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THE d-BAR FORMALISM FOR THE MODIFIED VESELOV–NOVIKOV EQUATION ON THE HALF-PLANE

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Abstract. We study the modified Veselov–Novikov equation (mVN) posed on the half-plane via the Fokas method, considered as an extension of the inverse scattering transform for boundary value problems. The mVN equation is one of the most natural (2+1)-dimensional generalization of the (1+1)-dimensional modified Korteweg-de Vries equation in the sense as to how the Novikov–Veselov equation is related to the Korteweg–de Vries equation. In this paper, by means of the Fokas method, we present the so-called global relation for the mVN equation, which is an algebraic equation coupled with the spectral functions, and the d-bar formalism, also known as Pompieu's formula. In addition, we characterize the d-bar derivatives and the relevant jumps across certain domains of the complex plane in terms of the spectral functions.

Keywords: initial-boundary value problem, integrable nonlinear PDE, spectral analysis, d-bar.

Mathematics Subject Classification: 35G31, 35Q53, 37K15.

1. INTRODUCTION

In this paper, we present a unified transform method, also known as the Fokas method, for the initial-boundary problem for the modified Veselov–Novikov equation (mVN) [24, 26, 29] posed on the half-plane

$$q_t + q_{zzz} + q_{\bar{z}\bar{z}\bar{z}} + 6\left(q_z v + q_{\bar{z}}\bar{v}\right) + 3q\left(v_z + \bar{v}_{\bar{z}}\right) = 0, \quad (z,t) \in \Omega, \quad z = x + iy,$$
$$2v_{\bar{z}} = \left(q^2\right)_z, \tag{1.1}$$

where $q(x, y, t)$ is a real-valued function and

$$\Omega = \{-\infty < x < \infty,\, 0 < y < \infty,\, 0 < t < T\}.$$

The mVN equation is a modification of the Novikov–Veselov equation (NV) in a similar manner as the connection between the modified Korteweg–de Vries equation (mKdV) and the Korteweg–de Vries equation (KdV) [3, 6]. The NV equation can be viewed as one of the most natural (2+1)-dimensional (i.e. two spatial and one temporal) generalization of the (1+1)-dimensional KdV equation. Hence, the mVN equation can be considered as the (2+1)-dimensional generalization of the (1+1)-dimensional mKdV equation. Both the KdV and mKdV equations are well known to be completely integrable, but play an important role in describing motions of shallow water waves. Thus, the NV and mVN equations are not only notable for their physical applications, for example the dispersionless NV equation describes the propagation model of high frequency electromagnetic wave in nonlinear media [6], but also of great interest for their integrability [19, 20, 29]. The analogy with the relationship between the KdV and mKdV equations also serves that the Novikov equation can be considered as a modified Degasperis–Procesi equation [4, 24] (cf. [5, 25] for recent results in the study of the Novikov equation).

It should be noted that the inverse scattering transform can be used for solving integrable nonlinear equations in (2+1)-dimension, such as the Davey–Stewartson (DS) and Kadomtsev–Petviashvili (KP) equations [1, 2, 14, 15, 30]. The NV and mVN equations, like the DS and KP equations, also can be analyzed by the inverse scattering transform [19, 29], due to their integrability. Recently, initial-boundary value problems for the DS and Kadomtsev–Petviashvili II equations on the half-plane [13, 23] can be solved by the Fokas method, considered as a significant extension of the inverse scattering transform for boundary value problems [9, 10, 12] (see also [16–18, 21, 22] for recent development of the method and reference therein).

Here, we study equation (1.1) the half-plane via the Fokas method, based on a Lax pair approach and a d-bar formalism, known as the Pompieu formula or the Cauchy–Green formula. The rigorous implementation of the Fokas method involves the following steps.

(i) *Analysis of the Lax pair*: We analyze the Lax pair for some sectionally analytic function

$$\mu = \mu(x, y, t, k_1, k_2)$$

for the spectral variable $k = k_1 + ik_2$ ($k_1, k_2 \in \mathbb{R}$). We derive the so-called global relation, which is an algebraic equation coupled with the spectral functions.

(ii) *Direct problem*: Assuming that a smooth solution $q(x, y, t)$ of equation (1.1) exists, we define a bounded function μ for $k \in \mathbb{C}$, which has different non-analytic representations in different domains in the complex k-plane.

(iii) *Inverse problem*: We formulate a d-bar problem for a sectionally non-analytic function. As an inverse problem, we characterize the d-bar derivatives and the relevant jumps across the different domains in the complex k-plane.

Throughout the paper, we will assume that there exists a sufficient smooth solution $q(x, y, t)$, which decays as $y \to \infty$ for all fixed (x, t) and as $|x| \to \infty$ for all fixed (y, t).

We will denote the initial and boundary values as

$$q_0(x,y) = q(x,y,0), \quad x \in \mathbb{R},\ 0 < y < \infty, \tag{1.2}$$

and

$$g_0(x,t) = q(x,0,t), \quad g_1(x,t) = q_y(x,0,t), \quad g_2(x,t) = q_{yy}(x,0,t), \quad x \in \mathbb{R}, 0 < t < T. \tag{1.3}$$

We will also assume that q_0 is a Schwartz class function, that is,

$$q_0 \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^+).$$

We will denote

$$k = k_1 + ik_2, \quad z = x + iy, \quad \zeta = \xi + i\eta,$$

where k_1, k_2, x, y, ξ and η are real variables.

2. THE LINEARIZED MODIFIED VESELOV–NOVIKOV EQUATION

Before studying the mVN equation, we begin by analyzing the linearized mVN equation posed on Ω

$$q_t + q_{zzz} + q_{\bar{z}\bar{z}\bar{z}} = 0. \tag{2.1}$$

The linearized mNV equation admits the following Lax pair

$$\mu_{\bar{z}} + ik\mu = -q, \tag{2.2a}$$

$$\mu_t + ik^3\mu + \mu_{zzz} = q_{\bar{z}\bar{z}} - ikq_{\bar{z}} - k^2 q, \tag{2.2b}$$

where $\mu = \mu(x,y,t,k_1,k_2)$ is a scalar function and bounded for $k \in \mathbb{C}$. In fact, equation (2.1) is compatible if and only if

$$\left(\partial_t + ik^3 + \partial^3\right) q + \left(\bar{\partial} + ik\right)\left(q_{\bar{z}\bar{z}} - ikq_{\bar{z}} - k^2 q\right) = 0,$$

where

$$\partial = \frac{1}{2}\left(\partial_x - i\partial_y\right), \quad \bar{\partial} = \frac{1}{2}\left(\partial_x + i\partial_y\right).$$

We note that equations (2.2) are equivalent to the following divergence form

$$\begin{aligned}\left(\mathrm{e}^{i(\lambda z + k\bar{z}) + i(k^3+\lambda^3)t} q\right)_t = {} & \left[\mathrm{e}^{i(\lambda z + k\bar{z}) + i(k^3+\lambda^3)t}\left(i\lambda q_z + \lambda^2 q - q_{zz}\right)\right]_z \\ & + \left[\mathrm{e}^{i(\lambda z + k\bar{z}) + i(k^3+\lambda^3)t}\left(ik\, q_{\bar{z}} + k^2 q - q_{\bar{z}\bar{z}}\right)\right]_{\bar{z}}.\end{aligned} \tag{2.3}$$

Thus, applying the Poincaré lemma [13], we find the global relation (cf. see also equation (2.23))

$$\begin{aligned}& -2i\left(\iint_{\Omega_0} \mathrm{d}x\mathrm{d}y\, \mathrm{e}^{i(\lambda z + k\bar{z}) + i(k^3+\lambda^3)t} q(x,y,t)\right)_t \\ & = \int_{\Omega_0} \mathrm{e}^{i(\lambda z + k\bar{z}) + i(k^3+\lambda^3)t} \\ & \quad \times \left[(i\lambda q_z + \lambda^2 q - q_{zz})\mathrm{d}\bar{z} - (ikq_{\bar{z}} + k^2 q - q_{\bar{z}\bar{z}})\mathrm{d}z\right], \quad \lambda, k \in \mathbb{C},\end{aligned} \tag{2.4}$$

where $\Omega_0 \subset \mathbb{R}^2$ is a bounded piecewise smooth domain.

On the other hand, we write equation (2.2a) as

$$\left(\mathrm{e}^{i\bar{k}z+ik\bar{z}}\mu\right)_{\bar{z}} = -\mathrm{e}^{i\bar{k}z+ik\bar{z}}q$$

and then the above equation can be solved by Pompieu's formula [13, 23]

$$\mathrm{e}^{i\bar{k}z+ik\bar{z}}\mu = \frac{1}{\pi}\iint_D \frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta - z}\mathrm{e}^{i\bar{k}\zeta+ik\bar{\zeta}}q(\xi,\eta,t) + \frac{1}{2i\pi}\int_{\partial D}\frac{\mathrm{d}\xi}{\xi - z}\mathrm{e}^{i\bar{k}\xi+ik\xi}\mu(\xi,0,t,k_1,k_2), \tag{2.5}$$

where $D = \{-\infty < \xi < \infty,\ 0 < \eta < \infty\}$. Letting

$$\mu(x,0,t,k_1,k_2) = \varphi(x,t,k_1,k_2),$$

we write equation (2.5) as

$$\begin{aligned}\mu(x,y,t,k_1,k_2) = {}& \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\frac{\mathrm{d}\eta}{\zeta - z}\mathrm{e}^{ik(\bar{\zeta}-\bar{z})+i\bar{k}(\zeta-z)}q(\xi,\eta,t) \\ & + \frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi - z}\mathrm{e}^{2ik_1\xi - i(k\bar{z}+\bar{k}z)}\varphi(\xi,t,k_1,k_2).\end{aligned} \tag{2.6}$$

We will determine $\varphi(x,t,k_1,k_2)$ by analyzing equation (2.2b). Note that equation (2.2) can be written in terms of derivatives with respect to x and y

$$i\mu_y = -\mu_x - 2ik\mu - 2q \tag{2.7a}$$

$$\begin{aligned}&\mu_t + ik^3\mu + \frac{1}{8}\left(\mu_{xxx} + i\mu_{yyy} - 3i\mu_{xxy} - 3\mu_{xyy}\right) \\ &= \frac{1}{4}\left(q_{xx} + 2iq_{xy} - q_{yy}\right) - \frac{ik}{2}\left(q_x + iq_y\right) - k^2 q.\end{aligned} \tag{2.7b}$$

From equation (2.7a) it follows that

$$\begin{aligned}i\mu_{yy} &= -i\mu_{xx} + 4k\mu_x - 2iq_x + 4ik^2\mu + 4kq - 2q_y, \\ i\mu_{yyy} &= \mu_{xxx} + 6ik\mu_{xx} - 12k^2\mu_x - 8ik^3\mu - 8k^2 q + 8ikq_x \\ &\quad - 2iq_{xy} + 4kq_y + 2q_{xx} - 2q_{yy}.\end{aligned}$$

Substituting the above equations into equation (2.7b) and evaluating the resulting equation at $y = 0$, we know that $\varphi(x,t,k_1,k_2)$ solves

$$\varphi_t + 3ik\varphi_{xx} - 3k^2\varphi_x + \varphi_{xxx} = Q(x,t,k), \tag{2.8}$$

where

$$Q(x,t,k) = -3ikg_{0_x} - \frac{3}{2}g_{0_{xx}} + \frac{3i}{2}g_{1_x}. \tag{2.9}$$

We introduce the Fourier transform pair

$$\begin{aligned}\varphi(x,t,k_1,k_2) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}l\, \mathrm{e}^{ilx}\hat{\varphi}(l,t,k_1,k_2),\\ \hat{\varphi}(l,t,k_1,k_2) &= \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ilx}\varphi(x,t,k_1,k_2).\end{aligned} \tag{2.10}$$

Then equation (2.8) can be written in terms of $\hat{\varphi}(l,t,k_1,k_2)$ as

$$\hat{\varphi}_t - il(l^2+3k+3k^2)\hat{\varphi} = \hat{Q}(l,t,k), \tag{2.11}$$

where

$$\hat{Q}(l,t,k) = \int_{-\infty}^{\infty} \mathrm{d}l\, \mathrm{e}^{-ilx}Q(x,t,k).$$

Using the following identity for the second integral in equation (2.6) together with equation (2.10),

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{\xi - z}\,\mathrm{e}^{i(2k_1+l)\xi} = \begin{cases} 2i\pi\mathrm{e}^{i(2k_1+l)z}, & l > -2k_1,\\ 0, & l < -2k_1,\end{cases}$$

equation (2.6) can be written as

$$\begin{aligned}\mu(x,y,t,k_1,k_2) = {} & \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\zeta - z}\mathrm{e}^{ik(\bar{\zeta}-\bar{z})+i\bar{k}(\zeta - z)}q(\xi,\eta,t)\\ & + \frac{\mathrm{e}^{-2ky}}{2\pi}\int_{-2k_1}^{\infty} \mathrm{d}l\, \mathrm{e}^{ilz}\hat{\varphi}(l,t,k_1,k_2).\end{aligned} \tag{2.12}$$

Evaluating equation (2.12) at $y = 0$, we find

$$\begin{aligned}\varphi(x,t,k_1,k_2) = {} & \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\zeta - x}\mathrm{e}^{2i(k_1\xi + k_2\eta)-2ik_1x}q(\xi,\eta,t)\\ & + \frac{1}{2\pi}\int_{-2k_1}^{\infty} \mathrm{d}l\, \mathrm{e}^{ilx}\hat{\varphi}(l,t,k_1,k_2).\end{aligned}$$

Note that

$$\int_{-\infty}^{-2k_1} \mathrm{d}l\, \mathrm{e}^{il(x-\zeta)} = \frac{i}{\zeta - x}\mathrm{e}^{2ik_1\xi - 2k_1\eta - 2ik_1x}$$

and then we find

$$\begin{aligned}\varphi(x,t,k_1,k_2) = \frac{1}{i\pi}\int\limits_{-\infty}^{-2k_1} \mathrm{d}l \int\limits_{-\infty}^{\infty} \mathrm{d}\xi \int\limits_{0}^{\infty} \mathrm{d}\eta\, \mathrm{e}^{il(x-\zeta)+2k\eta} q(\xi,\eta,t) \\ + \frac{1}{2\pi}\int\limits_{-2k_1}^{\infty} \mathrm{d}l\, \mathrm{e}^{ilx}\hat{\varphi}(l,t,k_1,k_2).\end{aligned} \tag{2.13}$$

Comparing equation (2.13) with equation (2.10), we know that for $-\infty < l < -2k_1$, $\hat{\varphi}$ is given by

$$\hat{\varphi}(l,t,k_1,k_2) = -2i\int\limits_{-\infty}^{\infty} \mathrm{d}\xi \int\limits_{0}^{\infty} \mathrm{d}\eta\, \mathrm{e}^{-il\zeta+2k\eta} q(\xi,\eta,t), \quad -\infty < l < -2k_1,$$

while $\hat{\varphi}(l,t,k_1,k_2)$ does not satisfy any restriction for $-2k_1 < l < \infty$.

