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SPECTRAL PROPERTIES OF CERTAIN OPERATORS ON THE FREE HILBERT SPACE $\mathfrak{F}\,[H_1, \ldots, H_N]$ AND THE SEMICIRCULAR LAW

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Abstract. In this paper, we fix N-many l^2-Hilbert spaces H_k whose dimensions are $n_k \in \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$, for $k = 1, \ldots, N$, for $N \in \mathbb{N} \setminus \{1\}$. And then, construct a Hilbert space $\mathfrak{F} = \mathfrak{F}[H_1, \ldots, H_N]$ induced by $H_1, \ldots, H_N$, and study certain types of operators on $\mathfrak{F}$. In particular, we are interested in so-called jump-shift operators. The main results (i) characterize the spectral properties of these operators, and (ii) show how such operators affect the semicircular law induced by $\bigcup_{k=1}^{N} \mathcal{B}_k$, where $\mathcal{B}_k$ are the orthonormal bases of H_k, for $k = 1, \ldots, N$.

Keywords: separable Hilbert spaces, free Hilbert spaces, jump operator, shift operators, jump-shift operators, semicircular elements.

Mathematics Subject Classification: 46L10, 46L54, 47L30, 47L55.

1. INTRODUCTION

In this paper, we study spectral properties of three types of (*bounded linear*) *operators* on a *Hilbert space*,

$$\mathfrak{F} \overset{denote}{=} \mathfrak{F}[H_1, \ldots, H_N],$$

generated by N-many multi l^2-*Hilbert spaces*

$$H_k = l^2(n_k), \quad \text{for} \quad n_k \in \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\},$$

equipped with their *orthonormal bases* (in short, ONBs, from below),

$$\mathcal{B}_k = \{\xi_1^{(k)}, \xi_2^{(k)}, \ldots, \xi_{n_k}^{(k)}\},$$

for $k = 1, \ldots, N$, for $N \in \mathbb{N} \setminus\{1\}$, i.e., the given multi Hilbert spaces H_k are n_k-dimensional *separable Hilbert spaces*, for $k = 1, \ldots, N$.

In particular, we are interested in operators on $\mathfrak{F}$ assigning ONB vectors to other ONB vectors, preserving patterns of them. We here not only characterize the operator-theoretic spectral properties of the operators in the operator algebra $B(\mathfrak{F})$, but also consider how these operators affect *the semicircular law* induced by some ONB vectors of $\mathfrak{F}$.

Recall that the spectral properties of elements of the operator algebra $B(H)$ on an arbitrary Hilbert space H are characterized by the types of them:

(i) an operator T is said to be self-adjoint if $T^* = T$,
(ii) an operator T is normal if $T^*T = TT^*$,
(iii) an operator T is an isometry if $T^*T = 1_H$,
(iv) an operator T is a unitary if $T^*T = 1_H = TT^*$,
(v) an operator T is a partial isometry if T^*T is a projection on H, where an operator P is a projection on H if it is self-adjoint and idempotent in the sense that $P^2 = P$ on H.

Above T^* is the adjoint of T in $B(H)$. The spectra of such operators are well-characterized generally (e.g., see [15]). So, by characterizing the spectral types (or the spectral properties) of operators, one can verify the spectra of operators in a universal manner.

1.1. MOTIVATION

For more about fundamental *operator theory* and its applications, e.g., see [14, 15] and [19], and see [2–5, 7, 8, 20–22] and [23] for the connections between classical *measure theory* and *free probability theory.* Also, for more about semicircularity of operators, or the semicircular law, see [1, 6, 16–18] and [24].

In [10] and [9], we considered how to construct *semicircular elements*, whose free distributions are *the semicircular law*, from mutually orthogonal $|\mathbb{Z}|$-many *projections* in a *C^*-probability space*, and studied the *Banach $*$-algebras* generated by such semicircular elements. As an application, the author and Jorgensen constructed semicircular elements, as *Banach-space operators* (in the sense of [14]) acting on an *operator algebra* $B(H)$, from mutually orthogonal $|\mathbb{N}|$-many *rank*-1 *projections* in [13], where H is the $|\mathbb{N}|$-dimensional Hilbert space.

Independently, the *joint free distributions* of multi semicircular elements are not only characterized, but also estimated and asymptotically estimated, in [12], by computing the *joint free moments* of them. The main results of [12] demonstrate that if a topological $*$-algebra is generated by multi, mutually free semicircular elements, and if the free distributions of the generators (which are the semicircular law) are preserved by certain multiplicative actions, then the original free-distributional data are preserved by such actions. e.g., see [11].

These series of works motivate us to find-or-construct certain operators in the sense of [14], or of [15], preserving-or-distorting the semicircular law, or the *free-distributional data* on $*$-algebras generated by semicircular elements.

Remark that, in those previous works, we handled adjointable Banach-space operators instead of considering Hilbert-space operators, not to be involved with Hilbert-space representations.

In this paper, different from the above previous researches, we are interested in Hilbert-space operators deforming the semicircular law. Even though our works are motivated by [11], the constructions, ideas, and results are different from those of [11], conceptually, and theoretically. We here study certain operators on $\mathfrak{F}$, and consider how these operators affect the semicircular law generated by some canonical rank-1 projections of the operator algebra $B(\mathfrak{F})$, i.e., meanwhile the main results of [11] show how the semicircular law is deformed by the actions "from outside", our results explain how the semicircular law is affected by the actions "from inside". In particular, the spectral properties of the acting operators on $\mathfrak{F}$ are characterized, and we show how these properties decide the deformations of the semicircular law.

1.2. OVERVIEW

In Section 2, we define the *free Hilbert space* $\mathfrak{F}$ from given N-many separable l^2-Hilbert spaces $H_1, \ldots, H_N$, for $N \in \mathbb{N} \setminus \{1\}$, whose dimensions $n_1, \ldots, n_N$ satisfy the ordering

$$n_1 \leq n_2 \leq \ldots \leq n_N \text{ in } \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\},$$

where $\infty = |\mathbb{N}|$.

In Sections 3 and 4, certain types of operators on $\mathfrak{F}$ are defined and studied. In particular, operators on $\mathfrak{F}$ assigning the subspace vectors of H_j to those of H_i in $\mathfrak{F}$ (by fixing all other vectors), for any arbitrarily fixed $i < j \in \{1, \ldots, N\}$, are studied in Section 3. We call them the *jump operators* from H_j to H_i in $\mathfrak{F}$, for $i < j \in \{1, \ldots, N\}$; and, the *shift operators* on $\mathfrak{F}$, assigning all basis vectors to other basis vectors under suitable shifting processes on the ONBs of $H_1, \ldots, H_N$, are considered in Section 4. Operator-theoretic spectral properties of those operators are characterized in the operator algebra $B(\mathfrak{F})$.

In Section 5, based on the main results of Sections 3 and 4, a new type of operators of $B(\mathfrak{F})$, called the *jump-shift operators*, is introduced. These operators are defined to be operator-products of shift operators and jump operators on $\mathfrak{F}$. Spectral properties of these operators are characterized case-by-case.

In Section 6, with help of [9] and [11], semicircular elements are constructed from some mutually orthogonal, rank-1 projections on the free Hilbert space $\mathfrak{F}$. They assign Banach-space operators, in the sense of [14], acting on $B(\mathfrak{F})$. The main results of Section 6 generalize those of [13].

In Section 7, we study how our jump-shift operators (and hence, how the jump operators, or the shift operators) deform the free probability on the Banach $*$-probability space $\mathbf{L}_Q[H_1, \ldots, H_N]$ generated by the semicircular elements of Section 6.

2. HILBERT SPACES $\mathfrak{F}[H_1, \ldots, H_N]$

For more about Hilbert spaces, and corresponding operator theory, see [15]. In this section, we construct and study a Hilbert space,

$$\mathfrak{F} = \mathfrak{F}[H_1, \ldots, H_N],$$

generated by N-many multi l^2-Hilbert spaces

$$H_k = l^2(n_k), \quad \text{with} \quad \dim_{\mathbb{C}} H_k = n_k \in \mathbb{N}^{\infty}, \tag{2.1}$$

for $k = 1, \ldots, N$, for $N \in \mathbb{N} \setminus \{1\}$, where $\dim_{\mathbb{C}} H$ means the "*dimension* of a Hilbert space H" over $\mathbb{C}$. Assume further that

$$n_1 \leq n_2 \leq \ldots \leq n_N \quad \text{in} \quad \mathbb{N}^{\infty}. \tag{2.2}$$

Since each Hilbert space H_k of (2.1) is n_k-dimensional, it has the corresponding ONB,

$$\mathcal{B}_k = \{\xi_1^{(k)}, \ldots, \xi_{n_k}^{(k)}\},$$

satisfying

$$\left\langle \xi_i^{(k)},\ \xi_j^{(k)} \right\rangle_k = \delta_{ij}, \quad \forall i, j = 1, \ldots, n_k, \tag{2.3}$$

and

$$\left\| \xi_i^{(k)} \right\|_k = \sqrt{\left\langle \xi_i^{(k)},\ \xi_i^{(k)} \right\rangle_k} = 1, \quad \forall i = 1, \ldots, n_k,$$

where $\langle \cdot, \cdot \rangle_k$ are the *inner products of* H_k, and $\|.\|_k$ are the *norms on* H_k induced by $\langle \cdot, \cdot \rangle_k$, and where δ is the Kronecker delta.

By definition, every vector $h \in H_k$ is expressed by

$$h = \sum_{l=1}^{n_k} t_l \xi_l^{(k)}, \quad \text{with} \quad t_l \in \mathbb{C},$$

satisfying

$$\|h\|_k^2 = \sum_{l=1}^{n_k} |t_l|^2 < \infty,$$

for $k = 1, \ldots, N$.

Let H_i and H_j be the above Hilbert spaces (2.1), for $i, j \in \{1, \ldots, N\}$. Then one can construct the corresponding *tensor product Hilbert space* $H_{i,j}$,

$$H_{i,j} \overset{def}{=} H_i \otimes H_j,$$

equipped with its ONB,

$$\mathcal{B}_{i,j} = \left\{ \xi_k^{(i)} \otimes \xi_l^{(j)} \,\middle|\, \xi_k^{(i)} \in \mathcal{B}_i,\ \xi_l^{(j)} \in \mathcal{B}_j,\ \text{for } k = 1, \ldots, n_i \text{ and } l = 1, \ldots, n_j \right\},$$

and the corresponding inner product $\langle \cdot, \cdot \rangle_{i,j}$ satisfying that

$$\langle x_1 \otimes y_1,\ x_2 \otimes y_2 \rangle_{i,j} = \langle x_1,\ x_2 \rangle_i \langle y_1,\ y_2 \rangle_j ,$$

for all $x_1, x_2 \in H_i$, and $y_1, y_2 \in H_j$, where $\otimes$ is the tensor product of Hilbert spaces.

Inductively, for $H_1, \ldots, H_N$, one can construct the Hilbert spaces

$$H_{i_1,\ldots,i_k} = \overset{k}{\underset{l=1}{\otimes}} H_{i_l}, \quad \text{for} \quad k \in \mathbb{N}, \tag{2.4}$$

where $(i_1, \ldots, i_k)$ are the k-tuples of $\{1, \ldots, N\}$.

Similarly, if one can define subspaces H_k^o of H_k by

$$H_k^o = H_k \ominus \mathbb{C}, \quad \text{for} \quad k = 1, \ldots, N, \tag{2.5}$$

where $\ominus$ is the orthogonal complement in terms of the direct product $\oplus$ of Banach spaces (recall that all Hilbert spaces are Banach spaces equipped with Hilbert-norm), then we have the subspaces

$$H_{i_1,\ldots,i_k}^o = \overset{k}{\underset{l=1}{\otimes}} H_{i_l}^o \quad \text{of} \quad H_{i_1,\ldots,i_k}, \tag{2.6}$$

where $H_{i_l}^o$ are the subspaces (2.5) of H_{i_l}, and $H_{i_1,\ldots,i_k}$ are the tensor product Hilbert spaces (2.4), for all $(i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$, for all $k \in \mathbb{N}$, i.e., the subspaces $H_{i_1,\ldots,i_k}^o$ of (2.6) are well-determined as Banach subspaces (or closed subspaces) of the Hilbert spaces $H_{i_1,\ldots,i_k}$ of (2.4).

Let $H_1, \ldots, H_N$ be given Hilbert spaces (2.1). Define a new Hilbert space

$$\mathfrak{F} \overset{denote}{=} \mathfrak{F}[H_1, \ldots, H_N]$$

by the Hilbert space

$$\mathfrak{F} \overset{\text{def}}{=} \underset{k \in \mathbb{N}}{\oplus} \left(\underset{(i_1,\ldots,i_k) \in Alt(\{1,\ldots,N\}^k)}{\oplus} H_{i_1,\ldots,i_k} \right),$$

(where $\oplus$ is the direct product of Hilbert spaces) having its “Banach space” expression

$$\mathfrak{F} = \mathbb{C} \oplus \left(\underset{k \in \mathbb{N}}{\oplus} \left(\underset{(i_1,\ldots,i_k) \in Alt(\{1,\ldots,N\}^k)}{\oplus} H_{i_1,\ldots,i_k}^o \right) \right), \tag{2.7}$$

where $H_{i_1,\ldots,i_k}^o$ are the subspaces (2.6) of the tensor product Hilbert spaces $H_{i_1,\ldots,i_k}$ of (2.4) (where $\oplus$, here, is the direct product of Banach spaces), and

$$Alt\left(\{1, \ldots, N\}^k\right) = \left\{ (i_1, \ldots, i_k) \;\middle|\; \begin{array}{c} (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k \\ \text{are } k\text{-tuples, satisfying} \\ i_1 \neq i_2,\ i_2 \neq i_3,\ \ldots,\ i_{k-1} \neq i_k \end{array} \right\}, \tag{2.8}$$

for all $k \in \mathbb{N}$.

By (2.8), all k-tuples $(i_1, \ldots, i_k) \in Alt\left(\{1, \ldots, N\}^k\right)$ in (2.7) are *alternating k-tuples*. For example,

$$(1,2,1,3,1,2,1) \in Alt\left(\{1,\ 2,\ 3,\ 4\}^7\right),$$

while

$$(3,3,2) \notin Alt\left(\{1,\ 2,\ 3,\ 4\}^3\right),$$

etc.

Definition 2.1. We call the Hilbert space $\mathfrak{F} = \mathfrak{F}[H_1, \ldots, H_N]$ of (2.7), generated by N-many Hilbert spaces $H_1, \ldots, H_N$ of (2.1), the free Hilbert space of $H_1, \ldots, H_N$. Remark that it is a generalized Fock space induced by $H_1, \ldots, H_N$ in the sense of [7] (e.g., [23]).

By the definition (2.7) of the free Hilbert space $\mathfrak{F}$, all nonzero vectors $\eta \in \mathfrak{F}$ are the limits of linear combinations of vector tensors formed by

$$\xi_{l_1}^{(i_1)} \otimes \xi_{l_2}^{(i_2)} \otimes \ldots \otimes \xi_{l_k}^{(i_k)}, \tag{2.9}$$

with

$$\xi_{l_j}^{(i_j)} \in \mathcal{B}_{i_j} \subset H_{i_j}, \quad \forall l_j \in \{1, \ldots, n_{i_j}\},$$

for $(i_1, \ldots, i_k) \in Alt\left(\{1,\ \ldots,\ N\}^k\right)$, for $k \in \mathbb{N}$.

For convenience, denote the vector tensors (2.9) simply by $\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}$, i.e.,

$$\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k} \overset{denote}{=} \xi_{l_1}^{(i_1)} \otimes \ldots \otimes \xi_{l_k}^{(i_k)}, \tag{2.10}$$

for all $(i_1, \ldots, i_k) \in Alt\left(\{1, \ldots, N\}^k\right)$, for all $k \in \mathbb{N}$.

It is not difficult to check that the set

$$\mathfrak{B} \overset{def}{=} \bigcup_{k\in\mathbb{N}} \Big(\bigcup_{(i_1,\ldots,i_k)\in Alt(\{1,\ldots,N\}^k)} \{\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k} \in \mathfrak{F} \mid \xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k} \text{ is a vector (2.10),} \quad \text{where } l_j \in \{1, \ldots, n_{i_j}\},\ \forall j = 1, \ldots, k\} \Big) \tag{2.11}$$

forms the ONB of the free Hilbert space $\mathfrak{F}$, satisfying

$$\begin{aligned}
\left\langle \xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k},\ \xi_{j_1,\ldots,j_n}^{k_1,\ldots,k_n} \right\rangle &= \delta_{k,n} \left\langle \xi_{i_1,\ldots,i_n}^{l_1,\ldots,l_n},\ \xi_{j_1,\ldots,j_n}^{k_1,\ldots,k_n} \right\rangle \\
&= \delta_{k,n} \prod_{t=1}^{n} \left(\delta_{i_t,j_t} \left\langle \xi_{l_t}^{(i_t)},\ \xi_{k_t}^{(i_t)} \right\rangle_{i_t} \right) \\
&= \delta_{k,n} \left(\prod_{t=1}^{n} \delta_{i_t,j_t} \delta_{l_t,k_t} \right) \\
&= \delta_{\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k},\, \xi_{j_1,\ldots,j_n}^{k_1,\ldots,k_n}}
\end{aligned}$$

and

$$\left\| \xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k} \right\| = \sqrt{\left\langle \xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k},\ \xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_l} \right\rangle} = 1,$$

for all $\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}, \xi_{j_1,\ldots,j_n}^{k_1,\ldots,k_n} \in \mathfrak{B}$, where $\mathfrak{B}$ is in the sense of (2.11), and $\langle \cdot, \cdot \rangle$ is the inner product of $\mathfrak{F}$.

3. JUMP OPERATORS ON $\mathfrak{F}[H_1, \ldots, H_N]$

In this section, we construct-and-study certain operators on the free Hilbert space $\mathfrak{F}$ of (2.7) of the fixed l^2-Hilbert spaces $H_1, \ldots, H_N$ of (2.1), satisfying condition (2.2).

Let H and K be arbitrary Hilbert spaces. Then the *operator space* (as a topological vector space), consisting of all (bounded linear) operators from H to K is denoted by $B(H, K)$, i.e.,

$$B(H, K) = \{T : H \to K \mid T \text{ is an operator from } H \text{ to } K\},$$

under the *operator-norm*

$$\|T\| = \sup \{\|T(h)\|_K \mid h \in H, \ \|h\|_H = 1\},$$

where $\|.\|_H$ and $\|.\|_K$ are the Hilbert-space norms for H and K, respectively. If $H = K$, we denote the operator space $B(H, H)$ by $B(H)$. In such a case, $B(H)$ forms an topological $*$-algebra, called the *operator algebra* (e.g., [14] and [15]). In particular, under the operator-norm topology, $B(H)$ forms a C^*-algebra.

3.1. OPERATORS $J_{i+1}^{\sigma} \in B(H_{i+1}, H_i)$ FOR $\sigma \in S_{n_i}$

Let $H_k = l^2(n_k)$ be the fixed l^2-Hilbert spaces (2.1) with their ONBs,

$$\mathcal{B}_k = \{\xi_1^{(k)}, \ldots, \xi_{n_k}^{(k)}\},$$

for all $k = 1, \ldots, N$, under the ordering,

$$n_1 \leq n_2 \leq \ldots \leq n_N \text{ in } \mathbb{N}^{\infty}. \tag{3.1}$$

For any fixed $k \in \{1, \ldots, N\}$, take the index set X_k,

$$X_k = \{1, \ldots, n_k\},$$

of the ONB $\mathcal{B}_k$, and define the *symmetric group* S_{n_k},

$$S_{n_k} = \{\sigma : X_k \to X_k \mid \sigma \text{ is bijective}\}, \tag{3.2}$$

under the usual functional-composition ($\circ$).

Now, take arbitrary $i \in \{1, \ldots, N-1\}$, and consider the quantities

$$i, i+1 \in \{1, \ldots, N\}. \tag{3.3}$$

For the chosen quantities i and $i+1$ of (3.3), take the corresponding Hilbert spaces H_i and H_{i+1}. Then, by (3.1),

$$\dim_{\mathbb{C}} H_i = n_i \leq n_{i+1} = \dim_{\mathbb{C}} H_{i+1},$$

in $\mathbb{N}^{\infty}$.

For an arbitrarily fixed $\sigma \in S_{n_i}$, define an operator $J_{i+1}^{\sigma} \in B\,(H_{i+1},\ H_i)$ by

$$J_{i+1}^{\sigma}\left(\sum_{l=1}^{n_{i+1}} t_l \xi_l^{(i+1)}\right) \overset{def}{=} \sum_{l=1}^{n_i} t_{\sigma(l)} \xi_{\sigma(l)}^{(i)}, \tag{3.4}$$

where S_{n_i} is the symmetric group (3.2) on X_{n_i}. Then it is a well-defined operator from H_{i+1} onto H_i, assigning

$$\begin{aligned} J_{i+1}^{\sigma} &: \xi_l^{(i+1)} \in \mathcal{B}_{i+1} \mapsto \xi_{\sigma(l)}^{(i)} \in \mathcal{B}_i, \quad \forall l = 1, \ldots, n_i, \\ J_{i+1}^{\sigma} &: \xi_l^{(i+1)} \in \mathcal{B}_{i+1} \mapsto 0_{H_i} \in H_i, \quad \forall l = n_i + 1, \ldots, n_{i+1}, \end{aligned} \tag{3.5}$$

in $B(H_{i+1}, H_i)$, for $i = 1, \ldots, N-1$, where 0_H mean the *zero vectors* of Hilbert spaces H.

