2\xi_i$ and then we have

$$\xi_i = \frac{a_i}{2}, \quad (i = 1, 2, 3).$$

The condition $\xi_i^2 = 1$ leads to (6.31).

Next we show the assertion (6.32). We first assume that all $i \in \{1, 2, 3\}$ satisfies

$$a_i a_j - 4\varepsilon_k = 0. \tag{6.35}$$

By considering similar transformation, we can make M_i and M_j into the lower triangular and triangular matrix, respectively. That is, we can take

$$M_i = \begin{pmatrix} \frac{a_i}{2} & \\ u_{21} & \frac{a_i}{2} \end{pmatrix}, \quad M_j = \begin{pmatrix} \frac{a_j}{2} & v_{12} \\ v_{21} & \frac{a_j}{2} \end{pmatrix}, \quad M_k = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

where $v_{12}v_{21} = 0$. From (6.34), the matrix M_i is not a diagonal matrix. Hence we have $u_{21} \neq 0$ and then can normalize

$$u_{21} = 1$$

by considering the conjugation by the suitable diagonal matrix. The condition $\operatorname{tr}(M_j M_i) = x_k = 2\varepsilon_k$ yields

$$\frac{a_i a_j}{2} + v_{12} = 2\varepsilon_k. \tag{6.36}$$

Since we assume (6.35) holds, then we have $v_{12} = 0$. Since we assume that the matrix M_j is not a diagonal matrix now, we obtain

$$v_{21} \neq 0. \tag{6.37}$$

Next we consider the conditions $\operatorname{tr} M_k = a_k$ and $\operatorname{tr}(M_j M_k) = x_i = 2\varepsilon_i$, which yield

$$\begin{cases} w_{11} + w_{22} = a_k, \\ \dfrac{a_j}{2}(w_{11} + w_{22}) + v_{21} w_{12} = 2\varepsilon_i. \end{cases}$$

From these equation we have

$$\frac{a_j a_k}{2} + v_{21} w_{12} = 2\varepsilon_i.$$

The assumption (6.35) yields $v_{21} w_{12} = 0$. Then we obtain $w_{12} = 0$ from (6.37). But it makes (M_i, M_j, M_k) reducible, which leads to a contradiction. Hence we see that there exists some $i \in \{1, 2, 3\}$ such that (6.32) holds.

We shall show the assertion (6.33). We return to (6.36) and obtain

$$v_{12} = -\frac{a_i a_j - 4\varepsilon_k}{2}. \tag{6.38}$$

From the conditions $\operatorname{tr} M_k = a_k$ and $\operatorname{tr}(M_k M_i) = x_j = 2\varepsilon_j$, we have

$$\begin{cases} w_{11} + w_{22} = a_k, \\ \dfrac{a_i}{2}(w_{11} + w_{22}) + w_{12} = 2\varepsilon_j. \end{cases} \tag{6.39}$$

Then we obtain

$$w_{12} = -\frac{a_k a_i - 4\varepsilon_j}{2}, \quad w_{22} = a_k - w_{11}. \tag{6.40}$$

On the other hand, the conditions $\operatorname{tr}(M_j M_k M_i) = a_4$ and $\operatorname{tr}(M_k M_j) = x_i = 2\varepsilon_i$ yields

$$\begin{cases} \dfrac{a_i a_j}{4}(w_{11} + w_{22}) + v_{12} w_{11} + \dfrac{a_j}{2} w_{12} + \dfrac{a_i}{2} v_{12} w_{21} = a_4, \\ \dfrac{a_i}{2}(w_{11} + w_{22}) + w_{21} v_{12} = 2\varepsilon_i. \end{cases} \tag{6.41}$$

Substituting (6.38) and (6.40) into the second equation, we have

$$w_{21} = \frac{a_j a_k - 4\varepsilon_i}{a_i a_j - 4\varepsilon_k}. \tag{6.42}$$

Next by substituting (6.38), (6.40) and (6.42) into the first equation and solving with respect to w_{11}, we have

$$w_{11} = \frac{-4a_4 - a_i(a_j a_k - 4\varepsilon_i) + 4a_j\varepsilon_j}{2(a_i a_j - 4\varepsilon_k)}$$

and complete the proof. □

Adding the assumption $a_4 \in \mathbb{R}$, in the same way as Proposition 5.3, we can show that any irreducible elements treated in Proposition 6.7 has an invariant non-degenerate Hermitian form.

