Opuscula Math. 41, no. 3 (2021), 395–412
https://doi.org/10.7494/OpMath.2021.41.3.395 Opuscula Mathematica

SPECTRUM LOCALIZATION OF A PERTURBED OPERATOR IN A STRIP AND APPLICATIONS

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Communicated by P.A. Cojuhari

Abstract. Let A and $\tilde{A}$ be bounded operators in a Hilbert space. We consider the following problem: let the spectrum of A lie in some strip. In what strip the spectrum of $\tilde{A}$ lies if A and $\tilde{A}$ are "close"? Applications of the obtained results to integral operators and matrices are also discussed. In addition, we apply our perturbation results to approximate the spectral strip of a Hilbert–Schmidt operator by the spectral strips of finite matrices.

Keywords: operator, spectrum, perturbation, approximation, integral operator, matrix.

Mathematics Subject Classification: 47A10, 47A55, 47B10.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Let $\mathcal{H}$ be a complex separable Hilbert space with a scalar product $(\cdot,\cdot)$, the norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ and unit operator I. By $\mathcal{B}(\mathcal{H})$ we denote the set of all bounded linear operators in $\mathcal{H}$. For an $A \in \mathcal{B}(\mathcal{H})$, A^* is the adjoint operator,

$\|A\|$ is the operator norm, $\sigma(A)$ is the spectrum,

$$\alpha(A) := \sup_{s \in \sigma(A)} \text{Re } \sigma(A) \quad \text{and} \quad \beta(A) := \inf_{s \in \sigma(A)} \text{Re } \sigma(A).$$

So $\sigma(A)$ lies in the strip $\{z \in \mathbb{C} : \beta(A) \le \text{Re } z \le \alpha(A)\}$, which will be called *the spectral strip of A.*

We consider the following problem: Let $\tilde{A} \in \mathcal{B}(\mathcal{H})$. In what strip $\sigma(\tilde{A})$ lies, if the spectral strip of A is known, and $\tilde{A}$ and A are sufficiently "close"? The perturbation theory of operators is very rich. The classical results are presented in the book [17], the interesting recent results can be found in [1–6, 11, 18, 22–24] and references, which are given therein, but to the best of our knowledge the above-pointed problem, was not investigated in the available literature although it is important for the localization of the spectrum and various other applications, cf. [8].

Throughout this paper c and b are real constants, satisfying the inequalities

$$c < \beta(A) \quad \text{and} \quad b > \alpha(A). \tag{1.1}$$

Below we check that the integrals

$$X_c := 2 \int_0^\infty e^{-(A^*-cI)t} e^{-(A-cI)t} dt$$

and

$$Y_b := 2 \int_0^\infty e^{-(bI-A^*)t} e^{-(bI-A)t} dt$$

converge in the operator norm.

Now we are in a position to formulate our main result.

Theorem 1.1. *Let $A, \tilde{A} \in \mathcal{B}(\mathcal{H})$ and $q = \|\tilde{A} - A\|$. Then $\beta(\tilde{A}) \geq c$, provided $q\|X_c\| < 1$. In addition, $\alpha(\tilde{A}) \leq b$, provided $q\|Y_b\| < 1$.*

The proof of this theorem is presented in the next section.

Obviously,

$$\|X_c\| \leq J_c(A) := 2 \int_0^\infty e^{2ct} \|e^{-At}\|^2 dt$$

and

$$\|Y_b\| \leq \hat{J}_b(A) := 2 \int_0^\infty e^{-2bt} \|e^{At}\|^2 dt.$$

Let us check that $J_c(A)$ and $\hat{J}_b(A)$ are finite. To this end apply the representation

$$e^{At} = \frac{1}{2\pi i} \int_L e^{zt} (zI - A)^{-1} dz,$$

where L is a closed Jordan contour surrounding $\sigma(A)$, cf. [8]. Taking a positive $\epsilon < b - \alpha(A)$, with a fitting L, we easily have

$$e^{bt} \|e^{At}\| \leq m_\epsilon e^{(-b+\alpha(A)+\epsilon)t} \quad (t \geq 0),$$

where

$$m_\epsilon = \frac{1}{2\pi} \int_L \|(zI - A)^{-1}\| dz.$$

Since $-b + \alpha(A) + \epsilon < 0$, it is not hard to check that $\hat{J}_b(A) < \infty$.

Similarly, taking a positive $\epsilon < \beta(A) - c$, we have

$$e^{ct} \|e^{-At}\| \leq \text{const } e^{(c-\beta(A)+\epsilon)t}.$$

Since $c - \beta(A) + \epsilon < 0$, we obtain $J_c(A) < \infty$.

