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QUADRATIC INEQUALITIES FOR FUNCTIONALS IN l^∞

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Abstract. For a class of operators T on l^∞ and T-invariant functionals φ we prove inequalities between $\varphi(x)$, $\varphi(x^2)$ and the upper density of the sets

$$P_r := \{n \in \mathbb{N}_0 : \varphi((T^n x) \cdot x) > r\}.$$

Applications are given to Banach limits and integrals.

Keywords: Banach algebras of bounded functions, operator-invariant functionals, Banach limits.

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1. INTRODUCTION

Let Γ be a nonempty set and let $l^\infty(\Gamma, \mathbb{R})$ denote the real Banach algebra of all bounded functions $x : \Gamma \to \mathbb{R}$ endowed with the supremum norm $\|\cdot\|$. Let $\mathcal{A}$ be a closed subalgebra of $l^\infty(\Gamma, \mathbb{R})$ containing the unit e, $e(\gamma) = 1$ $(\gamma \in \Gamma)$. Moreover let $\mathcal{A}$ be ordered by the cone $K := \{x \in \mathcal{A} : x(\gamma) \geq 0 \ (\gamma \in \Gamma)\}$, that is $x \leq y :\Leftrightarrow y - x \in K$. Let K^* denote the dual cone of K, that is $K^* := \{\varphi \in \mathcal{A}^* : \varphi(x) \geq 0 \ (x \in K)\}$. For $x \in \mathcal{A}$ and a continuous function $h : \overline{x(\Gamma)} \to \mathbb{R}$ we have $h \circ x \in \mathcal{A}$ and for short we set $h(x) := h \circ x$. Next, let $T : \mathcal{A} \to \mathcal{A}$, $T \neq 0$ be a linear operator such that

$$\forall x, y \in \mathcal{A} : \ T(x \cdot y) = (Tx) \cdot (Ty). \tag{1.1}$$

Note that T is monotone as $Tx = (T\sqrt{x}) \cdot (T\sqrt{x}) \geq 0$ $(x \in K)$, thus T is continuous. In particular $Te = (Te) \cdot (Te)$, thus $0 \leq Te \leq e$ and T has operator norm $|||T||| = \|Te\| = 1$. We set

$$\mathcal{B}(T) := \{\varphi \in K^* : \varphi(e) = 1, \ \varphi \circ T = \varphi\}.$$

For $\varphi \in \mathcal{B}(T)$ and $x \in \mathcal{A}$ we are interested in the size of the sets

$$P_r := \{n \in \mathbb{N}_0 : \varphi((T^n x) \cdot x) > r\},$$

and we will prove inequalities between $\varphi(x)$, $\varphi(x^2)$ and the upper density of P_r.

As an introducing example let $\mathcal{A} = l^\infty(\mathbb{N}, \mathbb{R})$ and let S denote the left shift operator $Sx = (x_{k+1})_{k \in \mathbb{N}}$. Recall that a functional $L \in (l^\infty(\mathbb{N}, \mathbb{R}))^*$ is called a Banach limit if it has the following three properties:

$$L \in K^*, \quad L(e) = 1, \quad L \circ S = L.$$

Let $\mathcal{L}$ denote the set of all Banach limits. In this case we have $\mathcal{L} = \mathcal{B}(S)$. More general, let $\sigma : \mathbb{N} \to \mathbb{N}$ be any function and let $T_\sigma : l^\infty(\mathbb{N}, \mathbb{R}) \to l^\infty(\mathbb{N}, \mathbb{R})$, $T_\sigma x = (x_{\sigma(k)})_{k \in \mathbb{N}}$. Clearly T_σ satisfies (1.1). In [5] sufficient conditions for $\mathcal{B}(T_\sigma) \cap \mathcal{L} \neq \emptyset$ are given. The dilation operator

$$T_\sigma x = (x_1, x_1, x_2, x_2, x_3, x_3, x_4, x_4, \ldots) \quad (x \in l^\infty(\mathbb{N}, \mathbb{R}))$$

is an example with $\emptyset \neq \mathcal{B}(T_\sigma) \cap \mathcal{L} \neq \mathcal{L}$, see [1].

