Opuscula Math. 41, no. 2 (2021), 205–226
https://doi.org/10.7494/OpMath.2021.41.2.205 Opuscula Mathematica

ON THE GAUGE-NATURAL OPERATORS SIMILAR TO THE TWISTED DORFMAN–COURANT BRACKET

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Communicated by P.A. Cojuhari

Abstract. All $\mathcal{VB}_{m,n}$-gauge-natural operators C sending linear 3-forms $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ on a smooth ($\mathcal{C}^\infty$) vector bundle E into $\mathbf{R}$-bilinear operators

$$C_H : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$$

transforming pairs of linear sections of $TE \oplus T^*E \to E$ into linear sections of $TE \oplus T^*E \to E$ are completely described. The complete descriptions is given of all generalized twisted Dorfman–Courant brackets C (i.e. C as above such that C_0 is the Dorfman–Courant bracket) satisfying the Jacobi identity for closed linear 3-forms H. An interesting natural characterization of the (usual) twisted Dorfman–Courant bracket is presented.

Keywords: natural operator, linear vector field, linear form, twisted Dorfman-Courant bracket, the Jacobi identity in Leibniz form.

Mathematics Subject Classification: 53A55, 53A45, 53A99.

1. INTRODUCTION

All manifolds considered in the paper are assumed to be Hausdorff, second countable, finite dimensional, without boundary, and smooth (of class $\mathcal{C}^\infty$). Maps between manifolds are assumed to be $\mathcal{C}^\infty$.

In [3], the authors completely described bilinear operators on sections of the Whitney sum $TN \oplus T^*N \to N$ of the tangent and cotangent bundles (for N a smooth manifold), which are $\mathcal{M}f_m$-natural, i.e. invariant under the morphisms in the category $\mathcal{M}f_m$ of m-dimensional manifolds and their submersions. The Courant bracket, defined in [2], is an example of such operators and it is of particular interest, because it involves in the concepts of Dirac and generalized complex structures on N, see [2, 4, 5]

A simple (but very important) modification of the Courant bracket is the so called twisted (or H-twisted) Courant bracket $[-,-]_H$ on sections of $TN \oplus T^*N \to N$

for any 3-form H on a smooth manifold N. The properties of $[-,-]_H$ (for closed H) were used in [8, 12] to define the concept of exact Courant algebroid. In [9], we completely described all $\mathcal{M}f_m$-natural operators which send 3-forms H on N into bilinear operators on sections of $TN \oplus T^*N \to N$ (for N a smooth manifold).

The restriction of the Courant bracket to linear sections of $TE \oplus T^*E \to E$ (for $E \to M$ a smooth vector bundle) is called the Dorfman–Courant bracket, see [6]. It is of particular interest, because $(TE \oplus T^*E; E, TM \oplus E^*; M)$ is the *standard VB-Courant algebroid* and the Dorfman–Courant bracket is the part of this structure. (The Dorfman–Courant bracket can be also interpreted as the bracket of the Omni-Lie algebroid $Der(E^*) \oplus J^1(E^*)$, studied in [1].)

In [10], we completely described all bilinear operators on linear sections of $TE \oplus T^*E \to E$ (for $E \to M$ a smooth vector bundle), which are $\mathcal{VB}_{m,n}$-gauge-natural, i.e. invariant under the morphisms in the category $\mathcal{VB}_{m,n}$ of rank-n vector bundles over m-dimensional bases and their vector bundle isomorphisms onto images. The Dorfman–Courant bracket is an example of such a $\mathcal{VB}_{m,n}$-gauge natural bilinear operator

$$A : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E),$$

where $\Gamma^l_E(TE \oplus T^*E)$ is the space of linear sections of $TE \oplus T^*E \to E$.

In [11], we completely described all $\mathcal{VB}_{m,n}$-gauge-natural (i.e. invariant under the morphisms in the category $\mathcal{VB}_{m,n}$) operators

$$C : \Gamma^{l-\mathrm{clos}}(\bigwedge^3 T^*) \rightsquigarrow Lin_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$$

which, like the twisted Dorfman–Courant bracket, transform closed linear 3-forms

$$H \in \Gamma^{l-\mathrm{clos}}_E(\bigwedge^3 T^*E)$$

on E into bilinear operators

$$C_H : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$$

(for E a $\mathcal{VB}_{m,n}$-object).

In the present paper, we completely describe all $\mathcal{VB}_{m,n}$-gauge-natural operators of the same type as in [11], but with bigger domain. Namely, we classify all $\mathcal{VB}_{m,n}$-gauge-natural operators

$$C : \Gamma^l(\bigwedge^3 T^*) \rightsquigarrow Lin_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$$

transforming linear 3-forms $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ on E into bilinear operators

$$C_H : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$$

(for E a $\mathcal{VB}_{m,n}$-object). Thus, the main result of the paper is the following theorem.

Theorem 1.1. *Let $m \geq 3$ and $n \geq 1$ be fixed integers. Any $\mathcal{VB}_{m,n}$-gauge-natural operator $C : \Gamma^l(\bigwedge^3 T^*) \rightsquigarrow Lin_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$ is of the form*

$$\begin{aligned} C_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus \{&b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 + b_4 d i_{X^2}\omega^1 \\ &+ b_5\mathcal{L}_{X^1} d i_L\omega^2 + b_6\mathcal{L}_{X^2} d i_L\omega^1 + c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH \\ &+ c_3 i_L i_{X^2} d i_{X^1} H + c_4 i_L i_{X^1} d i_{X^2} H + c_5 i_L d i_{X^2} i_{X^1} H\} \end{aligned} \tag{1.1}$$

for arbitrary (uniquely determined by C) reals $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5$, where $\rho^i = X^i \oplus \omega^i \in \Gamma^l_E(TE \oplus T^*E)$, $H \in \Gamma^l_E(\bigwedge^3 T^*E)$, and where $[-,-]$ is the usual bracket on vector fields, $\mathcal{L}$ is the Lie derivative, d is the exterior derivative, i is the insertion derivative and L is the Euler vector field.