We now analyze equation (2.11) to determine $\hat{\varphi}(l,t,k)$. Let

$$\mathcal{E}(k,l,t) = \mathrm{e}^{-il(l^2+3kl+3k^2)t}. \tag{2.14}$$

and then we write equation (2.11) as

$$(\mathcal{E}(k,l,t)\hat{\varphi})_t = \int\limits_{-\infty}^{\infty} \mathrm{d}\xi\, \mathcal{E}(k,l,t)\mathrm{e}^{-il\xi}\left(-3ikg_{0_\xi} - \frac{3}{2}g_{0_{\xi\xi}} + \frac{3i}{2}g_{1_\xi}\right). \tag{2.15}$$

Using the integration by parts, equation (2.15) can be written as

$$(\mathcal{E}(k,l,t)\hat{\varphi})_t = \int\limits_{-\infty}^{\infty} \mathrm{d}\xi\, \mathcal{E}(k,l,t)\mathrm{e}^{-il\xi} U(\xi,t,k,l), \tag{2.16}$$

where

$$U(\xi,t,k,l) = 3klg_0 + \frac{3}{2}l^2 g_0 - \frac{3}{2}lg_1.$$

Thus, the solution of equation (2.16) can be found in the form

$$\hat{\varphi}(l,t,k_1,k_2) = \begin{cases} \int_0^t \mathrm{d}\tau\, \mathcal{E}(k,l,\tau-t)\hat{U}(\tau,k,l) + \mathcal{E}(k,l,-t)\hat{\varphi}(l,0,k_1,k_2), \\ -\int_t^T \mathrm{d}\tau\, \mathcal{E}(k,l,\tau-t)\hat{U}(\tau,k,l) + \mathcal{E}(k,l,T-t)\hat{\varphi}(l,T,k_1,k_2), \end{cases}$$

where

$$\hat{U}(t,k,l) = \int\limits_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-il\xi} U(\xi,t,k,l).$$

Note that the real part of $\mathcal{E}(k,l,\tau-t)$ is given by

$$\operatorname{Re}\mathcal{E}(k,l,\tau-t) = \mathrm{e}^{3k_2 l(l+2k_1)(\tau-t)}, \tag{2.17}$$

and also note that

$$l(l+2k_1)\geq 0 \iff \begin{cases} l\in(-\infty,-2k_1]\cup[0,\infty), & k_1\geq 0,\\ l\in(-\infty,0]\cup[-2k_1,\infty), & k_1\leq 0.\end{cases}$$

We seek for a bounded solution $\hat{\varphi}$ for all $k\in\mathbb{C}$. Note that since there is no restriction imposed on $\hat{\varphi}(l,t,k_1,k_2)$ for $-2k_1<l<\infty$, we can choose

$$\hat{\varphi}(l,0,k_1,k_2)=\hat{\varphi}(l,T,k_1,k_2)=0$$

for $-2k_1<l<\infty$. Thus, according to the real part of $\mathcal{E}(k,l,\tau-t)$, we define $\hat{\varphi}(l,t,k_1,k_2)$ for $l\in[-2k_1,\infty)$ as

$$\hat{\varphi}_1^+(l,t,k_1,k_2)=\begin{cases}\int_0^t \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\hat{U}(\tau,k,l), & l\in[0,\infty),k\in\mathbb{C}_{\mathrm{I}},\\ -\int_t^T \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\hat{U}(\tau,k,l), & l\in[-2k_1,0],k\in\mathbb{C}_{\mathrm{I}},\end{cases}$$

$$\hat{\varphi}_1^-(l,t,k_1,k_2)=\begin{cases}-\int_t^T \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\hat{U}(\tau,k,l), & l\in[0,\infty),k\in\mathbb{C}_{\mathrm{IV}},\\ \int_0^t \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\hat{U}(\tau,k,l), & l\in[-2k_1,0],k\in\mathbb{C}_{\mathrm{IV}},\end{cases}$$

$$\hat{\varphi}_2^+(l,t,k_1,k_2)=\begin{cases}\int_0^t \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\hat{U}(\tau,k,l), & l\in[-2k_1,\infty),k\in\mathbb{C}_{\mathrm{II}},\\ -\int_t^T \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\hat{U}(\tau,k,l), & l\in[0,-2k_1],k\in\mathbb{C}_{\mathrm{II}},\end{cases}$$

$$\hat{\varphi}_2^-(l,t,k_1,k_2)=\begin{cases}-\int_t^T \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\hat{U}(\tau,k,l), & l\in[-2k_1,\infty),k\in\mathbb{C}_{\mathrm{III}},\\ \int_0^t \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\hat{U}(\tau,k,l), & l\in[0,-2k_1],k\in\mathbb{C}_{\mathrm{III}},\end{cases}$$

where $\mathbb{C}_{\mathrm{I-IV}}$ denote the four quadrants of the complex k-plane. Therefore, the solution $\mu(x,y,t,k_1,k_2)$ of equation (2.2) can be represented in the form

$$\mu(x,y,t,k_1,k_2)=\begin{cases}\mu_1^+(x,y,t,k_1,k_2), & k\in\mathbb{C}_{\mathrm{I}},\\ \mu_1^-(x,y,t,k_1,k_2), & k\in\mathbb{C}_{\mathrm{IV}},\\ \mu_2^+(x,y,t,k_1,k_2), & k\in\mathbb{C}_{\mathrm{II}},\\ \mu_2^-(x,y,t,k_1,k_2), & k\in\mathbb{C}_{\mathrm{III}},\end{cases} \tag{2.18}$$

where $\mu_{1,2}^{\pm}$ are given by (see Figure 1)

$$\begin{aligned}\mu_1^{\pm}(x,y,t,k_1,k_2)={}&\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\frac{\mathrm{d}\eta}{\zeta-z}\mathrm{e}^{ik(\bar{\zeta}-\bar{z})+i\bar{k}(\zeta-z)}q(\xi,\eta,t)+\frac{\mathrm{e}^{-2ky}}{2\pi}\\ &\times\begin{bmatrix}\int_0^{\infty}\mathrm{d}l\int_0^t\mathrm{d}\tau-\int_{-2k_1}^0\mathrm{d}l\int_t^T\mathrm{d}\tau\\ -\int_0^{\infty}\mathrm{d}l\int_t^T\mathrm{d}\tau+\int_{-2k_1}^0\mathrm{d}l\int_0^t\mathrm{d}\tau\end{bmatrix}\mathcal{E}(k,l,\tau-t)\\ &\times\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{il(z-\xi)}\left(3klg_0+\frac{3}{2}l^2g_0-\frac{3}{2}lg_1\right)\end{aligned} \tag{2.19a}$$

and

$$\begin{aligned}
&\mu_2^{\pm}(x,y,t,k_1,k_2) \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0}^{\infty} \frac{\mathrm{d}\eta}{\zeta - z} \mathrm{e}^{ik(\bar{\zeta}-\bar{z})+i\bar{k}(\zeta-z)} q(\xi,\eta,t) + \frac{\mathrm{e}^{-2ky}}{2\pi}\int_{-2k_1}^{\infty} \mathrm{d}l \\
&\times \begin{bmatrix} \int_0^t \mathrm{d}\tau \\ -\int_t^T \mathrm{d}\tau \end{bmatrix} \mathcal{E}(k,l,\tau-t) \int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(z-\xi)} \left(3klg_0 + \frac{3}{2}l^2 g_0 - \frac{3}{2} l g_1\right).
\end{aligned} \tag{2.19b}$$

Evaluating equation (2.19a) at $k_1 = 0$, we find

$$\left(\mu_1^+ - \mu_2^+\right)|_{k_1=0} = 0, \quad \left(\mu_1^- - \mu_2^-\right)|_{k_1=0} = 0,$$

which imply that μ does not have a discontinuity across the imaginary axis $k_1 = 0$.

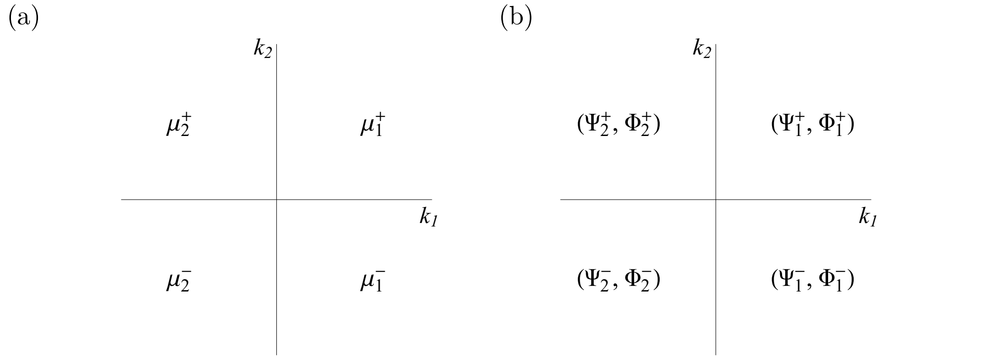

Fig. 1. (a) The function μ for the linearized mVN equation. (b) The function (Ψ, Φ) for the mVN equation.

Thus, $\mu(x,y,t,k_1,k_2)$ can be found by using Pompieu's formula

$$\begin{aligned}
\mu(x,y,t,k_1,k_2) =& \frac{1}{2i\pi}\int_0^{\infty} \frac{\mathrm{d}k_1'}{k_1'-k}\left(\mu_1^+ - \mu_1^-\right)|_{k_2'=0} + \frac{1}{2i\pi}\int_{-\infty}^{0} \frac{\mathrm{d}k_1'}{k_1'-k}\left(\mu_2^+ - \mu_2^-\right)|_{k_2'=0} \\
&- \frac{1}{\pi}\int_0^{\infty} \mathrm{d}k_1' \int_0^{\infty} \frac{\mathrm{d}k_2'}{k'-k}\frac{\partial \mu_1^+}{\partial \bar{k'}} - \frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}k_1' \int_0^{\infty} \frac{\mathrm{d}k_2'}{k'-k}\frac{\partial \mu_2^+}{\partial \bar{k'}} \\
&- \frac{1}{\pi}\int_0^{\infty} \mathrm{d}k_1' \int_{-\infty}^{0} \frac{\mathrm{d}k_2'}{k'-k}\frac{\partial \mu_1^-}{\partial \bar{k'}} - \frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}k_1' \int_{-\infty}^{0} \frac{\mathrm{d}k_2'}{k'-k}\frac{\partial \mu_2^-}{\partial \bar{k'}},
\end{aligned} \tag{2.20}$$

where $k' = k_1' + ik_2'$ with $k_1', k_2' \in \mathbb{R}$.

We will characterize $\frac{\partial \mu^{\pm}_{1,2}}{\partial \bar{k}}$ and $\left(\mu^{+}_{1,2} - \mu^{-}_{1,2}\right)|_{k_2=0}$. Let μ_0 denote the first integral in equations (2.19a), namely,

$$\mu_0(x, y, t, k_1, k_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0}^{\infty} \frac{\mathrm{d}\eta}{\zeta - z} \mathrm{e}^{ik(\bar{\zeta}-\bar{z})+i\bar{k}(\zeta-z)} q(\xi, \eta, t)$$

and then we find

$$\frac{\partial \mu_0}{\partial \bar{k}} = \frac{i}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0}^{\infty} \mathrm{d}\eta\, \mathrm{e}^{ik(\bar{\zeta}-\bar{z})+i\bar{k}(\zeta-z)} q(\xi, \eta, t).$$

Thus, differentiating μ_1^- with respect to $\bar{k}$ yields

$$\frac{\partial \mu_1^-}{\partial \bar{k}} = \frac{\partial \mu_0}{\partial \bar{k}} + \frac{\mathrm{e}^{-2ky}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0}^{t} \mathrm{d}\tau\, \mathrm{e}^{2i(\xi-z)k_1+2ik_1(k_1^2-3k_2^2)(\tau-t)} U(\xi, t, k, -2k_1), \quad (2.21)$$

where we have used the fact that

$$\mathrm{e}^{il(z-\xi)} \mathcal{E}(k, l, \tau - t)|_{l=-2k_1} = \mathrm{e}^{2i(\xi-z)k_1+2ik_1(k_1^2-3k_2^2)(\tau-t)}.$$

Let

$$\hat{q}(k, t) = \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0}^{\infty} \mathrm{d}\eta\, \mathrm{e}^{i(k\bar{\zeta}+\bar{k}\zeta)} q(\xi, \eta, t).$$

Since $\mathrm{e}^{-2ky-2ik_1 z} = \mathrm{e}^{-i(k\bar{z}+\bar{k}z)}$, we write equation (2.21) as

$$\frac{\partial \mu_1^-}{\partial \bar{k}} = \mathrm{e}^{-i(k\bar{z}+\bar{k}z)} \left[\frac{i}{\pi}\hat{q}(k, t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0}^{t} \mathrm{d}\tau\, \mathrm{e}^{2ik_1\xi+2ik_1(k_1^2-3k_2^2)(\tau-t)} U(\xi, t, k, -2k_1) \right]. \quad (2.22)$$

The equation (2.22) involves the term of the solution $\hat{q}(k, t)$, which can be eliminated by using the global relation (2.4) with

$$\Omega_0 = \{-\infty < x < \infty,\ 0 < y < \infty\}.$$

Replacing λ by $\bar{k}$ and evaluating at $y = 0$, the divergence form of the integral in equation (2.4) yields

$$(i\bar{k}q_z + \bar{k}^2 q - q_{zz})\mathrm{d}\bar{z} - (ikq_{\bar{z}} + k^2 q - q_{\bar{z}\bar{z}})\mathrm{d}z = (k_2 q_x + iq_{xy} - 4ik_1k_2 q + k_1 q_y)\mathrm{d}x.$$

Using the integration by parts, we find

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{2ik_1x+2ik_1(k_1^2-3k_2^2)t} \left(k_2 q_x + iq_{xy}\right)$$
$$= -\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{2ik_1x+2ik_1(k_1^2-3k_2^2)t} \left(2ik_1k_2 q - 2k_1 q_y\right),$$

and hence the global relation can be written as

$$-2i\left(\mathrm{e}^{2ik_1(k_1^2-3k_2^2)t}\hat{q}(k,t)\right)_t=\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{2ik_1x+2ik_1(k_1^2-3k_2^2)t}U(x,t,k,-2k_1),\qquad(2.23)$$

where we note that $U(x,t,k,-2k_1)=-6ik_1k_2g_0+3k_1g_1$. Integrating equation (2.23) from $\tau=0$ to $\tau=t$ with respect to τ, we find

$$i\hat{q}(k,t)=i\mathrm{e}^{-2ik_1x(k_1^2-3k_2^2)t}\hat{q}_0(k)-\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^t\mathrm{d}\tau\,\mathrm{e}^{2ik_1x+2ik_1(k_1^2-3k_2^2)(\tau-t)}U(x,t,k,-2k_1),$$

where $\hat{q}_0(k)=\hat{q}(k,0)$. Therefore, we derive

$$\frac{\partial\mu_1^-}{\partial\bar{k}}=\frac{i}{\pi}\mathrm{e}^{-2i(k_1x+k_2y)-2ik_1(k_1^2-3k_2^2)t}\hat{q}_0(k),\quad k_1\ge 0,k_2\le 0.\qquad(2.24)$$

Regarding $\frac{\partial\mu_1^+}{\partial\bar{k}}$, we note that $\int_0^t\mathrm{d}\tau=\int_0^T\mathrm{d}\tau-\int_t^T\mathrm{d}\tau$, and hence we find

$$\frac{\partial\mu_1^+}{\partial\bar{k}}=\mathrm{e}^{-2i(k_1x+k_2y)-2ik_1(k_1^2-3k_2^2)t}\left[\frac{i}{\pi}\hat{q}_0(k)-\frac{1}{2\pi}g(k_1,k_2)\right],\quad k_1\ge 0,k_2\ge 0,\quad(2.25)$$

where

$$g(k_1,k_2)=\int_{-\infty}^{\infty}\mathrm{d}x\int_0^T\mathrm{d}\tau\,\mathrm{e}^{2ik_1x+2ik_1(k_1^2-3k_2^2)\tau}\left(-6ik_1k_2g_0+3k_1g_1\right).$$

Similarly, we find

$$\frac{\partial\mu_2^+}{\partial\bar{k}}=\frac{i}{\pi}\mathrm{e}^{-2i(k_1x+k_2y)-2ik_1(k_1^2-3k_2^2)t}\hat{q}_0(k),\quad k_1\le 0,k_2\ge 0,\qquad(2.26)$$

and

$$\frac{\partial\mu_2^-}{\partial\bar{k}}=\mathrm{e}^{-2i(k_1x+k_2y)-2ik_1(k_1^2-3k_2^2)t}\left[\frac{i}{\pi}\hat{q}_0(k)-\frac{1}{2\pi}g(k_1,k_2)\right],\quad k_1\le 0,k_2\le 0.\quad(2.27)$$

We now compute $\left(\mu_{1,2}^+-\mu_{1,2}^-\right)|_{k_2=0}$. Let

$$\Delta\mu_{1,2}=\left(\mu_{1,2}^+-\mu_{1,2}^-\right)(x,y,t,k_1,0)$$

and then the first two integrals in equation (2.20) denoted by $\Delta\mu(x,y,t)$ can be written as

$$\Delta\mu(x,y,t)=\frac{1}{2i\pi}\int_0^{\infty}\frac{\mathrm{d}k_1'}{k_1'-k}\Delta\mu_1(x,y,t,k_1',0)+\frac{1}{2i\pi}\int_{-\infty}^{0}\frac{\mathrm{d}k_1'}{k_1'-k}\Delta\mu_2(x,y,t,k_1',0),$$
$$(2.28)$$

where

$$\Delta\mu_1 = \frac{\mathrm{e}^{-2k_1'y}}{2\pi}\left(\int_0^\infty \mathrm{d}l - \int_{-2k_1'}^0 \mathrm{d}l\right)\int_0^T \mathrm{d}\tau\,\mathcal{E}(k_1', l, \tau - t)\int_{-\infty}^\infty \mathrm{d}\xi\,\mathrm{e}^{il(z-\xi)}U(\xi, t, k_1', l),$$

and

$$\Delta\mu_2 = \frac{\mathrm{e}^{-2k_1'y}}{2\pi}\int_{-2k_1'}^\infty \mathrm{d}l\int_0^T \mathrm{d}\tau\,\mathcal{E}(k_1', l, \tau - t)\int_{-\infty}^\infty \mathrm{d}\xi\,\mathrm{e}^{il(z-\xi)}U(\xi, t, k_1', l).$$

Letting $\frac{l}{2} \to \alpha$ and $k_1' + \frac{l}{2} \to \beta$, equation (2.28) becomes

$$\begin{aligned}\Delta\mu(x,y,t) = \frac{1}{2i\pi^2}\int_0^\infty \mathrm{d}\beta\left(\int_0^\beta \mathrm{d}\alpha - \int_{-\infty}^0 \mathrm{d}\alpha + \int_\beta^\infty \mathrm{d}\alpha\right)\frac{\mathrm{e}^{-2\beta y}}{\beta - \alpha - k}\\ \times\int_0^T \mathrm{d}\tau\,\mathrm{e}^{-2i\alpha(\alpha^2+3\beta^2)(\tau-t)}\int_{-\infty}^\infty \mathrm{d}\xi\,\mathrm{e}^{2i\alpha(x-\xi)}\left(6\alpha\beta g_0 - 3\alpha g_1\right).\end{aligned} \tag{2.29}$$

Then letting $\alpha \to -k_1'$ and $\beta \to ik_2'$, equation (2.29) can be written as

$$\Delta\mu(x,y,t) = \frac{1}{2\pi^2}\left(\int_{-\infty}^0 \mathrm{d}k_1' - \int_0^\infty \mathrm{d}k_1'\right)\int_0^{-i\infty}\frac{\mathrm{d}k_2'}{k'-k}\mathrm{e}^{-2i(k_1'x+k_2'y)-2ik_1'(k_1'^2-3k_2'^2)t}g(k_1', k_2').$$