Now put

$$w_c(A) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \|(A + (is - c)I)^{-1}\|^2 ds$$

and

$$\hat{w}_b(A) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \|(A - (is + b)I)^{-1}\|^2 ds.$$

By the classical Parseval–Planscherel equality, for any $x \in \mathcal{H}$ we have

$$(X_c x, x) = \left(\int_0^{\infty} e^{(Ic - A^*)t} e^{(Ic - A)t} x \, dt, x \right) = \int_0^{\infty} \|e^{-(A - Ic)t} x\|^2 dt$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \|(A + (is - c)I)^{-1} x\|^2 ds.$$

Hence,

$$\|X_c\| \le w_c(A). \tag{1.2}$$

Similarly,

$$\|Y_b\| \le \hat{w}_b(A). \tag{1.3}$$

If A is normal (i.e. $AA^* = A^*A$), then by the spectral representation (see, for instance, [17] and the references therein), we easily have

$$\|e^{At}\| = e^{\alpha(A)t} \quad (t \ge 0).$$

Hence,

$$\|e^{-At}\| \le e^{\alpha(-A)t} = e^{-\beta(A)t}.$$

Therefore,

$$J_c(A) \le 2 \int_0^{\infty} e^{2(c - \beta(A)t} dt = \frac{1}{\beta(A) - c}$$

and

$$\hat{J}_b(A) \le 2 \int_0^{\infty} e^{-2(b - \alpha(A))t} dt = \frac{1}{b - \alpha(A)}.$$

Hence, making use of Theorem 1.1, we obtain the following result.

Corollary 1.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be normal. Then*

$$\beta(\tilde{A}) \ge \beta(A) - q \quad \text{and} \quad \alpha(\tilde{A}) \le \alpha(A) + q.$$

2. PROOF OF THEOREM 1.1

For a self-adjoint operator $Y \in \mathcal{B}(\mathcal{H})$ we write $Y > 0$, if Y is positive definite, i.e.

$$\inf_{x\in\mathcal{H}, \|x\|=1} (Yx, x) > 0.$$

By the Lyapunov theorem, cf. [8, Theorem I.5.1], the inequality $\alpha(B) < 0$ holds if and only if there exists a positive definite selfadjoint operator $X \in \mathcal{B}(\mathcal{H})$, such that the operator $XB + B^*X$ is negative definite. Consider the equation

$$XB + B^*X = -2I. \tag{2.1}$$

As is well-known [8, Section 1.5], the solution X_0 of (2.1) is representable as

$$X_0 = 2\int_0^\infty e^{B^*t}e^{Bt}dt \tag{2.2}$$

and the integral converges in the operator norm.

If $\beta(A) > c$, then $\beta(A - cI) > 0$ and $\alpha(-A + cI) < 0$. According to (2.2) X_c is a solution of the equation

$$X(-A + cI) + (-A + cI)^*X = 2I. \tag{2.3}$$

If $\alpha(A) < b$, then $\alpha(A - bI) < 0$, and according to (2.2) Y_b is a solution to the equation

$$X(A - bI) + (A^* - bI)X = -2I. \tag{2.4}$$

Furthermore, put $E = \tilde{A} - A$. Then from (2.3) we have

$$\begin{aligned}(\tilde{A} - cI)X_c + X_c(\tilde{A} - cI) &= (A - cI)X_c + X_c(A - cI) + EX_c + X_cE\\ &= 2I + EX_c + X_cE.\end{aligned}$$

If $q\|X_c\| < 1$, then $\mathrm{Re}\,(cI - \tilde{A})X_c < 0$. Here and below $\mathrm{Re}\,B = (B + B^*)/2$. By the Lyapunov theorem we have $\alpha(cI - \tilde{A}) < 0$, or $\beta(\tilde{A} - cI) > 0$. This proves that $\beta(\tilde{A}) > c$. In addition, (2.4) implies

$$\begin{aligned}(\tilde{A} - bI)Y_b + Y_b(\tilde{A} - bI) &= (A - bI)Y_b + Y_b(A - bI)EY_b + Y_bE\\ &= -2I + EY_b + Y_bE.\end{aligned}$$

If $q\|Y_b\| < 1$, then $\mathrm{Re}\,(\tilde{A} - bI)Y_b < 0$. By the Lyapunov theorem

$$\alpha(\tilde{A} - bI) = \alpha(\tilde{A}) - b < 0.$$

So $\alpha(\tilde{A}) < b$. The theorem is proved.

3. FINITE DIMENSIONAL OPERATORS

In this section $\mathcal{H} = \mathbb{C}^n$ is the n-dimensional complex Euclidean space. The set of all $n \times n$ matrices is denoted by $\mathbb{C}^{n\times n}$. Besides $\|A\|$ $(A \in \mathbb{C}^{n\times n})$ is the spectral norm:

$$\|A\|^2 = r_s(A^*A),$$

where $r_s(\cdot)$ means the spectral radius.