We have non-trivial operator $0 \oplus i_L i_{X^1} i_{X^2} dH$, which is 0 on closed linear 3-forms H. So, the present paper is an essential extension of [11].

The second result of the paper is the following.

Theorem 1.2. *Let $m \geq 4$ and $n \geq 1$. Any generalized twisted Dorfman–Courant bracket C (i.e. operator C as above such that C_0 is the usual Dorfman–Courant bracket) satisfying the Jacobi identity in Leibniz form for closed linear 3-forms (i.e.*

$$C_H(\rho^1, C_H(\rho^2, \rho^3)) = C_H(C_H(\rho^1, \rho^2), \rho^3) + C_H(\rho^2, C_H(\rho^1, \rho^3))$$

for all closed linear 3-forms $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ and all linear sections $\rho^i = X^i \oplus \omega^i \in \Gamma^l_E(TE \oplus T^*E)$ for $i = 1, 2, 3$ and all $\mathcal{VB}_{m,n}$-objects E) is of the form

$$\begin{aligned} C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [X^1, X^2] \oplus \{&\mathcal{L}_{X^1}\omega^2 - i_{X^2} d\omega^1 \\ &+ c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH\} \end{aligned} \tag{1.2}$$

for any (not necessarily closed) linear 3-form $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ and any $X^1 \oplus \omega^1$, $X^2 \oplus \omega^2 \in \Gamma^l_E(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$-object E, where c_1, c_2 are arbitrary (uniquely determined by C) real numbers.

From Theorem 1.2, we have the following interesting natural characterization of the (usual) twisted Dorfman–Courant bracket.

Corollary 1.3. *Let $m \geq 4$ and $n \geq 1$. Any generalized twisted Dorfman–Courant bracket C_H satisfying the Jacobi identity in Leibniz form for closed linear 3-forms satisfies*

$$C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_{cH} \tag{1.3}$$

for any closed linear 3-form $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_E(TE \oplus T^*E)$, where $[[-,-]]_H$ is the usual twisted (H-twisted) Dorfman–Courant bracket and c is an arbitrary (uniquely determined by C) real number.

Roughly speaking, the above corollary means that (for $m \geq 4$) the (usual) twisted Dorfman–Courant bracket $[[-,-]]_H$ (for closed linear 3-forms H) is the unique (up to multiplication of H by a real number c) $\mathcal{VB}_{m,n}$-gauge-natural extension of the (usual) Dorfman–Courant bracket $[[-,-]]_0$ (by means of closed linear 3-forms H) satisfying the Jacobi identity in Leibniz form.

From now on, let $\mathbf{R}^{m,n}$ be the trivial vector bundle over $\mathbf{R}^m$ with the standard fibre $\mathbf{R}^n$ and let $x^1,\dots,x^m,y^1,\dots,y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$.

2. THE DORFMAN–COURANT LIKE BRACKETS

Let $E=(E\to M)$ be a vector bundle.

Applying the tangent and the cotangent functors to $E\to M$, we obtain double vector bundles $(TE;E,TM;M)$ and $(T^*E;E,E^*;M)$.

A vector field X on E is called linear if it is a vector bundle map $X:E\to TE$ between $E\to M$ and $TE\to TM$. Equivalently, a vector field X on E is linear iff it has expression

$$X=\sum_{i=1}^m a^i(x^1,\dots,x^m)\frac{\partial}{\partial x^i}+\sum_{j,k=1}^n b^k_j(x^1,\dots,x^m)y^j\frac{\partial}{\partial y^k}$$

in any local vector bundle trivialization $x^1,\dots,x^m,y^1,\dots,y^n$ on E. The Euler vector field L on E is an example of a linear vector field on E. (We recall that the coordinate expression of L is $L=\sum_{j=1}^n y^j\frac{\partial}{\partial y^j}$.) Equivalently, a vector field X on E is linear iff $\mathcal{L}_L X=0$, where $\mathcal{L}$ denotes the Lie derivative.

A 1-form ω on E is called linear if it is a vector bundle map $\omega:E\to T^*E$ between $E\to M$ and $T^*E\to E^*$. Equivalently, a 1-form ω on E is linear iff it has expression

$$\omega=\sum_{i=1}^m\sum_{j=1}^n a_{ij}(x^1,\dots,x^m)y^j dx^i+\sum_{j=1}^n b_j(x^1,\dots,x^m)dy^j$$

in any local vector bundle trivialization $x^1,\dots,x^m,y^1,\dots,y^n$ on E. Equivalently, a 1-form ω on E is linear iff $\mathcal{L}_L\omega=\omega$, where L is the Euler vector field on E.

We have the following definition being respective modification of the general one from the fundamental monograph [7].

Definition 2.1. A $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator

$$A:\Gamma^l(T\oplus T^*)\times\Gamma^l(T\oplus T^*)\rightsquigarrow\Gamma^l(T\oplus T^*)$$

is a $\mathcal{VB}_{m,n}$-invariant family of $\mathbf{R}$-bilinear operators

$$A:\Gamma^l_E(TE\oplus T^*E)\times\Gamma^l_E(TE\oplus T^*E)\to\Gamma^l_E(TE\oplus T^*E)$$

for all $\mathcal{VB}_{m,n}$-objects E, where $\Gamma^l_E(TE\oplus T^*E)$ is the vector space of linear sections of $TE\oplus T^*E$ (couples $X\oplus\omega$ of linear vector fields X and linear 1-forms ω on E).