Therefore, from equation (2.20) it follows that the function $\mu(x,y,t,k_1,k_2)$ is given by

$$\begin{aligned}\mu(x,y,t,k_1,k_2) = \frac{1}{i\pi^2}\int_{-\infty}^\infty \mathrm{d}k_1'\int_{-\infty}^\infty \frac{\mathrm{d}k_2'}{k'-k}\mathrm{e}^{-2i(k_1'x+k_2'y)-2ik_1'(k_1'^2-3k_2'^2)t}\hat{q}_0(k')\\ + \frac{1}{2\pi^2}\int_{-\infty}^0 \mathrm{d}k_1'\int_{\partial\mathbb{C}_{\mathrm{III}}}\frac{\mathrm{d}k_2'}{k'-k}\mathrm{e}^{-2i(k_1'x+k_2'y)-2ik_1'(k_1'^2-3k_2'^2)t}g(k_1', k_2')\\ - \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k_1'\int_{\partial\mathbb{C}_{\mathrm{IV}}}\frac{\mathrm{d}k_2'}{k'-k}\mathrm{e}^{-2i(k_1'x+k_2'y)-2ik_1'(k_1'^2-3k_2'^2)t}g(k_1', k_2').\end{aligned}$$

Finally, using the Lax pair (2.2b) and $\mu = O(1/k)$ as $k \to \infty$, we find the solution $q(x,y,t)$ as

$$\begin{aligned} q(x,y,t) = &\frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{-2i(k_1x+k_2y)-2ik_1(k_1^2-3k_2^2)t}\hat{q}_0(k) \\ &- \frac{1}{2i\pi^2}\int_{-\infty}^{0} \mathrm{d}k_1 \int_{\partial\mathbb{C}_{\mathrm{III}}} \mathrm{d}k_2\, \mathrm{e}^{-2i(k_1x+k_2y)-2ik_1(k_1^2-3k_2^2)t} g(k_1,k_2) \\ &+ \frac{1}{2i\pi^2}\int_{0}^{\infty} \mathrm{d}k_1 \int_{\partial\mathbb{C}_{\mathrm{IV}}} \mathrm{d}k_2\, \mathrm{e}^{-2i(k_1x+k_2y)-2ik_1(k_1^2-3k_2^2)t} g(k_1,k_2). \end{aligned}$$

3. THE MODIFIED VESELOV–NOVIKOV EQUATION

3.1. THE LAX PAIR FORMULATION

We now study the mVN equation (1.1). First note that the mVN equation admits the following Lax pair [29]

$$\mu_{1_{\bar z}} = -q\mu_2, \quad \mu_{2_z} = q\mu_1, \tag{3.1a}$$
$$\mu_{1_t} = -\left(\mu_{1_{zzz}} + \mu_{1_{\bar z\bar z\bar z}} + 6v\mu_{1_z} + 3q_{\bar z}\mu_{2_{\bar z}} + 3\left(v_z\mu_1 - 2q\bar v\mu_2\right)\right), \tag{3.1b}$$
$$\mu_{2_t} = -\left(\mu_{2_{zzz}} + \mu_{2_{\bar z\bar z\bar z}} + 6\bar v\mu_{2_{\bar z}} - 3q_z\mu_{1_z} + 3\left(\bar v_{\bar z}\mu_2 + 2qv\mu_1\right)\right). \tag{3.1c}$$

Letting $\mu_1 = \Psi\mathrm{e}^{ikz+ik^3t}$ and $\mu_2 = \Phi\mathrm{e}^{ikz+ik^3t}$, we find the modified Lax pair

$$\Psi_{\bar z} = -q\Phi, \quad \Phi_z + ik\Phi = q\Psi, \tag{3.2a}$$
$$\Psi_t = -\left(\Psi_{zzz} + \Psi_{\bar z\bar z\bar z} + 3ik\Psi_{zz} - 3k^2\Psi_z + 6v\Psi_z + 3q_{\bar z}\Phi_{\bar z} + 6ikv\Psi + 3\left(v_z\Psi - 2q\bar v\Phi\right)\right), \tag{3.2b}$$
$$\Phi_t = -\left(\Phi_{zzz} + \Phi_{\bar z\bar z\bar z} + 3ik\Phi_{zz} - 3k^2\Phi_z + 6\bar v\Phi_{\bar z} - 3q_z\Psi_z - 3ikq_z\Psi + 3\left(\bar v_{\bar z}\Phi + 2qv\Psi\right)\right). \tag{3.2c}$$

Lax pair (3.2a) can be written in terms of derivatives with respect to x and y

$$\Psi_y = i\Psi_x + 2iq\Phi, \quad \Phi_y = -i\Phi_x + 2k\Phi + 2iq\Psi. \tag{3.3a}$$

Note that

$$\begin{aligned} \Psi_{yy} &= -\Psi_{xx} - 4q^2\Psi + 4ik\Phi - 2q_x\Phi + 2iq_y\Phi, \\ \Phi_{yy} &= -\Phi_{xx} - 4ik\Phi_x + 4k^2\Phi - 4q^2\Phi + 4ikq\Psi + 2q_x\Psi + 2iq_y\Psi. \end{aligned}$$

Using the above equations, Lax pair (3.2b) and (3.2c) also can be written in terms of derivatives with respect to x and y

$$\begin{aligned}\Psi_t + \Psi_{xxx} + 3ik\Psi_{xx} - 3k^2\Psi_x &= -3\left[qq_x + v_x + ik(q^2+2v)\right]\Psi \\ &\quad - \left[3ikq_x + 6q(v-\bar{v}) + \frac{3}{2}(q_{xx} - iq_{xy})\right]\Phi \qquad (3.3\text{b}) \\ &\quad - 3(q^2+2v)\Psi_x - 3q_x\Phi_x, \\ \Phi_t + \Phi_{xxx} + 3ik\Phi_{xx} - 3k^2\Phi_x &= -3\left[qq_x + \bar{v}_x + ik(q^2+2\bar{v})\right]\Phi \\ &\quad + \left[3ikq_x + 6q(\bar{v}-v) + \frac{3}{2}(q_{xx} + iq_{xy})\right]\Psi \qquad (3.3\text{c}) \\ &\quad - 3(q^2+2\bar{v})\Phi_x + 3q_x\Psi_x.\end{aligned}$$

Also, from the Lax pair (3.2), we can derive the following global relation, similar to equations (2.4) and (2.23) in Section 2.

Proposition 3.1. *Let q and v satisfy equation* (1.1)*. Suppose that equation* (1.1) *is valid for $(x, y) \in \Omega_0$, where Ω_0 is a bounded piecewise smooth domain in $\mathbb{R}^2$. Then,*

$$\begin{aligned}&-2i\left(\iint_{\Omega_0} \mathrm{d}x\mathrm{d}y\, \mathrm{e}^{i(kz+\bar{k}\bar{z})+2ik_1(k_1^2-3k_2^2)t} q\Psi\right)_t \\ &= \int_{\partial\Omega_0} \mathrm{e}^{i(kz+\bar{k}\bar{z})+2ik_1(k_1^2-3k_2^2)t} \\ &\quad \times\left\{\left[(q_{\bar{z}\bar{z}} - i\bar{k}q_{\bar{z}} + 6q\bar{v} - \bar{k}^2 q)\Psi + q\Psi_{\bar{z}\bar{z}} - (q_{\bar{z}} + i\bar{k}q)\Psi_{\bar{z}} + 3\bar{v}_z\Phi\right]\mathrm{d}z\right. \\ &\quad \left. - \left[(q_{zz} - ikq_z + 6qv - k^2 q)\Psi + q\Psi_{zz} - (q_z - 2ikq)\Psi_z - 3(\bar{v}_{\bar{z}} + 2i\bar{k}\bar{v})\Phi\right]\mathrm{d}\bar{z}\right\}. \qquad (3.4)\end{aligned}$$

Proof. For convenience, we let

$$e = \mathrm{e}^{-i(kz+\bar{k}\bar{z})-i(k^3+\bar{k}^3)t} = \mathrm{e}^{-2i(k_1x-k_2y)-2ik_1(k_1^2-3k_2^2)t}. \qquad (3.5)$$

Note that

$$(\bar{e}q\Psi)_t = \bar{e}q_t\Psi + \bar{e}q\Psi_t + i(k^3+\bar{k}^3)\bar{e}q\Psi.$$

Using equation (1.1) and the Lax pair (3.2b), we find

$$\begin{aligned}(\bar{e}q\Psi)_t &= -\bar{e}\Psi\left[q_{zzz} + q_{\bar{z}\bar{z}\bar{z}} + 6(q_zv + q_{\bar{z}}\bar{v}) + 3(qv_z + q\bar{v}_{\bar{z}})\right] \\ &\quad - \bar{e}q\big[\Psi_{zzz} + \Psi_{\bar{z}\bar{z}\bar{z}} + 3ik\Psi_{zz} - 3k^2\Psi_z + 6v\Psi_z + 3q_{\bar{z}}\Phi_{\bar{z}} + 6ikv\Psi \qquad (3.6) \\ &\quad + 3(v_z\Psi - 2q\bar{v}\Phi)\big] + i(k^3+\bar{k}^3)\bar{e}q\Psi.\end{aligned}$$

Using the following equations

$$\begin{aligned}\bar{e}q_{zzz}\Psi &= \left(\bar{e}q_{zz}\Psi - ik\bar{e}q_z\Psi\right)_z - \bar{e}\left(q_{zz} - ikq_z\right)\Psi_z - k^2\bar{e}q_z\Psi, \\ \bar{e}q\Psi_{zzz} &= \left(\bar{e}q\Psi_{zz} - \bar{e}q_z\Psi_z - ik\bar{e}q\Psi_z - k^2\bar{e}q\Psi\right)_z + \bar{e}\left(q_{zz} + 2ikeq_z\right)\Psi_z \\ &\quad + \bar{e}\left(k^2q_z + ik^3q\right)\Psi, \\ \bar{e}q\Psi_{zz} &= \left(\bar{e}q\Psi_z\right)_z - \bar{e}\left(q_z + ikq\right)\Psi_z, \quad \bar{e}qv\Psi_z = \left(\bar{e}qv\Psi\right)_z - \bar{e}\left(q_zv + qv_z + ikqv\right)\Psi\end{aligned}$$

and similar equations for $\bar{e}q_{\bar{z}\bar{z}\bar{z}}\Psi$, $\bar{e}q\Psi_{\bar{z}\bar{z}\bar{z}}$, $\bar{e}q_{\bar{z}}\bar{v}\Psi$ and $\bar{e}qq_{\bar{z}}\Phi_{\bar{z}}$, equation (3.6) can be written in the divergence form

$$\begin{aligned}(\bar{e}q\Psi)_t = &- \left\{\bar{e}[(q_{zz} - ikq_z - k^2q + 6qv)\Psi + q\Psi_{zz} - (q_z - 2ikq)\Psi_z - 3(\bar{v}_{\bar{z}} + 2i\bar{k}\bar{v})\Phi]\right\}_z \\ &- \{\bar{e}[(q_{\bar{z}\bar{z}} - i\bar{k}q_{\bar{z}} - \bar{k}^2q + 6q\bar{v})\Psi + q\Psi_{\bar{z}\bar{z}} - (q_{\bar{z}} + i\bar{k}q)\Psi_{\bar{z}} + 3\bar{v}_z\Phi]\}_{\bar{z}}.\end{aligned} \tag{3.7}$$

By applying the Poincaré lemma in equation (3.7), we find equation (3.4). □

Proposition 3.2. *Let* q *and* v *satisfy equation* (1.1). *Then*

$$\begin{aligned}&-2i\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{i(kz+\bar{k}\bar{z})}q\Psi + 2i\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{i(kz+\bar{k}\bar{z})-2ik_1(k_1^2-3k_2^2)t}q_0\Psi_0 \\ &= \mathrm{e}^{-2ik_1(k_1^2-3k_2^2)t}\int_0^t\mathrm{d}\tau\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{2ik_1x+2ik_1(k_1^2-3k_2^2)\tau}\left[U_{21}\psi + U_{22}\varphi\right]|_{l=-2k_1},\end{aligned} \tag{3.8}$$

where $\Psi_0(x,y,k_1,k_2) = \Psi(x,y,0,k_1,k_2)$.

Proof. Let $\Omega_0 = \{-\infty < x < \infty,\ 0 < y < \infty\}$ in Proposition 3.1. Then integrating equation (3.4) from $\tau = 0$ to $\tau = t$ with respect to $\mathrm{d}\tau$, we find

$$\begin{aligned}&-2i\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{i(kz+\bar{k}\bar{z})+2ik_1(k_1^2-3k_2^2)t}q\Psi + 2i\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{i(kz+\bar{k}\bar{z})}q_0\Psi_0 \\ &= \int_0^t\mathrm{d}\tau\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{2ik_1x+2ik_1(k_1^2-3k_2^2)\tau}\Big\{[ig_{1_x} + k_1g_1 - k_2g_{0_x} + 6g_0(\bar{v}_0 - v_0) \\ &\quad + 4ik_1k_2g_0]\psi + 3(\bar{v}_{0_x} + 2i\bar{k}\bar{v}_0)\varphi - \frac{i}{2}\left[g_1 + \bar{k}g_0 + 2kg_0\right]\psi_x \\ &\quad + ig_0\psi_{xy} - \frac{1}{2}\left[ig_{0_x} - \bar{k}g_0 + 2kg_0\right]\psi_y\Big\}.\end{aligned} \tag{3.9}$$

In order to simplify the right-hand-side of equation (3.9), we use integrating by parts

$$\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{2ik_1x}g_0\psi_{xy} = \int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{2ik_1x}\left(g_{0_x} - 2ik_1g_0\right)\psi_y,$$

which implies

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{2ik_1x}\left\{-\frac{i}{2}\left[g_1 + \bar{k}g_0 + 2kg_0\right]\psi_x + ig_0\psi_{xy} - \frac{1}{2}\left[ig_{0_x} - \bar{k}g_0 + 2kg_0\right]\psi_y\right\} \\ &= \int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{2ik_1x}\left[\left(-\frac{i}{2}g_1 + \frac{3}{2}g_{0_x} + 2k_2g_0\right)\psi_x + \left(3g_{0_x} + 3i\bar{k}g_0\right)g_0\varphi\right].\end{aligned}$$

Integrating by parts the terms involving ψ_x in the above equation and substituting the resulting equation into the right-hand-side of equation (3.9), we obtain equation (3.8). □

For later analysis and use, we also present the other version of the global relation.

Proposition 3.3. *Let* q *and* v *satisfy equation* (1.1). *Suppose that equation* (1.1) *is valid for* $(x, y) \in \Omega_0$, *where* Ω_0 *is a bounded piecewise smooth domain in* $\mathbb{R}^2$. *Then*

$$\begin{aligned}
&-2i\left(\iint\limits_{\Omega_0} \mathrm{d}x\mathrm{d}y\, \mathrm{e}^{-il\bar{z}-il(l^2-3\lambda l+3\lambda^2-3k_1l+6\lambda k_1+3k_1^2)t} q\bar{\Phi}\right)_t \\
&= \int\limits_{\partial\Omega_0} \mathrm{e}^{-il\bar{z}-il(l^2-3\lambda l+3\lambda^2-3k_1l+6\lambda k_1+3k_1^2)t} \\
&\quad\times\Big\{\Big[(q_{\bar{z}\bar{z}}+ilq_{\bar{z}}-l^2q+3l(k_1+\lambda)q-3(k_1+\lambda)^2q+6q\bar{v})\bar{\Phi}-3\bar{v}_z\bar{\Psi} \\
&\qquad+q\bar{\Phi}_{\bar{z}\bar{z}}+(ilq-q_{\bar{z}}-3i(k_1+\lambda)q)\bar{\Phi}_{\bar{z}}\Big]\mathrm{d}z \\
&\qquad-\big[(q_{zz}+6qv)\bar{\Phi}+q\bar{\Phi}_{zz}-q_z\bar{\Phi}_z+(6i(k_1+\lambda)\bar{v}+3\bar{v}_{\bar{z}}-6il\bar{v})\bar{\Psi}\big]\,\mathrm{d}\bar{z}\Big\}.
\end{aligned} \tag{3.10}$$

Proof. For convenience, we let

$$E = \mathrm{e}^{-il\bar{z}-il(l^2-3\lambda l+3\lambda^2-3k_1l+6\lambda k_1+3k_1^2)t}. \tag{3.11}$$

Using equation (1.1) and (3.2c), we find

$$\begin{aligned}
\left(Eq\bar{\Phi}\right)_t &= -E\bar{\Phi}\left[q_{zzz}+q_{\bar{z}\bar{z}\bar{z}}+6(q_zv+q_{\bar{z}}\bar{v})+3(qv_z+q\bar{v}_{\bar{z}})\right] \\
&\quad -Eq\big[\bar{\Phi}_{zzz}+\bar{\Phi}_{\bar{z}\bar{z}\bar{z}}-3i(k_1+\lambda)\bar{\Phi}_{\bar{z}\bar{z}}-3(k_1+\lambda)^2\bar{\Phi}_{\bar{z}}+6v\bar{\Phi}_z \\
&\qquad -3q_{\bar{z}}\bar{\Psi}_{\bar{z}}+3i(k_1+\lambda)q_{\bar{z}}\bar{\Psi}+3(v_z\bar{\Phi}+2q\bar{v}\bar{\Psi})\big] \\
&\quad -il(l^2-3\lambda l+3\lambda^2-3k_1l+6\lambda k_1+3k_1^2)Eq\bar{\Phi}.
\end{aligned} \tag{3.12}$$