In this section we are going to obtain estimates for $J_c(A)$ and $\hat{J}_b(A)$ for $A \in \mathbb{C}^{n\times n}$. Let $N_2(A)$ be the Hilbert–Schmidt (Frobenius) norm of A:

$$N_2(A) := (\text{trace } (A^*A))^{1/2}.$$

The following quantity (the departure of normality) plays a key role in this section:

$$g(A) := \left[N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2 \right]^{1/2},$$

where $\lambda_k(A)$ $(k = 1, \ldots, n)$ are the eigenvalues of A taken with their multiplicities. Since

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^2 \geq \left| \sum_{k=1}^{\infty} \lambda_k^2(A) \right| = |\text{trace } A^2|,$$

one can write

$$g^2(A) \leq N_2^2(A) - |\text{trace } A^2|.$$

If A is a normal matrix: $AA^* = A^*A$, then $g(A) = 0$, since

$$N_2^2(A) = \sum_{k=1}^{n} |\lambda_k(A)|^2$$

in this case.

The following properties of $g(A)$ are checked in [15, Section 3.1]. The inequality

$$g^2(A) \leq 2N_2^2(A_I) \quad (A_I = (A - A^*)/2i).$$

is valid, and for any all real number t and any complex number z, one has $g(A) = g(Ae^{it} + zI)$. Moreover, if A_1 and A_2 are commuting $n \times n$-matrices, then

$$g(A_1 + A_2) \leq g(A_1) + g(A_2).$$

In addition, by the inequality between geometric and arithmetic mean values, we have

$$\left(\frac{1}{n} \sum_{k=1}^{n} |\lambda_k(A)|^2 \right)^n \geq \left(\prod_{k=1}^{n} |\lambda_k(A)| \right)^2.$$

Hence,

$$g^2(A) \leq N_2^2(A) - n(\det A)^{2/n}.$$

Lemma 3.1. *Let $A \in \mathbb{C}^{n\times n}$. Then $J_c(A) \le M_n(A,c)$ and $\hat{J}_b(A) \le \hat{M}_n(A,b)$, where*

$$M_n(A,c) := \sum_{j,k=0}^{n-1} \frac{g^{j+k}(A)(k+j)!}{2^{j+k}(\beta(A)-c)^{j+k+1}(j!\ k!)^{3/2}}$$

and

$$\hat{M}_n(A,b) := \sum_{j,k=0}^{n-1} \frac{g^{j+k}(A)(k+j)!}{2^{j+k}(b-\alpha(A))^{j+k+1}(j!\ k!)^{3/2}}.$$

Proof. By Theorem 3.5 from [15], for any $B \in \mathbb{C}^{n\times n}$ we have

$$\|e^{Bt}\| \le \exp[\alpha(B)t] \sum_{k=0}^{n-1} \frac{g^k(B)t^k}{(k!)^{3/2}} \quad (t \ge 0). \tag{3.1}$$

Since $\alpha(-A) = -\beta(A)$, we can write

$$\begin{aligned} J_c(A) &\le 2\int_0^\infty e^{2(c-\beta(A))t} \left(\sum_{k=0}^{n-1} \frac{g^k(A)t^k}{(k!)^{3/2}}\right)^2 dt \\ &= 2\int_0^\infty \exp[2(c-\beta(A))t] \left(\sum_{j,k=0}^{n-1} \frac{g_I^{k+j}(A)t^{k+j}}{(j!k!)^{3/2}}\right) dt \\ &= \sum_{j,k=0}^{n-1} \frac{2(k+j)!g_I^{j+k}(A)}{(2(\beta(A)-c))^{j+k+1}(j!\ k!)^{3/2}}. \end{aligned}$$

So we have proved that $J_c(A) \le M_n(A,c)$.

Similarly, due to (3.1)

$$\begin{aligned} \hat{J}_b(A) &\le 2\int_0^\infty \exp[2(-b+\alpha(A))t] \left(\sum_{k=0}^{n-1} \frac{g^k(A)t^k}{(k!)^{3/2}}\right)^2 dt \\ &= 2\int_0^\infty \exp[2(-b+\alpha(A))t] \left(\sum_{j,k=0}^{n-1} \frac{g_I^{k+j}(A)t^{k+j}}{(j!k!)^{3/2}}\right) dt \\ &= \sum_{j,k=0}^{n-1} \frac{2(k+j)!g_I^{j+k}(A)}{(2(b-\alpha(A))^{j+k+1}(j!\ k!)^{3/2}}, \end{aligned}$$

and thus, the inequality $\hat{J}_b(A) \le \hat{M}_n(A,b)$ is also valid, as claimed. □

If A is normal, then $g(A) = 0$ and with $0^0 = 1$ we have

$$M_n(A,c) = \frac{1}{\beta(A)-c} \quad \text{and} \quad \hat{M}_n(A,b) = \frac{1}{b-\alpha(A)}.$$

The latter lemma and Theorem 1.1 imply the following corollary.