Remark 2.2. The $\mathcal{VB}_{m,n}$-invariance of A means that if

$$(X^1\oplus\omega^1,X^2\oplus\omega^2)\in\Gamma^l_E(TE\oplus T^*E)\times\Gamma^l_E(TE\oplus T^*E)$$

and

$$(\tilde X^1\oplus\tilde\omega^1,\tilde X^2\oplus\tilde\omega^2)\in\Gamma^l_{\tilde E}(T\tilde E\oplus T^*\tilde E)\times\Gamma^l_{\tilde E}(T\tilde E\oplus T^*\tilde E))$$

are φ-related by an $\mathcal{VB}_{m,n}$-map $\varphi:E\to\tilde E$ (i.e. $\tilde X^i\circ\varphi=T\varphi\circ X^i$ and $\tilde\omega^i\circ\varphi=T^*\varphi\circ\omega^i$ for $i=1,2$), then so are $A(X^1\oplus\omega^1,X^2\oplus\omega^2)$ and $A(\tilde X^1\oplus\tilde\omega^1,\tilde X^2\oplus\tilde\omega^2)$.

Remark 2.3. The Dorfman–Courant bracket

$$[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_0 := [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - i_{X^2}d\omega^1)$$

is an example of a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator

$$\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*).$$

Theorem 2.4 ([10]). *Let $m \geq 2$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator*

$$A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$$

is of the form

$$\begin{aligned} A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus \{&b_1\mathcal{L}_{X^1}\omega^2 \\ &+ b_2\mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 + b_4 d i_{X^2}\omega^1 \\ &+ b_5\mathcal{L}_{X^1} d i_L\omega^2 + b_6\mathcal{L}_{X^2} d i_L\omega^1\} \end{aligned} \tag{2.1}$$

for arbitrary (*uniquely determined by A*) *reals $a, b_1, b_2, b_3, b_4, b_5, b_6$, where $[-,-]$ is the usual bracket on vector fields, $\mathcal{L}$ is the Lie derivative, d is the exterior derivative, i is the insertion derivative and L is the Euler vector field.*

3. THE TWISTED DORFMAN–COURANT LIKE BRACKETS

A p-form ω on E is called linear if $\mathcal{L}_L\omega = \omega$, where L is the Euler vector field on E. Equivalently, a p-form ω on E is linear iff it has expression

$$\omega = \sum a_{i_1,\dots,i_p,j}(x)y^j dx^{i_1} \wedge \dots \wedge dx^{i_p} + \sum b_{i_1,\dots,i_{p-1},j}(x)dy^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}$$

in any local vector bundle trivialization $x^1, \dots, x^m, y^1, \dots, y^n$ on E.

Definition 3.1. A $\mathcal{VB}_{m,n}$-gauge-natural operator

$$C : \Gamma^l(\bigwedge^3 T^*) \rightsquigarrow Lin_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$$

sending linear 3-forms $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ on $\mathcal{VB}_{m,n}$-objects E into $\mathbf{R}$-bilinear operators

$$C_H : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$$

is a $\mathcal{VB}_{m,n}$-invariant family of regular operators (functions)

$$C : \Gamma^l_E(\bigwedge^3 T^*E) \to Lin_2(\Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E), \Gamma^l_E(TE \oplus T^*E))$$

for all $\mathcal{VB}_{m,n}$-objects E, where $Lin_2(U \times V, W)$ denotes the vector space of all bilinear (over $\mathbf{R}$) functions $U \times V \to W$ for any real vector spaces U, V, W.

Remark 3.2. The invariance of C means that if $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ and $\tilde{H} \in \Gamma^l_{\tilde{E}}(\bigwedge^3 T^*\tilde{E})$ are φ-related by a $\mathcal{VB}_{m,n}$-map $\varphi : E \to \tilde{E}$, and

$$(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E)$$

and

$$(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2) \in \Gamma^l_{\tilde{E}}(T\tilde{E} \oplus T^*\tilde{E}) \times \Gamma^l_{\tilde{E}}(T\tilde{E} \oplus T^*\tilde{E})$$

are also φ-related, then so are $C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $C_{\tilde{H}}(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2)$. The regularity of C means that C transforms smoothly parametrized families $(H_t, X^1_t \oplus \omega^1_t, X^2_t \oplus \omega^2_t)$ into smoothly parametrized families $C_{H_t}(X^1_t \oplus \omega^1_t, X^2_t \oplus \omega^2_t)$.

Definition 3.3. A $\mathcal{VB}_{m,n}$-gauge-natural operator C in the sense of Definition 3.1 is of order 1 if the following implication

$$(j^1_x H = j^1_x \tilde{H}, j^1_x \rho^1 = j^1_x \tilde{\rho}^1, j^1_x \rho^2 = j^1_x \tilde{\rho}^2) \Rightarrow C_H(\rho^1, \rho^2)_{|E_x} = C_{\tilde{H}}(\tilde{\rho}^1, \tilde{\rho}^2)_{|E_x}$$

holds for any $H, \tilde{H} \in \Gamma^l_E(\bigwedge^3 T^*E)$ and any $\rho^1, \rho^2, \tilde{\rho}^1, \tilde{\rho}^2 \in \Gamma^l_E(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$-object $E \to M$ and any $x \in M$.

Remark 3.4. The twisted Dorfman–Courant bracket

$$[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_H := [X^1, X^2] \oplus \{\mathcal{L}_{X^1}\omega^2 - i_{X^2}d\omega^1 + i_{X^1}i_{X^2}H\} \tag{3.1}$$

is a gauge natural operator (of order 1) in the sense of Definition 3.1.

The main result is the following classification theorem.

Theorem 3.5. *Let C be a $\mathcal{VB}_{m,n}$-gauge-natural operator in the sense of Definition 3.1. Assume that $m \geq 3$ and $n \geq 1$. Then there exist uniquely determined real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5$ such that*

$$\begin{aligned} C_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus \{&b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 \\ &+ b_3 d i_{X^1}\omega^2 + b_4 d i_{X^2}\omega^1 + b_5\mathcal{L}_{X^1} d i_L \omega^2 \\ &+ b_6\mathcal{L}_{X^2} d i_L\omega^1 + c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH \\ &+ c_3 i_L i_{X^2} d i_{X^1} H + c_4 i_L i_{X^1} d i_{X^2} H + c_5 i_L d i_{X^2} i_{X^1} H\} \end{aligned} \tag{3.2}$$

for any $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ and any $\rho^1, \rho^2 \in \Gamma^l_E(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$-object E, where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$.

