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BLOWUP PHENOMENA FOR SOME FOURTH-ORDER STRAIN WAVE EQUATIONS AT ARBITRARY POSITIVE INITIAL ENERGY LEVEL

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Abstract. In this paper, we study a series of fourth-order strain wave equations involving dissipative structure, which appears in elasto-plastic-microstructure models. By some differential inequalities, we derive the finite time blow up results and the estimates of the upper bound blowup time with arbitrary positive initial energy. We also discuss the influence mechanism of the linear weak damping and strong damping on blowup time, respectively.

Keywords: fourth-order strain wave equation, arbitrary positive initial energy, blowup, blowup time.

Mathematics Subject Classification: 35L05, 35A01, 35L55.

1. INTRODUCTION

In recent years, the research on high-order evolution problem including fourth-order partial differential equations has become more and more active. Now let us review some of the current results. Ghoul *et al.* [5] studied a semilinear parabolic equation in the whole space $\mathbb{R}^N$ with the high-order operator $(-\Delta)^m$, where m denotes an odd integer. In [3] for studying the higher-order Cahn–Hilliard and Allen–Cahn models, the authors obtained the existence of global attractor and gave numerical simulations to explore the effects of higher-order terms. Then for the case of fourth-order hyperbolic equation with strongly damping term was considered by Yang *et al.* [18], in which the asymptotic behavior, global existence and finite time blowup of weak solutions were obtained by using the variational method based on the concepts of invariant sets. Subsequently, the non-existence of global solution to the fourth-order hyperbolic equation with nonlinear strain term

$$v_{tt} + \Delta^2 v - \alpha\Delta v + \sum_i^n \frac{\partial}{\partial x_i}\sigma_i(v_{x_i}) = f(v), \quad \alpha \geq 0,\, x \in \Omega \subset \mathbb{R}^n$$

was studied by Lin *et al.* [13] in the critical and super-critical initial energy cases. Then, it is worth pointing that much interest was paid to study this type of damped wave equations with strain term [4,11,14,16,17,20], which are widely used in the description of the longitudinal motion of elasto-plastic bar in the mathematical investigation of the elasto-plastic microstructure model [1].

The classical fourth order hyperbolic equation with dissipation and strain terms has the following form

$$v_{tt} + \Delta^2 v + \sum_{i}^{n} \frac{\partial}{\partial x_i}\sigma_i(v_{x_i}) + mv_t = 0, \quad m \geq 0,\, x \in \Omega \subset \mathbb{R}^n, \tag{1.1}$$

which was considered in [14] as the model used to describe the elastoplastic bar with elasto-plastic-microstructure that causes the dissipative effects. By introducing a family of potential wells, they obtained the threshold conditions of global existence and finite time blowup of solutions in the sub-critical and the critical initial energy cases $E(0) \leq d$ (d denotes the mountain pass level), respectively. According to the same method as above, there has been some representative works [9, 12, 19] recently to discover the control mechanism of the initial data on the dynamical behavior of the solutions. For the more generalized strain function $\sigma_i(v_{x_i})$ compared to the works in [14], Han *et al.* [7] proved global existence, energy decay estimate and finite time blowup for the IBVP (i.e., initial boundary value problem) of (1.1). In particular, the authors in [7] gave some sufficient conditions for ensuring the finite time blowup solution to (1.1) for the arbitrary positive initial energy $E(0) > 0$.

Later, by taking advantage of potential well method, Wang *et al.* [16] turned to consider the following fourth-order wave equation with weak damping and nonlinear strain term

$$v_{tt} + \Delta^2 v - \alpha\Delta v + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(v_{x_i}) + v_t = f(v), \quad x \in \Omega \subset \mathbb{R}^n \tag{1.2}$$

for $\alpha \geq 0$, which has one more external force source $f(u)$ than Equation (1.1). The authors in [16] obtained the global solution and its exponential decay for $E(0) < d$ and pushed the finite time blowup result to the arbitrary initial energy.

Then Xu *et al.* [17] studied the dissipative model with strong damping

$$v_{tt} + \Delta^2 v - \Delta v + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(v_{x_i}) - \Delta v_t = f(v), \quad x \in \Omega,\, t > 0. \tag{1.3}$$

In the sub-critical initial energy case, i.e., $E(0) < d$, they obtained the sharp condition of global and non-global solutions for (1.3) with $f(v) = 0$ and $\Omega = (0, 1)$. Later, the arbitrarily positive initial energy blowup solution for (1.3) was considered in [20] by using the concavity method.

Recently, Lian *et al.* [11] studied the following fourth-order nonlinear wave equations with nonlinear strain term, strong damping and nonlinear weak damping

$$v_{tt} + \Delta^2 v - \Delta v + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(v_{x_i}) - \Delta v_t + |v_t|^{r-1}v_t = f(v), \quad x \in \Omega \subset \mathbb{R}^n. \tag{1.4}$$

They firstly proved the local existence of weak solution by fixed point theory. Then within the framework of variational method, the global existence and non-existence, asymptotic behavior of solutions for (1.4) were derived with $r \geq 1$ and $E(0) \leq d$. Ultimately, the arbitrarily positive initial energy blowup solution was also discussed to restrict $r = 1$.

As we can see from above works, the potential well theory is an effective tool to show the existence and non-existence of solution, which works for both the sub-critical initial energy case $E(0) < d$ and the critical initial energy case $E(0) = d$. However, for the sup-critical initial energy $E(0) > d$, only some sufficient conditions for the finite time blowup solution are currently available as shown in [7, 11] for $E(0) > 0$. Even so, this is the best way so far to investigate blowup results for the sup-critical initial energy case which included in the arbitrarily positive initial energy case. No exception, in this paper we discuss the unreached conclusions for the IBVP of (1.1)–(1.4), respectively, at arbitrarily positive initial energy level. In summary for the above fourth-order strain wave equations with damping terms, it can be seen that although the finite time blowup of solutions corresponding to Equations (1.1)–(1.4) were obtained in the case of $E(0) > 0$, there has been no estimates of the blowup time. So at what time t^* blow-up occurs? Motivated by the previous works, we aim to estimate the upper bounds for t^* to Equations (1.1)–(1.4) with $E(0) > 0$, respectively. Furthermore, we reveal the relationship between the finite time blowup and damping terms by comparing the different blowup time.