As a second example let $\mathcal{A} = C_{2\pi}(\mathbb{R}, \mathbb{R})$ or $\mathcal{A} = R_{2\pi}(\mathbb{R}, \mathbb{R})$ be the Banach algebra of all 2π-periodic continuous or regulated functions $x : \mathbb{R} \to \mathbb{R}$, respectively. In both cases $\mathcal{A}$ is a closed unital subalgebra of $l^\infty(\mathbb{R}, \mathbb{R})$. Let $\tau \in \mathbb{R}$, and let $T_\tau : \mathcal{A} \to \mathcal{A}$ be the translation operator $(T_\tau x)(t) = x(t + \tau)$. Then T_τ satisfies (1.1), and the functional

$$x \mapsto \varphi(x) = \frac{1}{2\pi} \int_0^{2\pi} x(t) dt$$

is in $\mathcal{B}(T_\tau)$.

2. MAIN RESULTS

For a set $M \subseteq \mathbb{N}_0$ the upper density of M is defined as

$$\overline{D}(M) := \limsup_{n \to \infty} \frac{|M \cap \{0, \ldots, n\}|}{n+1}.$$

If M is infinite and $M = \{n_j : j \in \mathbb{N}\}$ with $(n_j)_{j \in \mathbb{N}}$ strictly increasing, the upper density of M is

$$\overline{D}(M) = \limsup_{j \to \infty} \frac{j}{n_j}.$$

For finite sets clearly $\overline{D}(M) = 0$. We define the function $\rho : l^\infty(\mathbb{N}, \mathbb{R}) \to \mathbb{R}$ by

$$\rho(x) := \limsup_{\lambda \to 1+} (\lambda - 1) \sum_{k=1}^\infty \frac{x_k}{\lambda^k}.$$

Note that ρ is a sublinear functional on $l^\infty(\mathbb{N}, \mathbb{R})$.

Theorem 2.1. *Let $T : \mathcal{A} \to \mathcal{A}$ be a linear operator with property* (1.1)*, let $\varphi \in \mathcal{B}(T)$, $x \in \mathcal{A}$ and $r \in \mathbb{R}$. Then*

$$\varphi(x)^2 \le \rho\big((\varphi((T^n x)\cdot x))_{n\in\mathbb{N}}\big) \le (1-\overline{D}(P_r))r + \overline{D}(P_r)\varphi(x^2).$$

In particular, if $r < \varphi(x)^2$ then

$$0 < \frac{\varphi(x)^2 - r}{\varphi(x^2) - r} \le \overline{D}(P_r).$$

The following result specifies the quantity $\rho\big((\varphi((T^n x)\cdot x))_{n\in\mathbb{N}}\big)$ from Theorem 2.1 in some special cases.

Theorem 2.2. *If under the assumptions of Theorem 2.1 the sequence*

$$\left(\frac{1}{n}\sum_{k=1}^{n} T^k x\right)_{n\in\mathbb{N}}$$

has a weakly convergent subsequence, then it is norm convergent, and

$$\rho\big((\varphi((T^n x)\cdot x))_{n\in\mathbb{N}}\big) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\varphi((T^k x)\cdot x).$$

In the proof of Theorem 2.1 we use the following lemma (which was proved for Banach limits in [3]).

Lemma 2.3. *Let $\varphi \in K^*$, $\varphi(e) = 1$, and let $h : \mathbb{R} \to \mathbb{R}$ be convex. Then*

$$\forall x \in \mathcal{A} : \varphi(h(x)) \ge h(\varphi(x)).$$

Proof. Let $t_0 := \varphi(x)$. As h is convex it is continuous, hence $h(x) \in \mathcal{A}$, and there is a supporting straight line $t \mapsto h(t_0) + \alpha(t - t_0)$ such that