Proof. Operator $A := C_0$, where 0 is the zero linear 3-form, can be treated as the $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator in the sense of Definition 2.1. Then C_0 is described in Theorem 2.4. So, replacing C by $C - C_0$, we can assume

$$C_0 = 0. \tag{3.3}$$

We will keep this assumption in the rest of this section. The proof of our Theorem 3.5 will be continued after proving several lemmas.

By the $\mathcal{VB}_{m,n}$-invariance of C, such C is determined by the values

$$C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e \in T_e\mathbf{R}^{m,n} \oplus T_e^*\mathbf{R}^{m,n} \tag{3.4}$$

for all $H \in \Gamma^l_{\mathbf{R}^{m,n}}(\bigwedge^3 T^*\mathbf{R}^{m,n})$ and all $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_{\mathbf{R}^{m,n}}(T\mathbf{R}^{m,n} \oplus T^*\mathbf{R}^{m,n})$ and all $e \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n = \mathbf{R}^{m,n}_0$.

Given $e \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n = \mathbf{R}^{m,n}_0$, let $T_e(\mathbf{R}^m \times \mathbf{R}^n) = \mathbf{R}^m \times \mathbf{R}^n$ and $T_e^*(\mathbf{R}^m \times \mathbf{R}^n) = \mathbf{R}^{m*} \times \mathbf{R}^{n*}$ be the usual identifications. Let

$$\begin{aligned}
C^{1,1}_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e &= \text{the } \mathbf{R}^m\text{-part of } C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e,\\
C^{1,2}_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e &= \text{the } \mathbf{R}^n\text{-part of } C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e,\\
C^{2,1}_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e &= \text{the } \mathbf{R}^{m*}\text{-part of } C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e,\\
C^{2,2}_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e &= \text{the } \mathbf{R}^{n*}\text{-part of } C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e.
\end{aligned} \tag{3.5}$$

We will keep this notion (3.5) in the rest of this section.

Lemma 3.6. *C is of order* 1 *and $C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ is linear in H. Moreover, $C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ is independent of both ω^1 and ω^2.*

Proof. By the invariance of C with respect to $h_t = (\frac{1}{t}x^1, \dots, \frac{1}{t}x^m, y^1, \dots, y^n)$, we have the homogeneity conditions

$$\begin{aligned}
&t^{k(\mu,\nu)} C^{\mu,\nu}_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e\\
&= C^{\mu,\nu}_{(h_t)_*H}(t(h_t)_*X^1 \oplus t(h_t)_*\omega^1, t(h_t)_*X^2 \oplus t(h_t)_*\omega^2)_e
\end{aligned} \tag{3.6}$$

for $\mu, \nu = 1, 2$, where $k(1,1) = 1$, $k(1,2) = 2$, $k(2,1) = 3$, $k(2,2) = 2$. By Corollary 19.9 of the non-linear Petree theorem in [7], we may assume $H, X^1, X^2, \omega^1, \omega^2$ are polynomial of degree not more than r, where r is an arbitrary finite number. We can write

$$\begin{aligned}
(h_t)_*H &= a_2(H)t^2 + \ldots + a_{r+3}(H)t^{r+3},\\
t(h_t)_*X^1 &= b_0(X^1) + \ldots + b_{r+3}(X^1)t^{r+3},\\
t(h_t)_*\omega^1 &= c_1(\omega^1)t + \ldots + c_{r+3}(\omega^1)t^{r+3},\\
t(h_t)_*X^2 &= b_0(X^2) + \ldots + b_{r+3}(X^2)t^{r+3},\\
t(h_t)_*\omega^2 &= c_1(\omega^2)t + \ldots + c_{r+3}(\omega^2)t^{r+3}.
\end{aligned} \tag{3.7}$$

(The first above expression is because of H is a linear 3-form.) Then the homogeneous function theorem and the homogeneity condition (3.6) and the assumption $C_0 = 0$ complete the first sentence of the lemma. Moreover, they imply that $C^{1,1}_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e$ and $C^{1,2}_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e$ and $C^{2,2}_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e$ are independent of ω^1 and ω^2 for any e in question.

It remains to prove that $C^{2,1}_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e$ is independent of both ω^1 and ω^2, too. For, it is sufficient to show that $C^{2,1}_H(0 \oplus \omega^1, 0 \oplus \omega^2)_e = 0$ and $C^{2,1}_H(X^1 \oplus 0, 0 \oplus \omega^2)_e = 0$, and $C^{2,1}_H(0 \oplus \omega^1, X^2 \oplus 0)_e = 0$.

For any $\tau \in \mathbf{R}$, we can write

$$C^{2,1}_H(0 \oplus \omega^1, X^2 \oplus 0)_{\tau e} = \sum_{i=1}^{m} a_i \tau d_{\tau e} x^i,$$

where a_i are the real numbers (depending on ω^1 and X^2 and e and independent of τ). Using the invariance of C with respect to $(x^1, \dots, x^m, \frac{1}{t}y^1, \dots, \frac{1}{t}y^n)$ (preserving X^2 (as X^2 is linear) and sending H into tH (as H is linear) and ω^1 into $t\omega^1$ (as ω^1 is linear) and τe into $\frac{1}{t}\tau e$) and that $C_H(-,-)$ is linear in H, we get

$$t^2 \sum_{i=1}^{m} a_i \frac{1}{t} \tau d_{\frac{1}{t}\tau e} x^i = \sum_{i=1}^{m} a_i \tau d_{\frac{1}{t}\tau e} x^i \ .$$

Then $ta_i = a_i$, and then $a_i = 0$ for $i = 1, \dots, m$. Then $C^{2,1}_H(0 \oplus \omega^1, X^2 \oplus 0)_e = 0$, as well. The proofs of the two other equalities are quite similar.