The rest of this paper is organized as follows. In Section 2, we state some notations and preliminary lemmas. In Section 3, under some constraints on the initial data, we give an explicit expression of upper bound blowup time estimate for the above four classes of dissipative models with $E(0) > 0$. In addition, by comparing the upper bounds of blowup time corresponding to different strain wave equations, we clearly express the fact that the dissipative structure in the equation is beneficial to the global existence of the solution, but not to the blowup.

2. PRELIMINARY KNOWLEDGE

For simplicity, the norm of v in $L^2(\Omega)$ and $L^p(\Omega)$ are denoted by $\|v\|$ and $\|v\|_p$, respectively. The $H_0^1(\Omega)$ norm is defined by

$$\|v\|_{H_0^1(\Omega)} := \|\nabla v\|^2 + \|v\|^2,$$

which equivalents to $\|\nabla v\|$ in $\Omega \subset \mathbb{R}^N$. Moreover, we use $\langle \cdot, \cdot \rangle$ to denote the dual pairing between H and $H^{-2}(\Omega)$, where

$$H = \begin{cases} H_0^2(\Omega), & \text{when } v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ H^2(\Omega) \cap H_0^1(\Omega), & \text{when } v = \Delta v = 0 \text{ on } \partial\Omega, \end{cases}$$

where ν denotes the unit outer normal field. As we all know, $\|v\|_H$ is equivalent to $\|\Delta v\|$ for any $v \in H$.

Throughout the paper, we equip the initial data and Dirichlet boundary conditions of (1.1)–(1.4) in the same form

$$v(x,0)=v_0(x),\quad v_t(x,0)=v_1(x),\ x\in\Omega, \tag{2.1}$$

$$v=\frac{\partial v}{\partial \nu}=0,\ \text{or}\ v=\Delta v=0,\ x\in\partial\Omega,\ t>0. \tag{2.2}$$

In addition, $f(v)$ and $\sigma_i(v)$ $(1\le i\le n)$ satisfy

$$(H_1)\begin{cases}\text{(i)} & f(v)\in C^1 \text{ and } f(0)=f'(0)=0,\\ \text{(ii)} & f(v) \text{ is a monotone for } -\infty<v<\infty \text{ and is convex when } v>0,\\ & \text{concave when } v<0,\\ \text{(iii)} & (p_1+1)F(v)\le vf(v) \text{ and } F(v)=\int_0^v f(\tau)d\tau,\ 1<p_1<\infty \text{ if } n\le 4,\\ & 1<p_1<\frac{n+4}{n-4} \text{ if } n\ge 5.\end{cases}$$

$$(H_2)\begin{cases}\text{(i)} & \sigma_i(v)\in C^1 \text{ and } \sigma_i(0)=\sigma_i'(0)=0,\\ \text{(ii)} & \sigma_i(v) \text{ is a monotone for } -\infty<v<\infty \text{ and is convex when } v>0,\\ & \text{concave when } v<0,\\ \text{(iii)} & (p_2+1)G_i(v)\le v\sigma_i(v) \text{ and } G_i(v)=\int_0^v \sigma_i(\tau)d\tau, 1<p_2<\infty \text{ if } n\le 2,\\ & 1<p_2<\frac{n+2}{n-2} \text{ if } n\ge 3.\end{cases}$$

Clearly, we see that when $\sigma_i(v_{x_i})=|v_{x_i}|^{p-2}v_{x_i}$, it corresponds to the differential operator

$$\Delta_p v=\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\right),$$

for the recent developments in p-Laplacian can be found in [2, 6, 15].

Next we introduce a second-order differential inequality established in [8] to deal with the upper bound of blowup time.

Lemma 2.1 ([8]). *Suppose non-negative function $H(t)\in C^2([0,T])$, satisfying*

$$H(t)H''(t)-\xi(H'(t))^2+\gamma H'(t)H(t)+\beta H(t)\ge 0, \tag{2.3}$$

where $\beta\ge 0,\ \gamma\ge 0,\ \xi>1$ and $H(0)>0$,

$$H'(0)>\frac{\gamma}{\xi-1}H(0) \tag{2.4}$$

and

$$\left(H'(0)-\frac{\gamma}{\xi-1}H(0)\right)^2>\frac{2\beta}{2\xi-1}H(0). \tag{2.5}$$

Then there is a $T^*>0$ satisfying

$$\lim_{t\to T^*}\sup H(t)=+\infty,$$

here

$$T^* \le \frac{H^{1-\xi}(0)}{A} \tag{2.6}$$

and

$$A^2 = (\xi-1)^2 H^{-2\xi}(0)\left(\left(H'(0) - \frac{\gamma}{\xi-1}H(0)\right)^2 - \frac{2\beta}{2\xi-1}H(0)\right). \tag{2.7}$$

3. FINITE TIME BLOWUP FOR THE FOURTH-ORDER WAVE EQUATIONS WITH STRAIN AND DAMPING TERMS AT ARBITRARY POSITIVE INITIAL ENERGY LEVEL

With the appearance of damping terms, the energy structure will change from energy conservation to energy non-conservation, i.e., energy is decaying. At this point, the impact of the energy decaying on blowup is negative. In the following, we shall focus on the blowup dynamics of four classes damped wave equations with strain source, and explore the influence mechanism of the damping terms v_t and Δv_t on the blowup time.