Note that

$$\begin{aligned}
Eq_{\bar{z}\bar{z}\bar{z}}\bar{\Phi} &= \left(Eq_{\bar{z}\bar{z}}\bar{\Phi}+ilEq_{\bar{z}}\bar{\Phi}\right)_{\bar{z}} - E\left(q_{\bar{z}\bar{z}}+ilq_{\bar{z}}\right)\bar{\Phi}_{\bar{z}} - l^2Eq_{\bar{z}}\bar{\Phi}, \\
Eq\bar{\Phi}_{\bar{z}\bar{z}\bar{z}} &= \left(Eq\bar{\Phi}_{\bar{z}\bar{z}}-Eq_{\bar{z}}\bar{\Phi}_{\bar{z}}+ilEq\bar{\Phi}_{\bar{z}}-l^2Eq\bar{\Phi}\right)_{\bar{z}} + E\left(q_{\bar{z}\bar{z}}-2ilq_{\bar{z}}\right)\bar{\Phi}_{\bar{z}} \\
&\quad + E\left(l^2q_{\bar{z}}-il^3q\right)\bar{\Phi}, \\
Eq\bar{\Phi}_{\bar{z}\bar{z}} &= \left(Eq\bar{\Phi}_{\bar{z}}+ilEq\bar{\Phi}\right)_{\bar{z}} - Eq_{\bar{z}}\bar{\Phi}_{\bar{z}} - E\left(ilq_{\bar{z}}+l^2q\right)\bar{\Phi}, \\
Eq\bar{\Phi}_{\bar{z}} &= \left(Eq\bar{\Phi}\right)_{\bar{z}} - E\left(q_{\bar{z}}-ilq\right)\bar{\Phi}, \\
Eq_{zzz}\bar{\Phi} &= \left(Eq_{zz}\bar{\Phi}\right)_z - Eq_{zz}\bar{\Phi}_z, \\
Eq\bar{\Phi}_{zzz} &= \left(Eq\bar{\Phi}_{zz}-Eq_z\bar{\Phi}_z\right)_z + Eq_{zz}\bar{\Phi}_z.
\end{aligned}$$

Then straightforward calculations show that equation (3.12) can be written in the divergence form

$$\begin{aligned}\left(Eq\bar{\Phi}\right)_t =& -\left\{E\left[(q_{zz}+6qv)\bar{\Phi}+[6i(k_1+\lambda)\bar{v}+3\bar{v}_{\bar{z}}-6il\bar{v}]\bar{\Psi}+q\bar{\Phi}_{zz}-q_z\bar{\Phi}_z\right]\right\}_z \\ &-\Bigg\{E\Big[[q_{\bar{z}\bar{z}}+ilq_{\bar{z}}-l^2q+3l(k_1+\lambda)q-3(k_1+\lambda)^2q+6q\bar{v}]\bar{\Phi} \\ &-3\bar{v}_z\bar{\Psi}+q\bar{\Phi}_{\bar{z}\bar{z}}+(ilq-q_{\bar{z}}-3i(k_1+\lambda)q)\bar{\Phi}_{\bar{z}}\Big]\Bigg\}_{\bar{z}}.\end{aligned} \tag{3.13}$$

By employing the Poincaré lemma in equation (3.13), we find equation (3.10). □

Proposition 3.4. *Let q and v satisfy equation* (1.1)*. Then*

$$\begin{aligned}&-2i\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{-il\bar{z}-il(l^2-3\lambda l+3\lambda^2-3k_1l+6\lambda k_1+3k_1^2)t}q\bar{\Phi}+2i\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{-il\bar{z}}q_0\bar{\Phi}_0 \\ &=-\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{t}\mathrm{d}\tau\,\mathrm{e}^{-ilx-il(l^2-3\lambda l+3\lambda^2-3k_1l+6\lambda k_1+3k_1^2)\tau} \\ &\quad\times\left[U_{22}(x,\tau,k_1,l-2k_1-\lambda)\bar{\psi}-U_{21}(x,\tau,k_1,l-2k_1-\lambda)\bar{\varphi}\right],\end{aligned} \tag{3.14}$$

where $\Phi_0(x,y,k_1,k_2)=\Phi(x,y,0,k_1,k_2)$.

Proof. For $\Omega_0=\{-\infty<x<\infty,\ 0<y<\infty\}$ in equation (3.10), we know that

$$\begin{aligned}&-2i\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\left(Eq\bar{\Phi}\right)_t \\ &=\int_{-\infty}^{\infty}\mathrm{d}x\,\tilde{E}\Bigg\{[6i(l-k_1-\lambda)\bar{v}_0-3\bar{v}_{0_x}]\,\bar{\psi} \\ &\quad+\left[ig_{1_x}+\frac{il}{2}(g_{0_x}+ig_1)-l^2g_0+3(k_1+\lambda)(l-k_1-\lambda)g_0+6g_0(\bar{v}_0-v_0)\right]\bar{\varphi} \\ &\quad+\frac{i}{2}\left[lg_0-g_1-3(k_1+\lambda)g_0\right]\bar{\varphi}_x+\frac{i}{2}\left[ilg_0-g_{0_x}-3i(k_1+\lambda)g_0\right]\bar{\varphi}_y+ig_0\bar{\varphi}_{xy}\Bigg\},\end{aligned} \tag{3.15}$$

where E is given by equation (3.11) and $\tilde{E}=E|_{y=0}$. Using a similar way as in the proof of Proposition 3.2, we can simplify the expressions in the right-hand-side of equation (3.15) as

$$
\begin{aligned}
& -2i \int_{-\infty}^{\infty} \mathrm{d}x \int_{0}^{\infty} \mathrm{d}y \left(Eq\bar{\Phi}\right)_t \\
& = - \int_{-\infty}^{\infty} \mathrm{d}x\, \tilde{E} \Big\{ \left[3g_0 g_{0_x} + 3\bar{v}_{0_x} - 3i(l-k_1-\lambda)(g_0^2 + 2\bar{v}_0)\right] \bar{\psi} \\
& \quad - \left[6g_0(\bar{v}_0 - v_0) + 3i(l-k_1-\lambda) g_{0_x} - \frac{3}{2}(g_{0_{xx}} - i g_{1_x})\right] \bar{\varphi} \Big\}.
\end{aligned}
$$

Integrating the above equation from $\tau = 0$ to $\tau = t$ with respect to τ, we find equation (3.14). □

3.2. THE DIRECT PROBLEM

Proposition 3.5. *We assume that the solution of equation* (1.1) *exists. Then there exist the solutions of equation* (3.2), *which are bounded for all* $k \in \mathbb{C}$ *and given by*

$$
(\Psi(x,y,t,k_1,k_2), \Phi(x,y,t,k_1,k_2)) = \begin{cases} \left(\Psi_1^+, \Phi_1^+\right), & k \in \mathbb{C}_{\mathrm{I}}, \\ \left(\Psi_1^-, \Phi_1^-\right), & k \in \mathbb{C}_{\mathrm{IV}}, \\ \left(\Psi_2^+, \Phi_2^+\right), & k \in \mathbb{C}_{\mathrm{II}}, \\ \left(\Psi_2^-, \Phi_2^-\right), & k \in \mathbb{C}_{\mathrm{III}}, \end{cases} \tag{3.16}
$$

where $\Psi_j^{\pm}$ *and* $\Phi_j^{\pm}$, $j = 1, 2$, *are defined by* (*see Figure 1*)

$$
\begin{aligned}
\Psi_1^{\pm}(x,y,t,k_1,k_2) = 1 &+ \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0}^{\infty} \frac{\mathrm{d}\eta}{\zeta - z} q \Phi_1^{\pm} + \frac{1}{2\pi} \int_{0}^{\infty} \mathrm{d}l \begin{bmatrix} \int_0^t \mathrm{d}\tau \\ -\int_t^T \mathrm{d}\tau \end{bmatrix} \\
& \times \mathcal{E}(k,l,\tau - t) \int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(z-\xi)} \left[U_{11}\psi_1^{\pm} + U_{12}\varphi_1^{\pm}\right],
\end{aligned} \tag{3.17a}
$$

$$
\begin{aligned}
\Phi_1^{\pm}(x,y,t,k_1,k_2) = &-\frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0}^{\infty} \frac{\mathrm{d}\eta}{\bar{\zeta} - \bar{z}} \mathrm{e}^{ik(\zeta - z) + i\bar{k}(\bar{\zeta} - \bar{z})} q \Psi_1^{\pm} \\
&+ \frac{\mathrm{e}^{2ky}}{2\pi} \int_{-\infty}^{-2k_1} \mathrm{d}l \begin{bmatrix} \int_0^t \mathrm{d}\tau \\ -\int_t^T \mathrm{d}\tau \end{bmatrix} \\
& \times \mathcal{E}(k,l,\tau - t) \int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)} \left[U_{21}\psi_1^{\pm} + U_{22}\varphi_1^{\pm}\right],
\end{aligned} \tag{3.17b}
$$

$$\begin{aligned}\Psi_2^{\pm}(x,y,t,k_1,k_2) = 1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\zeta - z} q\Phi_2^{\pm} \\ + \frac{1}{2\pi}\begin{bmatrix} -\int_0^{-2k_1}\mathrm{d}l\int_t^T \mathrm{d}\tau + \int_{-2k_1}^{\infty}\mathrm{d}l\int_0^t\mathrm{d}\tau \\ \int_0^{-2k_1}\mathrm{d}l\int_0^t\mathrm{d}\tau - \int_{-2k_1}^{\infty}\mathrm{d}l\int_t^T\mathrm{d}\tau \end{bmatrix} \\ \times \mathcal{E}(k,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\, \mathrm{e}^{il(z-\xi)}\left[U_{11}\psi_2^{\pm} + U_{12}\varphi_2^{\pm}\right]\end{aligned} \tag{3.17c}$$

and

$$\begin{aligned}\Phi_2^{\pm}(x,y,t,k_1,k_2) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\frac{\mathrm{d}\eta}{\bar\zeta-\bar z}\mathrm{e}^{ik(\zeta-z)+i\bar k(\bar\zeta-\bar z)}q\Psi_2^{\pm} \\ + \frac{\mathrm{e}^{2ky}}{2\pi}\begin{bmatrix}\int_{-\infty}^0\mathrm{d}l\int_0^t\mathrm{d}\tau - \int_0^{-2k_1}\mathrm{d}l\int_t^T\mathrm{d}\tau \\ -\int_{-\infty}^0\mathrm{d}l\int_t^T\mathrm{d}\tau + \int_0^{-2k_1}\mathrm{d}l\int_0^t\mathrm{d}\tau\end{bmatrix}\mathcal{E}(k,l,\tau-t) \\ \times\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{il(\bar z-\xi)}\left[U_{21}\psi_2^{\pm}+U_{22}\varphi_2^{\pm}\right]\end{aligned} \tag{3.17d}$$

with $\mathcal{E}(k,l,t) = \mathrm{e}^{-il(l^2+3kl+3k^2)t}$, $v_0(x,t) = v(x,0,t)$,

$$U_{11}(x,t,k,l) = 3g_0g_{0_x} + 3v_{0_x} - 3i(l+k)(g_0^2+2v_0), \tag{3.18a}$$

$$U_{12}(x,t,k,l) = -3i(l+k)g_{0_x} - 6g_0(v_0-\bar v_0) + \frac{3}{2}(g_{0_{xx}} + ig_{1_x}), \tag{3.18b}$$

$$U_{21}(x,t,k,l) = 3i(l+k)g_{0_x} + 6g_0(\bar v_0 - v_0) - \frac{3}{2}(g_{0_{xx}} - ig_{1_x}), \tag{3.18c}$$

$$U_{22}(x,t,k,l) = 3g_0g_{0_x} + 3\bar v_{0_x} - 3i(l+k)(g_0^2+2\bar v_0) \tag{3.18d}$$

and

$$\psi_j^{\pm}(x,t,k_1,k_2) = \Psi_j^{\pm}(x,0,t,k_1,k_2), \quad \varphi_j^{\pm}(x,t,k_1,k_2) = \Phi_j^{\pm}(x,0,t,k_1,k_2), \quad j=1,2. \tag{3.19}$$

Proof. From the Lax pair (3.2), we seek for the solutions Ψ and Φ with the asymptotic behavior

$$(\Psi,\Phi) = \left(1, O\left(\frac{1}{z}\right)\right), \quad z\to\infty.$$

Thus, Pompieu's formula yields

$$\Psi(x,y,t,k_1,k_2) = 1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0}^{\infty} \frac{\mathrm{d}\eta}{\zeta - z} q\Phi + \frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{\xi - z}(\psi - 1), \tag{3.20}$$

$$\begin{aligned}\Phi(x,y,t,k_1,k_2) = &-\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_{0}^{\infty} \frac{\mathrm{d}\eta}{\bar\zeta - \bar z} \mathrm{e}^{ik(\zeta - z) + i\bar k(\bar\zeta - \bar z)} q\Psi \\ &- \frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{\xi - \bar z}\mathrm{e}^{2ik_1\xi - i(kz + \bar k \bar z)}\varphi,\end{aligned} \tag{3.21}$$

where

$$\psi(x,t,k_1,k_2) = \Psi(x,0,t,k_1,k_2), \quad \varphi(x,t,k_1,k_2) = \Phi(x,0,t,k_1,k_2).$$

We will determine the functions ψ and φ. In this respect, we use the supplement equations for the Lax pair (3.3b) and (3.3c). Evaluating (3.3b) and (3.3c) at $y = 0$, it follows that the functions ψ and φ solve

$$\psi_t + \psi_{xxx} + 3ik\psi_{xx} - 3k^2\psi_x = Q_1, \tag{3.22}$$

$$\varphi_t + \varphi_{xxx} + 3ik\varphi_{xx} - 3k^2\varphi_x = Q_2, \tag{3.23}$$

where

$$\begin{aligned} Q_1(x,t,k) = &-3\left[g_0 g_{0_x} + v_{0_x} + ik(g_0^2 + 2v_0)\right]\psi \\ &- \left[3ikg_{0_x} + 6g_0(v_0 - \bar v_0) + \frac{3}{2}(g_{0_{xx}} - ig_{1_x})\right]\varphi \\ &- 3(g_0^2 + 2v_0)\psi_x - 3g_{0_x}\varphi_x, \\ Q_2(x,t,k) = &\left[3ikg_{0_x} + 6g_0(\bar v_0 - v_0) + \frac{3}{2}(g_{0_{xx}} + ig_{1_x})\right]\psi \\ &- 3\left[g_0 g_{0_x} + \bar v_{0_x} + ik(g_0^2 + 2\bar v_0)\right]\varphi \\ &- 3(g_0^2 + 2\bar v_0)\varphi_x + 3g_{0_x}\psi_x. \end{aligned}$$

The spectral analysis of ψ. For equation (3.22), we take the Fourier transform in x

$$\psi(\xi,t,k_1,k_2) - 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}l\, \mathrm{e}^{il\xi}\hat\psi(l,t,k_1,k_2), \tag{3.24}$$

where

$$\hat\psi(l,t,k_1,k_2) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ilx}\left(\psi(x,t,k_1,k_2) - 1\right). \tag{3.25}$$

Note that

$$\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi-z}(\psi-1)=\frac{1}{2\pi}\int_{0}^{\infty}\mathrm{d}l\,\mathrm{e}^{ilz}\hat{\psi},\quad \operatorname{Im} z\geq 0,$$

which implies that

$$\Psi(x,y,t,k_1,k_2)=1+\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{0}^{\infty}\frac{\mathrm{d}\eta}{\zeta-z}q\Phi+\frac{1}{2\pi}\int_{0}^{\infty}\mathrm{d}l\,\mathrm{e}^{ilz}\hat{\psi}. \tag{3.26}$$

Thus, we will determine $\hat{\psi}$ for $0<l<\infty$. Evaluating equation (3.26) at $y=0$ and using the following identity

$$\int_{-\infty}^{0}\mathrm{d}l\,\mathrm{e}^{il(x-\zeta)}=\frac{i}{\zeta-x},\quad \zeta=\xi+i\eta,\quad \eta>0,$$

we find

$$\psi(x,t,k_1,k_2)=1+\frac{1}{i\pi}\int_{-\infty}^{0}\mathrm{d}l\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{0}^{\infty}\mathrm{d}\eta\,\mathrm{e}^{il(x-\zeta)}q\Phi+\frac{1}{2\pi}\int_{0}^{\infty}\mathrm{d}l\,\mathrm{e}^{ilx}\hat{\psi}. \tag{3.27}$$

Comparing equation (3.27) with equation (3.24), the function $\hat{\psi}$ satisfies the restriction

$$\hat{\psi}(l,t,k_1,k_2)=-2i\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{0}^{\infty}\mathrm{d}\eta\,\mathrm{e}^{-il\zeta}q\Phi,\quad -\infty<l<0,$$

while there is no restriction imposed on $\hat{\psi}$ for $0<l<\infty$.