The lemma is complete. □

Lemma 3.7. *The vector field part of $C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ is zero.*

Proof. Let $H \in \Gamma^l_{\mathbf{R}^{m,n}}(\bigwedge^3 T^*\mathbf{R}^{m,n})$ and $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_{\mathbf{R}^{m,n}}(T\mathbf{R}^{m,n} \oplus T^*\mathbf{R}^{m,n})$ and $e \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n = \mathbf{R}^{m,n}_0$. By the homogeneity condition from the proof of Lemma 3.6 and the homogeneous function theorem, we derive that $C^{1,1}_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e$ is independent of H, and then it is zero because of the assumption $C_0 = 0$. Further, for any $\tau \in \mathbf{R}$, we can write

$$C^{1,2}_H(X^1 \oplus 0, X^2 \oplus 0)_{\tau e} = \sum_{k=1}^{n} a^k \tau \frac{\partial}{\partial y^k}_{|\tau e},$$

where a^k are real numbers (depending on X^1 and X^2 and e and independent of τ). Then, using the invariance of C with respect to $\tilde{h}_t = (x^1, \dots, x^m, \frac{1}{t}y^1, \dots, \frac{1}{t}y^n)$ (preserving X^1 and X^2 as they are linear, sending H into tH as it is linear, and sending $\frac{\partial}{\partial y^k}_{|\tau e}$ into $\frac{1}{t}\frac{\partial}{\partial y^k}_{|\frac{1}{t}\tau e}$, and sending τe into $\frac{1}{t}\tau e$), since C_H is linear in H, we get

$$t \sum_{k=1}^{n} a^k \frac{1}{t} \tau \frac{\partial}{\partial y^k}_{|\frac{1}{t}\tau e} = \sum_{k=1}^{n} a^k \tau \frac{1}{t} \frac{\partial}{\partial y^k}_{|\frac{1}{t}\tau e} \ .$$

Then $a^k = 0$ for $k = 1, \dots, n$. Then, applying Lemma 3.6, we get

$$C^{1,2}_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e = C^{1,2}_H(X^1 \oplus 0, X^2 \oplus 0)_e = 0.$$

The lemma is complete. □

Lemma 3.8. *Under the assumption $m \geq 3$, C is determined by the collection*

$$\begin{aligned}
&C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}\Big(\frac{\partial}{\partial x^i}\oplus 0, \frac{\partial}{\partial x^{i_1}}\oplus 0\Big)_{e_1},\\
&C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i}\oplus 0, y^k\frac{\partial}{\partial y^{k_1}}\oplus 0\Big)_{e_1},\\
&C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(y^k\frac{\partial}{\partial y^{k_1}}\oplus 0, \frac{\partial}{\partial x^i}\oplus 0\Big)_{e_1},\\
&C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i}\oplus 0, x^3\frac{\partial}{\partial x^{i_1}}\oplus 0\Big)_{e_1},\\
&C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(x^3\frac{\partial}{\partial x^{i_1}}\oplus 0, \frac{\partial}{\partial x^i}\oplus 0\Big)_{e_1},\\
&C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i}\oplus 0, \frac{\partial}{\partial x^{i_1}}\oplus 0\Big)_{e_1}
\end{aligned} \tag{3.8}$$

for all $i, i_1 = 1, \ldots, m$ and $k, k_1 = 1, \ldots, n$, where $e_1 = (1, 0, \ldots, 0) \in \mathbf{R}^n = \mathbf{R}^{m,n}_0$.

Proof. By Lemmas 3.6 and 3.7 (and their proofs), C is determined by the collection

$$\begin{aligned}
&C_{\varphi(y)df^1(x)\wedge df^2(x)\wedge df^3(x)}(X^1\oplus 0, X^2\oplus 0)_e,\\
&C_{f^3(x)d\varphi(y)\wedge df^1(x)\wedge df^2(x)}(X^1\oplus 0, X^2\oplus 0)_e,\\
&C_{d\varphi(y)\wedge dg^1(x)\wedge dg^2(x)}(X^1\oplus 0, X^2\oplus 0)_e
\end{aligned} \tag{3.9}$$

for all $X^1, X^2 \in \Gamma^l_{\mathbf{R}^{m,n}}(T\mathbf{R}^{m,n})$ and all $e \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n \subset \mathbf{R}^{m,n}$ and all maps

$$f^1, f^2, f^3, g^1, g^2 : \mathbf{R}^m \to \mathbf{R}$$

with

$$f^1(0) = f^2(0) = f^3(0) = g^1(0) = g^2(0) = 0$$

and all linear maps $\varphi : \mathbf{R}^m \to \mathbf{R}$. Of course, we can assume $\varphi(e) = 1$ and the rank of (d_0f^1, d_0f^2, d_0f^3) is maximal and the rank of (d_0g^1, d_0g^2) is maximal. Then, using the $\mathcal{VB}_{m,n}$-invariance of C, we can assume $e = e_1$, $\varphi = y^1$, $f^1 = x^1$, $f^2 = x^2$, $f^3 = x^3$ (we use $m \geq 3$) and $g^1 = x^1$ and $g^2 = x^2$. Further, using the invariance of C with respect to $(x^1, \ldots, x^m, y^1 + x^3y^1, y^1, \ldots, y^n)^{-1}$, we can see that the values $C_{dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}$ determine the values $C_{d(y^1+x^3y^1)\wedge dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}$. Then the values $C_{dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}$ together with the values $C_{y^1dx^1\wedge dx^2\wedge dx^3}(X^1\oplus 0, X^2\oplus 0)_{e_1}$ determine the values $C_{x^3dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}$. So, the values $C_{x^3dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}$ may be omitted. So, C is determined by the collection of values