3.1. BLOWUP PHENOMENA FOR THE IBVP OF (1.1) WITH ARBITRARY POSITIVE INITIAL ENERGY

Firstly, for the IBVP of (1.1), we define the energy functional

$$E_{1.1}(t) := \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\Delta v\|^2 - \sum_{i=1}^{n}\int_{\Omega} G_i(v_{x_i})dx.$$

Based on Lemma 2.1, we obtain the following main result of this subsection.

Theorem 3.1. *Let $v_0 \in H$, $v_1 \in L^2(\Omega)$ and σ_i satisfy (H_2). Assume that*

$$\kappa := 2(v_0, v_1) - \frac{4m}{p_2 - 1}\|v_0\|^2 > 0 \tag{3.1}$$

and

$$\kappa^2 > 8E_{1.1}(0)\|v_0\|^2 > 0, \tag{3.2}$$

where $p_2 > 1$ is same as that in (H_2). Then there exists a time $T_{1.1}$ such that the IBVP of (1.1) admits a blowup solution satisfying

$$\lim_{t\to T_{1.1}} \|v(t)\|^2 = +\infty, \tag{3.3}$$

with

$$T_{1.1} \le \frac{1}{A_{1.1}}\|v_0\|^{\frac{1-p_2}{2}} \tag{3.4}$$

and

$$A_{1.1}^2 = \frac{(p_2-1)^2}{4}\|v_0\|^{-p_2-3}\left(\kappa^2 - 8E_{1.1}(0)\|v_0\|^2\right).$$

Proof. Let

$$H(t) := \|v(t)\|^2.$$

Then by a simple calculation, we have

$$H'(t) = 2(v, v_t) \tag{3.5}$$

and

$$H''(t) = 2\langle v_{tt}, v\rangle + 2\|v_t\|^2. \tag{3.6}$$

Multiplying the Equation (1.1) by v and integrating over Ω, it gives

$$\langle v_{tt}, v\rangle + m(v_t, v) + \|\Delta v\|^2 - \sum_{i=1}^{n} \int_{\Omega} v_{x_i}\sigma_i(v_{x_i})dx = 0. \tag{3.7}$$

Moreover, we give another auxiliary function

$$\Phi(t) := \|v_t\|^2.$$

Together with (3.6) and (3.7), it gives

$$\frac{1}{2}H''(t) + \frac{m}{2}H'(t) - \Phi(t) + \|\Delta v\|^2 = \sum_{i=1}^{n} \int v_{x_i}\sigma_i(v_{x_i})dx. \tag{3.8}$$

In addition, multiplying (1.1) by v_t yields

$$\frac{d}{dt}\left(\frac{1}{2}\Phi(t) + \frac{1}{2}\|\Delta v\|^2\right) + m\|v_t\|^2 = \frac{d}{dt}\left(\sum_{i=1}^{n} \int_{\Omega} G_i(v_{x_i})dx\right). \tag{3.9}$$

Integrating (3.9) over $(0, t)$ and combining (H_2) we have

$$\begin{aligned} \frac{1}{2}\Phi(t) + \frac{1}{2}\|\Delta v\|^2 + m\int_0^t \|v_\tau\|^2 d\tau - E_{1.1}(0) &= \sum_{i=1}^{n} \int_{\Omega} G_i(v_{x_i})dx \\ &\le \frac{1}{p_2+1}\left(\sum_{i=1}^{n} \int_{\Omega} v_{x_i}\sigma_i(v_{x_i})dx\right), \end{aligned} \tag{3.10}$$

which together with (3.8) gives

$$\frac{1}{2}H''(t) + \frac{m}{2}H'(t) + (p_2+1)E_{1.1}(0) \ge \frac{p_2+3}{2}\Phi(t). \tag{3.11}$$

Using the Cauchy–Schwarz inequality we obtain

$$(H'(t))^2 \leq 4H(t)\Phi(t). \tag{3.12}$$

Further from (3.12) and (3.11), it follows that

$$H''(t)H(t) - \frac{p_2+3}{4}(H'(t))^2 + mH'(t)H(t) + 2(p_2+1)E_{1.1}(0)H(t) \geq 0. \tag{3.13}$$

Comparing (3.13) with (2.3), it is easy to see that

$$\xi_{1.1} := \frac{p_2+3}{4} > 1, \quad \gamma_{1.1} := m, \quad \beta_{1.1} := 2(p_2+1)E_{1.1}(0) > 0. \tag{3.14}$$

Moreover, we can verify that

$$H(0) = \|v_0\|^2 > 0, \;\; H'(0) = 2(v_0, v_1) > 0. \tag{3.15}$$

Obviously, by using Lemma 2.1 along with (3.13)–(3.15) and initial conditions (3.1), (3.2), we can deduce that there exists a $T_{1.1}$ satisfying (3.4) such that (3.3) holds. □

3.2. BLOWUP PHENOMENA FOR THE IBVP OF (1.2) WITH ARBITRARY POSITIVE INITIAL ENERGY

For the IBVP of (1.2), we also give the following total energy functional

$$E_{1.2}(t) = \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\Delta v\|^2 + \frac{\alpha}{2}\|\nabla v\|^2 - \sum_{i=1}^{n}\int_\Omega G_i(v_{x_i})dx - \int_\Omega F(v)dx, \tag{3.16}$$

the Nehari functional

$$I_{1.2}(v) = \|\Delta v\|^2 + \alpha\|\nabla v\|^2 - \sum_{i=1}^{n}\int_\Omega v_{x_i}\sigma_i(v_{x_i})dx - \int_\Omega vf(v)dx \tag{3.17}$$

and unstable manifold

$$\mathcal{V}_{1.2} := \{v \in H \,|\, I_{1.2}(v) < 0\}.$$

From the process of constructing the differential inequality for blowup in Theorem 3.1, it can be found that the nonlinear strain term $\sum_i^n \frac{\partial}{\partial x_i}\sigma_i(v_{x_i})$ in (1.1) is actually a bridge linking the auxiliary functions $H(t)$ and $\Phi(t)$. By the same argument as Theorem 3.1, we can also get the finite time blowup result for the IBVP of (1.2) with an explicit upper bound estimate of blowup time, even if the structure of (1.2) contains both $\sum_i^n \frac{\partial}{\partial x_i}\sigma_i(v_{x_i})$ and $f(v)$.