Proposition 6.8. *In addition to the assumptions in Proposition 6.7, we assume* $a_4 \in \mathbb{R}$. *Then the representative* (6.33) *of the element* $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ *has an invariant non-degenerate Hermitian form and associate Hermitian matrix* H *is given by*

$$H = h \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \tag{6.43}$$

where h *is an arbitrary real number.*

6.2. CASE (II)

We consider the case (II), that is, we assume $v(a)w(a) = 0$ and there exists $i \in \{1, 2, 3\}$ such that $x_i^2 \neq 4$ for $(M_1, M_2, M_3) \in \mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$. In this case it holds that

$$\begin{cases} \psi(x_i, a_i, a_4) = x_i^2 + a_i^2 + a_4^2 - x_i a_i a_4 - 4 = 0, \\ \psi(x_i, a_j, a_k) = x_i^2 + a_j^2 + a_k^2 - x_i a_j a_k - 4 = 0. \end{cases} \tag{6.44}$$

We shall give a parametrization of (M_1, M_2, M_3).

Proposition 6.9. *Take an irreducible element* $[(M_1, M_2, M_3)] \in \mathcal{M}(a) \setminus \mathcal{M}^\circ(a)$ *satisfying* $x_i^2 \neq 4$ *and* (6.44). *Then it holds that*

$$x_j = \frac{(a_j a_4 + a_k a_i)(x_i^2 - r_i x_i - 4) + 2(a_k a_4 + a_i a_j)r_i + (r_i - x_i)(x_i^2 - 4)x_k}{2(x_i^2 - 4)}, \tag{6.45}$$

where $r_i = \sqrt{x_i^2 - 4}$. *Moreover, we can take a representative of the element* $[(M_i, M_j, M_k)]$ *either one of*

$$M_i = \begin{pmatrix} \dfrac{a_4 - a_i\lambda_i^-}{r_i} & x_k + \dfrac{2(a_k a_4 + a_i a_j) - (a_j a_4 + a_k a_i)x_i}{x_i^2 - 4} \\ 0 & -\dfrac{a_4 - a_i\lambda_i^+}{r_i} \end{pmatrix},$$
$$M_j = \begin{pmatrix} -\dfrac{a_k - a_j\lambda_i^+}{r_i} & 0 \\ 1 & \dfrac{a_k - a_j\lambda_i^-}{r_i} \end{pmatrix}, \qquad (6.46)$$
$$M_k = \begin{pmatrix} -\dfrac{a_j - a_k\lambda_i^+}{r_i} & 0 \\ -\lambda_i^- & \dfrac{a_j - a_k\lambda_i^-}{r_i} \end{pmatrix}$$

or

$$M_i = \begin{pmatrix} \dfrac{a_4 - a_i\lambda_i^-}{r_i} & 0 \\ x_k + \dfrac{2(a_k a_4 + a_i a_j) - (a_j a_4 + a_k a_i)x_i}{x_i^2 - 4} & -\dfrac{a_4 - a_i\lambda_i^+}{r_i} \end{pmatrix},$$
$$M_j = \begin{pmatrix} -\dfrac{a_k - a_j\lambda_i^+}{r_i} & 1 \\ 0 & \dfrac{a_k - a_j\lambda_i^-}{r_i} \end{pmatrix}, \qquad (6.47)$$
$$M_k = \begin{pmatrix} -\dfrac{a_j - a_k\lambda_i^+}{r_i} & -\lambda_i^+ \\ 0 & \dfrac{a_j - a_k\lambda_i^-}{r_i} \end{pmatrix},$$

where $\lambda_i^\pm = (x_i \pm r_i)/2$. *By taking the other branch in* $r_i = \sqrt{x_i^2 - 4}$, *the representative* (6.46) *changed into the representative equivalent with* (6.47), *and vice versa.*

Proof. Since $x_i = \mathrm{tr}(M_k M_j) \neq \pm 2$, the matrix $M_k M_j$ has distinct eigenvalues $\lambda_i^\pm$. By considering a similar transformation, we may assume $M_k M_j$ is already diagonalized;

$$M_k M_j = \begin{pmatrix} \lambda_i^+ & \\ & \lambda_i^- \end{pmatrix}, \qquad (6.48)$$

where

$$\lambda_i^+ + \lambda_i^- = a_i, \qquad \lambda_i^+ \lambda_i^- = 1.$$

Hence we can put

$$\lambda_i^\pm = \frac{x_i \pm r_i}{2}, \qquad r_i = \sqrt{x_i^2 - 4}.$$

Next we set

$$M_i = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \quad M_j = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}, \quad M_k = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}. \tag{6.49}$$