$$h(t) \ge h(t_0) + \alpha(t - t_0) \quad (t \in \mathbb{R}).$$

Hence

$$\varphi(h(x)) \ge \varphi(h(t_0)e + \alpha(x - t_0 e)) = h(\varphi(x)).$$

□

Remark 2.4. If $\varphi(x) \ne 0$ in Theorem 2.1 we can set $r(\alpha) := \alpha\varphi(x)^2$ $(\alpha < 1)$ and obtain the scaled inequality

$$\varphi(x)^2 \le \frac{\overline{D}(P_{r(\alpha)})}{1 - \alpha + \alpha\overline{D}(P_{r(\alpha)})}\varphi(x^2) \quad (\alpha < 1),$$

which is an improvement of $\varphi(x)^2 \le \varphi(x^2)$ coming from Lemma 2.3. If in addition $x \in K$ we have $\varphi(x^2) \le \varphi(x)\|x\|$ and obtain

$$\varphi(x) \le \frac{\overline{D}(P_{r(\alpha)})}{1 - \alpha + \alpha\overline{D}(P_{r(\alpha)})}\|x\| \quad (\alpha < 1).$$

We will now give the proofs of Theorem 2.1 and Theorem 2.2.

Proof. Recall that $|||T||| = 1$. By (1.1) we have

$$\varphi((T^{n+m}x)\cdot(T^m x)) = \varphi(T^m((T^n x)\cdot x)) = \varphi((T^n x)\cdot x) \quad (n, m \in \mathbb{N}_0). \tag{2.1}$$

Moreover note that $(y, z) \mapsto \varphi(y \cdot z)$ is a semi-definite bilinear form on $\mathcal{A}$, and therefore the Cauchy–Schwarz inequality is valid:

$$\varphi(y\cdot z)^2 \le \varphi(y^2)\varphi(z^2) \quad (y, z \in \mathcal{A}).$$

In particular, for $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$ we have

$$|\varphi((T^n x)\cdot x)| \le \sqrt{\varphi((T^n x)\cdot(T^n x))}\sqrt{\varphi(x^2)} = \varphi(x^2). \tag{2.2}$$

By Lemma 2.3, we have for $x \in \mathcal{A}$ and $\lambda > 1$:

$$\begin{aligned}
\big(\varphi((I - T/\lambda)^{-1}x)\big)^2 &\le \varphi(((I - T/\lambda)^{-1}x)\cdot((I - T/\lambda)^{-1}x))\\
&= \varphi\left(\sum_{n=0}^{\infty}\frac{T^n x}{\lambda^n}\cdot\sum_{n=0}^{\infty}\frac{T^n x}{\lambda^n}\right)\\
&= \sum_{n,m=0}^{\infty}\frac{1}{\lambda^{n+m}}\varphi((T^n x)\cdot(T^m x))\\
&= \sum_{n=0}^{\infty}\frac{1}{\lambda^{2n}}\varphi((T^n x)\cdot(T^n x))\\
&\quad + 2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda^{n+2m}}\varphi((T^{n+m}x)\cdot(T^m x))\\
&\overset{(2.1)}{=} \frac{1}{1-1/\lambda^2}\varphi(x\cdot x) + 2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda^{n+2m}}\varphi((T^n x)\cdot x)\\
&= \frac{1}{1-1/\lambda^2}\varphi(x\cdot x)\\
&\quad + 2\left(\sum_{m=0}^{\infty}\frac{1}{\lambda^{2m}}\right)\left(\sum_{n=1}^{\infty}\frac{1}{\lambda^n}\varphi((T^n x)\cdot x)\right)\\
&= \frac{1}{1-1/\lambda^2}\varphi(x\cdot x) + \frac{2}{1-1/\lambda^2}\sum_{n=1}^{\infty}\frac{1}{\lambda^n}\varphi((T^n x)\cdot x).
\end{aligned}$$