Theorem 3.2. *Let $v_0 \in H$, $v_1 \in L^2(\Omega)$, (H_1) and (H_2) hold. If*

$$\alpha := 2(v_0, v_1) - \frac{4}{p-1}\|v_0\|^2 > 0 \tag{3.18}$$

and

$$\alpha^2 > 8E_{1.2}(0)\|v_0\|^2 > 0 \tag{3.19}$$

with $p := \min\{p_1, p_2\}$. Then there exists a time $T_{1.2}$ which makes the solution of the IBVP of (1.2) *blow up in finite time, i.e.,*

$$\lim_{t\to T_{1.2}} \|v(t)\|^2 = +\infty, \tag{3.20}$$

here

$$T_{1.2} \le \frac{1}{A_{1.2}}\|v_0\|^{\frac{1-p}{2}} \tag{3.21}$$

and

$$A_{1.2}^2 = \frac{(p-1)^2}{4}\|v_0\|^{-p-3}\left(\alpha^2 - 8E_{1.2}(0)\|v_0\|^2\right).$$

Proof. We multiply both sides of (1.2) by v and integrate on Ω, it follows that

$$\langle v_{tt}, v\rangle + (v_t, v) + \|\Delta v\|^2 - \sum_{i=1}^n \int_\Omega v_{x_i}\sigma_i(v_{x_i})dx - \int_\Omega vf(v)dx = 0, \tag{3.22}$$

which together with (3.5), (3.6) and the definition of $\Phi(t)$ gives that

$$\frac{1}{2}H''(t) - \Phi(t) + \frac{1}{2}H'(t) + \|\Delta v\|^2 = \sum_{i=1}^n \int_\Omega v_{x_i}\sigma_i(v_{x_i})dx + \int_\Omega vf(v)dx. \tag{3.23}$$

Then integrating both sides of (1.2) by v_t, we have

$$\frac{d}{dt}\left(\frac{1}{2}\Phi(t) + \frac{1}{2}\|\Delta v\|^2\right) + \|v_t\|^2 = \frac{d}{dt}\left(\sum_{i=1}^n \int_\Omega G_i(v_{x_i})dx + \int_\Omega F(v)dx\right). \tag{3.24}$$

Therefore, by (3.24) and conditions (H_1), (H_2), we can derive

$$\begin{aligned}
&\frac{1}{2}\Phi(t) + \frac{1}{2}\|\Delta v\|^2 + \int_0^t \|v_\tau\|^2 d\tau - E_{1.2}(0) \\
&= \sum_{i=1}^{n} \int_\Omega G_i(v_{x_i})dx + \int_\Omega F(v)dx \\
&\le \frac{1}{p_2+1}\left(\sum_{i=1}^{n}\int_\Omega v_{x_i}\sigma_i(v_{x_i})dx\right) + \frac{1}{p_1+1}\int_\Omega vf(v)dx \\
&\le \frac{1}{p+1}\left(\sum_{i=1}^{n}\int_\Omega v_{x_i}\sigma_i(v_{x_i})dx + \int_\Omega vf(v)dx\right),
\end{aligned} \tag{3.25}$$

where $p = \min\{p_1, p_2\}$. Combining (3.25) and (3.23) to get

$$\frac{1}{2}H''(t) + \frac{1}{2}H'(t) + (p+1)E_{1.2}(0) \ge \frac{p+3}{2}\Phi(t). \tag{3.26}$$

Due to the Cauchy–Schwarz inequality (3.12), the above inequality becomes

$$H''(t)H(t) - \frac{p+3}{4}(H'(t))^2 + H'(t)H(t) + 2(p+1)E_{1.2}(0)H(t) \ge 0. \tag{3.27}$$

Hence, we can utilize Lemma 2.1 in a same way to get

$$\xi_{1.2} := \frac{p+3}{4} > 1, \quad \gamma_{1.2} := 1, \quad \beta_{1.2} := 2(p+1)E_{1.2}(0) > 0. \tag{3.28}$$

Further, it follows Lemma 2.1, (3.27), (3.28) and initial conditions (3.18), (3.19) that there exists a $T_{1.2}$ satisfying (3.21) such that (3.20) holds. □

3.3. BLOWUP PHENOMENA FOR THE IBVP OF (1.3) WITH ARBITRARY POSITIVE INITIAL ENERGY

In this subsection, we begin to consider the strong damping Δv_t contained in the fourth-order strain equation. Different from the nonlinear strain wave equation with only linear weak damping v_t, if we still use $\Phi(t) = \|v_t\|^2$ and $H(t) = \|v\|^2$ as the auxiliary functions to prove the arbitrary positive initial energy blowup based on Lemma 2.1, then there will be a new term $(\nabla v_t, \nabla v)$ caused by strong damping Δv_t that we cannot handle. In order to establish the arbitrary positive initial energy blowup theorem for the IBVP of (1.3), we establish a new auxiliary function $M(t)$ and choose the following differential inequality actually contained in Lemma 2.1.

Lemma 3.3 ([10]). *Suppose that a positive, twice differentiable function $\psi(t)$ satisfies the inequality*

$$\psi''(t)\psi(t) - (1+\mu)(\psi'(t))^2 \geq 0, \quad t > 0, \tag{3.29}$$

where $\mu > 0$. If $\psi(0) > 0$ and $\psi'(0) > 0$, then there exists $0 < t_1 \leq \frac{\psi(0)}{\mu\psi'(0)}$ such that $\psi(t)$ tends to ∞ as $t \to t_1$.