$$\begin{aligned}
&C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}(X^1\oplus 0, X^2\oplus 0)_{e_1},\\
&C^{2,2}_{y^1dx^1\wedge dx^2\wedge dx^3}(X^1\oplus 0, X^2\oplus 0)_{e_1},\\
&C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1},\\
&C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}
\end{aligned} \tag{3.10}$$

for all $\alpha, \beta, \gamma, \delta \in (\mathbf{N} \cup \{0\})^m$ and $i, i_1 = 1, \dots, m$ and $j, k, j_1, k_1 = 1, \dots, n$, where $(X^1 = x^\alpha \frac{\partial}{\partial x^i}$ or $X^1 = x^\beta y^j \frac{\partial}{\partial y^k})$ and $(X^2 = x^\gamma \frac{\partial}{\partial x^{i_1}}$ or $X^2 = x^\delta y^{j_1} \frac{\partial}{\partial y^{k_1}})$, where (of course) $x^\alpha := (x^1)^{\alpha_1} \dots (x^m)^{\alpha_m}$. We are going to study this collection (3.10).

(i) We start with $C^{2,1}_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$. If $X^1 = x^\alpha \frac{\partial}{\partial x^i}$ and $X^2 = x^\gamma \frac{\partial}{\partial x^{i_1}}$ then by invariance of C with respect to $h_t = (\frac{1}{t}x^1, \dots, \frac{1}{t}x^m, y^1, \dots, y^n)$, we get

$$t^{3+|\alpha|+|\gamma|-2} C^{2,1}_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = t C^{2,1}_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$$

and then

$$C^{2,1}_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

if $|\alpha| + |\gamma| \neq 0$. Quite similarly,

$$C^{2,1}_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

in the rest three sub-cases.

(ii) Now, we pass to $C^{2,2}_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$. If $X^1 = x^\alpha \frac{\partial}{\partial x^i}$ and $X^2 = x^\gamma \frac{\partial}{\partial x^{i_1}}$ then by invariance of C with respect to h_t (as above), we get

$$t^{3+|\alpha|+|\gamma|-2} C^{2,2}_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = C^{2,2}_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1},$$

and then we have

$$C^{2,2}_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0.$$

Quite similarly, we get

$$C^{2,2}_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

in the rest three sub-cases.

(iii) Now, we study $C^{2,1}_{dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$. If $X^1 = x^\alpha \frac{\partial}{\partial x^i}$ and $X^2 = x^\gamma \frac{\partial}{\partial x^{i_1}}$ then by the invariance of C with respect to h_t, we get

$$t^{2+|\alpha|+|\gamma|-2} C^{2,1}_{dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = t C^{2,1}_{dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1},$$

and then

$$C^{2,1}_{dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

if $|\alpha| + |\gamma| \neq 1$. Similarly, if $X^1 = x^\alpha \frac{\partial}{\partial x^i}$ and $X^2 = x^\delta y^{j_1} \frac{\partial}{\partial y^{k_1}}$ then

$$C^{2,1}_{dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

if $|\alpha| + |\delta| \neq 0$. Similarly, if $X^1 = x^\beta y^j \frac{\partial}{\partial y^k}$ and $X^2 = x^\gamma \frac{\partial}{\partial x^{i_1}}$, then

$$C^{2,1}_{dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

if $|\beta| + |\gamma| \neq 0$. Similarly,

$$C^{2,1}_{dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

in the rest sub-case. Further, we can see that the values $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^i}\oplus 0, x^{i_2}\frac{\partial}{\partial x^{i_1}}\oplus 0)_{e_1}$ are determined by the values $C^{2,1}_{dy^1\wedge df\wedge dg}(X^1\oplus 0, hX^2\oplus 0)_{e_1}$ for all "constant" vector fields X^1 and X^2 on $\mathbf{R}^m$ (treated as linear vector fields on $\mathbf{R}^{m,n}$) and all linear maps $f,g,h:\mathbf{R}^m\to\mathbf{R}$. Then (of course) we can assume that f, g, h are linearly independent (we use $m\geq 3$). Then, using the invariance of C with respect to $(\varphi(x^1,\dots,x^m),y^1,\dots,y^n)$ for a linear isomorphism $\varphi:\mathbf{R}^m\to\mathbf{R}^m$, we can assume $f=x^1$, $g=x^2$ and $h=x^3$. Because of the bi-linearity of C_H, we can else assume that $X^1=\frac{\partial}{\partial x^i}$ and $X^2=\frac{\partial}{\partial x^{i_1}}$. Quite similarly, one can proceed with $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(x^{i_2}\frac{\partial}{\partial x^{i_1}}\oplus 0,\frac{\partial}{\partial x^i}\oplus 0)_{e_1}$ instead of $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^i}\oplus 0, x^{i_2}\frac{\partial}{\partial x^{i_1}}\oplus 0)_{e_1}$.

(iv) Finally, we study $C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0,X^2\oplus 0)_{e_1}$. If $X^1=x^\alpha\frac{\partial}{\partial x^i}$ and $X^2=x^\gamma\frac{\partial}{\partial x^{i_1}}$ then by the invariance of C with respect to h_t we get

$$t^{2+|\alpha|+|\gamma|-2}C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0,X^2\oplus 0)_{e_1}=C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0,X^2\oplus 0)_{e_1},$$

and then

$$C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0,X^2\oplus 0)_{e_1}=0$$

if $|\alpha|+|\gamma|\neq 0$. Quite similarly, $C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(X^1\oplus 0,X^2\oplus 0)_{e_1}=0$ in the rest three sub-cases.