We have

$$\varphi((I - T/\lambda)y) = (1 - 1/\lambda)\varphi(y) \quad (y \in \mathcal{A}, \lambda > 1),$$

thus

$$\varphi(x)^2 = (1 - 1/\lambda)^2\big(\varphi((I - T/\lambda)^{-1}x)\big)^2,$$

which together with the previous calculations yields

$$\begin{aligned}\varphi(x)^2 &\le \frac{\lambda-1}{\lambda+1}\varphi(x\cdot x) + 2\frac{\lambda-1}{\lambda+1}\sum_{n=1}^{\infty}\frac{1}{\lambda^n}\varphi((T^n x)\cdot x)\\ &\le 2\frac{\lambda-1}{\lambda+1}\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\varphi((T^n x)\cdot x)\\ &\le 2\frac{\lambda-1}{\lambda+1}\left(\sum_{n\in P_r}\frac{1}{\lambda^n}\varphi((T^n x)\cdot x) + r\sum_{n\notin P_r}\frac{1}{\lambda^n}\right) \qquad (2.3)\\ &= 2\frac{\lambda-1}{\lambda+1}\left(\sum_{n\in P_r}\frac{1}{\lambda^n}(\varphi((T^n x)\cdot x) - r) + r\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\right)\\ &= \frac{2r\lambda}{\lambda+1} + 2\frac{\lambda-1}{\lambda+1}\sum_{n\in P_r}\frac{1}{\lambda^n}(\varphi((T^n x)\cdot x) - r).\end{aligned}$$

As $\lambda \to 1+$ we obtain (see (2.3))

$$\varphi(x)^2 \le \rho\big((\varphi((T^n x)\cdot x))_{n\in\mathbb{N}}\big) \le r + \limsup_{\lambda\to 1+} 2\frac{\lambda-1}{\lambda+1}\sum_{n\in P_r}\frac{1}{\lambda^n}(\varphi((T^n x)\cdot x) - r).$$

If $|P_r| < \infty$ (then $\overline{D}(P_r) = 0$) we get

$$\varphi(x)^2 \le \rho\big((\varphi((T^n x)\cdot x))_{n\in\mathbb{N}}\big) \le r = (1-\overline{D}(P_r))r + \overline{D}(P_r)\varphi(x^2).$$

Thus, let $P_r = \{n_j : j\in\mathbb{N}\}$ with $(n_j)_{j\in\mathbb{N}}$ strictly increasing. Let $\varepsilon > 0$. Then

$$\exists j_0\in\mathbb{N}\ \forall j\ge j_0 :\ \frac{j}{n_j} \le \overline{D}(P_r) + \varepsilon,$$

and

$$\sum_{j=j_0}^{\infty}\frac{1}{\lambda^{n_j}} \le \sum_{j=j_0}^{\infty}\frac{1}{\lambda^{j/(\overline{D}(P_r)+\varepsilon)}} \le \frac{1}{1-\lambda^{-1/(\overline{D}(P_r)+\varepsilon)}}.$$

We now have

$$\begin{aligned}&\frac{2r\lambda}{\lambda+1} + 2\frac{\lambda-1}{\lambda+1}\sum_{n\in P_r}\frac{1}{\lambda^n}(\varphi((T^n x)\cdot x) - r)\\ &\overset{(2.2)}{\le} \frac{2r\lambda}{\lambda+1} + \left(2\frac{\lambda-1}{\lambda+1}\sum_{j=1}^{\infty}\frac{1}{\lambda^{n_j}}\right)(\varphi(x^2) - r)\\ &\le \frac{2r\lambda}{\lambda+1} + 2\frac{\lambda-1}{\lambda+1}\left(\sum_{j=1}^{j_0-1}\frac{1}{\lambda^{n_j}} + \frac{1}{1-\lambda^{-1/(\overline{D}(P_r)+\varepsilon)}}\right)(\varphi(x^2) - r),\end{aligned}$$

and from

$$\lim_{\lambda\to 1+} \frac{\lambda-1}{1-\lambda^{-1/\alpha}} = \alpha \quad (\alpha > 0)$$

we conclude that

$$r + \limsup_{\lambda\to 1+} 2\frac{\lambda-1}{\lambda+1} \sum_{n\in P_r} \frac{1}{\lambda^n}(\varphi((T^n x)\cdot x) - r) \le r + (\overline{D}(P_r)+\varepsilon)(\varphi(x^2)-r).$$

