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A NOTE ON POSSIBLE DENSITY AND DIAMETER OF COUNTEREXAMPLES TO THE SEYMOUR'S SECOND NEIGHBORHOOD CONJECTURE

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Abstract. Seymour's second neighborhood conjecture states that every simple digraph without loops or 2-cycles contains a vertex whose second neighborhood is at least as large as its first. In this paper we show, that from falsity of Seymour's second neighborhood conjecture it follows that there exist strongly-connected counterexamples with both low and high density (dense and sparse graph). Moreover, we show that if there is a counterexample to conjecture, then it is possible to construct counterexample with any diameter $k \geq 3$.

Keywords: graph theory, Seymour's second neighborhood conjecture, density of graph, diameter of graph.

Mathematics Subject Classification: 05C12, 05C20, 05C42.

1. INTRODUCTION

All digraphs considered in this paper are oriented simple graphs, thus do not contain two-cycles or loops. We use $V(D)$ and $E(D)$ to denote the set of vertices and the set of arcs of a digraph D respectively. By $N^+(v)$ (resp. $N^-(v)$) we denote the set of vertices u such that $d(v,u) = 1$ (resp. $d(u,v) = 1$), where $d(v,u)$ is the length of the shortest directed path from v to u. If there is no such path, we set $d(v,u) = \infty$. By $N^{++}(v)$ we denote the set of vertices u such that $d(v,u) = 2$. We also denote $d^+(v) = |N^+(v)|$, $d^-(v) = |N^-(v)|$, $d^{++}(v) = |N^{++}(v)|$.

Conjecture 1.1 (Seymour's second neighborhood conjecture)**.** Every digraph without loops or two-cycles contains a vertex v for which $d^+(v) \leq d^{++}(v)$.

We will use SNC to refer to Seymour's second neighborhood conjecture throughout this note.

Fisher [2] showed that SNC is true if D is a tournament, what was conjectured to be true by Dean [1]. Kaneko and Locke [3] proved the SNC for all digraphs with minimum outdegree at most 6.

Definition 1.2. The order $|D|$ of the digraph $D = (V(D); E(D))$ is the number of its vertices. The density $\rho(D)$ of D is then $\rho(D) = \frac{2|E(D)|}{n(n-1)}$ (the ratio of the size of $|E(D)|$ of D and the number of possible arcs).

Definition 1.3. $d(i, j, S)$ is the distance from vertex i to vertex j of the subgraph S.

Definition 1.4. $d^+(v, S)$ is the number of vertices u inside subgraph S such that $d(v, u, S) = 1$. $d^{++}(v, S)$ is the number of vertices u inside subgraph S such that $d(v, u, S) = 2$.

Definition 1.5. The eccentricity of vertex v is the maximum of the distance to any vertex in the graph. The diameter $diam(D)$ is the maximum of the eccentricity of any vertex in the graph D.

2. DENSITY OF COUNTEREXAMPLES

For every $k \geq 3$ ($k \in \mathbb{N}$) and an orgraph D of order n, let $D^{(1)}, D^{(2)}, \ldots, D^{(k)}$ be vertex disjoint copies of D. By $D(k)$ we denote the orgraph with the set of vertices

$$V(D(k)) = \bigcup_{i=1}^{k} V(D^{(i)})$$

and the arc set equal to

$$E(D(k)) = \bigcup_{i=1}^{k} E(D^i) \cup \{(u, v) : u \in V(D^{(i)}), v \in V(D^{(i+1)}), i = 1, 2, \ldots, k-1\}$$
$$\cup \{(u, v) : u \in V(D^{(k)}), v \in V(D^{(1)})\}.$$

$D^{(i)}$ we shall call the i-th level of $D(k)$. Note that orgraph $D(k)$ is strongly connected. $|D(k)| = kn$, $d^+(v, D(k)) = d^+(v, D) + n$ and $d^{++}(v, D(k)) = d^{++}(v, D) + n$ for every vertex $v \in V(D)$ (we neglect here in which level the vertex v is situated). If D contains a Seymour's vertex then the same holds for $D(k)$. Moreover, if v is a Seymour's vertex of D then $D(k)$ has at least k Seymour's vertices.

Lemma 2.1. *If D is a counterexample to the SNC then also $D(k)$ is a counterexample to the SNC.*

Proof. Let $D = (V; E)$ be a counterexample to the SNC of order n. Consider arbitrary vertex $v \in V(D(k))$. Without loss of generality we may assume that v belongs to the first level of $D(k)$. Since the first level is a counterexample, then the following inequality holds:

$$d^+(v, D) > d^{++}(v, D).$$

Since $d^+(v, D(k)) = d^+(v, D) + n$ and $d^{++}(v, D(k)) = d^{++}(v, D) + n$, we have

$$d^+(v, D(k)) > d^{++}(v, D(k))$$

for every vertex v of the orgraph $D(k)$, so the digraph $D(k)$ is a counterexample to the SNC. □

Theorem 2.2. *If there exists a counterexample D to the SNC, then for any $\varepsilon > 0$ there is a strongly connected counterexample D^* with $\rho(D^*) < \varepsilon$.*

Proof. Let $D = (V; E)$ be a counterexample to the SNC of order n. From Lemma 2.1 it follows that digraph $D^* = D(k)$ is a counterexample to the SNC too. Since $|D^*| = kn$ and $|E(D^*)| \leq (\binom{n}{2} + n^2)k$, we have

$$\rho(D^*) \leq \frac{2(\frac{n(n-1)}{2} + n^2)k}{kn(kn-1)} = \frac{3n-1}{kn-1}$$

and $\rho(D^*) < \varepsilon$ for $k > \frac{3n+\varepsilon-1}{\varepsilon n}$. This finishes the proof of Theorem 2.2. □