Next, for the IBVP of (1.3), the energy functional is defined by

$$E_{1.3}(t) := \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\Delta v\|^2 + \frac{1}{2}\|\nabla v\|^2 - \int_\Omega F(v)dx - \sum_{i=1}^{n} \int_\Omega G_i(v_{x_i})dx, \tag{3.30}$$

and the Nehari functional

$$I_{1.3}(v) := \|\Delta v\|^2 + \|\nabla v\|^2 - \int_\Omega vf(v)dx - \sum_{i=1}^{n} \int_\Omega v_{x_i}\sigma_i(v_{x_i})dx. \tag{3.31}$$

By a simple calculation, it follows that

$$E_{1.3}(t) + \int_0^t \|\nabla v_\tau\|^2 d\tau = E_{1.3}(0). \tag{3.32}$$

Besides, we give the following unstable set

$$\mathcal{V}_{1.3} := \{v \in H \,|\, I_{1.3}(v) < 0\}.$$

By the discussion similar to Lemma 6.1 in [11], we derive the following conclusion.

Lemma 3.4 (Invariant set $\mathcal{V}_{1.3}$ for $E_{1.3}(0) > 0$). *Let v be the local solution to the IBVP of (1.3) with $v_0 \in H$ and $v_1 \in L^2(\Omega)$, $f(v)$ and $\sigma_i(v_{x_i})$ satisfy (H_1) and (H_2), respectively. Assume that $E_{1.3}(0) > 0$ and*

$$\|\nabla v_0\|^2 + 2(v_0, v_1) > \frac{2(C+1)(p+1)}{C(p-1)} E_{1.3}(0), \tag{3.33}$$

then $\mathcal{V}_{1.3}$ is a invariant set provided $v_0 \in \mathcal{V}_{1.3}$. Here $p = \min\{p_1, p_2\} > 1$ and C denote the constant in Poincaré inequality

$$\|\nabla v\|^2 \geq C\|v\|^2. \tag{3.34}$$

Proof. Let T be the maximal interval of existence time for v. Arguing by contradiction, we assume that there exists a first time $\bar{t} \in (0, T]$ such that

$$I_{1.3}(v(\bar{t})) = 0 \tag{3.35}$$

and

$$I_{1.3}(v(t)) < 0, \quad t \in [0, \bar{t}). \tag{3.36}$$

Integrating both sides of (1.3) by v with respect to Ω, we derive

$$\langle v_{tt}, v\rangle + (\nabla v_t, \nabla v) + \|\Delta v\|^2 + \|\nabla v\|^2 - \sum_{i=1}^{n} \int_{\Omega} v_{x_i}\sigma_i(v_{x_i})dx - \int_{\Omega} vf(v)dx = 0. \tag{3.37}$$

Then we define the auxiliary function

$$y(t) := \|\nabla v\|^2 + 2(v, v_t).$$

From (3.31) and (3.37), we gain

$$y'(t) = 2(\nabla v, \nabla v_t) + 2\langle v_{tt}, v\rangle + 2\|v_t\|^2 = 2\|v_t\|^2 - 2I_{1.3}(v). \tag{3.38}$$

Moreover, by (3.36) we have $y'(t) > 0$ on the interval $[0, \bar{t})$, which implies that $y(t)$ is a strictly increasing function. Hence

$$\|\nabla v\|^2 + 2(v, v_t) \geq \|\nabla v_0\|^2 + 2(v_0, v_1), \quad t \in [0, \bar{t}).$$

From (3.33) and the continuity of $y(t)$ in t, it yields

$$\|\nabla v(\bar{t})\|^2 + 2(v(\bar{t}), v_t(\bar{t})) > \frac{2(C+1)(p+1)}{C(p-1)} E_{1.3}(0), \quad t \in [0, \bar{t}). \tag{3.39}$$

Recalling (3.32), (3.30), (3.31) and conditions (H_1), (H_2), it gives

$$\begin{aligned}
E_{1.3}(0) &\geq E_{1.3}(t) \\
&= \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\left(\|\Delta v\|^2 + \|\nabla v\|^2\right) \\
&\quad - \int_{\Omega} F(v)dx - \sum_{i=1}^{n} \int_{\Omega} G_i(v_{x_i})dx \\
&\geq \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\left(\|\Delta v\|^2 + \|\nabla v\|^2\right) \\
&\quad - \frac{1}{p+1}\left(\int_{\Omega} vf(v)dx + \sum_{i=1}^{n} \int_{\Omega} v_{x_i}\sigma_i(v_{x_i})dx\right) \\
&= \frac{1}{2}\|v_t\|^2 + \frac{p-1}{2(p+1)}\left(\|\Delta v\|^2 + \|\nabla v\|^2\right) + \frac{1}{p+1} I_{1.3}(v),
\end{aligned} \tag{3.40}$$

which along with (3.35), (3.34) and the Cauchy–Schwarz inequality yields

$$\begin{aligned}
E_{1.3}(0) &\geq E_{1.3}(\bar{t}) \\
&\geq \frac{1}{2}\|v_t(\bar{t})\|^2 + \frac{p-1}{2(p+1)}\|\nabla v(\bar{t})\|^2 \\
&\geq \frac{C(p-1)}{2(C+1)(p+1)}\|v_t(\bar{t})\|^2 + \frac{C(p-1)}{2(C+1)(p+1)}\|\nabla v(\bar{t})\|^2 \\
&\quad + \frac{p-1}{2(C+1)(p+1)}\|\nabla v(\bar{t})\|^2 \\
&\geq \frac{C(p-1)}{2(C+1)(p+1)}\|v_t(\bar{t})\|^2 + \frac{C(p-1)}{2(C+1)(p+1)}\|\nabla v(\bar{t})\|^2 \\
&\quad + \frac{C(p-1)}{2(C+1)(p+1)}\|v(\bar{t})\|^2 \\
&\geq \frac{C(p-1)}{2(C+1)(p+1)}\left(\|\nabla v(\bar{t})\|^2 + 2(v(\bar{t}), v_t(\bar{t}))\right).
\end{aligned} \tag{3.41}$$