As $\varepsilon \to 0+$ we obtain

$$r + \limsup_{\lambda\to 1+} 2\frac{\lambda-1}{\lambda+1} \sum_{n\in P_r} \frac{1}{\lambda^n}(\varphi((T^n x)\cdot x) - r) \le (1-\overline{D}(P_r))r + \overline{D}(P_r)\varphi(x^2),$$

and summing up

$$\varphi(x)^2 \le \rho\big((\varphi((T^n x)\cdot x))_{n\in\mathbb{N}}\big) \le (1-\overline{D}(P_r))r + \overline{D}(P_r)\varphi(x^2).$$

If in addition $r < \varphi(x)^2$ (hence $r < \varphi(x)^2 \le \varphi(x^2)$) then

$$0 < \frac{\varphi(x)^2 - r}{\varphi(x^2) - r} \le \overline{D}(P_r).$$

□

Proof. To prove Theorem 2.2 we show in a first step that

$$\forall y \in l^\infty(\mathbb{N},\mathbb{R})\ \exists L \in \mathcal{L} : \rho(y) = L(y),$$

which is clear if y is convergent. Recall that ρ is sublinear on $l^\infty(\mathbb{N},\mathbb{R})$ and let $c(\mathbb{N},\mathbb{R})$ denote the subspace of all convergent sequences. Then $\rho(x) = \lim_{k\to\infty} x_k$ ($x \in c(\mathbb{N},\mathbb{R})$). Let $y \in l^\infty(\mathbb{N},\mathbb{R}) \setminus c(\mathbb{N},\mathbb{R})$. According to Hahn-Banach's Theorem there exists $L \in (l^\infty(\mathbb{N},\mathbb{R}))^*$ such that $L(x) = \rho(x)$ ($x \in c(\mathbb{N},\mathbb{R})$),

$$-\rho(-x) \le L(x) \le \rho(x) \quad (x \in l^\infty(\mathbb{N},\mathbb{R})),$$

and

$$L(y) = \inf\{\rho(y+x) - \rho(x) : x \in c(\mathbb{N},\mathbb{R})\} = \rho(y).$$

Clearly $L(e) = 1$ and $L(x) \ge -\rho(-x) \ge 0$ ($x \in K$). To see that $L(Sx) = L(x)$ ($x \in l^\infty(\mathbb{N},\mathbb{R})$) consider

$$\begin{aligned}\left|(\lambda-1)\sum_{k=1}^\infty \frac{x_{k+1}-x_k}{\lambda^k}\right| &= \left|(\lambda-1)\left(\frac{x_1}{\lambda} + (\frac{1}{\lambda}-1)\sum_{k=2}^\infty \frac{x_k}{\lambda^{k-1}}\right)\right| \\ &\le (\lambda-1)\left(\frac{2\|x\|}{\lambda}\right) \to 0 \quad (\lambda \to 1+).\end{aligned}$$

Thus $\rho(Sx - x) = \rho(x - Sx) = 0$, and therefore $L(Sx - x) = 0$ ($x \in l^\infty(\mathbb{N},\mathbb{R})$). If now y is in addition almost convergent in the sense of Lorentz [4], then

$$\rho(y) = L(y) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n y_k.$$

Thus, Theorem 2.2 is proved if we can show that $y = (\varphi((T^n x) \cdot x))_{n\in\mathbb{N}}$ is almost convergent. Under the assumptions of Theorem 2.2 the Mean Ergodic Theorem (see for example [2, Chapter 8.1]) proves that

$$\left(\frac{1}{n}\sum_{k=1}^{n} T^k x\right)_{n\in\mathbb{N}}$$

is convergent in norm towards a fixed point u of T. As $|||T||| = 1$ we have

$$\left\|\frac{1}{n}\sum_{k=m}^{n+m-1} T^k x - u\right\| \leq \left\|\frac{1}{n}\sum_{k=1}^{n} T^k x - u\right\| \quad (n, m \in \mathbb{N}).$$