From the conditions $\operatorname{tr} M_i = a_i$ and $\operatorname{tr}(M_k M_j M_i) = a_4$ we have

$$\begin{cases} u_{11} + u_{22} = a_i, \\ \lambda_i^+ u_{11} + \lambda_i^- u_{22} = a_4. \end{cases}$$

By solving these equations we get

$$u_{11} = \frac{a_4 - a_i \lambda_i^-}{r_i}, \qquad u_{22} = -\frac{a_4 - a_i \lambda_i^+}{r_i}. \tag{6.50}$$

Condition $M_k M_j = \operatorname{diag}\{\lambda_i^+, \lambda_i^-\}$ or equivalently $M_k = \operatorname{diag}\{\lambda_i^+, \lambda_i^-\} M_j^{-1}$ yields

$$\begin{cases} w_{11} = \lambda_i^+ v_{22}, \\ w_{12} = -\lambda_i^+ v_{12}, \\ w_{21} = -\lambda_i^- v_{21}, \\ w_{22} = \lambda_i^- v_{11}. \end{cases} \tag{6.51}$$

Therefore, from the conditions $\operatorname{tr} M_j = a_j$ and $\operatorname{tr} M_k = a_k$, we obtain

$$\begin{cases} v_{11} + v_{22} = a_j, \\ \lambda_i^- v_{11} + \lambda_i^+ v_{22} = a_k. \end{cases}$$

By solving these equations we get

$$v_{11} = -\frac{a_k - a_j \lambda_i^+}{r_i}, \qquad v_{22} = \frac{a_k - a_j \lambda_i^-}{r_i}. \tag{6.52}$$

Substituting these into (6.51), we obtain

$$w_{11} = -\frac{a_j - a_k \lambda_i^+}{r_i}, \qquad w_{22} = \frac{a_j - a_k \lambda_i^-}{r_i}. \tag{6.53}$$

The condition $\det M_i = 1$ yields

$$u_{11} u_{22} - u_{12} u_{21} = 1.$$

Substituting (6.50) into this equation, we have

$$u_{12} u_{21} = -\frac{\psi(x_i, a_i, a_4)}{x_i^2 - 4} = 0. \tag{6.54}$$

Similarly, from the condition $\det M_j = 1$ and (6.52), we obtain

$$v_{12} v_{21} = -\frac{\psi(x_i, a_j, a_k)}{x_i^2 - 4} = 0. \tag{6.55}$$

Clearly, the assumptions

(a) $u_{12} = u_{21} = 0$,
(b) $v_{12} = v_{21} = 0$,
(c) $u_{12} = v_{12} = 0$,
(d) $u_{21} = v_{21} = 0$

make our representative (M_i, M_j, M_k) reducible. Hence we assume that the all above four conditions do not hold. Then we see that one of v_{12} and v_{21} in non-zero and the other is zero.

Now we consider the case $v_{21} \neq 0$ and $v_{12} = 0$. Then considering the conjugation by the suitable diagonal matrix, we can take

$$v_{21} = 1.$$

From the irreducibility we have $u_{12} \neq 0$, which implies $u_{21} = 0$.

Next the conditions $\mathrm{tr}(M_j M_i) = x_k$ and $\mathrm{tr}(M_i M_k) = x_j$ lead to

$$\begin{cases} u_{11}v_{11} + u_{12} + u_{22}v_{22} = x_k, \\ \lambda_i^+ u_{11}v_{22} - \lambda_i^- u_{12} + \lambda_i v_{11}u_{22} = x_j. \end{cases}$$

By solving these equations with respect to u_{12} and x_j taking (6.52) and (6.53) into account, we obtain (6.45) and

$$u_{12} = x_k + \frac{2(a_k a_4 + a_i a_j) - (a_j a_4 + a_k a_i)x_i}{x_i^2 - 4}. \tag{6.56}$$

Hence we have the desired representative (6.46).