From below, we denote a Hilbert space of square-summable n-tuples,

$$\left\{ (z_i)_{i=1}^{n} : z_i \in \mathbb{C}, \sum_{i=1}^{n} |z_i|^2 < \infty \right\},$$

by $\mathbb{C}^{\times n}$, for all $n \in \mathbb{N}^{\infty}$, i.e., if $n < \infty$, then $\mathbb{C}^{\times n} = \mathbb{C}^n$, the usual n-dimensional $\mathbb{C}$-vector space, isomorphic to $l^2(n)$; and if $n = \infty$, then $\mathbb{C}^{\times \infty} = l^2(\mathbb{N})$, the canonical l^2-Hilbert space of square-summable $\mathbb{C}$-sequences. So, the Hilbert space $\mathbb{C}^{\times n}$ has its ONB,

$$\{e_1, \ldots, e_n\}, \tag{3.6}$$

with

$$e_l = (0,\ \ldots,\ 0,\ \underset{l\text{-th}}{1},\ 0,\ \ldots,\ 0),$$

for all $l = 1, \ldots, n$, for $n \in \mathbb{N}^{\infty}$. Note that if $k_1 \leq k_2$ in $\mathbb{N}^{\infty}$, then the Hilbert space $\mathbb{C}^{\times k_1}$ is understood as a (Hilbert-)subspace

$$K_{k_1,k_2} = \left\{ (\underbrace{z_1,\ \ldots,\ z_{k_1},\ 0,\ 0,\ \ldots,\ 0}_{k_2\text{-times}}) \,|\, z_i \in \mathbb{C} \text{ and } (z_1, \ldots, z_{k_1}) \in \mathbb{C}^{\times k_1} \right\} \tag{3.7}$$

of $\mathbb{C}^{\times k_2}$. From below, we use the term K_{k_1,k_2} of (3.7), whenever we regard the Hilbert space $\mathbb{C}^{\times k_1}$ as a subspace of $\mathbb{C}^{\times k_2}$. □

Remark that the operator J_{i+1}^{σ} of (3.4) is a "surjective" operator from H_{i+1} onto H_i, by (3.1) and (3.5). Moreover, it satisfies the following result.

Lemma 3.1. *Let* $J_{i+1}^{\sigma} \in B(H_{i+1}, H_i)$ *be an operator* (3.4)*, for* $\sigma \in S_{n_i}$*, for* $i = 1, \ldots, N-1$.

(i) *If* $n_i = n_{i+1}$ *in* $\mathbb{N}^{\infty}$*, then* J_{i+1}^{σ} *is unitarily equivalent to a unitary operator* $\mathbf{J}^{\sigma}$*, assigning*

$$\mathbf{J}^{\sigma} : e_l \in \mathbb{C}^{\times n_{i+1}} = \mathbb{C}^{\times n_i} \mapsto e_{\sigma(l)} \in \mathbb{C}^{\times n_i},$$

for all $l \in \{1, \ldots, n_{i+1} = n_i\}$*, in the operator algebra* $B(\mathbb{C}^{\times n_{i+1}})$*, where* e_l *are the basis vectors* (3.6) *of* $\mathbb{C}^{\times n_{i+1}}$.

(ii) *If* $n_i < n_{i+1}$ *in* $\mathbb{N}^\infty$, *then* J_{i+1}^σ *is unitarily equivalent to the operator* $\mathbf{J}^\sigma$ *in the operator algebra* $B\left(\mathbb{C}^{\times n_{i+1}}\right)$,

$$\mathbf{J}^\sigma = \begin{pmatrix} \sum_{n_i\times n_i} & O_{n_i\times(n_{i+1}-n_i)} \\ O_{(n_{i+1}-n_i)\times n_i} & O_{(n_{i+1}-n_i)\times(n_{i+1}-n_i)} \end{pmatrix},$$

the block-matrix operator on $\mathbb{C}^{\times n_{i+1}}$, *by regarding* $\mathbb{C}^{\times n_i}$ *as a closed subspace* $K_{n_i,n_{i+1}}$ *of* $\mathbb{C}^{\times n_{i+1}}$, *in the sense of* (3.7), *where* $O_{n\times k}$ *are the zero* $(n\times k)$*-matrices, for all* $n, k \in \mathbb{N}^\infty$. *Here, the operator* $\sum_{n_i\times n_i}$ *is an operator assigning,*

$$e_l \in \mathbb{C}^{\times n_{i+1}} \mapsto e_{\sigma(l)} \in \mathbb{C}^{\times n_{i+1}}, \quad \forall l = 1,\ldots,n_i,$$

which is a unitary operator on $K_{n_i,n_{i+1}} = \mathbb{C}^{\times n_i}$.

Proof. By (3.4) and (3.5), the proofs of (i) and (ii) are clear. Indeed, the Hilbert spaces H_i and H_{i+1} are isomorphic to $\mathbb{C}^{\times n_i}$, and $\mathbb{C}^{\times n_{i+1}}$, respectively (e.g., [15]), because there exist well-defined Hilbert-space isomorphisms from H_k onto $\mathbb{C}^{\times n_k}$, satisfying

$$\xi_l^{(k)} \in \mathcal{B}_k \mapsto e_l \in \mathbb{C}^{\times n_k},$$

for all $l = 1,\ldots,n_k$, for all $k = 1,\ldots, N$, where $\{e_1,\ldots,e_{n_k}\}$ is the canonical ONB, (3.6), of $\mathbb{C}^{\times n_k}$, for $n_k \in \mathbb{N}^\infty$. □

Let $J_{i+1}^\sigma \in B(H_{i+1},H_i)$ be as above for a fixed $i \in \{1,\ldots,N-1\}$, and for $\sigma \in S_{n_i}$. Then there exists a unique operator $J_{i+1}^{\sigma*}$ in the operator space $B(H_i,H_{i+1})$, such that

$$\begin{aligned} J_{i+1}^{\sigma*}\left(\sum_{l=1}^{n_i} t_l \xi_l^{(i)}\right) &= \sum_{l=1}^{n_i} t_{\sigma^{-1}(l)} \xi_{\sigma^{-1}(l)}^{(i+1)} \\ &= \sum_{l=1}^{n_i} t_{\sigma^{-1}(l)} \xi_{\sigma^{-1}(l)}^{(i+1)} + \sum_{t=n_i+1}^{n_{i+1}} 0\cdot \xi_t^{(i+1)}, \end{aligned}$$

assigning

$$J^* : \xi_l^{(i)} \in \mathcal{B}_i \mapsto \xi_{\sigma^{-1}(l)}^{(i+1)} \in \mathcal{B}_{i+1}, \quad \forall l = 1,\ldots,n_i, \tag{3.8}$$

from H_i into H_{i+1}, where σ^{-1} is the *inverse of* σ in S_{n_i}. By (3.8), such an operator $J_{i+1}^{\sigma*}$ is injective from H_i to H_{i+1}.

Proposition 3.2. *Let* $J_{i+1}^\sigma \in B(H_{i+1},H_i)$ *be an operator* (3.4), *for* $i \in \{1,\ldots,N-1\}$, *and* $\sigma \in S_{n_i}$, *and let* $J_{i+1}^{\sigma*} \in B(H_i,H_{i+1})$ *be an operator* (3.8). *Then* $J_{i+1}^{\sigma*}$ *is the adjoint of* J_{i+1}^σ *in the sense that*

$$\left\langle J_{i+1}^\sigma(\xi),\ \eta\right\rangle_i = \left\langle \xi,\ J_{i+1}^{\sigma*}(\eta)\right\rangle_{i+1}, \tag{3.9}$$

for all $\xi \in H_{i+1}, \eta \in H_i$.

Proof. Let $\xi = \sum_{l=1}^{n_{i+1}} t_l \xi_l^{(i+1)} \in H_{i+1}$, and $\eta = \sum_{t=1}^{n_i} s_t \xi_t^{(i)} \in H_i$, for the fixed $i \in \{1, \ldots, N-1\}$. Then

$$\begin{aligned}
\left\langle J_{i+1}^{\sigma}(\xi),\ \eta\right\rangle_i &= \left\langle \sum_{l=1}^{n_i} t_{\sigma(l)} \xi_{\sigma(l)}^{(i)},\ \sum_{l=1}^{n_i} s_l \xi_l^{(i)} \right\rangle_i = \sum_{l=1}^{n_i} t_l \overline{s_l} \\
&= \left\langle \sum_{l=1}^{n_i} t_l \xi^{(i+1)},\ \sum_{l=1}^{n_i} s_{\sigma^{-1}(l)} \xi_{\sigma^{-1}(l)}^{(i+1)} \right\rangle_{i+1} \\
&\quad + \left\langle \sum_{s=n_i+1}^{n_{i+1}} t_s \xi_s^{(i+1)},\ \sum_{t=n_i+1}^{n_{i+1}} 0 \cdot \xi_t^{(i+1)} \right\rangle_{i+1} \qquad (3.10)\\
&\text{by (3.8)}\\
&= \left\langle \sum_{l=1}^{n_{i+1}} t_l \xi_l^{(i+1)},\ \sum_{l=1}^{n_i} s_{\sigma^{-1}(l)} \xi_{\sigma^{-1}(l)}^{(i+1)} + \sum_{t=n_i+1}^{n_{i+1}} 0 \cdot \xi_t^{(i+1)} \right\rangle_{i+1} \\
&= \left\langle \xi,\ J_{i+1}^{\sigma *}(\eta)\right\rangle_{i+1}.
\end{aligned}$$

Since $\xi \in H_{i+1}$ and $\eta \in H_i$ are arbitrary, the operator $J_{i+1}^{\sigma *} \in B(H_i, H_{i+1})$ is the adjoint of $J_{i+1}^{\sigma} \in B(H_{i+1}, H_i)$, satisfying the relation (3.9), by (3.10). □

3.2. JUMPS $J_{j>i}^{\sigma} \in B(H_j, H_i)$ FOR $\sigma \in S_{n_i}$

Throughout this section, we use the same terminology and notations of Section 3.1. Now, take

$$i < j \in \{1, \ldots, N\}. \tag{3.11}$$

For a fixed permutation $\sigma \in S_{n_i}$, define now an operator $J_{j>i}^{\sigma}$ in the operator space $B(H_j,\ H_i)$ by

$$J_{j>i}^{\sigma} = J_{i+1}^{\sigma} \circ J_{i+2}^{\mathbf{1}_{n_{i+1}}} \circ \cdots \circ J_{j-1}^{\mathbf{1}_{n_{j-2}}} \circ J_j^{\mathbf{1}_{n_{j-1}}} : H_j \to H_i, \tag{3.12}$$

where $J_{i+1}^{\sigma}, J_{i+2}^{\mathbf{1}_{i+1}} \ldots,\ J_j^{\mathbf{1}_{j-1}}$ are in the sense of (3.4), and $\mathbf{1}_{n_k}$ are the group-identities of the symmetric groups S_{n_k} of (3.2), for all $k = 1, \ldots, N$. By the definition (3.12), and by the induction on (3.4), one has that

$$J_{j>i}^{\sigma}\left(\sum_{l=1}^{n_j} t_l \xi_l^{(j)}\right) = \sum_{l=1}^{n_i} t_{\sigma(l)} \xi_{\sigma(l)}^{(i)},$$

assigning

$$J_{j>i}^{\sigma} : \xi_l^{(j)} \in \mathcal{B}_j \mapsto \begin{cases} \xi_{\sigma(l)}^{(j)} \in \mathcal{B}_i & \text{if } l = 1, \ldots, n_i, \\ \\ 0_{H_i} \in H_i & \text{if } l = n_i + 1,\ \ldots,\ n_j, \end{cases} \tag{3.13}$$

for all $l = 1, \ldots, n_j$.

Definition 3.3. The operators $J^{\sigma}_{j>i} \in B(H_j, H_i)$ of (3.12) are said to be the σ-jumps from H_j to H_i, for $\sigma \in S_{n_i}$, for all $i < j \in \{1, \ldots, N\}$.

By (i) and (ii) of Lemma 3.1, (3.9) and (3.13), we obtain the following result.

Theorem 3.4. *Let $i < j \in \{1, \ldots, N\}$, and $J^{\sigma}_{j>i} \in B(H_j, H_i)$, the corresponding σ-jump* (3.12) *from H_j to H_i, for $\sigma \in S_{n_i}$.*

(i) *If $n_i = n_j$ in $\mathbb{N}^{\infty}$, then $J^{\sigma}_{j>i}$ is unitarily equivalent to a unitary $\mathbf{J}^{\sigma} \in B(\mathbb{C}^{\times n_j})$, assigning*

$$\mathbf{J}^{\sigma} : e_l \in \mathbb{C}^{\times n_j} = \mathbb{C}^{\times n_i} \mapsto e_{\sigma(l)} \in \mathbb{C}^{\times n_i},$$

where e_l are the basis vectors (3.6) *of $\mathbb{C}^{\times n_j}$.*

(ii) *If $n_i < n_j$ in $\mathbb{N}^{\infty}$, then $J^{\sigma}_{j>i} \in B(H_j, H_i)$ is unitarily equivalent to the block operator $\mathbf{J}^{\sigma} \in B(\mathbb{C}^{\times n_j})$,*

$$\mathbf{J}^{\sigma} = \begin{pmatrix} \sum_{n_i \times n_i} & O_{n_i \times (n_j - n_i)} \\ O_{(n_j - n_i) \times n_i} & O_{(n_j - n_i) \times (n_j - n_i)} \end{pmatrix},$$

by regarding $\mathbb{C}^{\times n_i}$ as a subspace of $\mathbb{C}^{\times n_j}$, where $\sum_{n_i \times n_i}$ is in the sense of Lemma 3.1(ii).

(iii) *If $n_i \leq n_j$ in $\mathbb{N}^{\infty}$, there exists the adjoint $J^{\sigma *}_{j>i} \in B(H_i, H_j)$ of $J^{\sigma}_{j>i}$, such that*

$$J^{\sigma *}_{j>i}\left(\sum_{l=1}^{n_i} t_l \xi_l^{(i)}\right) = \sum_{l=1}^{n_i} t_{\sigma^{-1}(l)} \xi^{(j)}_{\sigma^{-1}(l)} + \sum_{t=n_i+1}^{n_j} 0 \cdot \xi_t^{(j)},$$

in H_j, where σ^{-1} is the inverse of σ in S_{n_i}.

Proof. If $n_i = n_j$ in $\mathbb{N}^{\infty}$, then

$$n_i = n_{i+1} = n_{i+2} = \ldots = n_j \text{ in } \mathbb{N}^{\infty},$$

by the condition (3.1). So, in such a case, the σ-jump $J^{\sigma}_{j>i}$ is unitarily equivalent to the operator $\mathbf{J}^{\sigma}$ of (i) by Lemma 3.1(i) and (3.12).

Meanwhile, if $n_i < n_j$ in $\mathbb{N}^{\infty}$, then there exists at lease one quantity k_0, such that $i < k_0 \leq j$, and

$$n_i \leq n_{i+k_0} < n_{i+k_0+1} \leq \ldots \leq n_j,$$

in $\mathbb{N}^{\infty}$, and hence, $J^{\sigma}_{j>i}$ is unitarily equivalent to the matrix $\mathbf{J}^{\sigma} \in B(\mathbb{C}^{\times n_j})$ of (ii), mapping $\mathbb{C}^{\times n_j}$ onto the subspace

$$K_{i,j} = \bigcap_{k=i}^{j-1} K_{n_k, n_{k+1}} \text{ of } \mathbb{C}^{\times n_j} \tag{3.14}$$

where $K_{n_k, n_{k+1}}$ are the subspaces (3.7) of $\mathbb{C}^{\times n_{k+1}}$. This subspace $K_{i,j}$ of (3.14) is isomorphic to $\mathbb{C}^{\times n_i}$, i.e.,

$$K_{i,j} \overset{\text{Hilbert}}{=} \mathbb{C}^{\times n_i} \overset{\text{Hilbert}}{=} H_i,$$

where "$\overset{\text{Hilbert}}{=}$" means "being Hilbert-space isomorphic".

By (3.12), one can naturally define the adjoint $J^{\sigma*}_{j>i} \in B(H_i, H_j)$ of $J^{\sigma}_{j>i} \in B(H_j, H_i)$ by

$$J^{\sigma*}_{j>i} = J_j^{\mathbf{1}_{n_{j-1}}*} \circ \ldots \circ J_{i+2}^{\mathbf{1}_{n_{i+1}}*} \circ J^{\sigma*}_{i+1},$$

where $J^{\beta*}_k$ are the adjoints of J^{β}_k (see Lemma 3.1(i)), for all $k = i+1, \ldots, j$, for all $\beta \in S_{n_{k-1}}$. So, inductively, this operator $J^{\sigma*}_{j>i}$ satisfies the formula in (iii), i.e.,

$$\left\langle J^{\sigma}_{j>i}(\xi),\ \eta \right\rangle_i = \left\langle \xi,\ J^{\sigma*}_{j>i}(\eta) \right\rangle_j,$$

by (3.9), for all $\xi \in H_j$ and $\eta \in H_i$. □

For $i < j \in \{1, \ldots, N\}$, let $J^{\sigma}_{j>i} \in B(H_j, H_i)$ be the σ-jump (3.12) for $\sigma \in S_{n_i}$, and $J^{\sigma*}_{j>i} \in B(H_i, H_j)$, the adjoint of $J^{\sigma}_{j>i}$ (see Theorem 3.4(iii)). Then one can get the following operators:

$$X_{j>i} = J^{\sigma*}_{j>i} \circ J^{\sigma}_{j>i} \in B(H_j),$$

and

$$Y_{j>i} = J^{\sigma}_{j>i} \circ J^{\sigma*}_{j>i} \in B(H_i). \tag{3.15}$$

Theorem 3.5. *Let $X_{j>i} \in B(H_j)$ be in the sense of* (3.15)*. Then*

$$X_{j>i} = \begin{cases} P_{j>i} & \text{if } n_i < n_j \text{ in } \mathbb{N}^{\infty}, \\ 1_{H_j} & \text{if } n_i = n_j \text{ in } \mathbb{N}^{\infty}, \end{cases} \tag{3.16}$$

where $P_{j>i} \in B(H_j)$ is the projection from H_j onto the subspace $K_{j>i}$

$$K_{j>i} = \overline{span_{\mathbb{C}}\{\xi^{(j)}_l \in \mathcal{B}_{i+1} : l = 1, \ldots, n_i\}}^{H_j},$$

where $span_{\mathbb{C}}(Z)$ are (*pure-algebraic*) *subspace spanned by subsets Z, and $\overline{Y}^{H_j}$ are the H_j-topology closures of subsets Y of H_j.*

Meanwhile, if $Y_{j>i} \in B(H_i)$ is in the sense of (3.15)*, then*

$$Y_{j>i} = 1_{H_i} \in B(H_i). \tag{3.17}$$

Proof. By (i)–(iii) of Theorem 3.4, the operator $X_{j>i}$ of (3.15) assigns

$$H_j \ni \sum_{l=1}^{n_j} t_l \xi^{(j)}_l \overset{J^{\sigma}_{j>i}}{\longmapsto} \sum_{l=1}^{n_i} t_{\sigma(l)} \xi^{(i)}_{\sigma(l)} \in H_i$$

$$\overset{J^{\sigma*}_{j>i}}{\longmapsto} \sum_{l=1}^{n_i} t_{\sigma^{-1}(\sigma(l))} \xi^{(i)}_{\sigma^{-1}(\sigma(l))} = \sum_{l=1}^{n_i} t_l \xi^{(j)}_l + \sum_{l=n_i+1}^{n_j} 0 \cdot \xi^{(j)}_l \in H_j,$$

in $B(H_j)$. It shows that $X_{j>i}$ is identified with the projection $P_{j>i} \in B(H_j)$, projecting H_j onto the subspace $K_{j>i}$ of H_j, where

$$K_{j>i} = \overline{\mathrm{span}_{\mathbb{C}}\{\xi^{(j)}_l \in \mathcal{B}_{i+1} : l = 1, \ldots, n_i\}},$$

in H_j. Remark that if $n_i = n_j$ in $\mathbb{N}^\infty$, then the projection $P_{j>i}$ is identical to the identity operator 1_{H_j} of $B(H_j)$, because $K_{j>i} = H_j$.

Now, let $Y_{j>i} \in B(H_i)$ be in the sense of (3.15). Then it assigns

$$H_i \ni \sum_{l=1}^{n_i} t_l \xi_l^{(i)} \overset{J_{j>i}^{\sigma *}}{\longmapsto} \sum_{l=1}^{n_i} t_{\sigma^{-1}(l)} \xi_{\sigma^{-1}(l)}^{(i+1)} + \sum_{l=n_i+1}^{n_j} 0 \cdot \xi_l^{(i+1)} \in H_j,$$

$$\overset{J_{j>i}^{\sigma}}{\longmapsto} \sum_{l=1}^{n_i} t_{\sigma(\sigma^{-1}(l))} \xi_{\sigma(\sigma^{-1}(l))}^{(i)} = \sum_{l=1}^{n_i} t_l \xi_l^{(i)} \in H_i,$$

in $B(H_i)$. Therefore, this operator $Y_{j>i}$ is identical to the identity operator 1_{H_i} of $B(H_i)$. □

3.3. JUMP OPERATORS ON $\mathfrak{F}[H_1, \ldots, H_N]$

Under the same hypothesis, let $J_{j>i}^{\sigma}$ be the σ-jumps from H_j to H_i, for any $i < j \in \{1, \ldots, N\}$, for $\sigma \in S_{n_i}$. Now, let

$$\mathfrak{F} = \mathfrak{F}[H_1, \ldots, H_N]$$

be the free Hilbert space (2.7) of $H_1, \ldots, H_N$.