Thus

$$\frac{1}{n}\sum_{k=m}^{n+m-1} T^k x \to u \quad (n \to \infty)$$

uniformly in $m \in \mathbb{N}$. Therefore

$$\frac{1}{n}\sum_{k=m}^{n+m-1} \varphi((T^k x) \cdot x) \to \varphi(u \cdot x) \quad (n \to \infty)$$

uniformly in $m \in \mathbb{N}$, that is $(\varphi((T^n x) \cdot x))_{n\in\mathbb{N}}$ is almost convergent. □

3. APPLICATIONS AND EXAMPLES

For $x \in l^\infty(\mathbb{N}, \mathbb{R})$ let

$$q(x) := \lim_{j\to\infty} \inf_{m\in\mathbb{N}_0} \frac{1}{j}\sum_{k=m+1}^{j+m} x_k, \quad p(x) := \lim_{j\to\infty} \sup_{m\in\mathbb{N}_0} \frac{1}{j}\sum_{k=m+1}^{j+m} x_k.$$

According to Sucheston [6]

$$\min\{L(x) : L \in \mathcal{L}\} = q(x), \quad \max\{L(x) : L \in \mathcal{L}\} = p(x),$$

thus, as $\mathcal{L}$ is convex, we have

$$\{L(x) : L \in \mathcal{L}\} = [q(x), p(x)] \quad (x \in l^\infty(\mathbb{N}, \mathbb{R})).$$

From Theorem 2.1 we obtain the following corollary for the shift operator S.

Corollary 3.1. *Let $x \in l^\infty(\mathbb{N}, \mathbb{R})$ with $[q(x), p(x)] \neq \{0\}$. Then $p(x^2) > 0$ and*

$$\overline{D}(\{n \in \mathbb{N}_0 : p((S^n x) \cdot x) > 0\}) \geq \frac{\max\{q(x)^2, p(x)^2\}}{p(x^2)} > 0.$$

If in addition $x \in K$, then

$$\overline{D}(\{n \in \mathbb{N}_0 : p((S^n x) \cdot x) > 0\}) \geq \frac{p(x)}{\|x\|}.$$

Proof. Set
$$Q := \{n \in \mathbb{N}_0 : p((S^n x) \cdot x) > 0\}.$$
For some $L \in \mathcal{L}$ we have
$$|L(x)| = \max\{|q(x)|, |p(x)|\} > 0.$$
Consider Theorem 2.1 with P_0 corresponding to $\varphi = L$ and $T = S$. We have $P_0 \subseteq Q$ and
$$0 < \max\{q(x)^2, p(x)^2\} = L(x)^2 \leq \overline{D}(P_0)L(x^2) \leq \overline{D}(Q)p(x^2).$$
If in addition $x \in K$, then $0 \leq q(x) \leq p(x)$ and $p(x^2) \leq p(x)\|x\|$. □

For our second corollary let $\mathcal{A} = C_{2\pi}(\mathbb{R}, \mathbb{R})$. Application of Theorem 2.1 to $T = T_\tau$ ($\tau \in \mathbb{R}$) and φ from the introduction leads to the following inequalities.

Corollary 3.2. *Let $x \in \mathcal{A}$ and $r < \varphi(x)^2$. Then*
$$\begin{aligned} 0 &< \frac{\left(\frac{1}{2\pi}\int_0^{2\pi} x(t)dt\right)^2 - r}{\frac{1}{2\pi}\int_0^{2\pi} x(t)^2 dt - r} \\ &\leq \overline{D}\left(\left\{n \in \mathbb{N}_0 : \frac{1}{2\pi}\int_0^{2\pi} x(t+n\tau)x(t)dt > r\right\}\right). \end{aligned}$$

Remark 3.3. In Corollary 3.2 the sequence
$$\left(\frac{1}{n}\sum_{k=1}^{n} T^k x\right)_{n\in\mathbb{N}}$$
has a norm convergent subsequence (according to Arzelà-Ascoli's Theorem) and Theorem 2.2 applies. In particular this sequence is convergent, its limit function is τ-periodic and 2π-periodic, hence constant $\varphi(x)$ if $\tau/\pi \notin \mathbb{Q}$. In this case
$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \varphi((T^k x) \cdot x) = \varphi(x)^2,$$
and we have equality in the first inequality of Theorem 2.1.