Obviously, (3.41) contradicts (3.39). This proves the invariance of $\mathcal{V}_{1.3}$ for arbitrary positive initial energy case $E_{1.3}(0) > 0$. □

Theorem 3.5. *Let $v_0 \in H$, $v_1 \in L^2(\Omega)$, $f(v)$ and $\sigma_i(v_{x_i})$ satisfy assumptions (H_1) and (H_2). Suppose that $v_0 \in \mathcal{V}_{1.3}$, $E_{1.3}(0) > 0$, $(v_0, v_1) > 0$ and* (3.33) *holds, then the IBVP of* (1.3) *admits a blowup solution with the maximum existence time*

$$T_{1.3} \leq \frac{2(C+1)\|v_0\|^2 + 2(C+1)T_0\|\nabla v_0\|^2}{(p-1)(v_0, v_1)}. \tag{3.42}$$

Proof. Arguing by contradiction, assume that the solution $v(t)$ of (1.3) exists globally, i.e., the maximum existence time $T = \infty$. Let

$$M(t) := \|v\|^2 + \int_0^t \|\nabla v\|^2 d\tau + (T_0 - t)\|\nabla v_0\|^2, \quad t \in [0, T_0]. \tag{3.43}$$

Obviously, for any $t \in [0, T_0]$ we have $M(t) > 0$. Due to the continuity of $M(t)$ in t, we can see that there exists a constant $\epsilon > 0$ such that

$$M(t) \geq \epsilon, \quad t \in [0, T_0]. \tag{3.44}$$

By a simple calculation, it follows that

$$\begin{aligned}
M'(t) &= 2(v, v_t) + \|\nabla v\|^2 - \|\nabla v_0\|^2 \\
&= 2(v, v_t) + 2\int_0^t \int_\Omega \nabla v_\tau \nabla v dx d\tau, \quad t \in [0, T_0].
\end{aligned} \tag{3.45}$$

According to (3.45), (3.31) and (3.37), we have

$$M''(t) = 2(\nabla v_t, \nabla v) + 2\langle v_{tt}, v\rangle + 2\|v_t\|^2 = 2\|v_t\|^2 - 2I_{1.3}(v), \quad t \in [0, T_0]. \tag{3.46}$$

Through elementary calculation and the Cauchy–Schwarz inequality, we show the following identities

$$(M'(t))^2 = 4\left((v_t, v)^2 + 2(v_t, v)\int_0^t \int_\Omega \nabla v_\tau \nabla v dx d\tau\right) + 4\left(\int_0^t \int_\Omega \nabla v_\tau \nabla v dx d\tau\right)^2. \tag{3.47}$$

Using the Cauchy–Schwarz inequality, it yields

$$\left(\int_0^t \int_\Omega \nabla v_\tau \nabla v dx d\tau\right)^2 \le \int_0^t \|\nabla v_\tau\| d\tau \int_0^t \|\nabla v\| d\tau$$

and

$$\begin{aligned} 2(v_t, v)\int_0^t \int_\Omega \nabla v_\tau \nabla v dx d\tau &\le 2\|v_t\|\|v\| \left(\int_0^t \|\nabla v_\tau\|^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^t \|\nabla v\|^2 d\tau\right)^{\frac{1}{2}} \\ &\le \|v_t\|^2 \int_0^t \|\nabla v\|^2 d\tau + \|v\|^2 \int_0^t \|\nabla v_\tau\|^2 d\tau. \end{aligned}$$

Hence, by the above inequalities, (3.47) becomes

$$\begin{aligned} (M'(t))^2 &= 4\left(\|v_t\|^2 + \int_0^t \|\nabla v_\tau\|^2 d\tau\right)\left(\|v\|^2 + \int_0^t \|\nabla v\|^2 d\tau\right) \\ &\le 4M(t)\left(\|v_\tau\|^2 + \int_0^t \|\nabla v_\tau\|^2 d\tau\right). \end{aligned} \tag{3.48}$$

Then from (3.46) and (3.48), we have

$$\begin{aligned} &M''(t)M(t) - \frac{\lambda+3}{4}(M'(t))^2 \\ &\ge M(t)\left(M''(t) - (\lambda+3)\left(\|v_t\|^2 + \int_0^t \|\nabla v_\tau\|^2 d\tau\right)\right) \\ &\ge M(t)\left(-(\lambda+1)\|v_t\|^2 - 2I_{1.3}(v) - (\lambda+3)\int_0^t \|\nabla v_\tau\|^2 d\tau\right). \end{aligned} \tag{3.49}$$

If we set

$$\rho(t) := -(\lambda+1)\|v_t\|^2 - 2I_{1.3}(v) - (\lambda+3)\int_0^t \|\nabla v_\tau\|^2 d\tau, \tag{3.50}$$

then by (3.40) and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
\rho(t) &\geq (p-\lambda)\|v_t\|^2 + (p-1)\left(\|\Delta v\|^2 + \|\nabla v\|^2\right) \\
&\quad - 2(p+1)E_{1.3}(0) + (2p-\lambda-1)\int_0^t \|\nabla v_\tau\|^2 d\tau \\
&\geq (p-\lambda)\|v_t\|^2 + \frac{2(p-\lambda)}{C}\|\nabla v\|^2 + \frac{(p-1)C - 2(p-\lambda)}{C}\|\nabla v\|^2 \\
&\quad - 2(p+1)E_{1.3}(0) + (2p-\lambda-1)\int_0^t \|\nabla v_\tau\|^2 d\tau \\
&\geq (p-\lambda)\|v_t\|^2 + 2(p-\lambda)\|v\|^2 + \frac{(p-1)C - 2(p-\lambda)}{C}\|\nabla v\|^2 \\
&\quad - 2(p+1)E_{1.3}(0) + (2p-\lambda-1)\int_0^t \|\nabla v_\tau\|^2 d\tau \\
&\geq (p-\lambda)\left(\|v_t\|^2 + 2\|v\|^2\right) + \frac{(p-1)C - 2(p-\lambda)}{C}\|\nabla v\|^2 \\
&\quad - 2(p+1)E_{1.3}(0) + (2p-\lambda-1)\int_0^t \|\nabla v_\tau\|^2 d\tau \\
&\geq (p-\lambda)\left(\|v\|^2 + 2(v_t, v)\right) + \frac{(p-1)C - 2(p-\lambda)}{C}\|\nabla v\|^2 \\
&\quad - 2(p+1)E_{1.3}(0) + (2p-\lambda-1)\int_0^t \|\nabla v_\tau\|^2 d\tau.
\end{aligned} \tag{3.51}$$