Fix $i < j \in \{1, \ldots, N\}$, and $\sigma \in S_{n_i}$, and define an operator $J^\sigma[j > i] \in B(\mathfrak{F})$ by a bounded linear transformation on $\mathfrak{F}$, satisfying the following three conditions (3.18), (3.19) and (3.20) on the ONB $\mathfrak{B}$ of (2.11) in $\mathfrak{F}$, where

$$J^\sigma[j > i]\left(\xi_l^{(k)}\right) = \begin{cases} J_{j>i}^{\sigma}\left(\xi_l^{(j)}\right) & \text{if } k = j, \\ \xi_l^{(k)} & \text{otherwise,} \end{cases} \tag{3.18}$$

for all $\xi_l^{(k)} \in \mathcal{B}_k \subset \left(\bigcup_{l=1}^{N} \mathcal{B}_l\right) \subset \mathfrak{B}$ in $\mathfrak{F}$, for $k = 1, \ldots, N$, and

$$J^\sigma[j > i]\left(\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}\right) = \xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}, \tag{3.19}$$

for $\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k} \in \mathfrak{B}$, if $(i_1, \ldots, i_k) \in Alt\left(\{1, \ldots, N\}^k\right)$ does not contain j as its entry, for $k \in \mathbb{N}$, and if the alternating k-tuple $(i_1, \ldots, i_k)$ contains at least one i_t-th entry j, i.e., if

$$(i_1, \ldots, i_{t-1}, \text{“}i_t = j\text{”}, i_{t+1}, \ldots, i_k)$$

in $Alt\left(\{1,\ldots,N\}^k\right)$, for $k \in \mathbb{N}$, then

$$
\begin{aligned}
&J^{\sigma}[j>i]\left(\xi_{i_1,\ldots,j,\ldots,i_k}^{l_1,\ldots,l,\ldots,l_k}\right)\\
&= J^{\sigma}[j>i]\left(\xi_{l_1}^{(i_1)} \otimes \ldots \otimes \xi_{l_{t-1}}^{(i_{t-1})} \otimes \xi_{l}^{(j)} \otimes \xi_{l_{t+1}}^{(i_{t+1})} \otimes \ldots \otimes \xi_{l_k}^{(i_k)}\right)\\
&= \xi_{l_1}^{(i_1)} \otimes \ldots \otimes \xi_{l_{t-1}}^{(i_{t-1})} \otimes J_{j>i}^{\sigma}\left(\xi_{l}^{(j)}\right) \otimes \xi_{l_{t+1}}^{(i_{t+1})} \otimes \ldots \otimes \xi_{l_k}^{(i_k)}\\
&= \begin{cases}
\xi_{l_1}^{(i_1)} \otimes \ldots \otimes \xi_{l_{t-1}}^{(i_{t-1})} \otimes J_{j>i}^{\sigma}\left(\xi_{l}^{(j)}\right) \otimes \xi_{l_{t+1}}^{(i_{t+1})} \otimes \ldots \otimes \xi_{l_k}^{(i_k)} \\
\qquad\qquad (i_{t-1} \neq i \neq i_{t+1}),\\
\xi_{l_1}^{(i_1)} \otimes \ldots \otimes \left\langle \xi_{l_{t-1}}^{(i_{t-1})}, J_{j>i}^{\sigma}\left(\xi_{l}^{(j)}\right)\right\rangle_i \xi_{l}^{(i)} \otimes \xi_{l_{t+1}}^{(i_{t+1})} \otimes \ldots \otimes \xi_{l_k}^{(i_k)} \\
\qquad\qquad (i_{t-1} = i \neq i_{t+1}),\\
\xi_{l_1}^{(i_1)} \otimes \ldots \otimes \xi_{l_{t-1}}^{(i_{t-1})} \otimes \left\langle J_{j>i}^{\sigma}\left(\xi_{l}^{(j)}\right), \xi_{l_{t+1}}^{(i_{t+1})}\right\rangle_i \xi_{l}^{(i)} \otimes \ldots \otimes \xi_{l_k}^{(i_k)} \\
\qquad\qquad (i_{t-1} \neq i = i_{t+1}),\\
\xi_{l_1}^{(i_1)} \otimes \ldots \otimes \left\langle \xi_{l_{t-1}}^{(i_{t-1})}, J_{j>i}^{\sigma}\left(\xi_{l}^{(j)}\right)\right\rangle_i \left\langle J_{j>i}^{\sigma}\left(\xi_{l}^{(j)}\right), \xi_{l_{t+1}}^{(i_{t+1})}\right\rangle_i \xi_{l}^{(i)} \otimes \ldots \otimes \xi_{l_k}^{(i_k)} \\
\qquad\qquad (i_{t-1} = i = i_{t+1})
\end{cases}\\
&= \begin{cases}
\chi_l \xi_{i_1,\ldots,i,\ldots,i_k}^{l_1,\ldots,\sigma(l),\ldots,l_k} & \text{if } i_{t-1} \neq i \neq i_{t+1},\\
\chi_l \xi_{l_1}^{(i_1)} \otimes \ldots \otimes \delta_{l_{t-1},\sigma(l)} \xi_{l}^{(i)} \otimes \xi_{l_{t+1}}^{(i_{t+1})} \otimes \ldots \otimes \xi_{l_k}^{(i_k)} & \text{if } i_{t-1} = i \neq i_{t+1},\\
\chi_l \xi_{l_1}^{(i_1)} \otimes \ldots \otimes \xi_{l_{t-1}}^{(i_{t-1})} \otimes \delta_{\sigma(l),l_{t+1}} \xi_{l}^{(i)} \otimes \ldots \otimes \xi_{l_k}^{(i_k)} & \text{if } i_{t-1} \neq i = i_{t+1},\\
\chi_l \xi_{l_1}^{(i_1)} \otimes \ldots \otimes \delta_{l_{t-1},\sigma(l)} \delta_{\sigma(l),l_{t+1}} \xi_{l}^{(i)} \otimes \ldots \otimes \xi_{l_k}^{(i_k)} & \text{if } i_{t-1} = i = i_{t+1}
\end{cases}
\end{aligned}
\tag{3.20}
$$

for all $\xi_{i_1,\ldots,j,\ldots,i_k}^{l_1,\ldots,l,\ldots,l_k} \in \mathfrak{B}$ in $\mathfrak{F}$, where

$$
\chi_l = \begin{cases} 1 & \text{if } l = 1, \ \ldots, \ n_i,\\ 0 & \text{if } l = n_i + 1, \ \ldots, \ n_j. \end{cases}
$$

In fact, the condition (3.20) implies both (3.18) and (3.19), but we independently put these special cases (3.18) and (3.19) of (3.20) to emphasize such cases. In (3.20), if $t = k$, then the fourth equality is not considered, similarly, if $t = 1$, then the second equality is not considered naturally. Also, it is possible there are multi-entries identified with j in $(i_1, \ldots, i_k)$. In such a case, do the above processes step-by-step until they end.

By definition, this operator $J^{\sigma}[j > i]$ assigns basis vectors of $\mathfrak{F}$ to either basis vectors, or the zero vector $O_{\mathfrak{F}}$ of $\mathfrak{F}$, i.e.,

$$
J^{\sigma}[j > i] \mid_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B} \cup \{O_{\mathfrak{F}}\},
$$

in $\mathfrak{F}$, by replacing the index j to the index i.

Example 3.6. Let $\mathfrak{F} = \mathfrak{F}[H_1, H_2, H_3]$ be the free Hilbert space of l^2-spaces $H_1 = l^2(n_1)$, $H_2 = l^2(n_2), H_3 = l^3(n_3)$, satisfying

$$15 \leq n_1 \leq n_2 \leq n_3 \text{ in } \mathbb{N}^\infty.$$

Then

$$J^{\mathbf{1}_{n_1}}[3 > 1]\left(\xi_2^{(3)}\right) = \xi_2^{(1)} \text{ and } J^{\mathbf{1}_{n_1}}[3 > 1]\left(\xi_7^{(2)}\right) = \xi_7^{(2)},$$

by (3.18), and

$$J^{\mathbf{1}_{n_1}}[3 > 1]\left(\xi_{2,1,2,1,2}^{5,6,1,2,1}\right) = \xi_{2,1,2,1,2}^{5,6,1,2,1},$$

by (3.19), and

$$J^{\mathbf{1}_{n_1}}[3 > 1]\left(\xi_{1,3,2,1}^{2,4,5,7}\right) = \xi_{1,1,2,1}^{2,4,5,7} = \delta_{2,4}\xi_{1,2,1}^{4,5,7} = O_{\mathfrak{F}}$$

and

$$J^{\mathbf{1}_{n_1}}[3 > 1]\left(\xi_{1,3,2,1}^{2,2,5,7}\right) = \delta_{2,2}\xi_{1,2,1}^{2,5,7} = \xi_{1,2,1}^{2,5,7},$$

and

$$J^{\mathbf{1}_{n_1}}[3 > 1]\left(\xi_{1,3,1}^{10,10,10}\right) = \delta_{10,10}\delta_{10,10}\xi_{10}^{(1)} = \xi_{10}^{(1)},$$

by (3.20).

By the definition of the operator $J^\sigma[j > i] \in B(\mathfrak{F})$, one can have the corresponding adjoint $J^\sigma[j > i]^* \in B(\mathfrak{F})$, as an operator satisfying the following three conditions (3.21), (3.22) and (3.23):

$$J^\sigma[j > i]^*\left(\xi_l^{(k)}\right) = \begin{cases} J_{j>i}^{\sigma*}\left(\xi_l^{(i)}\right) = \xi_{\sigma^{-1}(l)}^{(j)} & \text{if } k = i, \\ \xi_l^{(k)} & \text{if } k \neq i, \end{cases} \tag{3.21}$$

for all $\xi_l^{(k)} \in \mathcal{B}_k \subset \left(\overset{N}{\underset{l=1}{\cup}} \mathcal{B}_l\right) \subset \mathfrak{B}$ in $\mathfrak{F}$, where $J_{j>i}^{\sigma*} \in B(H_i, H_j)$ is the adjoint of the σ-jump $J_{j>i}^\sigma \in B(H_j, H_i)$ (see Theorem 3.4(iii)), and if a k-tuple

$$(i_1, \ldots, i_k) \in Alt\left(\{1, \ldots, N\}^k\right), \text{ for } k \in \mathbb{N},$$

does not contain i as its entry, then

$$J^\sigma[j > i]^*\left(\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}\right) = \xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k} = 1_{\mathfrak{F}}\left(\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}\right), \tag{3.22}$$

for $\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k} \in \mathfrak{B}$, in $\mathfrak{F}$; and if an alternating k-tuple is

$$(i_1, \ldots, i_{t-1}, \text{“}i_t = i\text{''}, i_{t+1}, \ldots, i_k),$$

then

$$\begin{aligned}
&J^{\sigma}[j>i]^{*}\left(\xi_{i_1,\ldots,i,\ldots,i_k}^{l_1,\ldots,l,\ldots,l_k}\right)\\
&=J^{\sigma}[j>i]^{*}\left(\xi_{l_1}^{(i_1)}\otimes\ldots\otimes\xi_{l_{t-1}}^{(i_{t-1})}\otimes\xi_{l}^{(i)}\otimes\xi_{l_{t+1}}^{(i_{t+1})}\otimes\ldots\otimes\xi_{l_k}^{(i_k)}\right)\\
&=\xi_{l_1}^{(i_1)}\otimes\ldots\otimes\xi_{l_{t-1}}^{(i_{t-1})}\otimes J_{j>i}^{\sigma *}(\xi_{l}^{(i)})\otimes\xi_{l_{t+1}}^{(i_{t+1})}\otimes\ldots\otimes\xi_{l_k}^{(i_k)}\\
&=\begin{cases}
\xi_{l_1}^{(i_1)}\otimes\ldots\otimes\xi_{l_{t-1}}^{(i_{t-1})}\otimes\xi_{l}^{(j)}\otimes\xi_{l_{t+1}}^{(i_{t+1})}\otimes\ldots\otimes\xi_{l_k}^{(i_k)} & \text{if } i_{t-1}\neq j\neq i_{t+1},\\
\xi_{l_1}^{(i_1)}\otimes\ldots\otimes\left(\delta_{l_{i-1},l}\xi_{l}^{(j)}\right)\otimes\xi_{l_{t+1}}^{(i_{t+1})}\otimes\ldots\otimes\xi_{l_k}^{(i_k)} & \text{if } i_{t-1}=j\neq i_{t+1},\\
\xi_{l_1}^{(i_1)}\otimes\ldots\otimes\xi_{l_{t-1}}^{(i_{t-1})}\otimes\left(\delta_{l,l_{t+1}}\xi_{l_{t+1}}^{(j)}\right)\otimes\ldots\otimes\xi_{l_k}^{(i_k)} & \text{if } i_{t-1}\neq j=i_{t+1},\\
\xi_{l_1}^{(i_1)}\otimes\ldots\otimes\left(\delta_{l_{t-1},l}\delta_{l,l_{t+1}}\xi_{l_{t+1}}^{(j)}\right)\otimes\ldots\otimes\xi_{l_k}^{(i_k)} & \text{if } i_{t-1}=j=i_{t+1},
\end{cases}
\end{aligned}\tag{3.23}$$

similar to (3.20).

The condition (3.23) implies both (3.21) and (3.22). Actually, the process of (3.23) will be done step-by-step until the above processes end if there are multi-entries identical to i, as in (3.20).

By (iii) of Theorem 3.4, and (3.18) through (3.23), we have

$$\langle J^{\sigma}[j>i](\xi),\ \eta\rangle=\langle\xi,\ J^{\sigma}[j>i]^{*}(\eta)\rangle\,,\tag{3.24}$$

for all $\xi,\eta\in\mathfrak{F}$, where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathfrak{F}$.

Definition 3.7. Let $J^{\sigma}[j>i]\in B(\mathfrak{F})$ be the operator satisfying (3.18), (3.19) and (3.20), for $\sigma\in S_{n_i}$. Then it is called the σ-jump operator from H_j onto H_i in $\mathfrak{F}$, for $i<j\in\{1,\ldots,N\}$.

By definition, we obtain the following result.

Lemma 3.8. *Let $J=J^{\sigma}[j>i]\in B(\mathfrak{F})$ be the σ-jump operator from H_j onto H_i, for $i<j\in\{1,\ldots,N\}$, and $\sigma\in S_{n_i}$.*

(i) *If $n_i=n_j$ in $\mathbb{N}^{\infty}$, then J is a unitary on $\mathfrak{F}$, i.e.,*

$$J^{*}J=1_{\mathfrak{F}}=JJ^{*},\ \text{on } \mathfrak{F}.$$

(ii) *If $n_i<n_j$ in $\mathbb{N}^{\infty}$, then J satisfies that*

$$J^{*}J=P_{j>i}=JJ^{*},$$

where $P_{j>i}\in B(\mathfrak{F})$ is the projection, projecting $\mathfrak{F}$ onto the subspace $K_{j>i}$ of $\mathfrak{F}$, where

$$K_{j>i}=\overline{span_{\mathbb{C}}\left(\left\{\xi\in\mathfrak{B}\ \middle|\ \begin{array}{c}\xi=\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k},\ \textit{if } (i_1,\ldots,i_k)\\ \textit{does not contain } j,\ \textit{or}\\ \xi=\xi_{i_1,\ldots,j,\ldots,i_k}^{l_1,\ldots,l,\ldots,l_k},\\ \textit{with } l\in\{1,\ldots,n_i\}\end{array}\right\}\right)}^{\mathfrak{F}}.$$

Proof. Assume first that $n_i = n_j$ in $\mathbb{N}^\infty$. Recall that, if $n_i = n_j$, then the σ-jump $J^\sigma_{j>i} \in B(H_j, H_i)$ is a unitary in the sense that:

$$J^{\sigma^*}_{j>i} J^\sigma_{j>i} = 1_{H_j} = 1_{H_i} = J^\sigma_{j>i} J^{\sigma*}_{j>i} \quad \text{in } B(H_j) = B(H_i),$$

by (3.23) and (3.24). Since the jump operator $J = J^\sigma[j > i] \in B(\mathfrak{F})$ is defined by an operator satisfying (3.18), (3.19) and (3.20), one has that

$$J^* J = 1_{\mathfrak{F}} = JJ^* \quad \text{on } \mathfrak{F},$$

by (3.23) and (3.24), whenever $n_i = n_j$. Therefore, the statement (i) holds.

Now, suppose that $n_i < n_j$ in $\mathbb{N}^\infty$. Define a subspace $K_{j>i}$ of $\mathfrak{F}$ by

$$K_{j>i} = \overline{span_{\mathbb{C}} \mathfrak{B}_{j>i}}^{\mathfrak{F}},$$

with

$$\mathfrak{B}_{j>i} = \big\{\xi \in \mathfrak{B} \,|\, \xi = \xi^{l_1,\ldots,l_k}_{i_1,\ldots,i_k}, \text{ if } (i_1, \ldots, i_k) \text{ does not contain } j, \\ \text{or } \xi = \xi^{l_1,\ldots,l,\ldots,l_k}_{i_1,\ldots,j,\ldots,i_k}, \text{ containing tensor factor } \xi^{(j)}_l \in \mathcal{B}_j, \\ \text{with } l \in \{1, \ldots, n_i\}\big\}.$$

Then

$$J^* J = P_{j>i} = JJ^* \text{ in } B(\mathfrak{F}),$$

where $P_{j>i}$ is a projection on $\mathfrak{F}$, projecting $\mathfrak{F}$ onto $K_{j>i}$, by (3.16) and (3.17). Therefore, the statement (ii) holds. □

The above lemma shows that the σ-jump operator $J^\sigma[j > i] \in B(\mathfrak{F})$ is a partial isometry on $\mathfrak{F}$, in general. Recall that an operator $T \in B(H)$ is a *partial isometry*, if $T^*T \in B(H)$ is a projection on a Hilbert space H. It is well-known that $T \in B(H)$ is a partial isometry, if and only if $T = TT^*T$, if and only if the adjoint $T^* \in B(H)$ of T is a partial isometry, if and only if TT^* is a projection, if and only if $T^* = T^*TT^*$ on H (e.g., see [15]).

Recall also that an operator $T \in B(H)$ is said to be *normal* if

$$T^*T = TT^* \text{ on } H.$$

Theorem 3.9. *Let $J^\sigma[j > i] \in B(\mathfrak{F})$ be the σ-jump operator from H_j to H_i, for $i < j \in \{1, \ldots, N\}$, and $\sigma \in S_{n_i}$.*

(i) *If $n_i = n_j$ in $\mathbb{N}^\infty$, then $J^\sigma[j > i]$ is a unitary on $\mathfrak{F}$.*
(ii) *If $n_i < n_j$ in $\mathbb{N}^\infty$, then $J^\sigma[j > i]$ is a normal partial isometry on $\mathfrak{F}$.*

Proof. The proof of the statement (i) is done by Lemma 3.8(i), i.e., if $n_i = n_j$, then the jump operator $J^\sigma[j > i]$ is a unitary on $\mathfrak{F}$.

The statement (ii) is proven by Lemma 3.8(ii). Indeed, if $n_i < n_j$, then

$$J^\sigma[j > i]^* J^\sigma[j > i] = P_{j>i} = J^\sigma[j > i] J^\sigma[j > i]^*,$$

on $\mathfrak{F}$, where $P_{j>i} \in B(\mathfrak{F})$ is the projection on $\mathfrak{F}$ (see Lemma 3.8(ii)), projecting $\mathfrak{F}$ onto the subspace $K_{j>i} \subset \mathfrak{F}$. These operator-equalities show that both $J^{\sigma}[j > i]$ and $J^{\sigma}[j > i]^{*}$ are partial isometries on $\mathfrak{F}$, because $P_{j>i} \in B(\mathfrak{F})$ is a projection. Moreover, they also show that $J^{\sigma}[j > i]$ is normal on $\mathfrak{F}$. □

The above theorem characterizes the operator-theoretic spectral properties of jump operators $J^{\sigma}[j > i]$ on the free Hilbert space $\mathfrak{F}$, as normal partial isometries, meanwhile these normal partial isometries can be unitaries if and only if $n_i = n_j$, for $i < j \in \{1, \ldots, N\}$.

4. SHIFT OPERATORS ON $\mathfrak{F}[H_1, \ldots, H_N]$

In this section, we construct-and-study shift operators on the free Hilbert space $\mathfrak{F}$ of (2.7) generated by the fixed l^2-Hilbert spaces $H_1, \ldots, H_N$ of (2.1).