The lemma is complete. □

Lemma 3.9. *All values* $C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^i}\oplus 0,\frac{\partial}{\partial x^{i_1}}\oplus 0)_{e_1}$ *are zero except* (*eventually*) *of* $C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0)_{e_1}$ *and* $C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^2}\oplus 0,\frac{\partial}{\partial x^1}\oplus 0)_{e_1}$. *Moreover, we have*

$$\begin{aligned}C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}&=\tilde a d_{e_1}y^1,\\ C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^2}\oplus 0,\frac{\partial}{\partial x^1}\oplus 0\Big)_{e_1}&=-\tilde a d_{e_1}y^1,\end{aligned}\tag{3.11}$$

where $\tilde a$ *is the real number* (*determined by the operator* C).

Proof. We can write

$$C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i}\oplus 0,\frac{\partial}{\partial x^{i_1}}\oplus 0\Big)_{e_1}=\sum_{k=1}^n a_{ii_1k}d_{e_1}y^k,$$

where $a_{ii_1k}\in\mathbf{R}$. Then by invariance of C with respect to $(\frac{1}{\tau^1}x^1,\dots,\frac{1}{\tau^m}x^m,y^1,\dots,y^n)$ we get $\tau^1\tau^2\frac{1}{\tau^i}\frac{1}{\tau^{i_1}}a_{ii_1k}=a_{ii_1k}$. Then $a_{ii_1k}=0$ if $\{i,i_1\}\neq\{1,2\}$. Further, by the invariance of C with respect to $(x^1,\dots,x^m,y^1,\frac{1}{t}y^2,\dots,\frac{1}{t}y^n)$ we get $a_{12k}=ta_{12k}$ for $k=2,\dots,n$. Then $a_{12k}=0$ for $k=2,\dots,n$. (If $n=1$, this is trivial.) Further, by the invariance of C with respect to $(x^2,x^1,x^3,\dots,x^m,y^1,\dots,y^n)$ (replacing x^1 by x^2 and vice-versa) we get $a_{12k}=-a_{21k}$ for $k=1,\dots,n$. Summing up, all values $C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^i}\oplus 0,\frac{\partial}{\partial x^{i_1}}\oplus 0)_{e_1}$ are zero except (eventually) of $C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0)_{e_1}$ $C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^2}\oplus 0,\frac{\partial}{\partial x^1}\oplus 0)_{e_1}$. Moreover,

$$C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=-C^{2,2}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^2}\oplus 0,\frac{\partial}{\partial x^1}\oplus 0\Big)_{e_1}=a_{121}d_{e_1}y^1.$$

The lemma is complete. □

Lemma 3.10. *Let $m\ge 3$. All values $C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}(\frac{\partial}{\partial x^i}\oplus 0,\frac{\partial}{\partial x^{i_1}}\oplus 0)_{e_1}$ are equal to zero except (eventually) of $C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0)_{e_1}$ and $C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}(\frac{\partial}{\partial x^2}\oplus 0,\frac{\partial}{\partial x^1}\oplus 0)_{e_1}$ and $C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}(\frac{\partial}{\partial x^3}\oplus 0,\frac{\partial}{\partial x^1}\oplus 0)_{e_1}$ and $C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^3}\oplus 0)_{e_1}$ and $C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}(\frac{\partial}{\partial x^2}\oplus 0,\frac{\partial}{\partial x^3}\oplus 0)_{e_1}$ and $C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}(\frac{\partial}{\partial x^3}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0)_{e_1}$. Moreover, we have*

$$\begin{aligned}
C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}&=\tilde b d_{e_1}x^3,\\
C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}\Big(\frac{\partial}{\partial x^2}\oplus 0,\frac{\partial}{\partial x^1}\oplus 0\Big)_{e_1}&=-\tilde b d_{e_1}x^3,\\
C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}\Big(\frac{\partial}{\partial x^3}\oplus 0,\frac{\partial}{\partial x^1}\oplus 0\Big)_{e_1}&=\tilde b d_{e_1}x^2,\\
C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^3}\oplus 0\Big)_{e_1}&=-\tilde b d_{e_1}x^2,\\
C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}\Big(\frac{\partial}{\partial x^2}\oplus 0,\frac{\partial}{\partial x^3}\oplus 0\Big)_{e_1}&=\tilde b d_{e_1}x^1,\\
C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}\Big(\frac{\partial}{\partial x^3}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}&=-\tilde b d_{e_1}x^1,
\end{aligned}\tag{3.12}$$

where $\tilde b$ is the real number (determined by the operator C).

Proof. We can write

$$C^{2,1}_{y^1dx^1\wedge dx^2\wedge dx^3}\Big(\frac{\partial}{\partial x^i}\oplus 0,\frac{\partial}{\partial x^{i_1}}\oplus 0\Big)_{e_1}=\sum_{j=1}^m b_{ii_1j}d_{e_1}x^j,$$

where $b_{ii_1j}\in\mathbf{R}$. Then by the invariance of C with respect to base homotheties $(\frac{1}{\tau^1}x^1,\dots,\frac{1}{\tau^m}x^m,y^1,\dots,y^n)$, we get $\tau^1\tau^2\tau^3\frac{1}{\tau^i}\frac{1}{\tau^{i_1}}b_{ii_1j}=\tau^j b_{ii_1j}$. Then $b_{ii_1j}=0$ if $\{i,i_1,j\}\neq\{1,2,3\}$. Further, applying the invariance of C with respect to the permutations of x^1,x^2,x^3 we easily see that $b_{123}=-b_{213}=b_{312}=-b_{132}=-b_{321}=b_{231}$. The lemma is complete. □

Lemma 3.11. *All values $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^i}\oplus 0,y^k\frac{\partial}{\partial y^{k_1}}\oplus 0)_{e_1}$ are equal to zero except (eventually) of $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^1}\oplus 0,y^1\frac{\partial}{\partial y^1}\oplus 0)_{e_1}$ and $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^2}\oplus 0,y^1\frac{\partial}{\partial y^1}\oplus 0)_{e_1}$. Moreover, we have*

$$\begin{aligned}
C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,y^1\frac{\partial}{\partial y^1}\oplus 0\Big)_{e_1}&=\tilde c d_{e_1}x^2\\
C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^2}\oplus 0,y^1\frac{\partial}{\partial y^1}\oplus 0\Big)_{e_1}&=-\tilde c d_{e_1}x^1,
\end{aligned}\tag{3.13}$$

where $\tilde c$ is the real number (determined by the operator C).