From equation (3.22), it follows that

$$\left(\mathrm{e}^{-il(l^2+3lk+3k^2)t}\hat{\psi}\right)_t=\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi-il(l^2+3lk+3k^2)t}Q_1.$$

which yields

$$\hat{\psi}(l,t,k_1,k_2)=\begin{cases}\int_0^t \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}Q_1+\mathcal{E}(k,l,-t)\hat{\psi}(l,0,k_1,k_2),\\ -\int_t^T \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}Q_1+\mathcal{E}(k,l,T-t)\hat{\psi}(l,T,k_1,k_2),\end{cases}$$

where $\mathcal{E}(k,l,t)=\mathrm{e}^{-il(l^2+3kl+3k^2)t}$. Note that by the integration by parts, we find

$$\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}Q_1=\int_{-\infty}^{\infty}\mathrm{d}\xi\mathrm{e}^{-il\xi}\left[U_{11}\psi+U_{12}\varphi\right],$$

where U_{11} and U_{12} are given in equations (3.18a) and (3.18b). According to the real part of $\mathcal{E}(k,l,\tau-t)$ given in equation (2.17), we define $\hat{\psi}_j^{\pm}$ $(j=1,2)$ as

$$\hat{\psi}_1^{\pm}(l,t,k_1,k_2)=\begin{bmatrix}\int_0^t \mathrm{d}\tau\\ -\int_t^T \mathrm{d}\tau\end{bmatrix}\mathcal{E}(k,l,\tau-t)\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}\left[U_{11}\psi_1^{\pm}+U_{12}\varphi_1^{\pm}\right],\quad l\in[0,\infty),\tag{3.28a}$$

and for $\hat{\psi}_2^{\pm}$, we consider two cases:

$$\hat{\psi}_2^{\pm}(l,t,k_1,k_2)=\begin{bmatrix}-\int_t^T \mathrm{d}\tau\\ \int_0^t \mathrm{d}\tau\end{bmatrix}\mathcal{E}(k,l,\tau-t)\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}\left[U_{11}\psi_2^{\pm}+U_{12}\varphi_2^{\pm}\right],\quad l\in[0,-2k_1],\tag{3.28b}$$

and

$$\hat{\psi}_2^{\pm}(l,t,k_1,k_2)=\begin{bmatrix}\int_0^t \mathrm{d}\tau\\ -\int_t^T \mathrm{d}\tau\end{bmatrix}\mathcal{E}(k,l,\tau-t)\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}\left[U_{11}\psi_2^{\pm}+U_{12}\varphi_2^{\pm}\right],\quad l\in[-2k_1,\infty).\tag{3.28c}$$

Substituting equations (3.28) into equation (3.26), we find equations (3.17a) and (3.17c).

The spectral analysis of φ. Let

$$\varphi(\xi,t,k_1,k_2)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}l\,\mathrm{e}^{il\xi}\hat{\varphi}(l,t,k_1,k_2).\tag{3.29}$$

Equation (3.23) yields

$$\left(\mathcal{E}(k,l,t)\hat{\varphi}\right)_t=\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\,\mathcal{E}(k,l,t)\mathrm{e}^{-il\xi}Q_2,$$

and then the function $\hat{\varphi}(l,t,k_1,k_2)$ is found by

$$\hat{\varphi}(l,t,k_1,k_2)=\begin{cases}\int_0^t \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}Q_2+\mathcal{E}(k,l,-t)\hat{\varphi}(l,0,k_1,k_2),\\ -\int_t^T \mathrm{d}\tau\,\mathcal{E}(k,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}Q_2+\mathcal{E}(k,l,T-t)\hat{\varphi}(l,T,k_1,k_2).\end{cases}$$

Note that by the integration by parts, we write

$$\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}Q_2=\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}\left[U_{21}\psi+U_{22}\varphi\right],$$

where U_{21} and U_{22} are given in equations (3.18c) and (3.18d). Using the following identity

$$-\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi-\bar{z}}\,\mathrm{e}^{2ik_1\xi}\varphi=\frac{1}{2\pi}\int_{-\infty}^{-2k_1}\mathrm{d}l\,\mathrm{e}^{i(l+2k_1)\bar{z}}\hat{\varphi},\quad \operatorname{Im} z\geq 0,$$

equation (3.21) can be written as

$$\Phi(x,y,t,k_1,k_2)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{0}^{\infty}\frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}}\mathrm{e}^{ik(\zeta-z)+i\bar{k}(\bar{\zeta}-\bar{z})}q\Psi+\frac{\mathrm{e}^{2ky}}{2\pi}\int_{-\infty}^{-2k_1}\mathrm{d}l\,\mathrm{e}^{il\bar{z}}\hat{\varphi},\quad (3.30)$$

and hence we will determine $\hat{\varphi}$ for $-\infty<l<-2k_1$.

Evaluating equation (3.30) at $y=0$ and using the following identity

$$\int_{-2k_1}^{\infty}\mathrm{d}l\,\mathrm{e}^{il(x-\bar{\zeta})}=\frac{\mathrm{e}^{-2ik_1(x-\bar{\zeta})}}{i(\bar{\zeta}-x)},\quad \zeta=\xi+i\eta,\quad \eta>0,$$

we find

$$\varphi(x,t,k_1,k_2)=\frac{1}{i\pi}\int_{-2k_1}^{\infty}\mathrm{d}l\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{0}^{\infty}\mathrm{d}\eta\,\mathrm{e}^{-2k\eta+il(x-\bar{\zeta})}q\Psi+\frac{1}{2\pi}\int_{-\infty}^{-2k_1}\mathrm{d}l\,\mathrm{e}^{ilx}\hat{\varphi}.\quad (3.31)$$

Comparing equation (3.31) with equation (3.29), the function $\hat{\varphi}$ satisfies the restriction

$$\hat{\varphi}(l,t,k_1,k_2)=-2i\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{0}^{\infty}\mathrm{d}\eta\,\mathrm{e}^{-2k\eta-il\bar{\zeta}}q\Psi,\quad -2k_1<l<\infty,$$

while there is no restriction imposed on $\hat{\varphi}$ for $-\infty<l<-2k_1$. Thus, taking into account of the real part of $\mathcal{E}(k,l,\tau-t)$, we define $\hat{\varphi}_j^{\pm}$ ($j=1,2$) as

$$\hat{\varphi}_1^{\pm}(l,t,k_1,k_2)=\left[\begin{matrix}\int_0^t\mathrm{d}\tau\\ -\int_t^T\mathrm{d}\tau\end{matrix}\right.\mathcal{E}(k,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}\left[U_{21}\psi_1^{\pm}+U_{22}\varphi_1^{\pm}\right],\ l\in(-\infty,-2k_1],\quad (3.32\text{a})$$

and regarding $\hat{\varphi}_2^{\pm}$, we consider two cases:

$$\hat{\varphi}_2^{\pm}(l,t,k_1,k_2)=\left[\begin{matrix}\int_0^t\mathrm{d}\tau\\ -\int_t^T\mathrm{d}\tau\end{matrix}\right.\mathcal{E}(k,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il\xi}\left[U_{21}\psi_2^{\pm}+U_{22}\varphi_2^{\pm}\right],\quad l\in(-\infty,0],\quad (3.32\text{b})$$

and

$$\hat{\varphi}_2^{\pm}(l,t,k_1,k_2) = \begin{bmatrix} -\int_t^T \mathrm{d}\tau \\ \int_0^t \mathrm{d}\tau \end{bmatrix} \mathcal{E}(k,l,\tau-t) \int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-il\xi} \left[U_{21}\psi_2^{\pm} + U_{22}\varphi_2^{\pm}\right], \quad l \in [0,-2k_1]. \tag{3.32c}$$

Substituting equations (3.32) into equation (3.30), we find equations (3.17b) and (3.17d). □

Remark 3.6. 1. Letting $l \to -\lambda - 2k_1$ (and then replacing λ by l), we find

$$U_{11} \to \bar{U}_{22}, \quad U_{12} \to -\bar{U}_{21}, \quad U_{21} \to -\bar{U}_{12}, \quad U_{22} \to \bar{U}_{11},$$
$$\mathrm{e}^{il(z-\xi)} \to \mathrm{e}^{-il(z-\xi)-2ik_1(z-\xi)}, \quad \mathcal{E} \to \bar{\mathcal{E}}\mathrm{e}^{2ik_1(k_1^2-3k_2^2)(\tau-t)},$$

where $\mathcal{E} = \mathcal{E}(k,l,\tau-t)$. Then equations (3.17) are equivalent to the following equations

$$\begin{aligned}\Psi_1^{\pm}(x,y,t,k_1,k_2) =& 1 + \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\frac{\mathrm{d}\eta}{\zeta-z}q\Phi_1^{\pm} + \frac{e\,\mathrm{e}^{2\bar{k}y}}{2\pi}\int_{-\infty}^{-2k_1}\mathrm{d}l\begin{bmatrix}\int_0^t \mathrm{d}\tau \\ -\int_t^T \mathrm{d}\tau\end{bmatrix} \\ &\times \bar{\mathcal{E}}\mathrm{e}^{2ik_1(k_1^2-3k_2^2)\tau}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il(z-\xi)+2ik_1\xi}\left[\bar{U}_{22}\psi_1^{\pm} - \bar{U}_{21}\varphi_1^{\pm}\right],\end{aligned} \tag{3.33a}$$

$$\begin{aligned}\Phi_1^{\pm}(x,y,t,k_1,k_2) =& -\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}}\mathrm{e}^{ik(\zeta-z)+i\bar{k}(\bar{\zeta}-\bar{z})}q\Psi_1^{\pm} \\ &+ \frac{e}{2\pi}\int_0^{\infty}\mathrm{d}l\begin{bmatrix}\int_0^t \mathrm{d}\tau \\ -\int_t^T \mathrm{d}\tau\end{bmatrix} \\ &\times \bar{\mathcal{E}}\mathrm{e}^{2ik_1(k_1^2-3k_2^2)\tau}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il(\bar{z}-\xi)+2ik_1\xi}\left[-\bar{U}_{12}\psi_1^{\pm} + \bar{U}_{11}\varphi_1^{\pm}\right],\end{aligned} \tag{3.33b}$$

$$\begin{aligned}\Psi_2^{\pm}(x,y,t,k_1,k_2) =& 1 + \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\frac{\mathrm{d}\eta}{\zeta-z}q\Phi_2^{\pm} \\ &+ \frac{e\,\mathrm{e}^{2\bar{k}y}}{2\pi}\begin{bmatrix}-\int_0^{-2k_1}\mathrm{d}l\int_t^T \mathrm{d}\tau + \int_{-\infty}^0 \mathrm{d}l\int_0^t \mathrm{d}\tau \\ \int_0^{-2k_1}\mathrm{d}l\int_0^t \mathrm{d}\tau - \int_{-\infty}^0 \mathrm{d}l\int_t^T \mathrm{d}\tau\end{bmatrix} \\ &\times \bar{\mathcal{E}}\mathrm{e}^{2ik_1(k_1^2-3k_2^2)\tau}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il(z-\xi)+2ik_1\xi}\left[\bar{U}_{22}\psi_2^{\pm} - \bar{U}_{21}\varphi_2^{\pm}\right]\end{aligned} \tag{3.33c}$$

and

$$\Phi_2^{\pm}(x,y,t,k_1,k_2) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}} \mathrm{e}^{ik(\zeta-z)+i\bar{k}(\bar{\zeta}-\bar{z})} q\Psi_2^{\pm}$$
$$+\frac{e}{2\pi}\begin{bmatrix} \int_{-2k_1}^{\infty}\mathrm{d}l\int_0^t \mathrm{d}\tau - \int_0^{-2k_1}\mathrm{d}l\int_t^T \mathrm{d}\tau \\ -\int_{-2k_1}^{\infty}\mathrm{d}l\int_t^T \mathrm{d}\tau + \int_0^{-2k_1}\mathrm{d}l\int_0^t \mathrm{d}\tau \end{bmatrix} \tag{3.33d}$$
$$\times \bar{\mathcal{E}}\mathrm{e}^{2ik_1(k_1^2-3k_2^2)\tau}\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-il(\bar{z}-\xi)+2ik_1\xi}\left[-\bar{U}_{12}\psi_2^{\pm}+\bar{U}_{11}\varphi_2^{\pm}\right],$$

where e is given in equation (3.5), that is, $e = \mathrm{e}^{-2i(k_1x-k_2y)-2ik_1(k_1^2-3k_2^2)t}$.

2. Since $\hat{\psi}_2$ is continuous for $l \in \mathbb{C}$, evaluating equations (3.28b) and (3.28c) at $l = -2k_1$, we find

$$\int_0^t \mathrm{d}\tau\, \mathcal{E}(k,-2k_1,\tau-t)\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-2ik_1(z-\xi)}\left[U_{11}\psi_2^{\pm}+U_{12}\varphi_2^{\pm}\right]|_{l=-2k_1}$$
$$= -\int_t^T \mathrm{d}\tau\, \mathcal{E}(k,-2k_1,\tau-t)\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-2ik_1(z-\xi)}\left[U_{11}\psi_2^{\pm}+U_{12}\varphi_2^{\pm}\right]|_{l=-2k_1}. \tag{3.34}$$

3.3. THE INVERSE PROBLEM

In this section, we describe the d-bar derivatives of the functions $\Psi_j^{\pm}$ and $\Phi_j^{\pm}$ $(j=1,2)$ and the jumps across the real and imaginary k-axes.

Proposition 3.7. *For $\Psi_j^{\pm}$ and $\Phi_j^{\pm}$, $j=1,2$, given in equations* (3.17)*,*

$$\frac{\partial \Psi_j^{\pm}}{\partial \bar{k}} = -\mathrm{e}^{-2i(k_1x-k_2y)-2ik_1(k_1^2-3k_2^2)t}\gamma_j^{\pm}\bar{\Phi}_j^{\pm}, \quad j=1,2, \tag{3.35a}$$
$$\frac{\partial \Phi_j^{\pm}}{\partial \bar{k}} = \mathrm{e}^{-2i(k_1x-k_2y)-2ik_1(k_1^2-3k_2^2)t}\gamma_j^{\pm}\bar{\Psi}_j^{\pm}, \quad j=1,2, \tag{3.35b}$$

where

$$\gamma_1^{+} = \beta_1^{+}, \quad \gamma_1^{-} = \beta_1^{-}+\alpha_1^{-}, \quad \gamma_2^{+} = \beta_2^{+}+\alpha_2^{+}, \quad \gamma_2^{-} = \beta_2^{-} \tag{3.36}$$

with

$$\alpha_j^{\pm} = \frac{1}{2\pi}\int_0^T \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{2ik_1\xi+2ik_1(k_1^2-3k_2^2)\tau}\left[U_{21}\psi_j^{\pm}+U_{22}\varphi_j^{\pm}\right]|_{l=-2k_1}, \quad j=1,2, \tag{3.37}$$

$$\beta_j^{\pm} = \frac{1}{i\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \mathrm{d}\eta\, \mathrm{e}^{i(k\zeta+\bar{k}\bar{\zeta})} q_0 \Psi_j^{\pm}(\xi,\eta,0,k_1,k_2), \quad j=1,2. \tag{3.38}$$

Proof. Differentiating equations (3.17a) and (3.17b) with respect to $\bar{k}$, we find

$$\begin{aligned}\frac{\partial \Psi_1^{\pm}}{\partial \bar{k}} = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\zeta - z}\, q\, \frac{\partial \Phi_1^{\pm}}{\partial \bar{k}} + \frac{1}{2\pi}\int_0^{\infty} \mathrm{d}l \begin{bmatrix} \int_0^t \mathrm{d}\tau \\ -\int_t^T \mathrm{d}\tau \end{bmatrix} \\ \times \mathcal{E}(k,l,\tau - t)\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(z-\xi)} \left[U_{11}\frac{\partial \psi_1^{\pm}}{\partial \bar{k}} + U_{12}\frac{\partial \varphi_1^{\pm}}{\partial \bar{k}}\right],\end{aligned} \tag{3.39a}$$

$$\begin{aligned}\frac{\partial \Phi_1^{\pm}}{\partial \bar{k}} = &-\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\bar{\zeta} - \bar{z}} \mathrm{e}^{ik(\zeta - z) + i\bar{k}(\bar{\zeta} - \bar{z})} q\, \frac{\partial \Psi_1^{\pm}}{\partial \bar{k}} \\ &+ \frac{\mathrm{e}^{2ky}}{2\pi}\int_{-\infty}^{-2k_1} \mathrm{d}l \begin{bmatrix} \int_0^t \mathrm{d}\tau \\ -\int_t^T \mathrm{d}\tau \end{bmatrix} \\ &\times \mathcal{E}(k,l,\tau - t)\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)} \left[U_{21}\frac{\partial \psi_1^{\pm}}{\partial \bar{k}} + U_{22}\frac{\partial \varphi_1^{\pm}}{\partial \bar{k}}\right] + \tilde{\gamma}_1^{\pm},\end{aligned} \tag{3.39b}$$

where the forcing terms $\tilde{\gamma}_1^{\pm}$ are given by

$$\begin{aligned}\tilde{\gamma}_1^{\pm}(x,y,t,k) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \mathrm{d}\eta\, \mathrm{e}^{ik(\zeta - z) + \bar{k}(\bar{\zeta} - \bar{z})} q \Psi_1^{\pm} - \frac{\mathrm{e}^{2ky}}{2\pi} \begin{bmatrix} \int_0^t \mathrm{d}\tau \\ \int_0^t \mathrm{d}\tau - \int_0^T \mathrm{d}\tau \end{bmatrix} \\ \times \mathcal{E}(k,-2k_1,\tau - t)\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-2ik_1(\bar{z}-\xi)} \left[U_{21}\psi_1^{\pm} + U_{22}\varphi_1^{\pm}\right]|_{l=-2k_1}.\end{aligned}$$

Noting that $\mathrm{e}^{2ky}\mathrm{e}^{-2ik_1\bar{z}} = \mathrm{e}^{-i(kz+\bar{k}\bar{z})}$ and $\mathcal{E}(k,-2k_1,\tau - t) = \mathrm{e}^{2ik_1(k_1^2 - 3k_2^2)(\tau - t)}$, we denote

$$\begin{aligned}\tilde{\beta}_1^{\pm} = \frac{1}{i\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \mathrm{d}\eta\, \mathrm{e}^{i(k\zeta + \bar{k}\bar{\zeta})} q \Psi_1^{\pm} - \frac{1}{2\pi}\mathrm{e}^{-2ik_1(k_1^2 - 3k_2^2)t}\int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi \\ \times \mathrm{e}^{2ik_1\xi + 2ik_1(k_1^2 - 3k_2^2)\tau}\left[U_{21}\psi_1^{\pm} + U_{22}\varphi_1^{\pm}\right]|_{l=-2k_1}.\end{aligned}$$

Using the global relation (3.8), $\tilde{\beta}_1^{\pm}$ can be written as

$$\tilde{\beta}_1^{\pm} = \mathrm{e}^{-2ik_1(k_1^2 - 3k_2^2)t}\beta_1^{\pm},$$

where $\beta_1^{\pm}$ is defined in equation (3.38). Thus, $\tilde{\gamma}_1^{\pm}$ can be written as

$$\tilde{\gamma}_1^{\pm} = \mathrm{e}^{-i(kz+\bar{k}\bar{z}) - 2ik_1(k_1^2 - 3k_2^2)t}\gamma_1^{\pm},$$

where $\gamma_1^{\pm}$ are given in equation (3.36).