Corollary 3.2. *Let $A, \tilde{A} \in \mathbb{C}^{n\times n}$. Then the condition*

$$q\hat{M}_n(A,b) < 1 \tag{3.2}$$

implies $\alpha(\tilde{A}) \le b$, and the condition

$$qM_n(A,c) < 1. \tag{3.3}$$

implies $\beta(\tilde{A}) \ge c$.

Put

$$F_n(A,x) := \sum_{j,k=0}^{n-1} \frac{g^{j+k}(A)(k+j)!}{2^{j+k}x^{j+k+1}(j!\ k!)^{3/2}} \quad (x>0).$$

Then we can write

$$M_n(A,c) = F_n(A,\beta(A)-c) \quad \text{and} \quad \hat{M}_n(A,b) = F_n(A,b-\alpha(A)).$$

Let $x_n = x_n(q,A)$ be the unique positive root of the equation

$$qF_n(A,x) = 1. \tag{3.4}$$

Then, taking

$$c = \beta(A) - x_n(q,A) - \epsilon \quad (\epsilon > 0),$$

we have

$$qM_n(A,c) = qF_n(A,\beta(A)-c) < qF_n(A,x_n) = 1.$$

Now Corollary 3.2 implies

$$\beta(\tilde{A}) > \beta(A) - x_n(q,A) - \epsilon.$$

Hence, letting $\epsilon \to 0$, we obtain

$$\beta(\tilde{A}) \ge \beta(A) - x_n(q,A). \tag{3.5}$$

Similarly, taking

$$b = x_n(q,A) + \alpha(A) + \epsilon,$$

we have

$$q\hat{M}_n(A,b) = qF_n(A,b-\alpha(A)) < qF_n(A,x_n) = 1.$$

Now Corollary 3.2 implies $\alpha(\tilde{A}) < \alpha(A) + x_n(q,A) + \epsilon$. Hence,

$$\alpha(\tilde{A}) \le \alpha(A) + x_n(q,A). \tag{3.6}$$

We thus have proved the following theorem.

Theorem 3.3. *Let $A, \tilde{A} \in \mathbb{C}^{n\times n}$ and let $x_n(q,A)$ be the unique positive root of the equation* (3.4)*. Then inequalities* (3.5) *and* (3.6) *are valid.*

If $g(A) = 0$, then with $0^0 = 1$ we can write $F_n(A,x) = \frac{1}{x}$ and thus $x_n(q,A) = q$. The following lemma gives us an estimate for $x_n(q,A)$ in the case $g(A) \neq 0$.

Lemma 3.4. *Let $A \in \mathbb{C}^{n\times n}$ and with the notation*

$$\eta(A) := \sum_{j,k=0}^{n-1} \frac{g^{j+k}(A)(k+j)!}{2^{j+k}(j!\ k!)^{3/2}},$$

let

$$q\eta(A) \leq 1. \tag{3.7}$$

Then

$$x_n(q,A) \leq \sqrt[2n]{q\eta(A)}.$$

Proof. In this proof for the brevity put $x_n(q,A) = x_0$ and $F_n(A,x) = F(x)$. Then by (3.7) we have

$$qF(x_0) = 1 \geq qF(1) = q\eta(A).$$

Since F monotonically decreases, hence it follows that $x_0 \leq 1$. Multiplying equation (3.4) by x_0^{2n} we have

$$x_0^{2n} = q\sum_{j,k=0}^{n-1} \frac{g^{j+k}(A)x_0^{2n-j-k-1}(k+j)!}{2^{j+k}(j!\ k!)^{3/2}} \leq q\eta(A).$$

This proves the lemma. □

About other estimates for the roots of polynomials see for instance the classical book [21] and the references, which are given therein.

Theorem 3.3 and the latter lemma imply the following result.

Corollary 3.5. *Let $A, \tilde{A} \in \mathbb{C}^{n\times n}$ and condition (3.7) hold. Then*

$$\alpha(\tilde{A}) \leq \alpha(A) + \sqrt[2n]{q\eta(A)}$$

and

$$\beta(\tilde{A}) \geq \beta(A) - \sqrt[2n]{q\eta(A)}.$$

4. SPECTRAL STRIPS OF MATRICES "CLOSE" TO TRIANGULAR ONES

Let V_+ and V_- be the strictly upper and lower triangular parts of a matrix

$$A = (a_{jk})_{j,k=1}^n,$$

respectively, i.e.,

$$V_+ = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ . & \dots & . & . \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad V_- = \begin{pmatrix} 0 & \dots & 0 & 0 \\ a_{21} & \dots & 0 & 0 \\ . & \dots & . & \\ a_{n1} & \dots & a_{n,n-1} & 0 \end{pmatrix}.$$

In addition, put

$$D = \text{diag } (a_{11}, a_{22}, \ldots, a_{nn}),$$

and $A_+ = D + V_+$. So

$$A = A_+ + V_- = D + V_+ + V_-.$$

We are going to apply the results of the previous section with $A = A_+$ and $\tilde{A} = A$. Since the eigenvalues of triangular matrices are the diagonal entries, the obtained results give us bounds for the spectral strip of A.