4.1. SHIFTINGS s_k ON $\mathcal{B}_k$

Let H_k be a Hilbert space $l^2(n_k)$ of (2.1), and $\mathcal{B}_k$, the ONB of H_k, for $k = 1, \ldots, N$. By regarding $\mathcal{B}_k$ as an independent countable discrete set, define a function

$$s_k : \mathcal{B}_k \to \mathcal{B}_k,$$

by a function satisfying certain conditions, for $k = 1, \ldots, N$. From below, we fix $k \in \{1, \ldots, N\}$, and the corresponding ONB $\mathcal{B}_k$ of H_k, and understand $\mathcal{B}_k$ as a countable discrete set. Suppose first that $n_k < \infty$ in $\mathbb{N}^{\infty}$. Then define a function s_k on $\mathcal{B}_k$ by

$$s_k\left(\xi_l^{(k)}\right) \overset{def}{=} \begin{cases} \xi_{l+1}^{(k)} & \text{if } l = 1, \ldots, n_k - 1, \\ \xi_1^{(k)} & \text{if } l = n_k, \end{cases} \tag{4.1}$$

in $\mathcal{B}_k$. Secondly, if $n_k = \infty$ in $\mathbb{N}^{\infty}$, then we define s_k on $\mathcal{B}_k$ by

$$s_k\left(\xi_l^{(k)}\right) \overset{def}{=} \xi_{l+1}^{(k)}, \tag{4.2}$$

on $\mathcal{B}$.

Lemma 4.1. *Let $s_k : \mathcal{B}_k \to \mathcal{B}_k$ be a function* (4.1)*, or* (4.2)*.*

(i) *If $n_k < \infty$, then the functions s_k of* (4.1) *is bijective.*
(ii) *If $n_k = \infty$, then the functions s_k of* (4.2) *is injections, but not surjective.*

Proof. The proofs of (i) and (ii) are clear by the very definitions (4.1) and (4.2). In particular, if s_k is in the sense of (4.2), then s_k is injective, but not surjective, since

$$s_k\left(\mathcal{B}_k\right) = \left\{\xi_l^{(k)} \in \mathcal{B}_k : l > 1 \text{ in } \mathbb{N}\right\} \subsetneq \mathcal{B}_k. \qquad \square$$

Definition 4.2. A function $s_k : \mathcal{B}_k \to \mathcal{B}_k$ of (4.1), or (4.2), is called the shifting on $\mathcal{B}_k$, for all $k = 1, \ldots, N$.

4.2. THE SHIFTS α_k ON H_k

Like in Section 4.1, let us fix $k \in \{1, \ldots, N\}$, and the corresponding Hilbert space H_k with its ONB $\mathcal{B}_k = \{\xi_l^{(k)}\}_{l=1}^{n_k}$. Let $\{s_k\}_{k=1}^{N}$ be the shiftings of Definition 4.2.

First, assume that $n_k = \dim_{\mathbb{C}} H_k < \infty$ in $\mathbb{N}^{\infty}$, and hence, the shifting s_k is in the sense of (4.1) on $\mathcal{B}_k$. Define now an operator α_k on H_k by the linear morphism on H_k:

$$\alpha_k \left(\sum_{l=1}^{n_k} t_l \xi_l^{(k)} \right) \overset{def}{=} \sum_{l=1}^{n_k} t_l \left(s_k(\xi_l^{(k)}) \right) = \left(\sum_{l=1}^{n_k - 1} t_l \xi_{l+1}^{(k)} \right) + t_{n_k} \xi_1^{(k)}. \tag{4.3}$$

This operator α_k of (4.3) is well-defined on H_k, and it assigns

$$\xi_l^{(k)} \in \mathcal{B}_k \ \text{ to } \ s_k(\xi_l^{(k)}) \in \mathcal{B}_k$$

under (4.1), for all $l = 1, \ldots, n_k < \infty$.

Lemma 4.3. *Let $n_k = \dim_{\mathbb{C}} H_k < \infty$ in $\mathbb{N}^{\infty}$, and let $\alpha_k \in B(H_k)$ be an operator* (4.3) *on H_k. Then α_k is unitarily equivalent to the matrix $U \in M_{n_k}(\mathbb{C})$*

$$U = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{4.4}$$

Proof. The Hilbert space H_k is isomorphic to the n_k-dimensional Hilbert space $\mathbb{C}^{\times n_k} = \mathbb{C}^{n_k}$, since their dimensions are same (e.g., see [15]), i.e.,

$$H_k \overset{\text{Hilbert}}{=} \bigoplus_{l=1}^{n_k} \left(\mathbb{C} \cdot \xi_l^{(k)} \right) \overset{\text{Hilbert}}{=} \bigoplus_{l=1}^{n_k} (\mathbb{C} \cdot e_l) = \mathbb{C}^{\times n_k} = \mathbb{C}^{n_k},$$

where $e_1, \ldots, e_{n_k}$ are in the sense of (3.6). Thus, the operator $\alpha_k \in B(H_k)$ of (4.3) is unitarily equivalent to the matrix $U \in M_{n_k}(\mathbb{C}) = B(\mathbb{C}^{n_k})$ of (4.4), where $M_k(\mathbb{C})$ are the matricial algebra of $(k \times k)$-matrices over $\mathbb{C}$. □

By the above lemma, the adjoint α_k^* of the operator α_k of (4.3) is unitarily equivalent to the matrix,

$$U^* = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{n_k}(\mathbb{C}). \tag{4.5}$$

The above unitarily equivalences (4.4) and (4.5) imply that the operator α_k is not self-adjoint on H_k. Also, they demonstrate that

$$\alpha_k^* \alpha_k = 1_{H_k} = \alpha_k \alpha_k^* \ \text{ on } \ H_k,$$

because

$$U^*U = I_{n_k} = UU^* \text{ on } \mathbb{C}^{n_k}, \tag{4.6}$$

where I_{n_k} is the $(n_k \times n_k)$-identity matrix of $M_{n_k}(\mathbb{C})$.

Proposition 4.4. *Let $\alpha_k \in B(H_k)$ be the operator* (4.3)*. Then*

$$n_k < \infty \implies \alpha_k \tag{4.7}$$

is a unitary on H_k.

Proof. The proof of (4.7) is done by Lemma 4.3, and (4.6). □

The above proposition characterizes the unitarity (4.7) of α_k under the condition $n_k < \infty$.

Now, assume that $n_k = \dim_{\mathbb{C}} H_k = \infty$ in $\mathbb{N}^\infty$. Define an operator $\alpha_k \in B(H_k)$ by a linear morphism

$$\alpha_k \left(\sum_{l=1}^{\infty} t_l \xi_l^{(k)} \right) \overset{def}{=} \sum_{l=1}^{\infty} t_l s_k \left(\xi_l^{(k)} \right) = \sum_{l=1}^{\infty} t_l \xi_{l+1}^{(k)}, \tag{4.8}$$

where the shifting s_k, here, is the injection (4.2).

Lemma 4.5. *Let $n_k = \dim_{\mathbb{C}} H_k = \infty$ in $\mathbb{N}^\infty$, and let α_k be an operator* (4.8) *on H_k. Then α_k is unitarily equivalent to the unilateral shift $U \in B\left(l^2(\mathbb{N})\right)$,*

$$U = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}. \tag{4.9}$$

Proof. The Hilbert space H_k is isomorphic to $\mathbb{C}^{\times\infty} = l^2(\mathbb{N})$, since $n_k = \infty$ in $\mathbb{N}^\infty$, i.e.,

$$H_k \overset{\text{Hilbert}}{=} \bigoplus_{l=1}^{\infty} \left(\mathbb{C} \cdot \xi_l^{(k)} \right) \overset{\text{Hilbert}}{=} \bigoplus_{l=1}^{\infty} \left(\mathbb{C} \cdot e_l \right) = \mathbb{C}^{\times\infty} = l^2\left(\mathbb{N}\right),$$

by (3.6). Thus, the operator $\alpha_k \in B(H_k)$ of (4.8) is unitarily equivalent to the unilateral shift $U \in B\left(l^2(\mathbb{N})\right)$ of (4.9). □

For more about the unilateral shift, weighted shifts, and Toeplitz operators, see e.g., [15]. By the above lemma, one obtains the following spectral properties of α_k of (4.8) in $B(H_k)$, for $k = 1, \ldots, N$.

Proposition 4.6. *Let $\alpha_k \in B(H_k)$ be the operator* (4.8)*, where $n_k = \infty$ in $\mathbb{N}^\infty$. Then:*

(i) *α_k is an isometry in the sense that: $\alpha^*\alpha = 1_{H_k}$ on H_k,*
(ii) *α_k^* is a partial isometry on H_k.*

Proof. Let α_k be the operator (4.8) on H_k. Since the operator $\alpha_k \in B(H_k)$ is unitarily equivalent to the unilateral shift $U \in B\left(l^2(\mathbb{N})\right)$ of (4.9), satisfying

$$U^*U = I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix} \neq \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix} = UU^*, \tag{4.10}$$

on $l^2(\mathbb{N})$, where I is the identity operator of $B\left(l^2(\mathbb{N})\right)$, and hence, we have

$$\begin{aligned} \alpha_k^*\alpha_k &= 1_{H_k}, \quad \text{on } H_k, \\ \alpha_k\alpha_k^* &= P_1 \in B\left(H_l\right), \end{aligned} \tag{4.11}$$

by (4.10), where P_1 is the projection on H_k, projecting H_k onto the subspace

$$\left\{ \sum_{i=2}^{\infty} t_i \xi_i^{(k)} \in H_k \,|\, t_i \in \mathbb{C}, \forall i = 2, 3, \ldots \right\}, \tag{4.12}$$

because P_1 is unitarily equivalent to the projection,

$$\begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix} \in B\left(l^2(\mathbb{N})\right). \tag{4.13}$$

By (4.11), (4.12) and (4.13), the operator α_k of (4.8) is an isometry on H_k, while the adjoint α_k^* is a partial isometry on H_k. □

The above proposition characterizes the spectral properties of the operator α_k of (4.8) on the infinite-dimensional Hilbert space H_k.

Definition 4.7. Let $\alpha_k \in B(H_k)$ be the operator (4.3) if $n_k < \infty$, respectively, the operator (4.8) if $n_k = \infty$. Then this operator α_k is called the shift on H_k.

4.3. THE SHIFT OPERATOR α ON $\mathfrak{F}$

Let $H_k = l^2(n_k)$ be the given l^2-Hilbert spaces with $n_k = \dim_{\mathbb{C}} H_k \in \mathbb{N}^{\infty}$, for all $k = 1, \ldots, N$, for $N \in \mathbb{N} \setminus \{1\}$, and let $\mathfrak{F} = \mathfrak{F}[H_1, \ldots, H_N]$ be the corresponding free Hilbert space (2.7) of $H_1, \ldots, H_N$, equipped with its ONB $\mathfrak{B}$ of (2.11). Also, let α_k be the shifts (4.3) (if $n_k < \infty$), or (4.8) (if $n_k = \infty$), on H_k, for all $k = 1, \ldots, N$.

Define an operator α on $\mathfrak{F}$ by the linear transformation satisfying

$$\begin{aligned} \alpha\left(\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}\right) &= \alpha\left(\xi_{l_1}^{(i_1)} \otimes \ldots \otimes \xi_{l_k}^{(i_k)}\right) \\ &\overset{def}{=} \alpha_{i_1}\left(\xi_{l_1}^{(i_1)}\right) \otimes \ldots \otimes \alpha_{i_k}\left(\xi_{l_k}^{(i_k)}\right), \end{aligned} \tag{4.14}$$

for all basis vectors $\xi_{i_1,\dots,i_k}^{(l_1,\dots,l_k)} \in \mathfrak{B}$ of $\mathfrak{F}$ in the sense of (2.10) and (2.11), for $(i_1, \dots, i_k) \in Alt\left(\{1, \dots, N\}^k\right)$, for all $k \in \mathbb{N}$, where $\alpha_{i_j} \in B\left(H_{i_j}\right)$ are the shifts, for all $j = 1, \dots, k$.

Recall that, if there exists $i_j \in \{i_1, \dots, i_k\}$, such that $n_{i_j} < \infty$ in $\mathbb{N}^\infty$, then

$$\alpha_{i_j}\left(\xi_{l_j}^{(i_j)}\right) = \begin{cases} \xi_{l_j+1}^{(i_j)} & \text{if } l_j = 1, \dots, n_{i_j} - 1, \\ \xi_1^{(i_j)} & \text{if } l_j = n_{i_j}, \end{cases} \tag{4.15}$$

in H_{i_j}, meanwhile, if $n_{i_j} = \infty$ in $\mathbb{N}^\infty$, then

$$\alpha_{i_j}\left(\xi_{l_j}^{(i_j)}\right) = \xi_{l_j+1}^{(i_j)} \text{ in } H_{i_j}, \tag{4.16}$$

in (4.14). We denote these two different cases (4.15) and (4.16) altogether, by writing

$$\alpha_{i_j}\left(\xi_{l_j}^{(i_j)}\right) = \xi_{[l_j+1]}^{(i_j)} \text{ in } H_{i_j}, \tag{4.17}$$

where the resulted basis vector $\xi_{[l_j+1]}^{(i_j)} \in \mathcal{B}_{i_j} \subset H_{i_j}$ is either (4.15), or (4.16), case-by-case.

By using this new notation (4.17), the equality (4.14) can be re-written by

$$\alpha\left(\xi_{i_1,\dots,i_k}^{l_1,\dots,l_k}\right) = \xi_{[l_1+1]}^{(i_1)} \otimes \dots \otimes \xi_{[l_k+1]}^{(i_k)} = \xi_{i_1,\dots,i_k}^{[l_1+1],\dots,[l_k+1]}, \tag{4.18}$$

in $\mathfrak{B} \subset \mathfrak{F}$, in terms of (2.10), i.e., the operator α of (4.14) is an operator assigning basis vectors

$$\xi_{i_1,\dots,i_k}^{l_1,\dots,l_k} \in \mathfrak{B} \longmapsto \xi_{i_1,\dots,i_k}^{[l_1+1],\ \dots,\ [l_k+1]} \in \mathfrak{B},$$

in $\mathfrak{F}$. And hence, it is well-defined in the operator algebra $B(\mathfrak{F})$.

Moreover, this operator α preserves the lengths of basis vectors in the sense that: if

$$\xi_{i_1,\dots,i_k}^{l_1,\dots,l_k} = \xi_{l_1}^{(i_1)} \otimes \dots \otimes \xi_{l_k}^{(i_k)} \in H_{i_1,\dots,i_k}$$

has its length-k, for $(i_1, \dots, i_k) \in Alt\left(\{1,\ \dots,\ N\}^k\right)$, then

$$\alpha\left(\xi_{i_1,\dots,i_k}^{l_1,\dots,l_k}\right) = \xi_{i_1,\dots,i_k}^{[l_1+1],\dots,[l_k+1]} \in H_{i_1,\dots,i_k}$$

is a length-k basis vector of $\mathfrak{F}$, too, by (4.18). It shows that the pattern of an arbitrary vector $h \in \mathfrak{F}$ and that of $\alpha(h) \in \mathfrak{F}$ are identified under the action of α, different from the (action of) jump operators of Section 3.

Definition 4.8. Let $\mathfrak{F}$ be the free Hilbert space, and let $\alpha \in B(\mathfrak{F})$ be an operator (4.14). Then we call α, the shift operator on $\mathfrak{F}$.

Consider the spectral properties of the shift operator α in $B(\mathfrak{F})$.

Theorem 4.9. *Let H_k be the fixed n_k-dimensional Hilbert spaces $l^2(n_k)$, and assume that*

$$n_k < \infty, \quad \textit{for all } k = 1, \dots, N.$$

Then the shift operator α is a unitary on $\mathfrak{F}$.

Proof. Let $\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k} \in \mathfrak{B}$ be an arbitrary basis vector of $\mathfrak{F}$, and let α be the shift operator (4.14) on $\mathfrak{F}$. Then

$$\begin{aligned}\alpha^*\alpha\left(\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}\right) &= \alpha^*\left(\xi_{i_1,\ldots,i_k}^{[l_1+1],\ldots,[l_k+1]}\right) = \xi_{i_1,\ldots,i_k}^{[l_1+1-1],\ldots,[l_k+1-1]} \\ &= \xi_{i_1,\ldots,i_k}^{[l_1],\ldots,[l_k]} = \xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k} = 1_{\mathfrak{F}}\left(\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}\right) \\ &= \xi_{i_1,\ldots,i_k}^{[l_1-1+1],\ldots,[l_k-1+1]} = \alpha\alpha^*\left(\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}\right),\end{aligned}$$

in $\mathfrak{B} \subset \mathfrak{F}$, by (4.7) and (4.18), because $n_1, \ldots, n_N < \infty$ in $\mathbb{N}^\infty$. So, by (2.7) and (2.11), the above relation implies that

$$\alpha^*\alpha = 1_{\mathfrak{F}} = \alpha\alpha^* \text{ on } \mathfrak{F},$$

and hence, the shift operator α is a unitary on $\mathfrak{F}$, whenever $n_k < \infty$, for all $k = 1, \ldots, N$. □

The above theorem shows that if $n_k < \infty$ in $\mathbb{N}^\infty$, for "all" $k = 1, \ldots, N$, then the shift operators α of (4.14) is a unitary on the free Hilbert space $\mathfrak{F}$.

Theorem 4.10. *Let H_k be the fixed l^2-spaces $l^2(n_k)$, for $k = 1, \ldots, N$, and $\mathfrak{F}$, the free Hilbert space of $H_1, \ldots, H_N$. Assume that there exists at least one $n_j \in \{n_1, \ldots, n_N\}$, such that $n_j = \infty$ in $\mathbb{N}^\infty$. Then the shift operators α is not unitary: it is an isometry on $\mathfrak{F}$, meanwhile, the adjoint α^* is a partial isometry on $\mathfrak{F}$.*

Proof. Let α be the shift operator (4.14) on $\mathfrak{F}$, and assume that there exists $n_j \in \{n_1, \ldots, n_N\}$ such that

$$n_j = \dim_{\mathbb{C}} H_j = \infty, \text{ in } \mathbb{N}^\infty.$$

Then the shift α_j of (4.8) is an isometry on H_j, while the adjoint α_j^* is a partial isometry satisfying

$$\left(\alpha_j^*\right)^*\left(\alpha_j^*\right) = \alpha_j\alpha_j^* = P_1,$$

the projection from H_j onto the subspace H_j^1 of H_j, by (i) and (ii) of Proposition 4.6, where

$$H_j^1 = H_j \ominus \left(\mathbb{C} \cdot \xi_1^{(j)}\right) \text{ in } H_j,$$

in the sense of (4.12), i.e.,

$$P_1\left(\sum_{l=1}^{n_j} t_l\xi_l^{(j)}\right) = \sum_{l=2}^{n_j} t_l\xi_l^{(j)} \text{ in } H_j.$$

By the definition (4.14), the shift operator α is generated by the shifts $\{\alpha_k \in B(H_k)\}_{k=1}^N$. So, under hypotheses, the operator α cannot be a unitary on $\mathfrak{F}$, because $n_j = \infty$ (and hence, α_j is not unitary on H_j). It satisfies that, for any basis vector $\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k} \in \mathfrak{B}$,

$$\alpha^*\alpha\left(\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}\right) = \alpha^*\left(\xi_{i_1,\ldots,i_k}^{[l_1+1],\ldots,[l_k+1]}\right) = \xi_{i_1,\ldots,i_k}^{[l_1+1-1],\ldots,[l_k+1-1]} = 1_{\mathfrak{F}}\left(\xi_{i_1,\ldots,i_k}^{l_1,\ldots,l_k}\right),$$

in $\mathfrak{F}$, implying that

$$\alpha^*\alpha = 1_{\mathfrak{F}} \text{ on } \mathfrak{F},$$

by (4.18). It shows that α is an isometry on the free Hilbert space $\mathfrak{F}$; however, if $\xi_{i_1,\ldots,i_p,\ldots,i_k}^{l_1,\ldots,l_{i_p},\ldots,l_k} \in \mathfrak{B}$, where $i_p = j$, such that $n_j = \infty$, and

$$l_{i_p} = l_j = 1 \in \mathbb{N},$$

then

$$\begin{aligned}\alpha^*\left(\xi_{i_1,\ldots,j,\ldots,i_k}^{l_1,\ldots,1,\ldots,l_k}\right) &= \xi_{[l_1-1]}^{(i_1)} \otimes \ldots \otimes \xi_{[l_p-1]}^{(i_{p-1})} \otimes \xi_{[1-1]}^{(j)} \otimes \xi_{[l_{p+1}-1]}^{(i_{p+1})} \otimes \ldots \otimes \xi_{[l_k-1]}^{(i_k)} \\ &= \xi_{[l_1-1]}^{(i_1)} \otimes \ldots \otimes \xi_{[l_p-1]}^{(i_{p-1})} \otimes O_j \otimes \xi_{[l_{p+1}-1]}^{(i_{p+1})} \otimes \ldots \otimes \xi_{[l_k-1]}^{(i_k)} = O_{\mathfrak{F}},\end{aligned}$$

by (4.10) and (4.11). It shows that all basis vectors $\xi \in \mathfrak{B}$ formed by $\xi_{i_1,\ldots,j,\ldots,i_k}^{l_1,\ldots,1,\ldots,l_k}$ satisfy

$$\alpha^*(\xi) = O_{\mathfrak{F}} \text{ in } \mathfrak{F}.$$

i.e., there exists a subspace $\mathfrak{F}_j = \overline{span_{\mathbb{C}}(\mathfrak{B}_j)}$ of $\mathfrak{F}$ spanned by the subset

$$\mathfrak{B}_j = \bigcup_{k\in\mathbb{N}} \left(\bigcup_{(i_1,\ldots,j,\ldots,i_k)\in Alt(\{1,\ldots,N\}^k)} \left(\{\xi_{i_1,\ldots,j,\ldots,i_k}^{l_1,\ldots,1,\ldots,l_k} \in \mathfrak{B}\} \right) \right), \tag{4.19}$$

of the ONB $\mathfrak{B}$ of $\mathfrak{F}$, such that

$$\alpha^*(\mathfrak{F}_j) = \{O_{\mathfrak{F}}\}, \text{ the trivial subspace of } \mathfrak{F}.$$

So, in general, if $j_1, \ldots, j_s \in \{1, \ldots, N\}$ $(s \leq N)$ are the quantities satisfying

$$n_{j_t} = \dim_{\mathbb{C}} H_{j_t} = \infty \text{ in } \mathbb{N}^\infty, \text{ for } t = 1, \ldots, s,$$

then there exists the maximal subspace $\mathfrak{F}_{j_1,\ldots,j_s}$ spanned by

$$\mathfrak{B}_{j_1,\ldots,j_s} = \bigcup_{t=1}^{s} \mathfrak{B}_{j_t} \tag{4.20}$$

in $\mathfrak{F}$, where $\mathfrak{B}_{j_t}$ in (4.20) are in the sense of (4.19), for all $t = 1, \ldots, s$, such that

$$\alpha^*(\mathfrak{F}_{j_1,\ldots,j_s}) = \{O_{\mathfrak{F}}\} \text{ in } \mathfrak{F},$$

and

$$\alpha\alpha^*(\mathfrak{F} \ominus \mathfrak{F}_{j_1,\ldots,j_t}) = \mathfrak{F} \ominus \mathfrak{F}_{j_1,\ldots,j_t} \text{ in } \mathfrak{F}. \tag{4.21}$$

Thus, one can get that

$$\alpha\alpha^* = P_{j_1,\ldots,j_s} \text{ on } \mathfrak{F}, \tag{4.22}$$

by (4.21), where $P_{j_1,\ldots,j_s} \in B(\mathfrak{F})$ is the projection, projecting $\mathfrak{F}$ onto $\mathfrak{F} \ominus \mathfrak{F}_{j_1,\ldots,j_s}$, i.e., the adjoints α^* of the shift operator α is a partial isometry on $\mathfrak{F}$, by (4.22). □

The above theorem characterizes the spectral property of our shift operator α and its adjoint α^* on the free Hilbert space $\mathfrak{F}$, whenever there exists at least one $j \in \{1,\ldots,N\}$, such that

$$n_j = \dim_{\mathbb{C}} H_j = \infty \ \text{ in } \ \mathbb{N}^{\infty}.$$

The main results of this section are summarized by the following corollary.