The other case $v_{12} = 0$ and $v_{21} \neq 0$ can be treated in a similar manner and we have the representation (6.47). □

Before considering the invariant non-degenerate Hermitian form, we give a lemma concerning the irreducibility of (6.46) and (6.47).

Lemma 6.10. *We set x_j as* (6.45) *and assume that $a_1, a_2, a_3, a_4, x_k, x_i \in \mathbb{R}^4$ with $x_i^2 < 4$. Then the triple* (6.46) *and* (6.47) *is reducible if and only if $x_j \in \mathbb{R}$.*

Proof. We show the assertion for the triple (6.46); the other case for (6.47) can be shown in a similar manner. Under the assumption $x_i^2 < 4$, we have

$$r_i = \sqrt{x_i^2 - 4} \in \sqrt{-1}\mathbb{R}$$

which implies $\bar{r}_i = -r_i$. Hence we have

$$x_j - \bar{x}_j = \frac{r_i\{(a_j a_4 + a_k a_i)x_i - 2(a_k a_4 + a_i a_j) - (x_i^2 - 4)x_k\}}{x_i^2 - 4}.$$

From this and (6.56), we see that $x_j = \bar{x}_j$ if and only if $u_{12} = 0$, which means that the triple (6.46) is reducible. □

From Lemma 6.10, we see that $x_i^2 > 4$ is necessary for the irreducibility of (6.46) and (6.47). Under this condition we give a proposition for the existence of invariant Hermitian forms, which can be shown in a similar way with Proposition 5.3.

Proposition 6.11. *In addition to the assumptions in Proposition 6.9, we assume* $a_1, a_2, a_3, a_4, x_k, x_i \in \mathbb{R}$ *and* $x_i^2 > 4$. *Then the each representatives* (6.46) *or* (6.47) *of the element* $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ *has an invariant non-degenerate Hermitian form and associate Hermitian matrix* H *is given by*

$$H = h \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \tag{6.57}$$

where h *is an arbitrary real number.*

Combining Corollary 4.3, Propositions 5.3, 6.2, 6.5, 6.8 and 6.11, we obtain the following.

Theorem 6.12. *Let* $a = (a_1, a_2, a_3, a_4)$ *be an element in* $\mathbb{R}^4$ *and we set* $x = (x_1, x_2, x_3)$ *for the irreducible element* $[(M_1, M_2, M_3)] \in \mathcal{M}(a)$ *by* (3.6). *Then* $[(M_1, M_2, M_3)]$ *has an invariant non-degenerate Hermitian form if and only if* $x \in \mathcal{S}(a) \cap \mathbb{R}^3$ *holds.*

Remark 6.13. Summing up the Propositions 5.3, 6.2, 6.5, 6.8 and 6.11, we see that the space of the invariant Hermitian forms for each irreducible monodromy is real one-dimensional.

7. FINAL NOTE

After the acceptance of this paper, the author noticed the article [1], in which Miguel Acosta studied the $\mathrm{SL}(n, \mathbb{C})$-character variety $\mathcal{M}$ of a finitely generated group Γ, and characterized the points of $\mathcal{M}$ corresponding to the representations taking values in $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(n/2, \mathbb{H})$ or $\mathrm{SU}(p, q)$.

Our problem we studied in the present paper falls into the particular case where $n = 2$ and $\Gamma = \pi_1(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}, b)$. In this case, Acosta's result can be stated as follows: *An irreducible* $\rho = (M_1, M_2, M_3)$ *has an invariant Hermitian form if and only if*

$$\operatorname{tr} M \in \mathbb{R} \quad \textit{for all}\ M \in \langle M_1, M_2, M_3 \rangle.$$

On the other hand, in our paper the criterion is refined to a finite number of conditions

$$\operatorname{tr} M_i,\ \ \operatorname{tr}(M_j M_k),\ \ \operatorname{tr}(M_3 M_2 M_1) \in \mathbb{R},$$

where $\{i, j, k\} = \{1, 2, 3\}$. Moreover, we showed the existence of such ρ, which seems not to be considered in Acosta's paper.

Acknowledgements

The author would like to express his gratitude to Professor Yoshishige Haraoka for many valuable comments and suggestions for this study and manuscript. He also thanks to Professor Hiroshi Ogawara for fruitful discussions.

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Received: December 17, 2021.
Accepted: January 23, 2022.