It is clear that

$$\alpha(A_+) = \alpha(D) = \max_{k=1,\ldots,n} \text{Re } a_{kk}$$

and

$$\beta(A_+) = \beta(D) = \min_{k=1,\ldots,n} \text{Re } a_{kk}.$$

In addition, $\|A - A_+\| = \|V_-\|$ and $g(A_+) = N_2(V_+)$, cf. [15, Lemma 3.1].

Thus, for all $\hat{c} < \beta(D)$ and $\hat{b} > \alpha(D)$ we have

$$M_n(A_+, \hat{c}) := \sum_{j,k=0}^{n-1} \frac{N_2^{j+k}(V_+)(k+j)!}{2^{j+k}(\beta(D) - \hat{c})^{j+k+1}(j!\ k!)^{3/2}},$$

and

$$\hat{M}_n(A_+, \hat{b}) := \sum_{j,k=0}^{n-1} \frac{N_2^{j+k}(V_+)(k+j)!}{2^{j+k}(\hat{b} - \alpha(D))^{j+k+1}(j!\ k!)^{3/2}}.$$

Now Corollary 3.2 implies the following result.

Corollary 4.1. *Let $A \in \mathbb{C}^{n\times n}$. Then the condition*

$$\|V_-\|\hat{M}_n(A_+, \hat{b}) < 1$$

implies $\alpha(A) \le \hat{b}$, and the condition

$$\|V_-\|M_n(A_+, \hat{c}) < 1$$

implies $\beta(A) \ge \hat{c}$.

Furthermore, we have

$$\eta(A_+) := \sum_{j,k=0}^{n-1} \frac{N_2^{j+k}(V_+)(k+j)!}{2^{j+k}(j!\ k!)^{3/2}}.$$

Therefore condition (3.7) with $A = A_+$ takes the form $\|V_-\|\eta(A_+) \le 1$. Under this condition by Lemma 3.4 we obtain

$$x_n(\|V_-\|, A_+) \le \sqrt[2n]{\|V_-\|\eta(A_+)}.$$

Hence, making use of Corollary 3.5, we arrive at the following result.

Corollary 4.2. *Let $A \in \mathbb{C}^{n\times n}$ and the condition $\|V_-\|\eta(A_+) \le 1$ hold. Then*

$$\alpha(A) \le \alpha(D) + \sqrt[2n]{\|V_-\|\eta(A_+)}$$

and

$$\beta(A) \ge \beta(D) - \sqrt[2n]{\|V_-\|\eta(A_+)}.$$

5. OPERATORS WITH HILBERT–SCHMIDT HERMITIAN COMPONENTS

In this section we obtain estimates for $J_c(A)$ and $\hat{J}_b(A)$ $(A \in \mathcal{B}(\mathcal{H}))$, assuming that

$$A_I = (A - A^*)/(2i) \text{ is a Hilbert–Schmidt operator,} \tag{5.1}$$

i.e.

$$N_2(A_I) := (\text{trace } (A_I^2))^{1/2} < \infty.$$

Numerous integral operators satisfy this condition. We introduce the quantity

$$g_I(A) := \left[2N_2^2(A_I) - 2\sum_{k=1}^{\infty} |\text{Im } \lambda_k(A)|^2\right]^{1/2} \le \sqrt{2}N_2(A_I),$$

where $\lambda_k(A)$ $(k = 1, 2, \dots)$ are the eigenvalues of A taken with their multiplicities and ordered as $|\text{Im } \lambda_{k+1}(A)| \le |\text{Im } \lambda_k(A)|$ $(k = 1, 2, \dots)$. If A is normal, then $g_I(A) = 0$, cf. [15, Lemma 9.3].

Lemma 5.1. *Let condition* (5.1) *hold. Then*

$$J_c(A) \le M_I(A, c) \quad \textit{and} \quad \hat{J}_b(A) \le \hat{M}_I(A, b),$$

where

$$M_I(A, c) := \sum_{j,k=0}^{\infty} \frac{g_I^{j+k}(A)(k + j)!}{2^{j+k}(\beta(A) - c)^{j+k+1}(j!\ k!)^{3/2}}$$

and

$$\hat{M}_I(A, b) := \sum_{j,k=0}^{\infty} \frac{g_I^{j+k}(A)(k + j)!}{2^{j+k}(b - \alpha(A))^{j+k+1}(j!\ k!)^{3/2}}.$$

Proof. By Theorem 10.1 from [15] for any $B \in \mathcal{B}(\mathcal{H})$ with the property: $B_I = (B - B^*)/(2i)$ is a Hilbert–Schmidt operator, we have

$$\|e^{Bt}\| \le \exp[\alpha(B)t] \sum_{k=0}^{\infty} \frac{g_I^k(B)t^k}{(k!)^{3/2}} \quad (t \ge 0). \tag{5.2}$$