Corollary 4.11. *The shift operators α of* (4.14) *is an isometry on $\mathfrak{F}$, and its adjoint α^* is a partial isometry on $\mathfrak{F}$ satisfying* (4.22)*, in general. However, if*

$$n_k = \dim_{\mathbb{C}} H_k < \infty, \quad \forall k = 1,\ldots,N,$$

then α (and hence, α^*) is a unitary on $\mathfrak{F}$, for all $n \in \mathbb{N}$.

Proof. The proof is done by Theorems 4.9 and 4.10. □

5. JUMP-SHIFT OPERATORS ON $\mathfrak{F}[H_1,\ldots,H_N]$

In this section, we define a new type of operators on the free Hilbert space $\mathfrak{F} = \mathfrak{F}[H_1,\ldots,H_N]$ of l^2-Hilbert spaces $H_1,\ldots,H_N$ of (2.1), satisfying (2.2), from the jump operators of Section 3, and the shift operators of Section 4. Let

$$J[j>i] = J^{\mathbf{1}_{n_i}}[j>i] \in B(\mathfrak{F})$$

be the $\mathbf{1}_{n_i}$-jump operator from H_j to H_i in $\mathfrak{F}$, for $i < j \in \{1,\ldots,N\}$, where $\mathbf{1}_{n_i}$ is the group-identity of the symmetric group S_{n_i} and let $\alpha \in B(\mathfrak{F})$ be the shift operator. Define a new operator $\sum[j>i]$ of $B(\mathfrak{F})$ by

$$\sum[j>i] = \alpha \cdot J[j>i] \ \text{ on } \ \mathfrak{F} \tag{5.1}$$

for $i < j \in \{1,\ldots,N\}$.

Definition 5.1. The operators $\sum[j>i] \in B(\mathfrak{F})$ of (5.1) are called the jump-shift operators (in short, j-s operators) on $\mathfrak{F}$, for all $i < j \in \{1,\ldots,N\}$.

Let $\sum[j>i]$ be the j-s operator for a fixed $i < j \in \{1, \ldots, N\}$. Observe first that

$$\begin{aligned}\left(\sum[j>i]\right)^* \left(\sum[j>i]\right) &= \left(J[j>i]^*\alpha^*\right)\left(\alpha J[j>i]\right)\\ &= J[j>i]^*\left(\alpha^*\alpha\right) J[j>i]\\ &= J[j>i]^* J[j>i]\end{aligned}$$

since α is an isometry on $\mathfrak{F}$ by Theorem 4.10

$$= \begin{cases} P_{j>i} & \text{if } n_i < n_j \text{ in } \mathbb{N}^{\infty},\\ 1_{\mathfrak{F}} & \text{if } n_i = n_j \text{ in } \mathbb{N}^{\infty},\end{cases} \tag{5.2}$$

since $J[j>i]$ is a normal partial isometry in general, by Proposition 4.6, and it can be a unitary if and only if $n_i = n_j$, by Proposition 4.6, where $P_{j>i}$ is the projection on $\mathfrak{F}$ (see Lemma 3.8(ii)).

Similarly,

$$\left(\sum[j>i]\right)\left(\sum[j>i]\right)^* = (\alpha J[j>i])(\alpha J[j>i])^* = \alpha\left(J[j>i]J[j>i]^*\right)\alpha^* = \begin{cases} \alpha\alpha^* & \text{if } n_i = n_j \text{ in } \mathbb{N}^\infty, \\ \alpha P_{j>i}\alpha^* & \text{if } n_i < n_j \text{ in } \mathbb{N}^\infty, \end{cases}$$

by Proposition 4.6, where $P_{j>i}$ is the projection from Lemma 3.8(ii)

$$= \begin{cases} 1_{\mathfrak{F}} & \text{if } n_i = n_j < \infty \text{ in } \mathbb{N}^\infty, \\ P_1 & \text{if } n_i = n_j = \infty \text{ in } \mathbb{N}^\infty, \\ P_{j>i} & \text{if } n_i < n_j < \infty \text{ in } \mathbb{N}^\infty, \\ P_{j>i}P_1 & \text{if } n_i < n_j = \infty \text{ in } \mathbb{N}^\infty, \end{cases}$$

where P_1 is the projection (4.13)

$$= \begin{cases} 1_{\mathfrak{F}} & \text{if } n_i = n_j < \infty \text{ in } \mathbb{N}^\infty, \\ P_1 & \text{if } n_i = n_j = \infty \text{ in } \mathbb{N}^\infty, \\ P_{j>i} & \text{if } n_i < n_j \text{ in } \mathbb{N}^\infty, \end{cases} \tag{5.3}$$

since

$$P_{j>i}P_1 = P_{j>i} \text{ on } \mathfrak{F},$$

where $n_i < \infty$ and $n_j = \infty$ in $\mathbb{N}^\infty$, by Lemma 3.8(ii) and (4.13).

By (5.2) and (5.3), we obtain the following lemma.

Lemma 5.2. *Let $\Sigma[j>i] \in B(\mathfrak{F})$ be the j-s operator for $i < j \in \{1, \ldots, N\}$. Then*

$$(\Sigma[j>i])^*(\Sigma[j>i]) = \begin{cases} 1_{\mathfrak{F}} & \textit{if } n_i = n_j \textit{ in } \mathbb{N}^\infty, \\ P_{j>i} & \textit{if } n_i < n_j \textit{ in } \mathbb{N}^\infty, \end{cases}$$
$$(\Sigma[j>i])(\Sigma[j>i])^* = \begin{cases} 1_{\mathfrak{F}} & \textit{if } n_i = n_j < \infty \textit{ in } \mathbb{N}^\infty, \\ P_1 & \textit{if } n_i = n_j = \infty \textit{ in } \mathbb{N}^\infty, \\ P_{j>i} & \textit{if } n_i < n_j \textit{ in } \mathbb{N}^\infty, \end{cases} \tag{5.4}$$

where $P_{j>i} \in B(\mathfrak{F})$ is the projection from Lemma 3.8(ii), and $P_1 \in B(\mathfrak{F})$ is the projection (4.13).

Proof. The first (resp., second) formula of (5.4) is shown by (5.2) (resp., (5.3)). □

By (5.4), one can verify the following operator-theoretic spectral properties of j-s operators in the operator algebra $B(\mathfrak{F})$.

Theorem 5.3. *Let* $\sum[j > i] = \alpha \cdot J[j > i]$ *be the j-s operator* (5.1) *on the free Hilbert space* $\mathfrak{F}$ *for* $i < j \in \{1, \ldots, N\}$.

(i) *If* $n_i = n_j < \infty$ *in* $\mathbb{N}^\infty$, *then* $\sum[j > i]$ *is a unitary on* $\mathfrak{F}$.
(ii) *If* $n_i = n_j = \infty$ *in* $\mathbb{N}^\infty$, *then* $\sum[j > i]$ *is an isometry on* $\mathfrak{F}$, *while, the adjoint* $\sum[j > i]^*$ *is a partial isometry on* $\mathfrak{F}$.
(iii) *If* $n_i < n_j$ *in* $\mathbb{N}^\infty$, *then* $\sum[j > i]$ *is a normal partial isometry on* $\mathfrak{F}$.

Proof. Assume first that $n_i = n_j < \infty$ in $\mathbb{N}^\infty$. Then the j-s operator $\sum[j > i]$ satisfies that

$$\left(\sum[j > i]\right)^* \left(\sum[j > i]\right) = 1_{\mathfrak{F}} = \left(\sum[j > i]\right) \left(\sum[j > i]\right)^*,$$

on $\mathfrak{F}$, by (5.4), i.e., it is a unitary on $\mathfrak{F}$. So, the statement (i) holds.

Suppose now that $n_i = n_j = \infty$ in $\mathbb{N}^\infty$. Then, the operator $\sum[j > i]$ satisfies

$$\left(\sum[j > i]\right)^* \left(\sum[j > i]\right) = 1_{\mathfrak{F}},$$

and

$$\left(\sum[j > i]\right) \left(\sum[j > i]\right)^* = P_1,$$

on $\mathfrak{F}$, where $P_1 \in B(\mathfrak{F})$ is the projection (4.13). It shows that $\sum[j > i]$ is an isometry on $\mathfrak{F}$, and its adjoint $\sum[j > i]^*$ is a partial isometry on $\mathfrak{F}$. Thus, the statement (ii) is proven.

Finally, let $n_i < n_j$ in $\mathbb{N}^\infty$. Then, by (5.4), one has that

$$\left(\sum[j > i]\right)^* \left(\sum[j > i]\right) = P_{j>i} = \left(\sum[j > i]\right) \left(\sum[j > i]\right)^*,$$

on $\mathfrak{F}$. So, the j-s operator $\sum[j > i]$ is not only normal, but also a partial isometry on $\mathfrak{F}$, and hence, the adjoint $\sum[j > i]^*$ is a normal partial isometry on $\mathfrak{F}$, too. So, the statement (iii) holds. □

6. SEMICIRCULAR ELEMENTS INDUCED BY SOME BASIS ELEMENTS OF $\mathfrak{F}$

In this section, we consider semicircular elements induced by the subset $\bigcup_{k=1}^{N} \mathcal{B}_k$ of the ONB $\mathfrak{B}$ of the free Hilbert space $\mathfrak{F} = \mathfrak{F}[H_1, \ldots, H_N]$, where $\mathcal{B}_k$ are the ONBs of the fixed Hilbert spaces $H_k = l^2(n_k)$, for $k = 1, \ldots, N$.

6.1. FREE PROBABILITY AND SEMICIRCULAR ELEMENTS

Free probability is the noncommutative operator-algebraic version of classical measure theory (including probability theory), and statistical analysis. A free-probabilistic counterpart of a *measure space* (X, μ) of a set X and a (bounded, or unbounded) measure μ on X is a mathematical pair (A, φ) of a noncommutative algebra (including a pure-algebraic, or topological $*$-algebra) A, and (unbounded, or bounded) linear functional φ on A. Such a pair (A, φ) is called a (*noncommutative*) *free probability* *space*. All elements $a \in A$ are said to be *free random variables*, if we regard them as elements of (A, φ). Free probability is not only an important field of operator algebra theory (e.g., [16, 18, 21] and [23]), but also an interesting application to related topics (e.g., [2, 5, 9–11] and [13]). For more about free probability theory, see [20–22] and [23], and cited papers therein.

Studying *semicircular elements*, whose free distributions are *the semicircular law*, plays a key role in free probability by the (*free*) *central limit theorem* (e.g., [1, 4, 6, 16, 17, 21] and [24]). Roughly speaking, the semicircular law in free probability theory is the noncommutative version of the *Gaussian* (or the *normal*) *distribution* in classical commutative theory.

Different from the earlier works, semicircular elements are constructed-and-studied as *Banach-space operators* (in the sense of [14]) acting on C^*-algebras (by regarding them as Banach spaces with C^*-norms) in [9, 10] and [11]. We here apply the definitions and results of these papers. In particular, we generalize the main results of [13] to develop the semicircularity from our free Hilbert space $\mathfrak{F}$.

Definition 6.1. Let (A, φ) be a topological $*$-probability space (for instance, C^*-probability space, or W^*-probability space, or Banach $*$-probability space etc.) of a topological $*$-algebra A (C^*-algebra, resp., von Neumann algebra, or Banach $*$-algebra, etc.), and a bounded linear functional φ on A. Let $a \in (A, \varphi)$ be a self-adjoint free random variable. This operator a is semicircular in (A, φ), if

$$\varphi(a^n) = \omega_n c_{\frac{n}{2}}, \quad \text{for all} \quad n \in \mathbb{N}, \tag{6.1}$$

where

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

and

$$c_k = \frac{1}{k+1}\binom{2k}{k} = \frac{(2k)!}{k!(k+1)!},$$

are the k-th Catalan numbers for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It is checked that, by the Möbius inversion of [20], the free-momental definition (6.1) is equivalent to the following free-cumulant characterization,

$$k_n^{\varphi}(\underbrace{a,\ a, \ldots, a}_{n\text{-times}}) = \delta_{n,2}, \tag{6.2}$$

for all $n \in \mathbb{N}$, where $k_n^{\varphi}(\cdot)$ is the free cumulant on A with respect to the linear functional φ, and δ is the Kronecker delta.

6.2. ON THE FREE HILBERT SPACE $\mathfrak{F}[l^2(\mathbb{Z}), \ldots, l^2(\mathbb{Z})]$

In [13], by applying the main results of [10] and [9], we showed that if H is an "infinite-dimensional" separable Hilbert space, then there exists so-called the *cloned Hilbert space* H^o, whose dimension is $|\mathbb{Z}| = \infty$, equivalently, whose ONB consists of $|\mathbb{Z}|$-many basis vectors $\{\xi_j\}_{j=-\infty}^{\infty}$, and the corresponding $|\mathbb{Z}|$-many orthogonal *rank-1 projections* $\{q_j\}_{j=-\infty}^{\infty}$, generated by the ONB, induce semicircular elements $\{u_j\}_{j=-\infty}^{\infty}$, as a *Banach-space operators* acting on the operator algebra $B(H)$.

Let $\mathcal{H}$ be the Hilbert space,

$$\mathcal{H} = \left\{ \sum_{l=-\infty}^{\infty} t_l \xi_l \mid \sum_{l=-\infty}^{\infty} |t_l|^2 < \infty, t_l \in \mathbb{C}, \ \forall l \in \mathbb{Z} \right\},$$

with its ONB

$$\mathcal{B} = \{\xi_l \in \mathcal{H} : l \in \mathbb{Z}\}, \tag{6.3}$$

satisfying

$$\dim_{\mathbb{C}} \mathcal{H} = |\mathbb{Z}| = \infty \ \text{ in } \ \mathbb{N}^{\infty}.$$

This Hilbert space $\mathcal{H}$ has its canonical inner product $\langle \cdot, \cdot \rangle_o$ defined by

$$\left\langle \sum_{l=-\infty}^{\infty} t_l \xi_l, \ \sum_{l=-\infty}^{\infty} s_l \xi_l \right\rangle_o = \sum_{l=-\infty}^{\infty} t_l \overline{s_l},$$

and the norm $\|\cdot\|_o$,

$$\|h\|_o = \sqrt{\langle h, \ h \rangle_o}, \ \text{ for all } \ h \in \mathcal{H}.$$

It is not difficult to check that $\mathcal{H}$ is isomorphic to the l^2-space,

$$l^2(\mathbb{Z}) = \left\{ (z_j)_{j=-\infty}^{\infty} \ \middle| \ \begin{array}{c} z_j \in \mathbb{C}, \ \forall j \in \mathbb{Z} \\ \sum_{j=-\infty}^{\infty} |z_j|^2 < \infty \end{array} \right\},$$

equipped with its ONB $\{e_j : j \in \mathbb{Z}\}$ with

$$e_j = \ (\cdots, \ 0, \ 0, \ \underset{j\text{-th}}{1}, \ 0, \ 0, \ \ldots).$$

Such a $|\mathbb{Z}|$-dimensional Hilbert space $\mathcal{H}$ of (6.3) is naturally considered in operator theory (e.g., see [15]: the unilateral shift U on $l^2(\mathbb{N})$ has its normal extension, *the bilateral shift,*

$$\widetilde{U} = \begin{pmatrix} \ddots & & & & 0 \\ \ddots & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

on $l^2(\mathbb{Z})$, etc.).

Now, let us fix $k \in \{1, \ldots, N\}$, and the corresponding Hilbert space $H_k = l^2(n_k)$, for $n_k \in \mathbb{N}^{\infty}$, from (2.1) satisfying the ordering (2.2). From the Hilbert space $\mathcal{H}$ of (6.3), choose arbitrary n_k-many basis vectors,

$$\mathcal{B}_{(k)} = \{\xi_{j_1}, \ldots, \xi_{j_{n_k}}\},$$

in the ONB $\mathcal{B}$ of $\mathcal{H}$, and then construct the subspace $\mathcal{H}_{(k)}$ of $\mathcal{H}$, spanned by $\mathcal{B}_{(k)}$, i.e.,

$$\mathcal{H}_{(k)} = \overline{span_{\mathbb{C}}(\mathcal{B}_{(k)})}^{\mathcal{H}} \text{ in } \mathcal{H}. \tag{6.4}$$

For convenience, let us take

$$\mathcal{B}_{(k)} = \{\xi_1, \ldots, \xi_{n_k}\} \text{ in } \mathcal{B}, \tag{6.5}$$

without loss of generality. Then the two Hilbert spaces H_k and $\mathcal{H}_{(k)}$ are isomorphic.

Lemma 6.2. *Let $\mathcal{H}_{(k)}$ be the subspace* (6.4) *of the Hilbert space $\mathcal{H}$ of* (6.3)*, spanned by the set $\mathcal{B}_{(k)}$ of* (6.5)*. Then*

$$\mathcal{H}_{(k)} \overset{Hilbert}{=} H_k, \tag{6.6}$$

where H_k is our fixed Hilbert space $l^2(n_k)$ of (2.1)*.*

Proof. Since

$$\dim_{\mathbb{C}} \mathcal{H}_{(k)} = \left|\mathcal{B}_{(k)}\right| = n_k = |\mathcal{B}_k| = \dim_{\mathbb{C}} H_k,$$

the Hilbert spaces $\mathcal{H}_{(k)}$ and H_k are isomorphic. □

Let $\mathcal{H}$ be the Hilbert space (6.3), and assume that we have N-many $\mathcal{H}$'s. One may understand we have N-many Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_N$, such that

$$\mathcal{H}_k \overset{\text{Hilbert}}{=} l^2(\mathbb{Z}) \overset{\text{Hilbert}}{=} \mathcal{H},$$

for $k = 1, \ldots, N$, where $N \in \mathbb{N} \setminus \{1\}$ is the quantity fixed in (2.1). From below, this situation is simply said that we *have N-copies of $\mathcal{H}$.*

For these N-copies of $\mathcal{H}$, one can define the corresponding free Hilbert space $\mathfrak{F}_N(\mathcal{H})$ by

$$\mathfrak{F}_N(\mathcal{H}) \overset{def}{=} \mathfrak{F}[\underbrace{\mathcal{H}, \mathcal{H}, \ldots, \mathcal{H}}_{N\text{-times}}], \tag{6.7}$$

where the right-hand side of (6.7) means the free Hilbert space (2.7) of N-copies of $\mathcal{H}$.