Thus, we find

$$\begin{aligned}\frac{\partial \Phi_1^{\pm}}{\partial \bar{k}} &= e\gamma_1^{\pm} - \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}} \mathrm{e}^{ik(\zeta-z)+i\bar{k}(\bar{\zeta}-\bar{z})} q \frac{\partial \Psi_1^{\pm}}{\partial \bar{k}} \\ &+ \frac{\mathrm{e}^{2ky}}{2\pi}\int_{-\infty}^{-2k_1} \mathrm{d}l \begin{bmatrix} \int_0^t \mathrm{d}\tau \\ -\int_t^T \mathrm{d}\tau \end{bmatrix} \\ &\times \mathcal{E}(k,l,\tau-t)\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)} \left[U_{21}\frac{\partial \psi_1^{\pm}}{\partial \bar{k}} + U_{22}\frac{\partial \varphi_1^{\pm}}{\partial \bar{k}}\right],\end{aligned} \tag{3.40}$$

where e is given in equation (3.5).

We will compare equations (3.39a) and (3.40) with equations (3.17a) and (3.17b). In this respect, it is convenient to use equations (3.33a) and (3.33b) instead of equations (3.17a) and (3.17b). Employing the complex conjugate of equation (3.33a) and then multiplying the resulting equation by e, we find

$$\begin{aligned}&e\bar{\Psi}_1^{\pm}(x,y,t,k_1,k_2) \\ &= e + \frac{e}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}} q\bar{\Phi}_1^{\pm} + \frac{\mathrm{e}^{2ky}}{2\pi}\int_{-\infty}^{-2k_1} \mathrm{d}l \begin{bmatrix} \int_0^t \mathrm{d}\tau \\ -\int_t^T \mathrm{d}\tau \end{bmatrix} \\ &\times \mathcal{E}(k,l,\tau-t)\mathrm{e}^{-2ik_1(k_1^2-3k_2^2)\tau}\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)-2ik_1\xi} \left[U_{22}\left(\bar{\psi}_1^{\pm}\right) + U_{21}\left(-\bar{\varphi}_1^{\pm}\right)\right].\end{aligned} \tag{3.41}$$

Using $\mathrm{e}^{ik(\zeta-z)+i\bar{k}(\bar{\zeta}-\bar{z})} = \mathrm{e}^{-2i(k_1\xi-k_2\eta)-2i(k_1x-k_2y)}$, we write

$$\begin{aligned}e\bar{\Psi}_1^{\pm}(x,y,t,k_1,k_2) &= e - \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}} \mathrm{e}^{ik(\zeta-z)+i\bar{k}(\bar{\zeta}-\bar{z})} q\left(-e\bar{\Phi}_1^{\pm}\right) \\ &+ \frac{\mathrm{e}^{2ky}}{2\pi}\int_{-\infty}^{-2k_1} \mathrm{d}l \begin{bmatrix} \int_0^t \mathrm{d}\tau \\ -\int_t^T \mathrm{d}\tau \end{bmatrix} \\ &\times \mathcal{E}(k,l,\tau-t)\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)} \left[U_{22}\left(\tilde{e}\bar{\psi}_1^{\pm}\right) + U_{21}\left(-\tilde{e}\bar{\varphi}_1^{\pm}\right)\right],\end{aligned} \tag{3.42}$$

where $\tilde{e} = e|_{\eta=0}$.

From equation (3.33b) it follows that

$$
\begin{aligned}
&e\bar{\Phi}_1^{\pm}(x,y,t,k_1,k_2)\\
&= -\frac{e}{\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\int\limits_{0}^{\infty}\frac{\mathrm{d}\eta}{\zeta-z}\mathrm{e}^{-ik(\zeta-z)-i\bar{k}(\bar{\zeta}-\bar{z})}q\bar{\Psi}_1^{\pm}+\frac{1}{2\pi}\int\limits_{0}^{\infty}\mathrm{d}l\begin{bmatrix}\int_0^t \mathrm{d}\tau\\ -\int_t^T \mathrm{d}\tau\end{bmatrix}\\
&\qquad\times\mathcal{E}(k,l,\tau-t)\mathrm{e}^{-2ik_1(k_1^2-3k_2^2)\tau}\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{il(z-\xi)-2ik_1\xi}\left[-U_{12}\bar{\psi}_1^{\pm}+U_{11}\bar{\varphi}_1^{\pm}\right].
\end{aligned}
\tag{3.43}
$$

Similarly, we write equation (3.43) as

$$
\begin{aligned}
-e\bar{\Phi}_1^{\pm}(x,y,t,k_1,k_2) &= \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\int\limits_{0}^{\infty}\frac{\mathrm{d}\eta}{\zeta-z}q\left(e\bar{\Psi}_1^{\pm}\right)+\frac{1}{2\pi}\int\limits_{0}^{\infty}\mathrm{d}l\begin{bmatrix}\int_0^t \mathrm{d}\tau\\ -\int_t^T \mathrm{d}\tau\end{bmatrix}\\
&\qquad\times\mathcal{E}(k,l,\tau-t)\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{il(z-\xi)}\left[U_{12}\left(\tilde{e}\bar{\psi}_1^{\pm}\right)+U_{11}\left(-\tilde{e}\bar{\varphi}_1^{\pm}\right)\right].
\end{aligned}
\tag{3.44}
$$

Multiplying equations (3.42) and (3.44) by $\gamma_1^{\pm}$, we know that the resulting equations are equivalent, respectively, with equations (3.39a) and (3.40). Therefore, equations (3.35a) and (3.35b) follow for $j=1$.

Regarding equations (3.35a) and (3.35b) for $j=2$, note that

$$
\begin{aligned}
\frac{\partial\Psi_2^{\pm}}{\partial\bar{k}} &= \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\int\limits_{0}^{\infty}\frac{\mathrm{d}\eta}{\zeta-z}\,q\,\frac{\partial\Phi_2^{\pm}}{\partial\bar{k}}+\frac{1}{2\pi}\begin{bmatrix}-\int_0^{-2k_1}\mathrm{d}l\int_t^T\mathrm{d}\tau+\int_{-2k_1}^{\infty}\mathrm{d}l\int_0^t\mathrm{d}\tau\\ \int_0^{-2k_1}\mathrm{d}l\int_0^t\mathrm{d}\tau-\int_{-2k_1}^{\infty}\mathrm{d}l\int_t^T\mathrm{d}\tau\end{bmatrix}\\
&\qquad\times\mathcal{E}(k,l,\tau-t)\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{il(z-\xi)}\left[U_{11}\frac{\partial\psi_2^{\pm}}{\partial\bar{k}}+U_{12}\frac{\partial\varphi_2^{\pm}}{\partial\bar{k}}\right],
\end{aligned}
\tag{3.45}
$$

where we have used equation (3.34) for the forcing term of $\frac{\partial\Psi_2^{\pm}}{\partial\bar{k}}$. Also, using $\int_t^T\mathrm{d}\tau=\int_0^T\mathrm{d}\tau-\int_0^t\mathrm{d}\tau$ for the forcing term of $\frac{\partial\Phi_2^{\pm}}{\partial\bar{k}}$, we find

$$
\begin{aligned}
\frac{\partial\Phi_2^{\pm}}{\partial\bar{k}} &= -\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\int\limits_{0}^{\infty}\frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}}\mathrm{e}^{ik(\zeta-z)+i\bar{k}(\bar{\zeta}-\bar{z})}q\,\frac{\partial\Psi_2^{\pm}}{\partial\bar{k}}\\
&\quad+\frac{\mathrm{e}^{2ky}}{2\pi}\begin{bmatrix}\int_{-\infty}^{0}\mathrm{d}l\int_0^t\mathrm{d}\tau-\int_0^{-2k_1}\mathrm{d}l\int_t^T\mathrm{d}\tau\\ -\int_{-\infty}^{0}\mathrm{d}l\int_t^T\mathrm{d}\tau+\int_0^{-2k_1}\mathrm{d}l\int_0^t\mathrm{d}\tau\end{bmatrix}\mathcal{E}(k,l,\tau-t)\\
&\quad\times\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{il(\bar{z}-\xi)}\left[U_{21}\frac{\partial\psi_2^{\pm}}{\partial\bar{k}}+U_{22}\frac{\partial\varphi_2^{\pm}}{\partial\bar{k}}\right]+\tilde{\gamma}_2^{\pm},
\end{aligned}
\tag{3.46}
$$

where the forcing term $\tilde{\gamma}_2^{\pm}$ are given by

$$\tilde{\gamma}_2^{\pm} = \frac{1}{i\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \mathrm{d}\eta\, \mathrm{e}^{ik(\zeta-z)+i\bar{k}(\bar{\zeta}-\bar{z})} q\Psi_2^{\pm} - \frac{\mathrm{e}^{2ky}}{2\pi}\begin{bmatrix} \int_0^t \mathrm{d}\tau - \int_0^T \mathrm{d}\tau \\ \int_0^t \mathrm{d}\tau \end{bmatrix}$$
$$\times\, \mathcal{E}(k,-2k_1,\tau-t)\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{-2ik_1(\bar{z}-\xi)}\left[U_{21}\psi_2^{\pm} + U_{22}\varphi_2^{\pm}\right]|_{l=-2k_1}.$$

Let

$$\tilde{\beta}_2^{\pm} = \frac{1}{i\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \mathrm{d}\eta\, \mathrm{e}^{i(k\zeta+\bar{k}\bar{\zeta})} q\Psi_2^{\pm} - \frac{1}{2\pi}\mathrm{e}^{-2ik_1(k_1^2-3k_2^2)t}\int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi$$
$$\times\, \mathrm{e}^{2ik_1\xi+2ik_1(k_1^2-3k_2^2)\tau}\left[U_{21}\psi_2^{\pm} + U_{22}\varphi_2^{\pm}\right]|_{l=-2k_1}.$$

By the global relation (3.8), $\tilde{\beta}_2^{\pm}$ can be written as

$$\tilde{\beta}_2^{\pm} = \mathrm{e}^{-2ik_1(k_1^2-3k_2^2)t}\beta_2^{\pm},$$

where β_2 is given in equation (3.38). Thus, we find

$$\tilde{\gamma}_2^{\pm} = \mathrm{e}^{-i(kz+\bar{k}\bar{z})-2ik_1(k_1^2-3k_2^2)t}\gamma_2^{\pm},$$

where $\gamma_2^{\pm}$ are given in equation (3.36). Thus, equation (3.46) becomes

$$\begin{aligned}\frac{\partial \Phi_2^{\pm}}{\partial \bar{k}} &= e\gamma_2^{\pm} - \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}} \mathrm{e}^{ik(\zeta-z)+i\bar{k}(\bar{\zeta}-\bar{z})} q\,\frac{\partial \Psi_2^{\pm}}{\partial \bar{k}} \\ &+ \frac{\mathrm{e}^{2ky}}{2\pi}\begin{bmatrix} \int_{-\infty}^0 \mathrm{d}l \int_0^t \mathrm{d}\tau - \int_0^{-2k_1} \mathrm{d}l \int_t^T \mathrm{d}\tau \\ -\int_{-\infty}^0 \mathrm{d}l \int_t^T \mathrm{d}\tau + \int_0^{-2k_1} \mathrm{d}l \int_0^t \mathrm{d}\tau \end{bmatrix} \mathcal{E}(k,l,\tau-t) \\ &\times \int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)}\left[U_{21}\frac{\partial \psi_2^{\pm}}{\partial \bar{k}} + U_{22}\frac{\partial \varphi_2^{\pm}}{\partial \bar{k}}\right].\end{aligned} \tag{3.47}$$

In order to relate equations (3.45) and (3.47) with equations (3.17c) and (3.17d) (or equivalently equations (3.33c) and (3.33d)), we take the complex conjugate of equations (3.33c) and (3.33d) and multiply the resulting equations by e defined in equation (3.5). Then, we find

$$\begin{aligned}e\bar{\Psi}_2^{\pm}(x,y,t,k_1,k_2) &= e - \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}} \mathrm{e}^{ik(\zeta-z)+i\bar{k}(\bar{\zeta}-\bar{z})} q\left(-e\bar{\Phi}_2^{\pm}\right) \\ &+ \frac{\mathrm{e}^{2ky}}{2\pi}\begin{bmatrix} -\int_0^{-2k_1} \mathrm{d}l \int_t^T \mathrm{d}\tau + \int_{-\infty}^0 \mathrm{d}l \int_0^t \mathrm{d}\tau \\ \int_0^{-2k_1} \mathrm{d}l \int_0^t \mathrm{d}\tau - \int_{-\infty}^0 \mathrm{d}l \int_t^T \mathrm{d}\tau \end{bmatrix} \\ &\times \mathcal{E}(k,l,\tau-t)\int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)}\left[U_{22}\left(\tilde{e}\bar{\psi}_2^{\pm}\right) + U_{21}\left(-\tilde{e}\bar{\varphi}_2^{\pm}\right)\right],\end{aligned} \tag{3.48}$$

and

$$
\begin{aligned}
-e\bar{\Phi}_2^{\pm}(x,y,t,k_1,k_2) = {} & \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\zeta - z} q\left(e\bar{\Psi}_2^{\pm}\right) \\
& + \frac{1}{2\pi}\begin{bmatrix} \int_{-2k_1}^{\infty} \mathrm{d}l \int_0^t \mathrm{d}\tau - \int_0^{-2k_1} \mathrm{d}l \int_t^T \mathrm{d}\tau \\ - \int_{-2k_1}^{\infty} \mathrm{d}l \int_t^T \mathrm{d}\tau + \int_0^{-2k_1} \mathrm{d}l \int_0^t \mathrm{d}\tau \end{bmatrix} \\
& \times \mathcal{E}(k,l,\tau - t) \int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(z-\xi)} \left[U_{12}\left(\tilde{e}\bar{\psi}_2^{\pm}\right) + U_{11}\left(-\tilde{e}\bar{\varphi}_2^{\pm}\right)\right].
\end{aligned}
\tag{3.49}
$$

Multiplying equations (3.48) and (3.49) by $\gamma_2^{\pm}$, the resulting equations are equivalent with eqs (3.45) and (3.47). Therefore, we prove eqs (3.35a) and (3.35b) for $j = 2$. □

We now compute the jumps of $\Psi_{1,2}^{\pm}$ and $\Phi_{1,2}^{\pm}$ across the real and imaginary k-axes. Evaluating functions $\Psi_{1,2}^{\pm}$ at $k_1 = 0$, we know that $\Psi_1^{\pm}$ and $\Psi_2^{\pm}$ are equivalent and $\Phi_1^{\pm}$ and $\Phi_2^{\pm}$ are also equivalent. Thus, we find

$$
\left(\Psi_1^{\pm} - \Psi_2^{\pm}\right)|_{k_1=0} = 0, \quad \left(\Phi_1^{\pm} - \Phi_2^{\pm}\right)|_{k_1=0} = 0
$$

and hence Ψ and Φ are continuous across the imaginary k-axis. However, we will show that Ψ and Φ are discontinuous across the real k-axis. In particular, we address the case for the jumps across the positive real k-axis.