Since $\alpha(-A) = -\beta(A)$, hence it follows

$$J_c(A) \leq 2 \int_0^\infty e^{2(c-\beta(A))t} \left(\sum_{k=0}^\infty \frac{g_I^k(A) t^k}{(k!)^{3/2}} \right)^2 dt$$
$$= 2 \int_0^\infty \exp[2(c - \beta(A))t] \left(\sum_{j,k=0}^\infty \frac{g_I^{k+j}(A) t^{k+j}}{(j!k!)^{3/2}} \right) dt$$
$$= \sum_{j,k=0}^\infty \frac{2(k+j)! g_I^{j+k}(A)}{(2(\beta(A) - c))^{j+k+1} (j!\, k!)^{3/2}}.$$

So we have proved that $J_c(A) \leq M_I(A, c)$.

Similarly, due to (5.2),

$$\hat{J}_b(A) \leq 2 \int_0^\infty \exp[2(-b + \alpha(A))t] \left(\sum_{k=0}^\infty \frac{g_I^k(A) t^k}{(k!)^{3/2}} \right)^2 dt$$
$$= 2 \int_0^\infty \exp[2(-b + \alpha(A))t] \left(\sum_{j,k=0}^\infty \frac{g_I^{k+j}(A) t^{k+j}}{(j!k!)^{3/2}} \right) dt$$
$$= \sum_{j,k=0}^\infty \frac{2(k+j)! g_I^{j+k}(A)}{(2(b - \alpha(A)))^{j+k+1} (j!\, k!)^{3/2}},$$

and thus, the inequality $\hat{J}_b(A) \leq \hat{M}_I(A, b)$ is also valid, as claimed. □

If A is normal, then $g_I(A) = 0$ and with $0^0 = 1$ we have

$$M_I(A, c) = \frac{1}{\beta(A) - c} \quad \text{and} \quad \hat{M}_I(A, b) = \frac{1}{b - \alpha(A)}.$$

The latter lemma and Theorem 1.1 imply the following corollary.

Corollary 5.2. *Let condition* (5.1) *hold. Then the inequality* $qM_I(A, c) < 1$ *implies* $\beta(\tilde{A}) \geq c$*, and the inequality* $q\hat{M}_I(A, b) < 1$ *implies* $\alpha(\tilde{A}) \leq b$*.*

Furthermore, put

$$G(A, x) := \sum_{j,k=0}^\infty \frac{g_I^{j+k}(A)(k+j)!}{2^{j+k} x^{j+k+1} (j!\, k!)^{3/2}} \quad (x > 0).$$

Then we can write

$$M_I(A, c) = G(A, \beta(A) - c) \quad \text{and} \quad \hat{M}_I(A, b) = G(A, b - \alpha(A)).$$

Let y_0 be the unique positive root of the equation

$$qG(A, x) = 1. \tag{5.3}$$

Taking $b = y_0 + \alpha(A) + \epsilon$, we have

$$q\hat{M}_I(A, b) = qG(A, b - \alpha(A)) < qG(A, y_0) = 1.$$

Now Corollary 5.2 implies

$$\alpha(\tilde{A}) \le y_0 + \alpha(A) + \epsilon.$$

Similarly, taking $c = \beta(A) - y_0 - \epsilon$, we have

$$qM_I(A, c) = qG(A, \beta(A) - c) < qG(A, y_0) = 1.$$

Due to Corollary 5.2, $\beta(\tilde{A}) \ge \beta(A) - y_0 - \epsilon$. Since $\epsilon > 0$ is arbitrary, we arrive at our next result.

Theorem 5.3. *Let the condition* (5.1) *hold and* y_0 *be the unique positive root of equation* (5.3). *Then* $\beta(\tilde{A}) \ge \beta(A) - y_0$ *and* $\alpha(\tilde{A}) \le \alpha(A) + y_0$.

If $g_I(A) = 0$, then with $0^0 = 1$ we can write $G(A, x) = \frac{1}{x}$ and thus $y_0 = q$. The following lemma gives us an estimate for y_0 in the case $g_I(A) \ne 0$.

Lemma 5.4. *Let*

$$q \le \sqrt{e} g_I(A). \tag{5.4}$$

Then

$$y_0 \le \zeta(q, g_I(A)),$$

where

$$\zeta(a, d) := \frac{2d}{[\ln (\sqrt{e} d/a)]^{1/2}} \quad (a, d > 0).$$

Proof. Since

$$2^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!\, k!} \quad (n = 2, 3, \ldots),$$

we have

$$2^n \ge \frac{n!}{(n-k)!\, k!} \quad \text{and} \quad 2^{j+k} \ge \frac{(j+k)!}{j!\, k!}.$$