Here $p > 1$ and $\lambda := \frac{C+p}{C+1} \in (1, p)$, this implies $2p - \lambda - 1 > 0$. Then by (3.51) and the monotonically increasing of $\left\{t \mapsto \|\nabla v\|^2 + 2(v, v_t)\right\}$ since $I_{1.3}(0) < 0$, we gain

$$\begin{aligned}
\rho(t) &\geq \frac{C(p-1)}{C+1}\left(2(v_t, v) + \|\nabla v\|^2\right) - 2(p+1)E_{1.3}(0) \\
&\geq \frac{C(p-1)}{C+1}\left(2(v_1, v_0) + \|\nabla v_0\|^2\right) - 2(p+1)E_{1.3}(0) \\
&=: \varrho.
\end{aligned} \tag{3.52}$$

Further by (3.33) we derive

$$\rho(t) \geq \varrho > 0. \tag{3.53}$$

Hence, from (3.49), (3.44) and (3.33), we arrive at

$$M''(t)M(t) - \left(1 + \frac{p-1}{4C+4}\right)(M'(t))^2 \geq \epsilon\varrho > 0, \quad t \in [0, T_0], \tag{3.54}$$

where $p = \min\{p_1, p_2\} > 1$. Since $(v_0, v_1) > 0$, we have

$$M(0) = \|v_0\|^2 + T_0\|\nabla v_0\|^2 > 0 \tag{3.55}$$

and

$$M'(0) = 2(v_0, v_1) > 0. \tag{3.56}$$

Obviously, by Lemma 3.3 along with (3.54)–(3.56), it implies that there exists a $T_{1.3} > 0$ such that (3.42) holds and

$$\lim_{t \to T_{1.3}} M(t) = \infty.$$

This completes the proof. □

3.4. BLOWUP PHENOMENA FOR THE IBVP OF (1.4) WITH ARBITRARY POSITIVE INITIAL ENERGY

For the IBVP of (1.4), we define the energy functional of the form

$$E_{1.4}(t) := \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\Delta v\|^2 + \frac{1}{2}\|\nabla v\|^2 - \int_\Omega F(v)dx - \sum_{i=1}^n \int_\Omega G_i(v_{x_i})dx, \tag{3.57}$$

the Nehari functional

$$I_{1.4}(v) := \|\Delta v\|^2 + \|\nabla v\|^2 - \int_\Omega vf(v)dx - \sum_{i=1}^n \int_\Omega v_{x_i}\sigma_i(v_{x_i})dx. \tag{3.58}$$

The unstable set $\mathcal{V}_{1.4}$ is given by

$$\mathcal{V}_{1.4} := \{v \in H \,|\, I_{1.4}(v) < 0\}.$$

As shown in [11, Theorem 6.2], the blow-up result of the solution for (1.4) at arbitrary positive initial energy level has been proved. Next we shall estimate the upper bound of the blow-up time based on the established conclusion in [11].

Theorem 3.6. *Let* $v_0 \in H$, $v_1 \in L^2(\Omega)$, (H_1) *and* (H_2) *hold. Assume that* $v_0 \in \mathcal{V}_{1.4}$, $E_{1.4}(0) > 0$, $(v_0, v_1) > 0$, $r = 1$ *and*

$$\|\nabla v_0\|^2 + \|v_0\|^2 + 2(v_0, v_1) > \frac{2(C+1)(p+1)}{C(p-1)} E_{1.4}(0).$$

Then there exists a $T_{1.4}$ *as*

$$T_{1.4} \leq \frac{2(C+1)(T_0+1)\|v_0\|^2 + 2(C+1)T_0\|\nabla v_0\|^2}{(p-1)(v_0, v_1)} \tag{3.59}$$

such that the solution to the IBVP of (1.4) *blows up in finite time. Here* C *and* p *are same as that in Lemma 3.4.*

Proof. As shown in [11, Theorem 6.2], the authors constructed the following differential inequality

$$B''(t)B(t) - \left(1 + \frac{p-1}{2C+4}\right)(B'(t))^2 > 0 \tag{3.60}$$

to prove the blowup solution of (1.4) with $r = 1$ and $E_{1.4}(0) > 0$, where C is an embedded constant shown in (3.34), $p > 1$ is same as that in Theorem 3.5. However, in order to facilitate the comparison of the upper bounds of blowup time for different models in subsequent Section 3.6, it is necessary to recalculate the coefficient in (3.60). With regards to this, we replace $C+2$ in (6.6) with $C+1$ in the proof of [11, Lemma 6.1] (the invariance of set $\mathcal{V}_{1.4}$ for $E_{1.4}(0) > 0$). Then in the proof of [11, Theorem 6.5], $\lambda := p - \frac{C(p-1)}{C+2}$ should be changed to $\lambda := p - \frac{C(p-1)}{C+1}$. In this way, we shall derive the second order differential inequality

$$B''(t)B(t) - \left(1 + \frac{p-1}{4C+4}\right)(B'(t))^2 > 0. \tag{3.61}$$

Moreover, the auxiliary function in (3.61) is as follows

$$B(t) := \|v\|^2 + \int_0^t \left(\|\nabla v(\tau)\|^2 + \|v(\tau)\|^2\right) d\tau + (T_0 - t)\left(\|\nabla v_0\|^2 + \|v_0\|^2\right)$$

same as (3.60) for $t \in [0, T_0]$. Naturally, we can deduce that

$$B(t) > 0, \quad t \in [0, T_0]. \tag{3.62}$$

By a direct calculation, it gives

$$B'(t) = 2(v, v_t) + (\|\nabla v\|^2 + \|v\|^2) - (\|\nabla v_0\|^2 + \|v_0\|^2), \quad t \in [0, T_0].$$