Proposition 6.3. *Let $\mathfrak{F}_N(\mathcal{H})$ be the free Hilbert space* (6.7) *of N-copies of the Hilbert space $\mathcal{H}$ of* (6.3)*, and let $\mathfrak{F} = \mathfrak{F}[H_1, \ldots, H_N]$ be the free Hilbert space* (2.7) *of $H_1, \ldots, H_N$. Then $\mathfrak{F}$ is a subspace of $\mathfrak{F}_N(\mathcal{H})$.*

Proof. By (6.6), our fixed Hilbert spaces H_k are Hilbert-space isomorphic to the subspaces $\mathcal{H}_{(k)}$ of (6.4) in $\mathcal{H}$, for all $k = 1, \ldots, N$. So,

$$\mathfrak{F} = \mathfrak{F}[H_1, \ldots, H_N] \overset{\text{Hilbert}}{=} \mathfrak{F}\left[\mathcal{H}_{(1)}, \ldots, \mathcal{H}_{(N)}\right],$$

which is a well-defined subspace of $\mathfrak{F}_N(\mathcal{H})$, by (6.7). Thus, $\mathfrak{F}$ is (isomorphic to) a subspace of $\mathfrak{F}_N(\mathcal{H})$. □

From below, the free Hilbert space $\mathfrak{F}$ of (2.7) will be regarded as a subspace of the Hilbert space $\mathfrak{F}_N(\mathcal{H})$ of (6.7).

6.3. SEMICIRCULAR ELEMENTS INDUCED BY $\mathcal{H}$

Let $\mathcal{H}$ be the Hilbert space (6.3), with its ONB

$$\mathcal{B} = \{\xi_j : j \in \mathbb{Z}\}.$$

Then, canonically, one can define the *rank*-1 *orthogonal projections* q_j by

$$q_j(h) = \langle h, \xi_j\rangle_o \xi_j, \quad \text{for all} \quad h \in \mathcal{H}, \tag{6.8}$$

i.e., q_j project $\mathcal{H}$ onto the 1-dimensional subspace $\mathbb{C} \cdot \xi_j$, for all $j \in \mathbb{Z}$.

It is not difficult to check that such projections q_j are unitarily equivalent to the projections

$$Q_j = \begin{pmatrix} \ddots & & & & & & 0 \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & \underset{(j,j)\text{-th}}{1} & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ 0 & & & & & & \ddots \end{pmatrix},$$

on $l^2(\mathbb{Z}) \overset{\text{Hilbert}}{=} \mathcal{H}$.

By (6.8), the family $\mathbf{Q} = \{q_j\}_{j\in\mathbb{Z}}$ satisfies that

$$q_j^* = q_j = q_j^2 \quad \text{on} \quad \mathcal{H},$$

and

$$q_i q_j = \delta_{ij} q_j \quad \text{in} \quad B(\mathcal{H}),$$

for all $i, j \in \mathbb{Z}$, because the unitarily equivalent projections $\{Q_j\}_{j\in\mathbb{Z}}$ are mutually orthogonal on $l^2(\mathbb{Z})$.

On the operator algebra $B(\mathcal{H})$, define the canonical trace φ by

$$\varphi(T) \overset{def}{=} \sum_{j\in\mathbb{Z}} \langle T\xi_j,\ \xi_j\rangle, \tag{6.9}$$

for all $T \in B(\mathcal{H})$.

We now sectionize (or filterize) the canonical trace φ of (6.9), and define a system $\{\varphi_j\}_{j\in\mathbb{Z}}$ of bounded linear functionals φ_j's by

$$\varphi_j(T) = \langle T\xi_j,\ \xi_j\rangle, \quad \text{for all } T \in B(\mathcal{H}), \tag{6.10}$$

for all $j \in \mathbb{Z}$. Then the pairs $(B(\mathcal{H}),\ \varphi_j)$ form well-defined C^*-probability spaces for all $j \in \mathbb{Z}$.

Let us focus on the $|\mathbb{Z}|$-many orthogonal projections,

$$\mathbf{Q} = \{q_j\}_{j\in\mathbb{Z}} \subset B(\mathcal{H}), \tag{6.11}$$

and define the C^*-subalgebra,

$$Q \overset{def}{=} C^*(\mathbf{Q}) \text{ of } B(\mathcal{H}), \tag{6.12}$$

by the C^*-algebra generated by the set $\mathbf{Q}$ of (6.12) in $B(\mathcal{H})$.

Lemma 6.4. *Let Q be a C^*-subalgebra* (6.12) *of the operator algebra $B(\mathcal{H})$. Then*

$$Q \overset{*\text{-}iso}{=} \mathbb{C}^{\oplus|\mathbb{Z}|}, \tag{6.13}$$

where "$\overset{*\text{-}iso}{=}$" means "being $*$-isomorphic", and $\oplus$ is the direct product of C^*-algebras.

Proof. By the definition (6.13), one has that

$$Q \overset{*\text{-iso}}{=} \underset{j\in\mathbb{Z}}{\oplus} (\mathbb{C} \cdot q_j) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus|\mathbb{Z}|},$$

because of the mutual orthogonality on the generating set $\mathbf{Q}$ of (6.11). □

Define Banach-space operators $\mathbf{c}$ and $\mathbf{a}$ acting on the C^*-algebra Q of (6.13) by the bounded linear transformations acting on Q satisfying

$$\mathbf{c}(q_j) \overset{def}{=} q_{j+1} \quad \text{and} \quad \mathbf{a}(q_j) \overset{def}{=} q_{j-1}, \tag{6.14}$$

for all $q_j \in \mathbf{Q}$ in Q. These Banach-space operators $\mathbf{c}$ and $\mathbf{a}$ are well-defined by regarding the C^*-algebra Q as a *Banach space* equipped with its C^*-norm (inherited by the operator-norm on $B(\mathcal{H})$), by (6.13), e.g., see [14]. They are elements of the *operator space* $B(Q)$, consisting of all bounded linear transformations on the Banach space Q.

Definition 6.5. We call the Banach-space operators $\mathbf{c}$ and $\mathbf{a}$ of (6.14), the creation, respectively, the annihilation on Q. Define a new Banach space operator $\mathbf{l} \in B(Q)$ by

$$\mathbf{l} = \mathbf{c} + \mathbf{a} \text{ on } Q. \tag{6.15}$$

Then this operator $\mathbf{l} \in B(Q)$ is called the radial operator on Q.

The creation $\mathbf{c}$ and the annihilation $\mathbf{a}$ on Q satisfy that

$$\mathbf{ca} = 1_Q = \mathbf{ac}, \text{ on } Q,$$

and hence,

$$\mathbf{c}^n\mathbf{a}^n = (\mathbf{ca})^n = 1_Q^n = 1_Q = (\mathbf{ac})^n = \mathbf{a}^n\mathbf{c}^n,$$

and

$$\mathbf{c}^{n_1}\mathbf{a}^{n_2} = \mathbf{a}^{n_2}\mathbf{c}^{n_1}, \text{ on } Q, \tag{6.16}$$

for all $n, n_1, n_2 \in \mathbb{N}$, (e.g., see [10]), where $1_Q \in B(Q)$ is the identity operator on Q,

$$1_Q(T) = T, \text{ for all } T \in Q.$$

By (6.16), the radial operator $\mathbf{l}$ satisfies that

$$\mathbf{l}^n = (\mathbf{c} + \mathbf{a})^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{c}^k \mathbf{a}^{n-k}, \tag{6.17}$$

for all $n \in \mathbb{N}$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \text{for all} \quad k \leq n \in \mathbb{N}_0.$$

Lemma 6.6. *Let* $\mathbf{l} = \mathbf{c} + \mathbf{a}$ *be the radial operator* (6.15) *on* Q. *Then, for any* $n \in \mathbb{N}$,

(i) $\mathbf{l}^{2n-1}$ *does not contain* 1_Q*-terms, and*
(ii) $\mathbf{l}^{2n}$ *contains its* 1_Q*-term,* $\binom{n}{k} \cdot 1_Q$.

Proof. The proofs of (i) and (ii) are done by (6.17). See [10] and [9] for details. □

Define now the Banach $*$-algebra $\mathfrak{L}$ generated by the radial operator $\mathbf{l}$, as a subalgebra embedded in $B(Q)$, by

$$\mathfrak{L} \overset{def}{=} \overline{\mathbb{C}[\{\mathbf{l}\}]}, \tag{6.18}$$

where $\overline{Z}$ means the Banach-topology closure of a subset Z of $B(Q)$, and $\mathbb{C}[\{\mathbf{l}\}]$ is the polynomial algebra in $\{\mathbf{l}\}$.

By defining an adjoint $(*)$ on $\mathfrak{L}$ by

$$\sum_{n=0}^{\infty} t_n \mathbf{l}^n \in \mathfrak{L} \longmapsto \sum_{n=0}^{\infty} \overline{t_n} \mathbf{l}^n \in \mathfrak{L},$$

with identity

$$\mathbf{l}^0 = 1_Q,$$

this closed subspace $\mathfrak{L}$ of $B(Q)$ forms a well-defined Banach $*$-algebra (e.g., [10]).

Construct the tensor-product Banach $*$-algebra $\mathfrak{L}_Q$ by

$$\mathfrak{L}_Q \overset{def}{=} \mathfrak{L} \otimes_{\mathbb{C}} Q, \tag{6.19}$$

where $\otimes_{\mathbb{C}}$ is the tensor product of Banach $*$-algebras. Remark that, since Q is a C^*-algebra, it is a well-determined Banach $*$-algebra equipped with its C^*-norm, and hence, the structure $\mathfrak{L}_Q$ of (6.19) is a well-defined Banach $*$-algebra.

By definition, all elements of $\mathfrak{L}_Q$ are the limits of linear combinations of the operators formed by

$$\mathbf{l}^n \otimes q_j = (\mathbf{l} \otimes q_j)^n, \quad \text{for all} \quad n \in \mathbb{N}_0, j \in \mathbb{Z},$$

by the cyclicity (6.18), and by the structure theorem (6.13) (see [9] and [11]). Equivalently, the operators $\{\mathbf{l} \otimes q_j\}_{j \in \mathbb{Z}}$ generate $\mathfrak{L}_Q$. From below, we denote these generating operators $\mathbf{l} \otimes q_j$ of $\mathfrak{L}_Q$ simply by u_j:

$$u_j \overset{denote}{=} \mathbf{l} \otimes q_j, \quad \text{for all} \quad j \in \mathbb{Z}. \tag{6.20}$$

Define now bounded linear transformations $E_{j,Q} : \mathfrak{L}_Q \to Q$ by the linear morphisms from $\mathfrak{L}_Q$ onto Q, satisfying that

$$E_{j,Q}\left(u_i^n\right) \overset{def}{=} \begin{cases} \frac{\varphi(q_j)^{n-1}}{\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)} \mathbf{1}^n\left(q_j\right) & \text{if } i = j, \\ O & \text{otherwise,} \end{cases} \tag{6.21}$$

for all $j \in \mathbb{Z}$, where O is the zero element of $\mathfrak{L}_Q$, and φ is the canonical trace (6.9) on $B(\mathcal{H})$, where $\left\lfloor \frac{n}{2} \right\rfloor$ are the *minimal integer* greater than or equal to $\frac{n}{2}$, for all $n \in \mathbb{N}$. Then these morphisms $E_{j,Q}$ are well-defined bounded linear transformations from $\mathfrak{L}_Q$ to Q, by (6.17), (6.18) and (6.19) (see [9, 11] and [13]).

Now, define linear functionals τ_j on $\mathfrak{L}_Q$, by

$$\tau_j \overset{def}{=} \varphi_j \circ E_{j,Q} \ \text{ on } \ \mathfrak{L}_Q, \forall j \in \mathbb{Z}, \tag{6.22}$$

where φ_j are the sectionized linear functionals (6.10) of the canonical trace φ of (6.9) on $B(\mathcal{H})$, and $E_{j,Q}$ are in the sense of (6.21).

Then one obtains well-defined Banach $*$-probability spaces

$$\mathfrak{L}_Q(j) \overset{denote}{=} \left(\mathfrak{L}_Q, \ \tau_j\right), \quad \forall j \in \mathbb{Z}, \tag{6.23}$$

where τ_j are the linear functionals (6.22) on $\mathfrak{L}_Q$.

Definition 6.7. We call the Banach $*$-probability spaces $\mathfrak{L}_Q(j)$ of (6.23), the j-th free filters of $B(\mathcal{H})$, for all $j \in \mathbb{Z}$.

We obtain the following free-distributional data on the j-th free filter $\mathfrak{L}_Q(j)$.

Lemma 6.8. *Let $\mathfrak{L}_Q(j)$ be the j-th free filter of $B(\mathcal{H})$, for $j \in \mathbb{Z}$. If $u_i = \mathbf{1} \otimes q_i$ are the generating operators* (6.20) *of $\mathfrak{L}_Q(j)$, then*

$$\tau_j\left(u_i^n\right) = \delta_{ij}\omega_n c_{\frac{n}{2}}, \quad \textit{for all} \ \ n \in \mathbb{N}, \tag{6.24}$$

where ω_n and $c_{\frac{n}{2}}$ are in the sense of (6.1).

Proof. The proof of (6.24) is done by straightforward computations, with help of (i) and (ii) of Lemma 6.6. See [13] for details. □

Let $\mathfrak{L}_Q(j)$ be the j-th free filters (6.23) of $B(\mathcal{H})$, for all $j \in \mathbb{Z}$. Construct the *free product Banach $*$-probability space of* $\{\mathfrak{L}_Q(j)\}_{j \in \mathbb{Z}}$, by

$$\mathfrak{L}_Q(\mathbb{Z}) \overset{denote}{=} \left(\mathfrak{L}_Q(\mathbb{Z}), \ \tau\right) \overset{def}{=} \underset{j \in \mathbb{Z}}{\star} \mathfrak{L}_Q(j) = \left(\underset{j \in \mathbb{Z}}{\star} \mathfrak{L}_Q, \ \underset{j \in \mathbb{Z}}{\star} \tau_j\right), \tag{6.25}$$

where $(\star)$ is the free product of [20, 21] and [23], i.e., Banach $*$-probability space $\mathfrak{L}_Q(\mathbb{Z})$ of (6.25) is the free product Banach $*$-probability space having its free blocks $\{\mathfrak{L}_Q(j)\}_{j \in \mathbb{Z}}$, the free filters (6.23).

Theorem 6.9. *Let* $u_j = \mathbf{1} \otimes q_j$ *be the "j-th" generating operator of the "j-th" free block* $\mathfrak{L}_Q(j)$ *of* $\mathfrak{L}_Q(\mathbb{Z})$*, for* $j \in \mathbb{Z}$*. Then it is semicircular in* $\mathfrak{L}_Q(\mathbb{Z})$*. Meanwhile, if* $u_i = \mathbf{1} \otimes q_i$ *are the i-th generating operators of the j-th free block* $\mathfrak{L}_Q(j)$ *of* $\mathfrak{L}(\mathbb{Z})$*, for* $i \neq j \in \mathbb{Z}$*, then they have the zero free distribution on* $\mathfrak{L}_Q(\mathbb{Z})$*.*

Proof. First of all, it is easy to check that the generating operators $u_k \in \mathfrak{L}_Q(j)$ are self-adjoint in $\mathfrak{L}_Q(\mathbb{Z})$, since

$$u_k^* = \mathbf{1}^* \otimes q_k^* = \mathbf{1} \otimes q_k = u_k,$$

since $\mathbf{1}^* = \mathbf{1}$ by (6.18), and $q_k^* = q_k$ on $\mathcal{H}$, for all $k, j \in \mathbb{Z}$.

Since the j-th generating operator $u_j \in \mathfrak{L}_Q(j)$ is self-adjoint in $\mathfrak{L}_Q(\mathbb{Z})$, the free distribution of u_j is fully characterized by the free-moment sequence $\left(\tau(u_j^n)\right)_{n=1}^{\infty}$, where τ is the free product linear functional of (6.25). Note that, since $u_j^n \in \mathfrak{L}_Q(j)$, as a free reduced word of $\mathfrak{L}(\mathbb{Z})$ with its length-1, we have

$$\tau\left(u_j^n\right) = \tau_j\left(u_j^n\right) = \omega_n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, by (6.24). Therefore, by the semicircularity (6.1), the self-adjoint free random variable u_j is semicircular in $\mathfrak{L}_Q(\mathbb{Z})$, for $j \in \mathbb{Z}$.

Now, let u_i be the i-th generating operators of the j-th free block $\mathfrak{L}_Q(j)$ of $\mathfrak{L}_Q(\mathbb{Z})$, for $i \neq j \in \mathbb{Z}$. It is shown that such generating operators u_i are self-adjoint in $\mathfrak{L}_Q(\mathbb{Z})$, and hence the free distribution of u_i are characterized by the free-moment sequence,

$$(\tau(u_i^n))_{n=1}^{\infty} = (\tau_j\left(u_i^n\right))_{n=1}^{\infty} = (0, 0, 0, 0, \ldots),$$

by (6.24). Therefore, they have the zero free distribution. See [9, 10] and [13] for details. □

6.4. SEMICIRCULAR ELEMENTS INDUCED BY $\{H_k\}_{k=1}^{N}$

In Section 6.3, we showed that, if $\mathcal{H} = l^2(\mathbb{Z})$ is in the sense of (6.3), and if $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ is the family of canonical orthogonal rank-1 projections of $B(\mathcal{H})$, then they induce semicircular elements

$$\mathcal{S} = \{u_j = \mathbf{1} \otimes q_j \in \mathfrak{L}_Q(j)\}_{j \in \mathbb{Z}} \subset \mathfrak{L}_Q(\mathbb{Z}), \tag{6.26}$$

as Banach-space operators, in the Banach $*$-probability space $\mathfrak{L}_Q(\mathbb{Z})$ of (6.25).

Moreover, the family $\mathcal{S}$ of (6.26) is a *free semicircular family* in the sense that: (i) it is a free family consisting of mutually free elements $u_j \in \mathfrak{L}_Q(j)$ in $\mathfrak{L}(\mathbb{Z})$, because $\{\mathfrak{L}_Q(j)\}_{j \in \mathbb{Z}}$ form the free blocks of $\mathfrak{L}_Q(\mathbb{Z})$, and (ii) all elements of $\mathcal{S}$ are semicircular in $\mathfrak{L}_Q(\mathbb{Z})$, by Theorem 6.9.

Naturally, one can define the Banach $*$-subalgebra $\mathbb{L}_Q$ of $\mathfrak{L}_Q(\mathbb{Z})$ generated by this free semicircular family $\mathcal{S}$ of (6.26), i.e.,

$$\mathbb{L}_Q \overset{def}{=} \overline{\mathbb{C}[\mathcal{S}]} \ \text{ in } \ \mathfrak{L}_Q(\mathbb{Z}), \tag{6.27}$$

where $\overline{Z}$ mean the Banach-topology closures of subsets Z of $\mathfrak{L}_Q(\mathbb{Z})$. And this Banach $*$-subalgebra $\mathbb{L}_Q$ of (6.27) induces the corresponding free-probabilistic sub-structure

$$\mathbb{L}_Q \overset{denote}{=} \left(\mathbb{L}_Q,\ \tau = \tau\mid_{\mathbb{L}_Q}\right) \text{ in } \mathfrak{L}_Q(\mathbb{Z}). \tag{6.28}$$

By the definition (6.27), this Banach $*$-probability space $\mathbb{L}_Q$ of (6.28) is generated by $|\mathbb{Z}|$-many, mutually-free semicircular elements. From now on, we focus on this Banach $*$-probability space $\mathbb{L}_Q$.

Recall that our fixed Hilbert spaces $H_k = l^2(n_k)$ of (2.1) are subspaces of $\mathcal{H}$, and hence, one can verify that the basis vectors,

$$\xi_l^{(k)} \in \mathcal{B}_k \text{ of } H_k, \text{ for all } l = 1, \ldots, n_k,$$

induce the mutually orthogonal n_k-many rank-1 projections

$$q_l^{(k)} \in B(H_k) \subset B(\mathcal{H}), \text{ for } l = 1, \ldots, n_k,$$

by identifying $q_l^{(k)}$ to $q_l \in \mathbf{Q} \subset Q \subset B(\mathcal{H})$, and they generates the semicircular elements,

$$u_l^{(k)} = \mathbf{1} \otimes q_l^{(k)} \in \mathfrak{L}_Q(l) \text{ in } \mathfrak{L}_Q(\mathbb{Z}),$$

by identifying $u_l^{(k)}$ as $u_l \in \mathfrak{L}_Q(l) \in \mathfrak{L}_Q(\mathbb{Z})$, equivalently,

$$u_l^{(k)} = \mathbf{1} \otimes q_l \in \mathbb{L}_Q, \tag{6.29}$$

for all $l = 1, \ldots, n_k$, where $\mathbb{L}_Q$ is the Banach $*$-probability space (6.28).

By (6.29), one can construct the free-probabilistic sub-structure,

$$\mathbb{L}_Q^{(k)} = \left(\mathbb{L}_Q^{(k)}, \tau = \tau\mid_{\mathbb{L}_Q^{(k)}}\right) \tag{6.30}$$

of the Banach $*$-probability space $\mathbb{L}_Q$, where the Banach $*$-subalgebra $\mathbb{L}_Q^{(k)}$ is generated by the free semicircular (sub-)family $\{u_l^{(k)}\}_{l=1}^{n_k}$ in $\mathbb{L}_Q$, i.e.,

$$\mathbb{L}_Q^{(k)} = \overline{\mathbb{C}\left[\{u_l^{(k)}\}_{l=1}^{n_k}\right]} \subset \mathbb{L}_Q, \tag{6.31}$$

for $k \in \{1, \ldots, N\}$.