Consider $x(t) = 5/(2 + \sin(t))$. Then Corollary 3.2 gives
$$\frac{\frac{25}{3} - r}{\frac{50}{\sqrt{27}} - r} \leq \overline{D}\left(\left\{n \in \mathbb{N}_0 : \frac{1}{2\pi}\int_0^{2\pi} \frac{25}{(2+\sin(t+n\tau))(2+\sin(t))}dt > r\right\}\right).$$

For $\tau = \sqrt{2}$ and $r = 8$ numerically this inequality reads $0.205 \leq 0.570$. The following Figure 1 shows $\varphi((T^n x) \cdot x)$ for $n = 1, \ldots, 1000$.

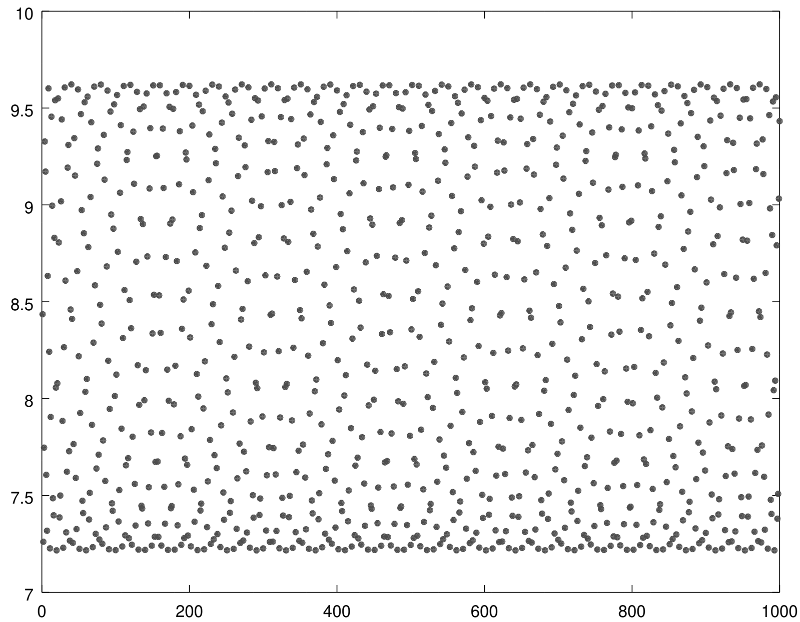

Fig. 1. $\varphi((T^n x)\cdot x)$ for $n = 1, \ldots, 1000$

As another example let $\mathcal{A} = C_{2\pi}(\mathbb{R},\mathbb{R})$ and $T : \mathcal{A} \to \mathcal{A}$ the dilation operator $(Tx)(t) = x(2t)$. In this case the functional

$$x \mapsto \varphi(x) = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt$$

is in $\mathcal{B}(T)$ as well. This situation has an extremal property:

$$\forall x, y \in \mathcal{A}: \ \varphi((T^n x)\cdot y) \to \varphi(x)\varphi(y)$$

(if $x \in K$ the Mean Value Theorem for Integrals leads to a sequence of Riemann sums converging to $\varphi(x)\varphi(y)$). In particular

$$\forall x \in \mathcal{A}: \ \varphi((T^n x)\cdot x) \to \varphi(x)^2.$$

Again we have equality in the first inequality of Theorem 2.1, now for each $x \in \mathcal{A}$, and moreover

$$\overline{D}(P_r) = 1 \ (r < \varphi(x)^2), \quad \overline{D}(P_r) = 0 \ (r > \varphi(x)^2)$$

for each $x \in \mathcal{A}$.

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