Thus

$$G(A, x) \le \sum_{j,k=0}^{\infty} \frac{g_I^{j+k}(A)}{x^{j+k+1}(j!\, k!)^{1/2}} = \frac{1}{x} H^2(A, x) \quad (x > 0),$$

where

$$H(A, x) := \sum_{j=0}^{\infty} \frac{g_I^j(A)}{x^j (j!)^{1/2}} \quad (x > 0).$$

By the Schwarz inequality,

$$H^2(A, x) = \left(\sum_{j=0}^{\infty} \frac{(\sqrt{2})^j g_I^j(A)}{(\sqrt{2})^j x^j (j!)^{1/2}} \right)^2 \le \sum_{j=0}^{\infty} \frac{1}{2^j} \sum_{j=0}^{\infty} \frac{2^j g_I^{2j}(A)}{x^{2j} j!} = 2 \exp\left[\frac{2g_I^2(A)}{x^2} \right].$$

Thus $y_0 \le y$, where y is is the unique positive root of the equation

$$1 = \frac{2q}{x} \exp\left[\frac{2g_I^2(A)}{x^2}\right].$$

Or

$$1 = \frac{4q^2}{x^2} \exp\left[\frac{4g_I^2(A)}{x^2}\right].$$

With

$$w = \frac{4g_I^2(A)}{y^2},$$

we obtain

$$1 = \frac{q^2}{g_I^2(A)} we^w.$$

It is simple to check that $we^{-w} \le e^{-1}$, and thus $w \le e^{w-1}$ $(w > 0)$. Therefore,

$$1 \le \frac{q^2}{g_I^2(A)} e^{2w-1} = \frac{q^2}{eg_I^2(A)} e^{2w}$$

and

$$\frac{1}{2} \ln \left[\frac{eg_I^2(A)}{q^2}\right] \le w,$$

provided $\sqrt{e} g_I(A) \ge q$. Hence

$$\frac{4g_I^2(A)}{y^2} \ge \frac{1}{2} \ln \left[\frac{eg_I^2(A)}{q^2}\right].$$

This implies

$$y_0^2 \le y^2 \le 8g_I^2(A)/\left(\ln \left[\frac{eg_I^2(A)}{q^2}\right]\right) = 4g_I^2(A)/\left[\ln\left(\sqrt{e} g_I/q\right)\right] = \zeta^2(q, g_I(A)),$$

as claimed.

Theorem 5.3 and the latter lemma imply the following corollary.

Corollary 5.5. *Let the conditions* (5.1) *and* (5.4) *hold. Then the inequalities*

$$\alpha(\tilde{A}) \le \alpha(A) + \zeta(q, g_I(A)) \quad \text{and} \quad \beta(\tilde{A}) \ge \beta(A) - \zeta(q, g_I(A))$$

are valid.

6. INTEGRAL OPERATORS

Let $L^2 = L^2(0,1)$ be the complex space of scalar functions h defined on $[0,1]$ and equipped with the norm

$$\|h\| = \left[\int_0^1 |h(x)|^2 dx\right]^{1/2}.$$

Let $\tilde{A}$ be the operator defined in $L^2(0,1)$ by

$$(\tilde{A}h)(x) = a(x)h(x) + \int_0^1 k(x,s)h(s)ds \quad (h \in L^2, x \in [0,1]), \tag{6.1}$$

where $a(x)$ is a real bounded measurable function and $k(x,s)$ is a complex kernel defined on $0 \le x, s \le 1$, and

$$\int_0^1 \int_0^1 |k(x,s)|^2 ds\, dx < \infty. \tag{6.2}$$

So the Volterra operator V defined by

$$(Vh)(x) = \int_x^1 k(x,s)h(s)ds \quad (h \in L^2, x \in [0,1]),$$

is a Hilbert–Schmidt one. Define operator A by

$$(Ah)(x) = a(x)h(x) + \int_x^1 k(x,s)h(s)ds \quad (h \in L^2, x \in [0,1]).$$

Then $A = D + V$, where D is defined by $(Dh)(x) = a(x)h(x)$. Due to [16, Lemma 7.1] and [16, Corollary 3.5] we have $\sigma(A) = \sigma(D)$. So $\sigma(A)$ is real and

$$\beta(A) = a_{\inf} := \inf_x a(x) \quad \text{and} \quad \alpha(A) = a_{\sup} := \sup_x a(x).$$

Moreover,

$$N_2(A_I) = N_2(V_I) \le N_2(V) = \left[\int_0^1 \int_x^1 |k(x,s)|^2 ds\, dx\right]^{1/2}.$$

Here $V_I = (V - V^*)/(2i)$. Thus,

$$g_I(A) \le \sqrt{2} N_2(V).$$

Note also, that

$$q = \|A - \tilde{A}\| \leq \left[\int_0^1 \int_0^x |k(x,s)|^2 ds\, dx \right]^{1/2}.$$

Making use of Corollary 5.5, we arrive at

Corollary 6.1. *Let* $\tilde{A}$ *be defined by* (6.1) *and let the conditions* (6.2) *and*

$$q \leq \sqrt{2e} N_2(V)$$

hold. Then the inequalities

$$\alpha(\tilde{A}) \leq a_{\sup} + \zeta(q, \sqrt{2} N_2(V))$$

and

$$\beta(\tilde{A}) \geq a_{\inf} - \zeta(q, \sqrt{2} N_2(V))$$

are valid.