Theorem 6.10. *The ONB $\mathcal{B}_k$ of a fixed Hilbert space H_k generates the Banach $*$-probability spaces $\mathbb{L}_Q^{(k)} = (\mathbb{L}_Q^{(k)}, \tau)$ of (6.30), generated by the free semicircular family,*

$$\mathcal{S}^{(k)} = \{u_l^{(k)} = \mathbf{1} \otimes q_l^{(k)}\}_{l=1}^{n_k}, \tag{6.32}$$

of the semicircular elements (6.29)*, for $k = 1, \ldots, N$.*

Proof. Since the family $\mathcal{S}^{(k)}$ of (6.32) is a free sub-family of the free semicircular family $\mathcal{S}$ of (6.26), generating the Banach $*$-probability space $\mathbb{L}_Q = (\mathbb{L}_Q, \tau)$, the Banach $*$-subalgebra $\mathbb{L}_Q^{(k)}$ of (6.31) is a well-determined free product Banach $*$-subalgebra,

$$\underset{l=1}{\overset{n_k}{\star}} \left(\overline{\mathbb{C}[\{u_l^{(k)}\}]}\right) \text{ of } \underset{j\in\mathbb{Z}}{\star} \left(\overline{\mathbb{C}[\{u_j\}]}\right) = \mathbb{L}_Q,$$

by (6.10) (e.g., see [11, 13, 20–22] and [23]). □

The above theorem, Theorem 6.10, generalizes the main result of [13], i.e., the natural rank-1 projections induced by the ONB vectors of an arbitrary (finite, or infinite dimensional) separable Hilbert space generate the corresponding semicircular elements.

Now, take such Banach $*$-probability spaces $\mathbb{L}_Q^{(1)}, \ldots, \mathbb{L}_Q^{(N)}$ of (6.30), and define a new free product Banach $*$-probability space

$$\begin{aligned} \mathbf{L}_Q[H_1, \ldots, H_N] &\overset{def}{=} \underset{k=1}{\overset{N}{\star}} \mathbb{L}_Q^{(k)} \\ &= \left(\underset{k=1}{\overset{N}{\star}} \mathbb{L}_Q^{(k)}, \ \underset{k=1}{\overset{N}{\star}} \left(\tau\mid_{\mathbb{L}_Q^{(k)}}\right) \overset{denote}{=} \tau^{\star N} \right) \end{aligned} \tag{6.33}$$

Then this Banach $*$-probability space $\mathbf{L}_Q[H_1, \ldots, H_N]$ of (6.33) is the free product Banach $*$-probability space generated by the free semicircular family,

$$\mathcal{S}[H_1, \ldots, H_N] = \bigsqcup_{k=1}^{N} \mathcal{S}^{(k)}, \tag{6.34}$$

where $\sqcup$ is the disjoint union, and $\mathcal{S}^{(k)}$ are the free semicircular families (6.32), for all $k = 1, \ldots, N$.

By the definition (6.30) and (6.33), the subsets $\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(N)}$ of the free semicircular family $\mathcal{S}[H_1, \ldots, H_N]$ of (6.34) are mutually free from each other in $\mathbf{L}_Q[H_1, \ldots, H_N]$.

Theorem 6.11. *Let $H_k = l^2(n_k)$ be our fixed Hilbert spaces* (2.1) *with their ONBs $\mathcal{B}_k$, for $k = 1, \ldots, N$. Then the subset $\bigcup_{k=1}^{N} \mathcal{B}_k$ of the free Hilbert space $\mathfrak{F}$ induces the free semicircular family $\mathcal{S}[H_1, \ldots, H_N]$ of* (6.34)*, generating the free product Banach $*$-probability space $\mathbf{L}_Q[H_1, \ldots, H_N]$ of* (6.33).

Proof. The proof is done by the very constructions of (6.32) and (6.33). □

The above theorem shows how to construct semicircular elements from N-many finite-or-infinite-dimensional l^2-Hilbert spaces, which also generalizes the main results of [13].

Definition 6.12. We call the Banach $*$-probability space $\mathbf{L}_Q[H_1, \ldots, H_N]$ of (6.33), the semicircular (Banach $*$-probability) space of $H_1, \ldots, H_N$.

7. JUMP-SHIFT OPERATORS ON $\mathfrak{F}$ AND SEMICIRCULARITY ON $\mathbf{L}_Q[H_1, \ldots, H_N]$

In this section, we study how our j-s operators $\sum[j > i] \in B(\mathfrak{F})$ of (5.1) affect the free probability on the semicircular space $\mathbf{L}_Q[H_1, \ldots, H_N]$ of (6.33), for $i < j \in \{1, \ldots, N\}$.

Recall and note that the j-s operator $\sum[j > i] \in B(\mathfrak{F})$ for $i < j \in \{1, \ldots, N\}$ satisfies the spectral property (i), or (ii), or (iii) of Theorem 5.3, by (5.4),

$$\begin{aligned} \left(\sum[j>i]\right)^* \left(\sum[j>i]\right) &= \begin{cases} 1_{\mathfrak{F}} & \text{if } n_i = n_j \text{ in } \mathbb{N}^\infty, \\ P_{j>i} & \text{if } n_i < n_j \text{ in } \mathbb{N}^\infty, \end{cases} \\ \left(\sum[j>i]\right) \left(\sum[j>i]\right)^* &= \begin{cases} 1_{\mathfrak{F}} & \text{if } n_i = n_j < \infty \text{ in } \mathbb{N}^\infty, \\ P_1 & \text{if } n_i = n_j = \infty \text{ in } \mathbb{N}^\infty, \\ P_{j>i} & \text{if } n_i < n_j \text{ in } \mathbb{N}^\infty, \end{cases} \end{aligned} \tag{7.1}$$

where P_1 and $P_{j>i}$ are the projections from Lemma 3.8(ii), respectively, (4.13) on the free Hilbert space $\mathfrak{F}$.

Also, note that the semicircular space $\mathbf{L}_Q[H_1, \ldots, H_N]$ is generated by the free semicircular family $\mathcal{S}[H_1, \ldots, H_N]$ of (6.34) in the Banach $*$-probability space $\underset{l=1}{\overset{N}{\star}} \mathbb{L}_Q$ of the N-copies of the Banach $*$-probability space $\mathbb{L}_Q$ of (6.28), for all $l = 1, \ldots, N$. From below, fix an arbitrary semicircular element,

$$u_j^{(k)} = \mathbf{1} \otimes q_j^{(k)} \in \mathcal{S}[H_1, \ldots, H_N], \tag{7.2}$$

generating $\mathbf{L}_Q[H_1, \ldots, H_N]$, for $j = 1, \ldots, n_k$ and $k = 1, \ldots, N$

7.1. BANACH-SPACE OPERATORS ON $\mathbf{L}_Q[H_1, \ldots, H_N]$ INDUCED BY JUMP-SHIFT OPERATORS

In this section, we construct-and-study certain Banach-space operators acting on the semicircular space $\mathbf{L}_Q[H_1, \ldots, H_N]$ of (6.33) (by regarding it as a Banach space), induced by our j-s operators of $B(\mathfrak{F})$.

Fix an arbitrary $i < j$ in $\{1, \ldots, N\}$, and take the corresponding j-s operator,

$$\sum[j > i] = \alpha \cdot J^{\mathbf{1}_{n_i}}[j > i] \in B(\mathfrak{F}) \tag{7.3}$$

where α is the shift operator, and $J^{\mathbf{1}_{n_i}}[j > i]$ is the $\mathbf{1}_{n_i}$-jump operator on the free Hilbert space $\mathfrak{F}$.

Observe now that, for any basis vectors,

$$\xi_l^{(k)} \in \mathcal{B}_k \subset \left(\bigcup_{i=1}^N \mathcal{B}_i\right) \subset \mathfrak{B} \text{ of } \mathfrak{F}, \tag{7.4}$$

the j-s operator $\sum[j > i]$ of (7.3) assigns it to

$$\sum[j>i]\left(\xi_l^{(k)}\right) = \alpha\left(J^{\mathbf{1}_{n_i}}[j>i]\left(\xi_l^{(k)}\right)\right)$$
$$= \begin{cases} \alpha\left(\xi_l^{(k)}\right) & \text{if } k \neq j, \\ \alpha\left(\xi_l^{(j)}\right) & \text{if } k = j, \text{ and } l = 1, \ldots, n_i, \\ \alpha\left(O_{\mathfrak{F}}\right) & \text{if } k = j, \text{ and } l = n_i + 1, \ldots, n_j \end{cases} \tag{7.5}$$
$$= \begin{cases} \xi_l^{(k)} & \text{if } k \neq j, \\ \xi_{[l+1]}^{(i)} & \text{if } k = j, \text{ and } l = 1, \ldots, n_i, \\ O_{\mathfrak{F}} & \text{if } k = j, \text{ and } l = n_i + 1, \ldots, n_j \end{cases}$$

for all $k \in \{1, \ldots, N\}$, and for all $l \in \{1, \ldots, n_j\}$.

Consider now that, by (6.33) and (6.34), an arbitrary basis vector $\xi_l^{(k)} \in \mathfrak{B} \subset \mathfrak{F}$ of (7.4) induces the corresponding semicircular element (7.2),

$$u_l^{(k)} = \mathbf{1} \otimes q_l^{(k)} \in \mathcal{S}[H_1, \ldots, H_N],$$

in $\mathbf{L}_Q[H_1, \ldots, H_N]$, i.e.,

$$\xi_l^{(k)} \overset{\text{induce}}{\Longrightarrow} q_l^{(k)}(\bullet) = \left\langle \bullet,\ \xi_l^{(k)} \right\rangle \xi_l^{(k)} \in B\left(\mathfrak{F}\right)$$
$$\overset{\text{induce}}{\Longrightarrow} \mathbf{1} \otimes q_l^{(k)} = u_l^{(k)} \in \mathcal{S}[H_1, \ldots, H_N], \tag{7.6}$$

in $\mathbf{L}_Q[H_1, \ldots, H_N]$.

So, by (7.5) and (7.6), one can define a Banach-space operator,

$$S_{j>i} \in B\left(\mathbf{L}_Q[H_1, \ldots, H_N]\right),$$

acting on the semicircular space $\mathbf{L}_Q[H_1, \ldots, H_N]$, by a bounded "multiplicative" linear morphism,

$$S_{j>i}\left(u_l^{(k)}\right) = S_{j>i}\left(\mathbf{1} \otimes q_l^{(k)}\right)$$
$$= S_{j>i}\left(\mathbf{1} \otimes \left(\left\langle \bullet, \xi_l^{(k)} \right\rangle \xi_l^{(k)}\right)\right) \tag{7.7}$$
$$= \mathbf{1} \otimes \left(\left\langle \bullet,\ \sum[j>i]\left(\xi_l^{(k)}\right)\right\rangle \sum[j>i]\left(\xi_l^{(k)}\right)\right),$$

for all $u_l^{(k)} \in \mathcal{S}[H_1, \ldots, H_N]$, where $\sum[j > i]\left(\xi_l^{(k)}\right) \in \mathfrak{F}$ satisfies (7.5).

By the multiplicativity of the Banach-space operator $S_{j>i}$ of (7.7), it generally satisfies that

$$S_{j>i}\left(\prod_{s=1}^{m}\left(u_{l_s}^{(k_s)}\right)^{n_s}\right) = \prod_{s=1}^{m} S_{j>i}\left(\left(u_{l_s}^{(k_s)}\right)^{n_s}\right)$$
$$= \prod_{s=1}^{m}\left(S_{j>i}\left(u_{l_s}^{(k_s)}\right)\right)^{n_s}, \tag{7.8}$$

for all $n_1, \ldots, n_m, m \in \mathbb{N}$, where $k_s \in \{1, \ldots, N\}$ and $l_s \in \{1, \ldots, n_{k_s}\}$, and where each factor $S_{j>i}\left(u_{l_s}^{(k_s)}\right)$ in (7.8) satisfies (7.7), for all $s = 1, \ldots, m$.

Such a multiplicativity of $S_{j>i}$ is in fact naturally determined by the very definitions of jump operators, shift operators, and j-s operators of $B(\mathfrak{F})$ (see (3.18), (3.19), (3.20), (4.3), Proposition 4.6(i) and (5.1)).

Consider the definition (7.7) of the Banach-space operator $S_{j>i}$, induced by the j-s operator $\sum[j > i]$, more precisely. Observe that

$$\begin{aligned} S_{j>i}\left(u_l^{(k)}\right) &= \mathbf{1} \otimes \left(\left\langle \bullet,\ \sum[j>i]\left(\xi_l^{(k)}\right)\right\rangle \sum[j>i]\left(\xi_l^{(k)}\right)\right) \\ &= \begin{cases} \mathbf{1} \otimes \left(\left\langle \bullet,\ \xi_l^{(k)}\right\rangle \xi_l^{(k)}\right) & \text{if } k \neq j, \\ \mathbf{1} \otimes \left(\left\langle \bullet,\ \xi_{[l+1]}^{(i)}\right\rangle \xi_{[l+1]}^{(i)}\right) & \text{if } k = j \text{ and } l = 1, \ldots, n_i, \\ \mathbf{1} \otimes O_{\mathfrak{F}} = O & \text{if } k = j \text{ and } l = n_i + 1, \ldots, n_j, \end{cases} \\ &= \begin{cases} u_l^{(k)} & \text{if } k \neq j, \\ u_{[l+1]}^{(i)} & \text{if } k = j, \text{ and } l = 1, \ldots, n_i, \\ O & \text{if } k = j, \text{ and } l = n_i + 1, \ldots, n_j. \end{cases} \end{aligned} \tag{7.9}$$

This computation (7.9) illustrates that the Banach-space operator $S_{j>i}$ on $\mathbf{L}_Q[H_1, \ldots, H_N]$, induced by the j-s operator $\sum[j > i] \in B(\mathfrak{F})$ of (7.3), assign the generating semicircular elements $u_l^{(k)} \in \mathcal{S}[H_1, \ldots, H_N]$ to the other semicircular element of $\mathcal{S}[H_1, \ldots, H_N]$, or the zero free random variable O of $\mathbf{L}_Q[H_1, \ldots, H_N]$.

Theorem 7.1. *For any $i < j \in \{1, \ldots, N\}$, the corresponding j-s operator $\sum[j > i] \in B(\mathfrak{F})$ of (7.3) induces a Banach-space operator $S_{j>i}$ of (7.7), acting on the semicircular space $\mathbf{L}_Q[H_1, \ldots, H_N]$, satisfying that*

$$S_{j>i}\left(u_l^{(k)}\right) = \begin{cases} u_l^{(k)} & \text{if } k \neq j, \\ u_{[l+1]}^{(i)} & \text{if } k = j, \text{ and } l = 1, \ldots, n_i, \\ O & \text{if } k = j, \text{ and } l = n_i + 1, \ldots, n_j, \end{cases} \tag{7.10}$$

for all $u_l^{(k)} \in \mathcal{S}[H_1, \ldots, H_N]$, for $l \in \{1, \ldots, n_k\}$, for $k \in \{1, \ldots, N\}$.

Proof. The well-definedness of multiplicative Banach-space operators $S_{j>i}$ of (7.7) from j-s operators $\sum[j > i]$ is guaranteed by (7.5) and (7.6). The formula (7.10) is obtained by (7.7) and (7.9). □

The above theorem shows that the j-s operators $\sum[j > i]$ on the free Hilbert space $\mathfrak{F} = \mathfrak{F}[H_1, \ldots, H_N]$ provide an action on the semicircular space $\mathbf{L}_Q[H_1, \ldots, H_N]$, and such an action is determined by the Banach-space operators $S_{j>i}$ of (7.7), for all $i < j \in \{1, \ldots, N\}$.

Now, let $\sum[j > i]^* \in B(\mathfrak{F})$ be the adjoint of the j-s operator $\sum[j > i]$ of (7.3), i.e.,

$$\sum[j > i]^* = \left(J^{\mathbf{1}_{n_i}}[j > i]\right)^* \alpha^* \ \text{ on } \ \mathfrak{F}. \tag{7.11}$$

Let $\xi_l^{(k)} \in \mathcal{B}_k \subset (\bigcup_{i=1}^N \mathcal{B}_i) \subset \mathfrak{B}$ be a basis vector of $\mathfrak{F}$. Then

$$\begin{aligned}\sum[j>i]^*\left(\xi_l^{(k)}\right) &= \left(J^{\mathbf{1}_{n_i}}[j>i]\right)^*\left(\alpha^*(\xi_l^{(k)})\right)\\ &\text{by (7.11)}\\ &= \left(J^{\mathbf{1}_{n_i}}[j>i]\right)^*\left(\xi_{[l-1]}^{(k)}\right)\\ &= \begin{cases}\xi_{[l-1]}^{(k)} & \text{if } k\neq i,\\ \xi_{[l-1]}^{(j)} & \text{if } k=i, \text{ and } [l-1]=1,\ldots,n_i,\\ O_{\mathfrak{F}} & \text{if } k=i, \text{ and } [l-1]=n_i+1,\ldots,n_j.\end{cases}\end{aligned} \tag{7.12}$$

Similar to the Banach-space operator $S_{j>i}$ of (7.7), one can define a multiplicative Banach-space operator $S_{j>i}^*$ on $\mathbf{L}_Q[H_1, \ldots, H_N]$ by the multiplicative linear morphism satisfying that

$$S_{j>i}^*\left(u_l^{(k)}\right) = \mathbf{1}\otimes\left(\left\langle\bullet,\ \sum[j>i]^*\left(\xi_l^{(k)}\right)\right\rangle\sum[j>i]^*\left(\xi_l^{(k)}\right)\right), \tag{7.13}$$

for all $u_l^{(k)} \in \mathcal{S}[H_1, \ldots, H_N]$.

Note here that we cannot say that the Banach-space operator $S_{j>i}^*$ of (7.13) is the adjoint of $S_{j>i}$. It is just a new Banach-space operator on $\mathbf{L}_Q[H_1, \ldots, H_N]$ induced by the adjoint $\sum[j>i]^* \in B(\mathfrak{F})$ of the j-s operator $\sum[j>i] \in B(\mathfrak{F})$.

This operator $S_{j>i}^*$ satisfies that

$$\begin{aligned}S_{j>i}^*\left(u_l^{(k)}\right) &= \mathbf{1}\otimes\left(\left\langle\bullet,\ \sum[j>i]^*\left(\xi_l^{(k)}\right)\right\rangle\sum[j>i]^*\left(\xi_l^{(k)}\right)\right)\\ &= \begin{cases}\mathbf{1}\otimes q_l^{(k)} & \text{if } k\neq i,\\ \mathbf{1}\otimes q_{[l-1]}^{(j)} & \text{if } k=i, \text{ and } [l-1]=1,\ldots,n_i,\\ \mathbf{1}\otimes O_{\mathfrak{F}} & \text{if } k=i, \text{ and } [l-1]=n_i+1,\ldots,n_j\end{cases}\\ &\text{by (7.12)}\\ &= \begin{cases}u_l^{(k)} & \text{if } k\neq i,\\ u_{[l-1]}^{(j)} & \text{if } k=i, \text{ and } [l-1]=1,\ldots,n_i,\\ O & \text{if } k=i, \text{ and } [l-1]=n_i+1,\ldots,n_j,\end{cases}\end{aligned} \tag{7.14}$$

for all $u_l^{(k)} \in \mathcal{S}[H_1, \ldots, H_N]$, in $\mathbf{L}_Q[H_1, \ldots, H_N]$.

Theorem 7.2. *For any $i < j \in \{1, \ldots, N\}$, the adjoint $\sum[j>i]^*$ of the j-s operator $\sum[j>i] \in B(\mathfrak{F})$ induces a Banach-space operator $S_{j>i}^*$ of* (7.13)*, acting on the semicircular space $\mathbf{L}_Q[H_1, \ldots, H_N]$, satisfying that*

$$S_{j>i}^*\left(u_l^{(k)}\right) = \begin{cases}u_l^{(k)} & \text{if } k\neq i,\\ u_{[l-1]}^{(j)} & \text{if } k=i, \text{ and } [l-1]=1,\ldots,n_i,\\ O & \text{if } k=i, \text{ and } [l-1]=n_i+1,\ldots,n_j,\end{cases} \tag{7.15}$$

for all $u_l^{(k)} \in \mathcal{S}[H_1, \ldots, H_N]$, for $l \in \{1, \ldots, n_k\}$, for $k \in \{1, \ldots, N\}$.

Proof. The well-definedness of multiplicative Banach-space operators $S^{*}_{j>i}$ of (7.13) from $\sum[j > i]^{*}$ is guaranteed by (7.6) and (7.12). The formula (7.15) is obtained by (7.14). □

The above theorem shows how the adjoint $\sum[j > i]^{*}$ of the j-s operator $\sum[j > i] \in B(\mathfrak{F})$ induces a corresponding Banach-space operator $S^{*}_{j>i}$ acting on $\mathbf{L}_Q[H_1, \ldots, H_N]$.

Definition 7.3. The Banach-space operator $S_{j>i}$ of (7.7), induced by the j-s operator $\sum[j > i] \in B(\mathfrak{F})$, is called the jump-shift on $\mathbf{L}_Q[H_1, \ldots, H_N]$ for $i < j \in \{1, \ldots, N\}$. Also, the Banach-space operator $S^{*}_{j>i}$ of (7.13) induced by the adjoint $\sum[j > i]^{*} \in B(\mathfrak{F})$ is said to be the jump-shift-adjoint (in short, the j-s-adjoint) on $\mathbf{L}_Q[H_1, \ldots, H_N]$ for $i < j \in \{1, \ldots, N\}$.