Theorem 2.3. *If there exists a counterexample D to the SNC, then for any $\varepsilon > 0$ there is a strongly connected counterexample D^{**} with $\rho(D^{**}) > 1 - \varepsilon$.*

Proof. Let $D = (V; E)$ be a counterexample to the SNC of order n. Then $D(3)$ is a counterexample to the SNC (from Lemma 2.1) of order $3n$ and size at least $3n^2$. Hence,

$$\rho(D(3)) \geq \frac{2 \cdot 3n^2}{3n(3n-1)} > \frac{2 \cdot 3n^2}{3n \cdot 3n} = \frac{2}{3}. \tag{2.1}$$

We define the sequence of orgraphs (H_i) recursively:

$$H_1 = D(3),$$
$$H_{k+1} = H_k(3) \quad \text{for } k \geq 1.$$

Every orgraph H_k is strongly connected counterexample to the SNC (from Lemma 2.1) of order $|V(H_k)| = 3^k n$. □

We shall prove the following lemma.

Lemma 2.4. *For every $k \in \mathbb{N}, \rho(H_k) \geq 1 - \frac{1}{3^k}$.*

Proof. For $k = 1$, Lemma 2.4 follows by (2.1).

For $k \geq 2$ we have

$$|E(H_k)| = \frac{1}{2}\left(\sum_{v \in H_k} d^+(v) + \sum_{v \in H_k} d^-(v)\right) = \frac{1}{2}\sum_{v \in H_k}(d^+(v) + d^-(v)).$$

By construction of H_k we have

$$d^+(v) + d^-(v) \geq 3^k n - n, \tag{2.2}$$

since vertex v can be not adjacent only to vertices of subgraph D.

From (2.2) we obtain

$$|E(H_k)| = \frac{1}{2}\sum_{v \in H_k}(d^+(v) + d^-(v)) \geq \frac{1}{2}\sum_{v \in H_k}(3^k n - n) = \frac{1}{2}3^k n(3^k n - n),$$

and finally

$$\rho(H_k) = \frac{2|E(H_k)|}{3^k n(3^k n - 1)} \geq \frac{2|E(H_k)|}{3^k n(3^k n)} \geq \frac{2\left(\frac{1}{2}3^k n(3^k n - n)\right)}{3^k n(3^k n)} = \frac{3^k n - n}{3^k n} = 1 - \frac{1}{3^k}.$$

This proves Lemma 2.4. □

Consider digraph $D^{**} = H_k$. The inequality $\rho(H_k) > 1 - \varepsilon$ is satisfied when $k > \log_3 \frac{1}{\varepsilon}$ and Theorem 2.3 is proved.

3. DIAMETER OF COUNTEREXAMPLES

Lemma 3.1. *Every orgraph D of diameter two has the Seymour's vertex.*

Proof. Let D be an orgraph of diameter two with the vertex set $V(D) = \{v_1, \ldots, v_n\}$. Consider arbitrary vertex $v_i \in V(D)$. For any other vertex $v_j \in V(D)$ $(j \neq i)$, $v_j \in N^+(v_i)$ or $v_j \in N^{++}(v_i)$. We also have $N^+(v_i) \cap N^-(v_i) = \emptyset$, so that $N^-(v_i) \subseteq N^{++}(v_i)$ implies $d^-(v_i) \leq d^{++}(v_i)$. Since last inequality holds for every v_i we obtain

$$\sum_{i=1}^{n} d^-(v_i) \leq \sum_{i=1}^{n} d^{++}(v_i) \Leftrightarrow \sum_{i=1}^{n} d^+(v_i) \leq \sum_{i=1}^{n} d^{++}(v_i). \tag{3.1}$$

Suppose that graph D does not contain Seymour's vertex. Then for any v_i $d^+(v_i) > d^{++}(v_i)$. By adding up this for all vertices of orgraph D we obtain

$$\sum_{i=1}^{n} d^+(v_i) > \sum_{i=1}^{n} d^{++}(v_i).$$

This is a contradiction to (3.1) and the proof is complete now. □

Theorem 3.2. *If there is a counterexample D to the SNC, then for any $k \geq 3$ $(k \in \mathbb{N})$ there is a counterexample D^{***} for which $diam(D^{***}) = k$.*

Proof. Let $D = (V; E)$ be a counterexample to the SNC. We consider two cases.

Case 1. $diam(D) \geq k$. Without loss of generality consider the first level of $D(k)$. For $v \in D^{(1)}, u \in D^{(i)}, 2 \leq i \leq k$ $d(v, u) < k$ and for $v, u \in D^{(1)}$, $d(v, u) \leq k$. Then from Lemma 2.1 $D^{***} = D(k)$ is a counterexample to SNC with $diam(D^{***}) = k$.

Case 2. $diam(D) < k$. Without loss of generality consider the first level of $D(k+1)$. For $v \in D^{(1)}$,$u \in D^{(i)}$,$2 \leq i \leq k+1$ $d(v, u) \leq k$ and for $v, u \in D^{(1)}$, $d(v, u) < k$. From Lemma 2.1 it follows that a counterexample to the SNC with diameter equal to k is $D^{***} = D(k+1)$. □

Remark 3.3. Proving the SNC for orgraphs of diameter 3 is equivalent to proving the SNC for all the orgraphs.

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