Proposition 3.8. *For Ψ_1 and Φ_1 defined by equations* (3.17)*,*

$$
\left(\Psi_1^{+} - \Psi_1^{-}\right)(x,y,t,k_1,0) = \left(\Delta\Psi_1\right)(x,y,t,k_1) + \left(\delta\Psi_1\right)(x,y,t,k_1), \tag{3.50a}
$$

$$
\left(\Phi_1^{+} - \Phi_1^{-}\right)(x,y,t,k_1,0) = \left(\Delta\Phi_1\right)(x,y,t,k_1) + \left(\delta\Phi_1\right)(x,y,t,k_1), \tag{3.50b}
$$

where

$$
\Delta\Psi_1 = -\int_0^{\infty} \mathrm{d}\lambda\, e_\lambda \chi_1^{(1)}(k_1,\lambda) \bar{\Phi}_1^{+}\left(x,y,t,k_1 + \frac{\lambda}{2}, \frac{i\lambda}{2}\right), \tag{3.51a}
$$

$$
\Delta\Phi_1 = \int_0^{\infty} \mathrm{d}\lambda\, e_\lambda \chi_1^{(1)}(k_1,\lambda) \bar{\Psi}_1^{+}\left(x,y,t,k_1 + \frac{\lambda}{2}, \frac{i\lambda}{2}\right), \tag{3.51b}
$$

$$
\delta\Psi_1 = \hat{e}\int_0^{\infty} \mathrm{d}\lambda\, E_\lambda \chi_1^{(2)}(k_1,\lambda) \Psi_1^{+}\left(x,y,t,k_1 + \frac{\lambda}{2}, -\frac{i\lambda}{2}\right), \tag{3.51c}
$$

$$
\delta\Phi_1 = \hat{e}\int_0^{\infty} \mathrm{d}\lambda\, E_\lambda \chi_1^{(2)}(k_1,\lambda) \Phi_1^{+}\left(x,y,t,k_1 + \frac{\lambda}{2}, -\frac{i\lambda}{2}\right), \tag{3.51d}
$$

and

$$e_\lambda = \mathrm{e}^{-i(2k_1+\lambda)x-\lambda y-i(\lambda^3+3k_1\lambda^2+3k_1^2\lambda+2k_1^3)t}, \tag{3.52}$$

$$\hat{e} = e|_{k_2=0} = \mathrm{e}^{-2ik_1x-2ik_1^3t}, \quad E_\lambda = \mathrm{e}^{i(2k_1+\lambda)x-\lambda y+i(\lambda^3+3\lambda^2k_1+3\lambda k_1^2+2k_1^3)t}, \tag{3.53}$$

$$A_1^{(j)}(k_1,\lambda) = \chi_1^{(j)}(k_1,\lambda) + \int_0^\infty \mathrm{d}l\,\chi_1^{(j)}(k_1,\lambda+l)f_1^{(j)}(k_1,\lambda+l,l), \quad j=1,2, \tag{3.54a}$$

$$A_1^{(1)}(k_1,\lambda) = \frac{1}{2\pi}\int_0^T \mathrm{d}\tau \int_{-\infty}^\infty \mathrm{d}x\,\mathrm{e}^{i(2k_1+\lambda)x+i(\lambda^3+3k_1\lambda^2+3k_1\lambda+2k_1^3)\tau} \tag{3.54b}$$
$$\times\left[-\bar{U}_{12}\psi_1^- + \bar{U}_{11}\varphi_1^-\right](x,t,k_1,\lambda),$$

$$A_1^{(2)}(k_1,\lambda) = \frac{1}{2\pi}\int_0^T \mathrm{d}\tau \int_{-\infty}^\infty \mathrm{d}x\,\mathrm{e}^{-i\lambda x-i\lambda(\lambda^2+3\lambda k_1+3k_1^2)\tau}\left[U_{11}\psi_1^- + U_{12}\varphi_1^-\right](x,t,k_1,\lambda), \tag{3.54c}$$

with

$$f_1^{(1)}(k_1,\lambda,l) = -\frac{1}{i\pi}\int_{-\infty}^\infty \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta\,\mathrm{e}^{-il\xi-l\eta}q_0\bar{\Phi}_1^+\left(\xi,\eta,0,k_1+\frac{\lambda}{2},\frac{i\lambda}{2}\right), \tag{3.55}$$

$$f_1^{(2)}(k_1,\lambda,l) = \frac{1}{i\pi}\int_{-\infty}^\infty \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta\,\mathrm{e}^{il\xi-l\eta}q_0\Phi_1^+\left(\xi,\eta,0,k_1+\frac{\lambda}{2},-\frac{i\lambda}{2}\right). \tag{3.56}$$

Proof. Evaluating equations (3.17a) and (3.17b) at $k_2=0$ and subtracting the resulting equations, we find the following forcing terms:

1. the forcing obtained from $\left(\Phi_1^+ - \Phi_1^-\right)|_{k_2=0}$

$$\frac{\mathrm{e}^{2k_1y}}{2\pi}\int_{-\infty}^{-2k_1}\mathrm{d}l\int_0^T \mathrm{d}\tau\,\mathcal{E}(k_1,l,\tau-t)\int_{-\infty}^\infty \mathrm{d}\xi\,\mathrm{e}^{il(\bar{z}-\xi)}\left[U_{21}\psi_1^- + U_{22}\varphi_1^-\right]|_{k_2=0} \tag{3.57}$$

2. the forcing obtained from $\left(\Psi_1^+ - \Psi_1^-\right)|_{k_2=0}$

$$\frac{1}{2\pi}\int_0^\infty \mathrm{d}l\int_0^T \mathrm{d}\tau\,\mathcal{E}(k_1,l,\tau-t)\int_{-\infty}^\infty \mathrm{d}\xi\,\mathrm{e}^{il(z-\xi)}\left[U_{11}\psi_1^- + U_{12}\varphi_1^-\right]|_{k_2=0}. \tag{3.58}$$

Let $(\Delta\Psi_1,\Delta\Phi_1)$ denote the solution of the equation by subtracting equations (3.17a) and (3.17b) evaluated at $k_2=0$ and by ignoring the forcing term (3.58). Similarly, let $(\delta\Psi_1,\delta\Phi_1)$ denote the corresponding solution obtained by ignoring the forcing term (3.57). Then, we can derive equations (3.50a) and (3.50b). Thus, in what follows we will characterize $(\Delta\Psi_1, \Delta\Phi_1)$ and $(\delta\Psi_1, \delta\Phi_1)$.

Analysis of $(\Delta\Psi_1, \Delta\Phi_1)$. Letting $l = -2k_1 - \lambda$ and using the equation

$$\begin{aligned}&\mathcal{E}(k_1, l, \tau - t)\mathrm{e}^{il(\bar{z}-\xi)}\big|_{l=-2k_1-\lambda}\\ &= \mathrm{e}^{-i(2k_1+\lambda)x-(2k_1+\lambda)y+i(2k_1+\lambda)\xi+i(\lambda^3+3\lambda^2k_1+3k_1^2\lambda+2k_1^3)(\tau-t)},\end{aligned}$$

equation (3.57) can be written in the form

$$\int_0^\infty \mathrm{d}\lambda\, e_\lambda A_1^{(1)}(k_1, \lambda),$$

where e_λ and $A_1^{(1)}$ are defined in equations (3.52) and (3.54b), respectively. Thus, by ignoring the forcing term (3.58), $(\Delta\Psi_1, \Delta\Phi_1)$ solves the following equations:

$$\begin{aligned}\Delta\Phi_1 =& \int_0^\infty \mathrm{d}\lambda\, e_\lambda A_1^{(1)} - \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\xi \int_0^\infty \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}}\, \mathrm{e}^{-2ik_1(x-\xi)} q\Delta\Psi_1\\ &+ \frac{\mathrm{e}^{2k_1 y}}{2\pi}\int_{-\infty}^{-2k_1} \mathrm{d}l \int_0^t \mathrm{d}\tau\, \mathcal{E}(k_1, l, \tau - t)\int_{-\infty}^\infty \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)}\left[U_{21}\Delta\psi_1 + U_{22}\Delta\varphi_1\right]|_{k_2=0},\end{aligned} \tag{3.59a}$$

and

$$\begin{aligned}\Delta\Psi_1 =& \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\xi \int_0^\infty \frac{\mathrm{d}\eta}{\zeta - z}\, q\Delta\Phi_1\\ &+ \frac{1}{2\pi}\int_0^\infty \mathrm{d}l \int_0^t \mathrm{d}\tau\, \mathcal{E}(k_1, l, \tau - t)\int_{-\infty}^\infty \mathrm{d}\xi\, \mathrm{e}^{il(z-\xi)}\left[U_{11}\Delta\psi_1 + U_{12}\Delta\varphi_1\right]|_{k_2=0}.\end{aligned} \tag{3.59b}$$

We will relate $(\Delta\Psi_1, \Delta\Phi_1)$ with $\left(\bar{\Phi}_1^+, \bar{\Psi}_1^+\right)$. In this respect, we write equation (3.42) as

$$\begin{aligned}e\bar{\Psi}_1^+ =& e + \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\xi \int_0^\infty \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}} \mathrm{e}^{i\lambda(\bar{z}-\bar{\zeta})-2ik_1(x-\xi)+2ik_2(y-\eta)} q\left(e\bar{\Phi}_1^+\right)\\ &+ \frac{\mathrm{e}^{2ky}}{2\pi}\int_{-\infty}^{-2k_1} \mathrm{d}l \int_0^t \mathrm{d}\tau\, \mathcal{E}(k, l, \tau - t)\int_{-\infty}^\infty \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)}\left[U_{22}\left(\tilde{e}\bar{\psi}_1^+\right) + U_{21}\left(-\tilde{e}\bar{\varphi}_1^+\right)\right]\\ &+ \frac{i}{\pi}\int_{-\infty}^\infty \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta \int_{-2k_1}^{-2k_1+\lambda} \mathrm{d}l\, \mathrm{e}^{i(l+2k_1)(\bar{z}-\bar{\zeta})-2ik_1(x-\xi)+2ik_2(y-\eta)} q\left(e\bar{\Phi}_1^+\right),\end{aligned} \tag{3.60}$$

where $\tilde{e} = e|_{\eta=0}$ and we have used the identity

$$\frac{1}{\zeta - z} = \frac{\mathrm{e}^{-i\lambda(z-\zeta)}}{\zeta - z} - i \int_{-2k_1}^{-2k_1+\lambda} \mathrm{d}l\, \mathrm{e}^{-i(l+2k_1)(z-\zeta)}.$$

Letting $k_1 \to k_1 + \lambda/2$ and $k_2 \to i\lambda/2$ in equation (3.60), where $\lambda \geq 0$, we note that under this transformation,

$$\begin{aligned} k = k_1 + ik_2 = k_1, \quad e = \mathrm{e}^{-2i(k_1x - k_2y) - 2ik_1(k_1^2 - 3k_2^2)t} &\to e_\lambda, \\ \mathrm{e}^{i\lambda(\bar{z}-\bar{\zeta}) - 2ik_1(x-\xi) + 2ik_2(y-\eta)} &\to \mathrm{e}^{-2ik_1(x-\xi)}, \\ \mathrm{e}^{i(l+2k_1)(\bar{z}-\bar{\zeta}) - 2ik_1(x-\xi) + 2ik_2(y-\eta)} &\to \mathrm{e}^{il(x-\xi) + (l+2k_1)(y-\eta)}, \\ \mathcal{E}(k, l, \tau - t) \to \mathcal{E}(k_1, l, \tau - t) &= \mathrm{e}^{-il(l^2+3lk_1+3k_1^2)(\tau-t)}. \end{aligned}$$

Moreover, since $-2k_1 \to -2k_1 - \lambda$, we know that

$$\int_{-\infty}^{-2k_1} \mathrm{d}l \to \int_{-\infty}^{-2k_1} \mathrm{d}l + \int_{-2k_1}^{-2k_1-\lambda} \mathrm{d}l, \qquad \int_{-2k_1}^{-2k_1+\lambda} \mathrm{d}l \to - \int_{-2k_1}^{-2k_1-\lambda} \mathrm{d}l.$$

Thus, letting

$$\begin{aligned} M_1 &= \bar{\Psi}_1^+ \left(x, y, t, k_1 + \frac{\lambda}{2}, \frac{i\lambda}{2} \right), \quad N_1 = \bar{\Phi}_1^+ \left(x, y, t, k_1 + \frac{\lambda}{2}, \frac{i\lambda}{2} \right), \\ m_1 &= \bar{\psi}_1^+ \left(x, t, k_1 + \frac{\lambda}{2}, \frac{i\lambda}{2} \right), \quad n_1 = \bar{\varphi}_1^+ \left(x, k_1 + \frac{\lambda}{2}, \frac{i\lambda}{2} \right) \end{aligned}$$

equation (3.60) yields

$$\begin{aligned} e_\lambda M_1 = e_\lambda + F_1^{(1)} + \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^\infty \frac{\mathrm{d}\eta}{\bar{\zeta} - \bar{z}} \mathrm{e}^{-2ik_1(x-\xi)} q\left(e_\lambda N_1\right) + \frac{\mathrm{e}^{2k_1 y}}{2\pi} \int_{-\infty}^{-2k_1} \mathrm{d}l \int_0^t \mathrm{d}\tau \\ \times \mathcal{E}(k_1, l, \tau - t) \int_{-\infty}^{\infty} \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)} \left[U_{22}\left(\tilde{e}_\lambda m_1\right) + U_{21}\left(-\tilde{e}_\lambda n_1\right)\right]|_{k_2=0}, \qquad (3.61) \end{aligned}$$

where $\tilde{e}_\lambda = e_\lambda|_{\eta=0}$ and

$$\begin{aligned} F_1^{(1)}(x, y, t, k_1, \lambda) = \frac{1}{2\pi} \int_{-2k_1}^{-2k_1-\lambda} \mathrm{d}l\, \mathrm{e}^{ilx + (l+2k_1)y + il(l^2+3lk_1+3k_1^2)t} \Bigg\{ -2i \int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta \\ \times \mathrm{e}^{-il\xi - (l+2k_1)\eta - il(l^2+3lk_1+3k_1^2)t} q\left(e_\lambda N_1\right) + \int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^t \mathrm{d}\tau \\ \times \mathrm{e}^{-il\xi - il(l^2+3lk_1+3k_1^2)\tau} \left[U_{22}\left(\tilde{e}_\lambda m_1\right) + U_{21}\left(-\tilde{e}_\lambda n_1\right)\right]|_{k_2=0} \Bigg\}. \qquad (3.62) \end{aligned}$$

Here, letting $l+2k_1+\lambda\to l$ in equation (3.62), $F_1^{(1)}$ can be written as

$$\begin{aligned}
&F_1^{(1)}(x,y,t,k_1,\lambda)\\
&=-\frac{1}{2\pi}\int_0^{\lambda}\mathrm{d}l\,\mathrm{e}^{i(l-2k_1-\lambda)x+(l-\lambda)y+i(l-2k_1-\lambda)(l^2-2\lambda l+\lambda^2-k_1l+\lambda k_1+k_1^2)t}\\
&\quad\times\left\{-2i\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\mathrm{d}\eta\,\mathrm{e}^{-il\bar{\zeta}-il(l^2-3\lambda l+3\lambda^2-3k_1l+6k_1\lambda+3k_1^2)t}qN_1\right.\\
&\quad\left.+\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{t}\mathrm{d}\tau\,\mathrm{e}^{-il\xi-il(l^2-3\lambda l+3\lambda^2-3k_1l+6k_1\lambda+3k_1^2)\tau}\left[\hat{U}_{22}m_1-\hat{U}_{21}n_1\right]\right\},
\end{aligned}\tag{3.63}$$

where

$$\hat{U}_{21}=U_{21}(\xi,\tau,k_1,l-2k_1-\lambda),\quad \hat{U}_{22}=U_{22}(\xi,\tau,k_1,l-2k_1-\lambda).$$

Similarly, replacing $k_1\to k_1+\lambda/2$ and $k_2\to i\lambda/2$ in equation (3.44), we find

$$\begin{aligned}
-e_\lambda N_1&=\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\frac{\mathrm{d}\eta}{\zeta-z}q\,(e_\lambda M_1)+\frac{1}{2\pi}\int_0^{\infty}\mathrm{d}l\int_0^{t}\mathrm{d}\tau\\
&\quad\times\mathcal{E}(k_1,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{il(z-\xi)}\left[U_{12}\,(\tilde{e}_\lambda m_1)+U_{11}\,(-\tilde{e}_\lambda n_1)\right]|_{k_2=0}.
\end{aligned}\tag{3.64}$$

We note that equations (3.61) and (3.64) are similar to equations (3.59a) and (3.59b), respectively, except for the forcing term $F_1^{(1)}$, which will be determined by using equation (3.14). Replacing $\lambda\to\lambda/2$ and then letting $k_1\to k_1+\lambda/2$ and $k_2\to i\lambda/2$ in equation (3.14), we find

$$E=\mathrm{e}^{-il\bar{z}-il(l^2-3\lambda l+3\lambda^2-3k_1l+6\lambda k_1+3k_1^2)t}\to\mathrm{e}^{-il\bar{z}-il\left(l^2-3l\lambda+3\lambda^2-3k_1l+6\lambda k_1+3k_1^2\right)t}.$$

Thus, from equation (3.14) it follows that the curly bracket $\{\cdots\}$ in $F_1^{(1)}$ given by equation (3.63) can be written as

$$-2i\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\mathrm{d}\eta\,\mathrm{e}^{-il\bar{\zeta}}q_0N_1(\xi,\eta,0,k_1,\lambda),$$

and hence we write

$$F_1^{(1)}(x,y,t,k_1,\lambda)=\int_0^{\lambda}\mathrm{d}l\,\hat{E}(x,y,t,k_1,l,\lambda)f_1^{(1)}(k_1,\lambda,l),$$

where $f_1^{(1)}$ is given in (3.55) and

$$\hat{E}(x,y,t,k_1,l,\lambda) = \mathrm{e}^{i(l-2k_1-\lambda)x+(l-\lambda)y+i(l-2k_1-\lambda)(l^2-2\lambda l+\lambda^2-lk_1+\lambda k_1+k_1^2)t}.$$

We now determine the function $\chi_1^{(1)}$ satisfying the equation

$$\int_0^\infty \mathrm{d}\lambda \left(e_\lambda \chi_1^{(1)} + F_1^{(1)}\chi_1^{(1)}\right) = \int_0^\infty \mathrm{d}\lambda\, e_\lambda A_1^{(1)}.$$

Letting $\lambda \to \lambda + l$, we note that $\hat{E}(k,\lambda+l,l) = e_\lambda$ and

$$\int_0^\infty \mathrm{d}\lambda\, F_1^{(1)}\chi_1^{(1)} = \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}\lambda\, e_\lambda \chi_1^{(1)}(k_1,\lambda+l) f_1^{(1)}(k_1,\lambda+l,l),$$

which implies that $\chi_1^{(1)}$ should be given in equation (3.54a). Thus, multiplying equations (3.61) and (3.64) by $\chi_1^{(1)}$ and then integrating the resulting equations from $\lambda = 0$ to $\lambda = \infty$ with respect to $\mathrm{d}\lambda$, we find

$$\begin{aligned}\int_0^\infty \mathrm{d}\lambda\, e_\lambda \chi_1^{(1)} M_1 = \int_0^\infty &\mathrm{d}\lambda\, e_\lambda A_1^{(1)}\\ &- \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\xi \int_0^\infty \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}} \mathrm{e}^{-2ik_1(x-\xi)} q \left(-\int_0^\infty \mathrm{d}\lambda\, \chi_1^{(1)} e_\lambda N_1\right)\\ &+ \frac{\mathrm{e}^{2k_1 y}}{2\pi}\int_{-\infty}^{-2k_1} \mathrm{d}l \int_0^t \mathrm{d}\tau\, \mathcal{E}(k_1,l,\tau-t) \int_{-\infty}^\infty \mathrm{d}\xi\, \mathrm{e}^{il(\bar{z}-\xi)}\\ &\times \left[U_{22}\left(\int_0^\infty \mathrm{d}\lambda\, \chi_1^{(1)}\tilde{e}_\lambda m_1\right) + U_{21}\left(-\int_0^\infty \mathrm{d}\lambda\, \chi_1^{(1)}\tilde{e}_\lambda n_1\right)\right]\Big|_{k_2=0},\end{aligned} \tag{3.65}$$

and

$$\begin{aligned}&-\int_0^\infty \mathrm{d}\lambda\, e_\lambda \chi_1^{(1)} N_1\\ &= \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\xi \int_0^\infty \frac{\mathrm{d}\eta}{\zeta - z} q \left(\int_0^\infty \mathrm{d}\lambda\, \chi_1^{(1)} e_\lambda M_1\right) + \frac{1}{2\pi}\int_0^\infty \mathrm{d}l \int_0^t \mathrm{d}\tau\, \mathcal{E}(k_1,l,\tau-t)\\ &\times \int_{-\infty}^\infty \mathrm{d}\xi\, \mathrm{e}^{il(z-\xi)} \left[U_{12}\left(\int_0^\infty \mathrm{d}\lambda\, \chi_1^{(1)}\tilde{e}_\lambda m_1\right) + U_{11}\left(-\int_0^\infty \mathrm{d}\lambda\, \chi_1^{(1)}\tilde{e}_\lambda n_1\right)\right]\Big|_{k_2=0}.\end{aligned} \tag{3.66}$$

Then equations (3.65) and (3.66) are equivalent to equations (3.59a) and (3.59b), and hence equations (3.51a) and (3.51b) for $(\Delta\Psi_1, \Delta\Phi_1)$ follow.