7.2. THE CASE WHERE $n_i = n_j < \infty$

Throughout this section, fix $i < j \in \{1, \ldots, N\}$, and suppose

$$n_i = n_j < \infty \quad \text{in} \quad \mathbb{N}^{\infty}, \tag{7.16}$$

where $n_l = \dim_{\mathbb{C}} H_l$, for all $l = 1, 2$.

By Theorem 5.3(i) and (ii), if the condition (7.16) holds, then the corresponding j-s operator $\sum[j > i] \in B(\mathfrak{F})$ is a unitary on the free Hilbert space $\mathfrak{F}$, i.e.,

$$\left(\sum[j > i]\right)^{*}\left(\sum[j > i]\right) = 1_{\mathfrak{F}} = \left(\sum[j > i]\right)\left(\sum[j > i]\right)^{*}, \tag{7.17}$$

on $\mathfrak{F}$.

By (7.10), (7.15) and (7.17), if the condition (7.16) holds, then we obtain the following characterization.

Lemma 7.4. *Suppose the condition* (7.16) *holds. If $S_{j>i}$ is the jump-shift* (7.7)*, and $S^{*}_{j>i}$ is the j-s-adjoint* (7.13) *for $i < j \in \{1, \ldots, N\}$. Then*

$$S^{*}_{j>i}S_{j>i} = 1_{\mathbf{L}_Q} = S_{j>i}S^{*}_{j>i}, \tag{7.18}$$

on $\mathbf{L}_Q[H_1, \ldots, H_N]$, where $1_{\mathbf{L}_Q}$ is the identity operator of $B\left(\mathbf{L}_Q[H_1, \ldots, H_N]\right)$.

Proof. The proof of (7.18) is done by (7.10), (7.15) and (7.17). Indeed, under (7.16),

$$S^{*}_{j>i}S_{j>i}\left(u_l^{(k)}\right) = \begin{cases} S^{*}_{j>i}\left(u_l^{(k)}\right) = u_l^{(k)} & \text{if } k \neq j, \\ S^{*}_{j>i}\left(u_{[l+1]}^{(i)}\right) = u_{[l+1-1]}^{(j)} = u_l^{(j)} & \text{if } k = j, \end{cases}$$

and

$$S_{j>i}S^{*}_{j>i}\left(u_l^{(k)}\right) = \begin{cases} S_{j>i}\left(u_l^{(k)}\right) = u_l^{(k)} & \text{if } k \neq i, \\ S_{j>i}\left(u_{[l-1]}^{(j)}\right) = u_{[l-1+1]}^{(i)} = u_l^{(i)} & \text{if } k = i, \end{cases}$$

implying that

$$S^*_{j>i}S_{j>i}\left(u_l^{(k)}\right) = u_l^{(k)} = S_{j>i}S^*_{j>i}\left(u_l^{(k)}\right), \tag{7.19}$$

for all $u_l^{(k)} \in \mathcal{S}[H_1,\ldots,H_N]$.

By (6.33) and (6.34), the relation (7.19) guarantees the relation (7.18). □

By the above lemma, we obtain the following result.

Theorem 7.5. *Let* $S_{j>i}, S^*_{j>i} \in B\left(\mathbf{L}_Q[H_1,\ldots,H_N]\right)$ *be in the sense of* (7.7) *and* (7.13)*, respectively, for* $i < j \in \{1,\ldots,N\}$*. If the condition* (7.16) *holds, then*

$$S_{j>i}\left(u_l^{(k)}\right) \quad \textit{and} \quad S^*_{j>i}\left(u_l^{(k)}\right) \quad \textit{are semicircular}, \tag{7.20}$$

in $\mathbf{L}_Q[H_1,\ldots,H_N]$*, for all semicircular elements* $u_l^{(k)} \in \mathcal{S}[H_1,\ldots,H_N]$.

Proof. Under the condition (7.16), one can realize that

$$S_{j>i}\left(u_l^{(k)}\right),\ S^*_{j>i}\left(u_l^{(k)}\right) \in \mathcal{S}[H_1,\ldots,H_N],$$

in $\mathbf{L}_Q[H_1,\ldots,H_N]$, by (7.19), implying that they are semicircular, too. Therefore, the relation (7.20) holds. □

By the multiplicativity of $\left\{S_{j>i},\ S^*_{j>i}\right\}$, the above relation (7.20) guarantees that if $n_i = n_j < \infty$ in $\mathbb{N}^\infty$, then the free probability on $\mathbf{L}_Q[H_1,\ldots,H_N]$ is preserved by the action of $\left\{S_{j>i},\ S^*_{j>i}\right\}$.

Corollary 7.6. *Let* $\sum[j>i] \in B(\mathfrak{F})$ *be the j-s operators for* $i < j \in \{1,\ldots,N\}$*. If* $n_i = n_j < \infty$ *in* $\mathbb{N}^\infty$*, then the action of*

$$\left\{\sum[j>i],\ \sum[j>i]^*\right\},$$

on the free Hilbert space $\mathfrak{F}$*, preserves the free probability on the semicircular space* $\mathbf{L}_Q[H_1,\ldots,H_N]$.

Proof. The proof is done by (7.7), (7.13) and (7.20). □

7.3. THE CASE WHERE $n_i = n_j = \infty$

In this section, fix $i < j \in \{1,\ldots,N\}$, and assume that

$$n_i = \dim_{\mathbb{C}} H_i = \dim_{\mathbb{C}} H_j = n_j = \infty \ \text{ in } \ \mathbb{N}^\infty, \tag{7.21}$$

and let $\sum[j>i] \in B(\mathfrak{F})$ be the corresponding j-s operator with its adjoint $\sum[j>i]^*$. Also, let $S_{j>i}$ be the jump-shift (7.7), and $S^*_{j>i}$, the j-s-adjoint (7.13) on $\mathbf{L}_Q[H_1,\ldots,H_N]$, induced by $\sum[j>i]$, respectively, by $\sum[j>i]^*$.

Under the condition (7.21), one has that

$$S^*_{j>i}S_{j>i}\left(u_l^{(k)}\right) = \begin{cases} S^*_{j>i}\left(u_l^{(k)}\right) = u_l^{(k)} & \text{if } k \neq j, \\ S^*_{j>i}\left(u_{l+1}^{(i)}\right) = u_l^{(j)} & \text{if } k = j, \end{cases}$$

by (7.10) and (7.15), i.e.,

$$S^*_{j>i}S_{j>i}\left(u_l^{(k)}\right) = u_l^{(k)} \ \text{ in } \ \mathbf{L}_Q[H_1,\ldots,H_N], \tag{7.22}$$

for all $u_l^{(k)} \in \mathcal{S}[H_1,\ldots,H_N]$.

By (6.33) and (6.34), the relation (7.22) implies that

$$S^*_{j>i}S_{j>i} = 1_{\mathbf{L}_Q} \ \text{ on } \ \mathbf{L}_Q[H_1,\ldots,H_N]. \tag{7.23}$$

Similarly, under (7.21),

$$S_{j>i}S^*_{j>i}\left(u_l^{(k)}\right) = \begin{cases} S_{j>i}\left(u_l^{(k)}\right) = u_l^{(k)} & \text{if } k \neq i, \\ S_{j>i}\left(u_{l-1}^{(j)}\right) = u_l^{(i)} & \text{if } k = i, \text{ and } l > 1, \\ S_{j>i}\left(O_{\mathfrak{F}}\right) = O & \text{if } k = i, \text{ and } l = 1, \end{cases}$$

by (7.10) and (7.15), i.e.,

$$S_{j>i}S^*_{j>i}\left(u_l^{(k)}\right) = \begin{cases} O & \text{if } k = i, \text{ and } l = 1, \\ u_l^{(k)} & \text{otherwise,} \end{cases} \tag{7.24}$$

for all $u_l^{(k)} \in \mathcal{S}[H_1,\ldots,H_N]$.

Define now a "multiplicative" Banach-space operator $P_{i,1}$ of $B\left(\mathbf{L}_Q[H_1,\ldots,H_N]\right)$ by a multiplicative linear morphism satisfying that

$$P_{i,1}\left(u_l^{(k)}\right) = \begin{cases} u_l^{(k)} & \text{if } k \neq i, \\ u_l^{(i)} & \text{if } k = i, \text{ and } l > 1, \\ O & \text{if } k = i, \text{ and } l = 1, \end{cases} \tag{7.25}$$

for all $u_l^{(k)} \in \mathcal{S}[H_1,\ldots,H_N]$. Then, by (6.33), (6.34), (7.24) and (7.25), one obtains that

$$S_{j>i}S^*_{j>i} = P_{i,1} \ \text{ on } \ \mathbf{L}_Q[H_1,\ldots,H_N]. \tag{7.26}$$

Recall that, under (7.21), we have that

$$\sum[j>i]^* \sum[j>i] = 1_{\mathfrak{F}} \quad \text{and} \quad \sum[j>i]\sum[j>i]^* = P_1, \tag{7.27}$$

in the operator algebra $B(\mathfrak{F})$, where P_1 is the projection (4.13) on $\mathfrak{F}$, i.e., if the condition (7.21) holds, then the j-s operator $\sum[j > i]$ is an isometry, but the adjoint $\sum[j > i]^*$ is a partial isometry on $\mathfrak{F}$, as in (7.27). If we compare the operator equalities (7.23) and (7.26), with (7.27), then these equalities are reasonable because $S_{j>i}$ and $S^*_{j>i}$ are induced by $\sum[j > i]$, and $\sum[j > i]^*$, respectively.

Lemma 7.7. *Let $S_{j>i}$ and $S^*_{j>i}$ be the jump-shift, and the j-s-adjoint on the semicircular Banach space $\mathbf{L}_Q[H_1,\ldots,H_N]$ for the fixed $i < j \in \{1,\ldots,N\}$. Then*

$$S^*_{j>i}S_{j>i} = 1_{\mathbf{L}_Q} \quad \textit{and} \quad S_{j>i}S^*_{j>i} = P_{i,1} \tag{7.28}$$

on $\mathbf{L}_Q[H_1,\ldots,H_N]$, where $P_{i,1}$ is in the sense of (7.25).

Proof. The proof of the relation (7.28) is done by (7.24) and (7.26) in the operator space $B(\mathbf{L}_Q[H_1,\ldots,H_N])$. □

By (7.28), the jump-shift $S_{j>i}$ is acting like an isometry, and the j-s-adjoint $S_{j>i}^*$ is acting like a partial isometry on $\mathbf{L}_Q[H_1,\ldots,H_N]$, as Banach-space operators. So, one can obtain the following result.

Theorem 7.8. *Let $S_{j>i}$ and $S_{j>i}^*$ be the jump-shift, and the j-s-adjoint on the semicircular space $\mathbf{L}_Q[H_1,\ldots,H_N]$ for fixed $i<j\in\{1,\ldots,N\}$. If the condition* (7.21) *holds, then*

$$S_{j>i}\left(u_l^{(k)}\right)\in\mathcal{S}\left[H_1,\ldots,H_N\right],$$

and

$$S_{j>i}^*\left(u_l^{(k)}\right)=\begin{cases}u_l^{(k)}\in\mathcal{S}[H_1,\ldots,H_N] & \text{if } k\neq i,\\ u_{l-1}^{(j)}\in\mathcal{S}[H_1,\ldots,H_N] & \text{if } k=i \text{ and } l>1,\\ O & \text{if } k=i \text{ and } l=1,\end{cases}\tag{7.29}$$

in $\mathbf{L}_Q[H_1,\ldots,H_N]$, for all $u_l^{(k)}\in\mathcal{S}[H_1,\ldots,H_N]$. Equivalently, $S_{j>i}(u_l^{(k)})$ are semicircular in $\mathbf{L}_Q[H_1,\ldots,H_N]$, for all $u_l^{(k)}\in\mathcal{S}[H_1,\ldots,H_N]$, and $S_{j>i}^*(u_l^{(k)})$ are semicircular in $\mathbf{L}_Q[H_1,\ldots,H_N]$, whenever "$k\neq i$", or "$k=i$ and $l>1$", and $S_{j>i}^*(u_1^{(i)})=O$, having the zero free distribution in $\mathbf{L}_Q[H_1,\ldots,H_N]$.

Proof. The formula (7.29) is obtained by (7.28). □

The above theorem shows how the action of $\{S_{j>i},\ S_{j>i}^*\}$ affect the original free-distributional data on $\mathbf{L}_Q[H_1,\ldots,H_N]$, by (6.33) and (6.34), under (7.21), i.e., if the condition (7.21) holds, then $S_{j>i}$ preserves the free probability on $\mathbf{L}_Q[H_1,\ldots,H_N]$, but $S_{j>i}^*$ deforms the free probability on $\mathbf{L}_Q[H_1,\ldots,H_N]$ in the sense that: $S_{j>i}^*(u_1^{(i)})$ follows the zero free distribution, meanwhile, $S_{j>i}^*\left(u_l^{(k)}\right)$ follows the semicircular law, whenever $l\neq 1$, in $\mathbf{L}_Q[H_1,\ldots,H_N]$.

Corollary 7.9. *For a fixed $i<j\in\{1,\ldots,N\}$, satisfying* (7.21)*, the action of the j-s operator $\sum[j>i]$ on the free Hilbert space $\mathfrak{F}$ preserves the free probability on the semicircular space $\mathbf{L}_Q[H_1,\ldots,H_N]$, but the action of the adjoint $\sum[j>i]^*$ in $B(\mathfrak{F})$ deforms the free probability on $\mathbf{L}_Q[H_1,\ldots,H_N]$; and the deformation is characterized by* (7.29).

Proof. This corollary is a re-expression of Theorem 7.8, by (7.6). □

7.4. THE CASE WHERE $n_i<n_j$

In this section, fix $i<j\in\{1,\ldots,N\}$, and assume that

$$n_i=\dim_{\mathbb{C}}H_i<\dim_{\mathbb{C}}H_j=n_j \text{ in } \mathbb{N}^\infty,\tag{7.30}$$

and let $S_{j>i}$ and $S_{j>i}^*$ be the corresponding jump-shift, and the j-s-adjoint on the semicircular space $\mathbf{L}_Q[H_1,\ldots,H_N]$, induced by the j-s operator $\sum[j>i]$, and its adjoint $\sum[j>i]^*$ of $B(\mathfrak{F})$, respectively, for $i<j$.

If the condition (7.30) holds, then

$$\left(\sum[j>i]\right)^{*}\left(\sum[j>i]\right) = P_{j>i} = \left(\sum[j>i]\right)\left(\sum[j>i]\right)^{*}, \tag{7.31}$$

by (5.4), where $P_{j>i}$ is the projection from Lemma 3.8(ii) of $B(\mathfrak{F})$. Similar to Section 7.3, we define the following Banach-space operator $\mathcal{P}_{j>i}$ on $\mathbf{L}_Q[H_1, \ldots, H_N]$ by the multiplicative linear transformation satisfying

$$\mathcal{P}_{j>i}\left(u_l^{(k)}\right) = \begin{cases} u_l^{(k)} & \text{if } k \neq j, \\ u_l^{(j)} & \text{if } k = j, \text{ and } l \in \{1, \ldots, n_i\}, \\ O & \text{if } k = j, \text{ and } l \in \{n_i+1, \ldots, n_j\}, \end{cases} \tag{7.32}$$

in $\mathbf{L}_Q[H_1, \ldots, H_N]$, for all $u_l^{(k)} \in \mathcal{S}[H_1, \ldots, H_N]$. It is easy to check that the projection $P_{j>i} \in B(\mathfrak{F})$ of Lemma 3.8(ii) induces the above Banach-space operator $\mathcal{P}_{j>i}$ of (7.32) as in (7.6).

Note that

$$S_{j>i}^{*}S_{j>i}\left(u_l^{(k)}\right) = \begin{cases} S_{j>i}^{*}\left(u_l^{(k)}\right) = u_l^{(k)} & \text{if } k \neq j, \\ S_{j>i}^{*}\left(u_l^{(i)}\right) = u_l^{(j)} & \begin{array}{l}\text{if } k = j, \text{ and} \\ l \in \{1, \ldots, n_i\},\end{array} \\ S_{j>i}^{*}(O) = O & \begin{array}{l}\text{if } k = j, \text{ and} \\ l \in \{n_i+1, \ldots, n_j\},\end{array} \end{cases} \tag{7.33}$$

$$S_{j>i}S_{j>i}^{*}\left(u_l^{(k)}\right) = \begin{cases} S_{j>i}\left(u_l^{(k)}\right) = u_l^{(k)} & \text{if } k \neq i, \\ S_{j>i}\left(u_l^{(j)}\right) = u_l^{(i)} & \text{if } k = i, \end{cases}$$

by (7.6) and (7.31).

By (7.32) and (7.33), one can obtain that

$$S_{j>i}^{*}S_{j>i}\left(u_l^{(k)}\right) = \mathcal{P}_{j>i}\left(u_l^{(k)}\right) = S_{j>i}S_{j>i}^{*}\left(u_l^{(k)}\right), \tag{7.34}$$

in $\mathbf{L}_Q[H_1, \ldots, H_N]$, for all $u_l^{(k)} \in \mathcal{S}[H_1, \ldots, H_N]$.

Lemma 7.10. *If the condition* (7.30) *holds, then the jump-shift $S_{j>i}$ and the j-s-adjoint $S_{j>i}^{*}$, for the fixed $i < j \in \{1, \ldots, N\}$, satisfy*

$$S_{j>i}^{*}S_{j>i} = \mathcal{P}_{j>i} = S_{j>i}S_{j>i}^{*}, \tag{7.35}$$

on $\mathbf{L}_Q[H_1, \ldots, H_N]$, where $\mathcal{P}_{j>i}$ is the Banach-space operator (7.32).

Proof. The relation (7.35) is obtained by (6.33), (6.34) and (7.34). □

By (7.35), one can obtain the following result.

Theorem 7.11. *Suppose the condition* (7.30) *holds. Then* $S_{j>i}\left(u_l^{(k)}\right)$ *and* $S_{j>i}^*\left(u_l^{(k)}\right)$ *are semicircular in the semicircular Banach space* $\mathbf{L}_Q[H_1,\ldots,H_N]$*, if*

$$\mathcal{P}_{j>i}\left(u_l^{(k)}\right)=u_l^{(k)} \quad in \;\; \mathcal{S}[H_1,\ldots,H_N], \tag{7.36}$$

in $\mathbf{L}_Q[H_1,\ldots,H_N]$*, and they are identical to the zero element* O *of* $\mathbf{L}_Q[H_1,\ldots,H_N]$*, otherwise, for all* $u_l^{(k)}\in\mathcal{S}[H_1,\ldots,H_N]$*.*

Proof. Under the condition (7.30), if the assumption (7.36) holds for $u_l^{(k)}\in\mathcal{S}[H_1,\ldots,H_N]$, then

$$S_{j>i}\left(u_l^{(k)}\right)\in\mathcal{S}\left[H_1,\ldots,H_N\right],$$

by (7.35), and hence they are semicircular in $\mathbf{L}_Q[H_1,\ldots,H_N]$. Otherwise, they are identical to the zero element O of $\mathbf{L}_Q[H_1,\ldots,H_N]$, by (7.35). $\square$

The above theorem characterizes how the Banach-space operators $\left\{S_{j>i},\ S_{j>i}^*\right\}$ act on $\mathbf{L}_Q[H_1,\ldots,H_N]$. Equivalently, one obtains the following corollary.

Corollary 7.12. *Let* $\sum[j>i]$ *be the j-s operator on the free Hilbert space* $\mathfrak{F}$ *for the fixed* $i<j\in\{1,\ldots,N\}$*, with its adjoint* $\sum[j>i]^*$*. Then the action of*

$$\{\Sigma[j>i],\ \Sigma[j>i]^*\}$$

on the free Hilbert space $\mathfrak{F}$ *deform the free probability on* $\mathbf{L}_Q[H_1,\ldots,H_N]$ *as follows:*

(i) *if* $P_{j>i}\left(\xi_l^{(k)}\right)=\xi_l^{(k)}$*, then* $\sum[j>i]\left(\xi_l^{(k)}\right)$ *and* $\sum[j>i]^*\left(\xi_l^{(k)}\right)$ *induce the semicircular elements in* $\mathcal{S}[H_1,\ldots,H_N]$*,*

(ii) *if* $P_{j>i}\left(\xi_l^{(k)}\right)=O_{\mathfrak{F}}$*, then they induce the zero element* O *of* $\mathbf{L}_Q[H_1,\ldots,H_N]$*.*

Proof. The proofs of the statements (i) and (ii) are done by Theorem 7.11, by (7.6). $\square$

The above corollary shows how our j-s operators of $B(\mathfrak{F})$ affect the free probability on $\mathbf{L}_Q[H_1,\ldots,H_N]$ under the condition (7.30).

In Sections 7.2, 7.3 and 7.4, we showed how our j-s operators $\sum[j>i]$ affect the semicircular law of the free generators of the semicircular space $\mathbf{L}_Q[H_1,\ldots,H_N]$, by considering how the free distributions of mutually free, generating semicircular elements of $\mathcal{S}[H_1,\ldots,H_N]$ are deformed by the actions of the Banach-space operators $S_{j>i}$ of (7.7), and $S_{j>i}^*$ of (7.13), for $i<j\in\{1,\ldots,N\}$. Actually, these results fully characterize how the j-s operators $\sum[j>i]\in B(\mathfrak{F})$, and their adjoints $\sum[j>i]^*\in B(\mathfrak{F})$, deform the free probability on $\mathbf{L}_Q[H_1,\ldots,H_N]$, for $i<j\in\{1,\ldots,N\}$, by the main results of [11] and [12].

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