Then by (3.62) and the fact that $(v_0, v_1) > 0$, we have $B(0) > 0$ and $B'(0) > 0$. Together with Lemma 3.3 and (3.61), it follows that there exists a $T_{1.4} > 0$ such that (3.59) holds and

$$\lim_{t \to T_{1.4}} B(t) = \infty,$$

which completes the proof. □

3.5. COROLLARY

In fact, the blowup theory in Lemma 3.3 not only can be used to the fourth-order strain wave equation with Δv_t, but also the fourth-order nonlinear strain wave equation (1.2) with only linear weak damping v_t. The key is to derive an second-order differential inequality consistent with the form of (3.29) shown in Lemma 3.3 by using the classical concavity function method. We illustrate this with the following corollary.

Corollary 3.7. *For the IBVP of* (1.2) *with* $\alpha = 1$ *and*

$$v_0 \in \mathcal{V}_{1.2}, \quad E_{1.2}(0) > 0, \quad (v_0, v_1) > 0.$$

Let

$$\eta(t) := \|v\|^2 + \int_0^t \|v\|^2 d\tau + (T_0 - t)\|v_0\|^2, \quad t \in [0, T_0],$$

it gives

$$\eta'(t) := 2(v, v_t) + \|v\|^2 - \|v_0\|^2, \quad t \in [0, T_0].$$

Replacing $M(t)$ *in Theorem 3.5 by* $\eta(t)$ *and repeating the discussion similar to the proof of Theorem 3.5, we can also easily get the following differential inequality*

$$\eta''(t)\eta(t) - \left(1 + \frac{p-1}{4C+4}\right)(\eta'(t))^2 > 0, \quad t \in [0, T_0], \tag{3.63}$$

where $p = \min\{p_1, p_2\} > 1$, C *is same as that in* (3.34). *Notice that*

$$\eta(0) = \|v_0\|^2 + T\|v_0\|^2 > 0, \quad \eta'(0) = 2(v_0, v_1) > 0. \tag{3.64}$$

By adopting Lemma 3.3 together with (3.63) *and* (3.64), *it follows that there exists a* $0 < T_* \leq T_0$ *such that*

$$\lim_{t \to T_*} \eta(t) = \infty,$$

moreover,

$$T_* \leq \frac{2(C+1)(1+T_0)\|v_0\|^2}{(p-1)(v_0, v_1)}. \tag{3.65}$$

3.6. EFFECTS OF THE DISSIPATIVE TERMS ON ARBITRARY POSITIVE INITIAL ENERGY BLOWUP

In the present subsection, we shall reveal the influence mechanism of the linear weak damping v_t and strong damping Δv_t on the arbitrary positive initial energy blowup. We discuss this issue by comparing the upper bound of blowup time. In general, the larger the upper bound of blowup time, the less likely it is that the blowup will occur.

3.6.1. Effect of linear weak damping v_t on arbitrary positive initial energy blowup

In order to compare the upper bounds of the arbitrary initial energy blowup time, we need to keep a single variable term v_t. Hence, we can consider the upper bounds of blowup time $T_{1.3}$ and $T_{1.4}$. In addition, we need to control the IBVP of (1.3) to have the same initial value as the IBVP of (1.4). By observing the initial conditions involved in Theorem 3.5 and Theorem 3.6, we can judge that this is easily achieved.

Through a simple comparison, we find that

$$\underbrace{\frac{2(C+1)\left(\|v_0\|^2+T_0\|\nabla v_0\|^2\right)}{(p-1)(v_0,v_1)}}_{\text{the upper bound of } T_{1.3},\ (1.3)\text{ with } \Delta v_t} \le \underbrace{\frac{2(C+1)\left((1+T_0)\|v_0\|^2+T_0\|\nabla v_0\|^2\right)}{(p-1)(v_0,v_1)}}_{\text{the upper bound of } T_{1.4},\ (1.4)\text{ with } \Delta v_t \text{ and } v_t}.$$

Obviously, the upper bound of $T_{1.3}$ is less than the upper bound of $T_{1.4}$. Therefore, we can draw a conclusion that weak damping v_t makes the blowup time longer. In other words, it is helpful for the global existence of solution.

3.6.2. Effect of linear strong damping Δv_t on arbitrary positive initial energy blowup

To investigate the effect of strong damping on arbitrary positive initial blowup, we keep the other structures in the equation except Δv_t consistent. Therefore, we choose the upper bound of the arbitrary positive initial energy blowup time corresponding to Equation (1.2) with $\alpha = 1$ and Equation (1.4) for comparison, i.e., (3.59) and (3.65). Obviously, when the IBVP of (1.2) has the same initial data as the IBVP of (1.4), the following inequality

$$\underbrace{\frac{2(C+1)(1+T_0)\|v_0\|^2}{(p-1)(v_0,v_1)}}_{\text{the upper bound of } T_*,\ \text{without } \Delta v_t} < \underbrace{\frac{2(C+1)(1+T_0)\|v_0\|^2+2(C+1)T_0\|\nabla v_0\|^2}{(p-1)(v_0,v_1)}}_{\text{the upper bound of } T_{1.4},\ \text{with } \Delta v_t}$$

holds, which implies that the strong damping Δv_t makes the blowup more difficult to happen.

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