Analysis of $(\delta\Psi_1, \delta\Phi_1)$. Letting $l \to \lambda$, equation (3.58) can be written as

$$\hat{e}\int_0^\infty \mathrm{d}\lambda\, E_\lambda A_1^{(2)}(k_1,\lambda), \quad \hat{e} = e|_{k_2=0} = e^{-2ik_1x-2ik_1^3t},$$

where E_λ and $A_1^{(2)}$ are defined by equations (3.53) and (3.54c). Then by ignoring the forcing (3.57), $(\delta\Psi_1, \delta\Phi_1)$ solves

$$\begin{aligned}\hat{e}^{-1}\delta\Psi_1 &= \int_0^\infty \mathrm{d}\lambda\, E_\lambda A_1^{(2)} \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^\infty \frac{\mathrm{d}\eta}{\zeta - z}\,\mathrm{e}^{2ik_1(x-\xi)}q\left(\hat{e}^{-1}\delta\Phi_1\right) + \frac{\mathrm{e}^{2k_1y}}{2\pi}\int_{-\infty}^{-2k_1}\mathrm{d}l\int_0^t \mathrm{d}\tau \\ &\quad \times \bar{\mathcal{E}}(k_1,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il(z-\xi)}\left[\bar{U}_{22}\left(\hat{e}^{-1}\delta\psi_1\right) + \bar{U}_{21}\left(-\hat{e}^{-1}\delta\varphi_1\right)\right]|_{k_2=0},\end{aligned} \tag{3.67a}$$

and

$$\begin{aligned}-\hat{e}^{-1}\delta\Phi_1 &= \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^\infty \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}}\,q\left(\hat{e}^{-1}\delta\Psi_1\right) + \frac{1}{2\pi}\int_0^\infty \mathrm{d}l\int_0^t \mathrm{d}\tau \\ &\quad \times \bar{\mathcal{E}}(k_1,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il(\bar{z}-\xi)}\left[\bar{U}_{12}\left(\hat{e}^{-1}\delta\psi_1\right) + \bar{U}_{11}\left(-\hat{e}^{-1}\delta\varphi_1\right)\right]|_{k_2=0}.\end{aligned} \tag{3.67b}$$

In order to relate $(\delta\Psi_1, \delta\Phi_1)$ with (Ψ_1^+, Φ_1^+), we first write equations (3.33a) and (3.33b) as

$$\begin{aligned}e^{-1}\Psi_1^+ &= e^{-1} + \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^\infty \frac{\mathrm{d}\eta}{\zeta - z}\mathrm{e}^{-i\lambda(z-\zeta)+2ik_1(x-\xi)-2ik_2(y-\eta)}q\left(e^{-1}\Phi_1^+\right) \\ &\quad + \frac{\mathrm{e}^{2\bar{k}y}}{2\pi}\int_{-\infty}^{-2k_1}\mathrm{d}l\int_0^t \mathrm{d}\tau \\ &\quad \times \bar{\mathcal{E}}(k,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il(z-\xi)}\left[\bar{U}_{22}\left(\tilde{e}^{-1}\psi_1^+\right) + \bar{U}_{21}\left(-\tilde{e}^{-1}\varphi_1^+\right)\right] \\ &\quad - \frac{i}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^\infty \mathrm{d}\eta\int_{-2k_1}^{-2k_1+\lambda}\mathrm{d}l\,\mathrm{e}^{-i(l+2k_1)(z-\zeta)+2ik_1(x-\xi)-2ik_2(y-\eta)}q\left(e^{-1}\Phi_1^+\right),\end{aligned} \tag{3.68}$$

and

$$-e^{-1}\Phi_1^+ = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^{\infty} \frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}} q\left(e^{-1}\Psi_1^+\right) + \frac{1}{2\pi}\int_0^{\infty}\mathrm{d}l\int_0^t \mathrm{d}\tau \times \bar{\mathcal{E}}(k,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\, \mathrm{e}^{-il(\bar{z}-\xi)}\left[\bar{U}_{12}\left(\tilde{e}^{-1}\psi_1^+\right)+\bar{U}_{11}\left(-\tilde{e}^{-1}\varphi_1^+\right)\right]. \quad (3.69)$$

Replacing $k_1 \to k_1 + \lambda/2$ and $k_1 \to -i\lambda/2$, where $\lambda \geq 0$, we find

$$\bar{k} = k_1 - ik_2 = k_1, \quad e^{-1} = \mathrm{e}^{2i(k_1x-k_2y)+2ik_1(k_1^2-3k_2^2)t} \to E_\lambda,$$
$$\mathrm{e}^{-i\lambda(z-\zeta)+2ik_1(x-\xi)-2ik_2(y-\eta)} \to \mathrm{e}^{2ik_1(x-\xi)},$$
$$\mathrm{e}^{-i(l+2k_1)(z-\zeta)+2ik_1(x-\xi)-2ik_2(y-\eta)} \to \mathrm{e}^{-il(x-\xi)+(l+2k_1)(y-\eta)},$$
$$\bar{\mathcal{E}}(k,l,\tau-t) \to \bar{\mathcal{E}}(k_1,l,\tau-t) = \mathrm{e}^{il(l^2+3lk_1+3k_1^2)(\tau-t)}.$$

Thus, letting

$$\tilde{M}_1 = \Psi_1^+\left(x,y,t,k_1+\frac{\lambda}{2},-\frac{i\lambda}{2}\right), \quad \tilde{N}_1 = \Phi_1^+\left(x,y,t,k_1+\frac{\lambda}{2},-\frac{i\lambda}{2}\right),$$
$$\tilde{m}_1 = \psi_1^+\left(x,t,k_1+\frac{\lambda}{2},-\frac{i\lambda}{2}\right), \quad \tilde{n}_1 = \varphi_1^+\left(x,t,k_1+\frac{\lambda}{2},-\frac{i\lambda}{2}\right),$$

equations (3.68) and (3.69) yield

$$E_\lambda\tilde{M}_1 = E_\lambda + F_1^{(2)} + \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\frac{\mathrm{d}\eta}{\zeta-z}\mathrm{e}^{2ik_1(x-\xi)}q\left(E_\lambda\tilde{N}_1\right) + \frac{\mathrm{e}^{2k_1y}}{2\pi}\int_{-\infty}^{-2k_1}\mathrm{d}l\int_0^t\mathrm{d}\tau \times \bar{\mathcal{E}}(k_1,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il(z-\xi)}\left[\bar{U}_{22}\left(\tilde{E}_\lambda\tilde{m}_1\right)+\bar{U}_{21}\left(-\tilde{E}_\lambda\tilde{n}_1\right)\right]|_{k_2=0},$$

and

$$-E_\lambda\tilde{N}_1 = \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^{\infty}\frac{\mathrm{d}\eta}{\bar{\zeta}-\bar{z}}q\left(E_\lambda\tilde{M}_1\right) + \frac{1}{2\pi}\int_0^{\infty}\mathrm{d}l\int_0^t\mathrm{d}\tau \times \bar{\mathcal{E}}(k_1,l,\tau-t)\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-il(\bar{z}-\xi)}\left[\bar{U}_{12}\left(\tilde{E}_\lambda\tilde{m}_1\right)+\bar{U}_{11}\left(-\tilde{E}_\lambda\tilde{n}_1\right)\right]|_{k_2=0},$$

where $\tilde{E}_\lambda = E_\lambda|_{\eta=0}$ and

$$F_1^{(2)}(x,y,t,k_1,\lambda) = \frac{1}{2\pi}\int\limits_{-2k_1}^{-2k_1-\lambda} \mathrm{d}l\, \mathrm{e}^{-ilx+(l+2k_1)y-il(l^2+3lk_1+3k_1^2)t}\Bigg\{2i\int\limits_{-\infty}^{\infty}\mathrm{d}\xi\int\limits_{0}^{\infty}\mathrm{d}\eta$$
$$\times\, \mathrm{e}^{il\xi-(l+2k_1)\eta+il(l^2+3lk_1+3k_1^2)t} q\left(E_\lambda \tilde{N}_1\right) + \int\limits_{-\infty}^{\infty}\mathrm{d}\xi\int\limits_{0}^{t}\mathrm{d}\tau$$
$$\times\, \mathrm{e}^{il\xi+il(l^2+3lk_1+3k_1^2)\tau}\left[\bar{U}_{22}\left(\tilde{E}_\lambda\tilde{m}_1\right)+\bar{U}_{21}\left(-\tilde{E}_\lambda\tilde{n}_1\right)\right]|_{k_2=0}\Bigg\}.$$

The rest of analysis is similar to the case of $(\Delta\Psi_1, \Delta\Phi_1)$ and then equations (3.51c) and (3.51d) follow. □

4. CONCLUDING REMARKS

We have presented the Fokas method for the mVN equation posed on the half-plane. Specifically, for the linearized mVN equation we have obtained the integral representation for the solution in terms of the functions $q_0(x,y)$, $g_0(x,t)$ and $g_1(x,t)$. For the nonlinear mVN equation, we have derived several version of the global relation and the d-bar formalism (or the so-called Pompieu's formula) for the sectionally defined non-analytic functions. It should be remarked that the main advantage of the Fokas method is that it provides integral representations of the solutions in the complex plane. As a consequence, it makes possible to study asymptotics of the solution by using the Deift–Zhou method [7] for the long-time behavior, or to study the small dispersion limit by using the Deift–Venakides–Zhou method [8] (see also [27, 28] for recent applications). In addition, we have characterized d-bar derivatives of the spectral functions and the jumps across the real k-axis (while the jumps along the imaginary k-axis vanish). It requires to characterize the jumps across the negative real k-axis, which can be done by the similar analysis as we discussed in Section. 3.3, but involving more complicated forcing terms. Moreover, it should be noted that in order to be effective in implementation of the Fokas method for the mVN equation posed in half-plane, it may be necessary to characterize the boundary values (cf. the linear case), since one of the boundary values in equation (1.3) is possibly unknown either in physical settings or for a well-posedness of the equation as discussed in the case for (1+1)-dimensional integrable systems [11]. We will discuss these issues in the near future.

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REFERENCES

[1] M.J. Ablowitz, D. BarYaacov, A.S. Fokas, *On the inverse scattering transform for the Kadomtsev–Petviasvhili equation*, Stud. Appl. Math. **69** (1983), 135–143.

[2] R. Beals, R.R. Coifman, *Linear spectral problems, nonlinear equations and the $\bar{\partial}$-method*, Inverse Problems **5** (1989), 87–130.

[3] L.V. Bogdanov, *The Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg-de Vries equation*, Theoret. and Math. Phys. **70** (1987), 219–223.

[4] A. Boutet de Monvel, D. Shepelsky, L. Zielinski, *A Riemann–Hilbert approach for the Novikov equation*, Symmetry Integr. Geom.: Methods Appl. **12** (2016), 095.

[5] R.M. Chen, W. Lian, D. Wang, R. Xu, *A rigidity property for the Novikov equation and the asymptotic stability of peakons*, Arch. Rational Mech. Anal. **241** (2021), 497–533.

[6] R. Croke, J.L. Muller, A. Stahel, *Transverse instability of plane wave soliton solutions of the Novikov–Veselov equation*, Contemporary Mathematics **635** (2015).

[7] P. Deift, X. Zhou, *A steepest descent method for oscillatory Riemann–Hilbert problems*, Bull. Amer. Math. Soc. **26** (1992), 119–123.

[8] P. Deift, S. Venakides, X. Zhou, *New results in small dispersion KdV by an extension of the steepest descent method for Riemann–Hilbert problems*, Int. Math. Res. Notices **6** (1997), 286–299.

[9] A.S. Fokas, *A unified transform method for solving linear and certain nonlinear PDEs*, Proc. Roy. Soc. London A **453** (1997), 1411–1443.

[10] A.S. Fokas, *Integrable nonlinear evolution equations on the half-line*, Comm. Math. Phys. **230** (2002), 1–39.

[11] A.S. Fokas, *The generalized Dirichlet-to-Neumann map for certain nonlinear evolution PDEs*, Comm. Pure Appl. Math. **LVIII** (2005), 639–670.

[12] A.S. Fokas, *A Unified Approach to Boundary Value Problems*, CBMS-NSF Regional Conference Series in Applied Mathematics SIAM, 2008.

[13] A.S. Fokas, *The Davey–Stewartson equation on the half-plane*, Commun. Math. Phys. **289** (2009), 957–993.

[14] A.S. Fokas, M.J. Ablowitz, *On the scattering transform of multi-dimensional nonlinear equations related to first order systems in the plane*, J. Math. Phys. **25** (1984), 2494–2505.

[15] A.S. Fokas, L.Y. Sung, *On the solvability of the N-wave the Davey–Stewartson and the Kadomtsev–Petviashvili equation*, Inverse Problems **8** (1992), 673–708.

[16] G. Hwang, *The modified Korteweg–de Vries equation on the quarter plane with t-periodic data*, J. Nonlinear Math. Phys. **24** (2017), 620–634.

[17] G. Hwang, *The elliptic sinh-Gordon equation in a semi-strip*, Adv. Nonlinear Anal. **8** (2019), 533–544.

[18] G. Hwang, A.S. Fokas, *The modified Korteweg-de Vries equation on the half-line with a sine-wave as Dirichlet datum*, J. Nonlinear Math. Phys. **20** (2013), 135–157.

[19] M. Lassas, J.L. Muller, S. Siltanen, A. Stahel, *The Novikov–Veselov equation and the inverse scattering method, Part I: Analysis*, Physica D **241** (2012), 1322–1335.

[20] M. Lassas, J.L. Muller, S. Siltanen, A. Stahel, *The Novikov–Veselov equation and the inverse scattering method: II. Computation*, Nonlinearity **25** (2012), 1799–1818.

[21] J. Lenells, A.S. Fokas, *The nonlinear Schrödinger equation with t-periodic data: I. Exact results*, Proc. R. Soc. A **471** (2015), 20140925.

[22] J. Lenells, A.S. Fokas, *The nonlinear Schrödinger equation with t-periodic data: II. Perturbative results*, Proc. R. Soc. A **471** (2015), 20140926.

[23] D. Mantzavinos, A.S. Fokas, *The Kadomtsev–Petviashvili II equation on the half-plane*, Physica D **240** (2011), 477–511.

[24] A.V. Mikhailov, V.S. Novikov, *Perturbative symmetry approach*, J. Phys. A: Math. Gen. **35** (2002), 4775–4790.

[25] B. Moon, *Single peaked traveling wave solutions to a generalized μ–Novikov equation*, Adv. Nonlinear Anal. **10** (2021), 66–75.

[26] I.A. Taimanov, *Modified Veselov–Novikov equation and differential geometry of surface*, Trans. Amer. Math. Soc. **197** (1997), 133–151.

[27] D.-S. Wang, X. Wang, *Long-time asymptotics and the bright N-soliton solutions of the Kundu–Eckhaus equation via the Riemann–Hilbert approach*, Nonlinear Anal.: Real World Appl. **41** (2018), 334–361.

[28] D.-S. Wang, B. Guo, X. Wang, *Long-time asymptotics of the focusing Kundu–Eckhaus equation with nonzero boundary conditions*, J. Differential Equations **266** (2019), 5209–5253.

[29] D. Yu, Q. P. Liu, S. Wang, *Darboux transformation for the modified Veselov–Novikov equation*, J. Phys. A: Math. Gen. **35** (2002), 3779–3785.

[30] X. Zhou, *Inverse scattering transform for the time dependent Schrödinger equation with application to the KPI equation*, Commun. Math. Phys. **128** (1990), 551–564.

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