The recent results on the spectral properties of integral operators can be found, for instance, in the papers [7, 9, 10, 12].

7. APPROXIMATION OF THE SPECTRAL STRIP OF A COMPACT OPERATOR BY THE SPECTRAL STRIPS OF FINITE MATRICES

Let $\{d_k\}$ be an orthonormal basis in $\mathcal{H}$ and let A be a Hilbert–Schmidt operator represented in that basis by the infinite matrix $A = (a_{jk})_{j,k=1}^{\infty}$. Introduce the projections

$$P_n = \sum_{k=1}^{n} (\cdot, d_k) d_k \quad (n = 1, 2, \ldots).$$

We will approximate the spectral strip of A by the spectral strips of the operators $A_n = P_n A P_n$ $(n = 1, 2, \ldots)$. A_n is representable in the mentioned basis by the matrix $A_n = (a_{jk})_{j,k=1}^{n}$, Put $B_n = A_n + C_n$, where

$$C_n = \sum_{k=n+1}^{\infty} \Delta P_k A \Delta P_k = \sum_{k=n+1}^{\infty} a_{kk} \Delta P_k \quad (\Delta P_k = P_k - P_{k-1}; P_0 = 0),$$

i.e., C_n is representable in $(I - P_n)\mathcal{H}$ by the diagonal matrix diag $(a_{kk})_{k=n+1}^{\infty}$.

Since A_n and C_n act in the mutually orthogonal subspaces, one has $A_n C_n = C_n A_n = 0$, and

$$\sigma(B_n) = \sigma(A_n) \cup \sigma(C_n).$$

Thus

$$\alpha(B_n) = \max\{\alpha(A_n), \alpha(C_n)\} = \max\left\{\alpha(A_n), \sup_{k>n} \operatorname{Re} a_{kk}\right\}$$

and

$$\beta(B_n) = \min\{\beta(A_n), \beta(C_n)\} = \min\left\{\beta(A_n), \inf_{k>n} \operatorname{Re} a_{kk}\right\}.$$

In addition, $g_I(B_n) = g_I(A_n) + g_I(C_n)$. But $g_I(C_n) = 0$, since C_n is normal. Thus,

$$g_I(B_n) = g_I(A_n) \le \sqrt{2} N_2(\operatorname{Im} A_n) \quad (\operatorname{Im} A_n = (A_n - A_n^*)/(2i)).$$

Put

$$\nu_n = \|A - B_n\|.$$

Since A is a Hilbert–Schmidt operator, we have $B_n \to A$ as $n \to \infty$ in the Hilbert–Schmidt norm, and therefore, $\nu_n \to 0$.

Now we can directly apply the results of Section 5 with $A = B_n$, $\tilde{A} = A$, $g_I(B_n) = g_I(A_n)$ and $q = \nu_n$. In particular, for real constants $\hat{c} < \beta(B_n)$ and $\hat{b} > \alpha(B_n)$, omitting simple calculations, we can write

$$M_I(B_n, \hat{c}) = \sum_{j,k=0}^{\infty} \frac{g_I^{j+k}(A_n)(k+j)!}{2^{j+k}(\beta(B_n) - \hat{c})^{j+k+1}(j!\ k!)^{3/2}}$$

and

$$\hat{M}_I(B_n, \hat{b}) = \sum_{j,k=0}^{\infty} \frac{g_I^{j+k}(A_n)(k+j)!}{2^{j+k}(\hat{b} - \alpha(B_n))^{j+k+1}(j!\ k!)^{3/2}}.$$

Now Corollary 5.2 implies the following result.

Corollary 7.1. *Let A be a Hilbert–Schmidt operator, represented in an orthogonal basis by matrix $A = (a_{jk})_{j,k=1}^{\infty}$ and let $A_n = (a_{jk})_{j,k=1}^{n}$. Assume that $\nu_n M_I(B_n, \hat{c}) < 1$. Then $\beta(A) \ge \hat{c}$. If, in addition, $\nu_n \hat{M}_I(B_n, \hat{b}) < 1$, then $\alpha(A) \le \hat{b}$.*

Since $\nu_n \to 0$ and $a_{nn} \to 0$ as $n \to \infty$, Corollary 7.1 gives us approximations of $\alpha(A)$ and $\beta(A)$ by $\alpha(A_n)$ and $\beta(A_n)$, respectively. About the recent results on spectral approximations of operators, for instance see the papers [13, 14, 19, 20] and the references given therein.

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Received: December 24, 2020.
Accepted: January 24, 2021.