Proof. We can write

$$C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i}\oplus 0, y^k\frac{\partial}{\partial y^{k_1}}\oplus 0\Big)_{e_1} = \sum_{j=1}^{m} c_{ikk_1j}d_{e_1}x^j,$$

where c_{ikk_1j} are the real numbers. Then by the invariance of C with respect to the base homotheties $(\frac{1}{\tau^1}x^1,\dots,\frac{1}{\tau^m}x^m,y^1,\dots,y^n)$ we get $\tau^1\tau^2\frac{1}{\tau^i}c_{ikk_1j}=\tau^j c_{ikk_1j}$. Then $c_{ikk_1j}=0$ if $\{i,j\}\neq\{1,2\}$. Further, by the invariance of C with respect to replacing x^1 by x^2 (and vice-versa) we get $c_{1kk_12}=-c_{2kk_11}$. Further, by invariance of C with respect to $(x^1,\dots,x^m,\frac{1}{\tau^1}y^1,\frac{1}{\tau^2}y^2,\dots,\frac{1}{\tau^n}y^n)$ with $\tau^1=1$, we get $\tau^k\frac{1}{\tau^{k_1}}c_{1kk_12}=c_{1kk_12}$. Then $c_{1kk_12}=0$ if $k\neq k_1$. Further, if $k\in\{2,\dots,n\}$, there exists a $\mathcal{VB}_m$-map $\psi=(x^1,\dots,x^m,y^1,\tilde{\psi}(x^2,\dots,x^m,y^2,\dots,y^n))$ sending $\frac{\partial}{\partial x^2}$ into $\frac{\partial}{\partial x^2}+y^k\frac{\partial}{\partial y^k}$. Then, using the invariance of C with respect to ψ, from

$$C_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=0\oplus\tilde{a}d_{e_1}y^1,$$

where $\tilde{a}$ is from Lemma 3.9, we get

$$C_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,\Big(\frac{\partial}{\partial x^2}+y^k\frac{\partial}{\partial y^k}\Big)\oplus 0\Big)_{e_1}=0\oplus\tilde{a}d_{e_1}y^1.$$

(That $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0)_{e_1}=0$, see the proof of Lemma 3.8.) Then

$$C_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,y^k\frac{\partial}{\partial y^k}\oplus 0\Big)_{e_1}=0\oplus 0$$

for $k=2,\dots,n$. The lemma is complete. □

Lemma 3.12. *All values* $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(y^k\frac{\partial}{\partial y^{k_1}}\oplus 0,\frac{\partial}{\partial x^i}\oplus 0)_{e_1}$ *are zero except (eventually) of* $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(y^1\frac{\partial}{\partial y^1}\oplus 0,\frac{\partial}{\partial x^1}\oplus 0)_{e_1}$ *and* $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(y^1\frac{\partial}{\partial y^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0)_{e_1}$. *Moreover, we have*

$$\begin{aligned} C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(y^1\frac{\partial}{\partial y^1}\oplus 0,\frac{\partial}{\partial x^1}\oplus 0\Big)_{e_1}&=\tilde{e}d_{e_1}x^2\\ C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(y^1\frac{\partial}{\partial y^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}&=-\tilde{e}d_{e_1}x^1,\end{aligned}\tag{3.14}$$

where $\tilde{e}$ is the real number (determined by C).

Proof. The proof is quite the same as the one of Lemma 3.11. In fact, this lemma is Lemma 3.11 for C^{op} instead of C, where

$$C^{\mathrm{op}}_H(X^1\oplus\omega^1,X^2\oplus\omega^2):=C_H(X^2\oplus\omega^2,X^1\oplus\omega^1).$$ □

Lemma 3.13. *Let $m \geq 3$. All values $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^i} \oplus 0, x^3\frac{\partial}{\partial x^{i_1}} \oplus 0)_{e_1}$ are equal to zero except (eventually) of $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^1} \oplus 0, x^3\frac{\partial}{\partial x^2} \oplus 0)_{e_1}$ and $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^2} \oplus 0, x^3\frac{\partial}{\partial x^1} \oplus 0)_{e_1}$ and $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^3} \oplus 0, x^3\frac{\partial}{\partial x^2} \oplus 0)_{e_1}$ and $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^3} \oplus 0, x^3\frac{\partial}{\partial x^1} \oplus 0)_{e_1}$. Moreover, we have*

$$\begin{aligned}
C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1} \oplus 0, x^3\frac{\partial}{\partial x^2} \oplus 0\Big)_{e_1} &= \tilde{f} d_{e_1} x^3,\\
C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^2} \oplus 0, x^3\frac{\partial}{\partial x^1} \oplus 0\Big)_{e_1} &= -\tilde{f} d_{e_1} x^3,\\
C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^3} \oplus 0, x^3\frac{\partial}{\partial x^2} \oplus 0\Big)_{e_1} &= \tilde{g} d_{e_1} x^1,\\
C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^3} \oplus 0, x^3\frac{\partial}{\partial x^1} \oplus 0\Big)_{e_1} &= -\tilde{g} d_{e_1} x^2,
\end{aligned} \tag{3.15}$$

where $\tilde{f}$ and $\tilde{g}$ are the real numbers (determined by the operator C).

Proof. We can write

$$C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i} \oplus 0, x^3\frac{\partial}{\partial x^{i_1}} \oplus 0\Big)_{e_1} = \sum_{j=1}^{m} q_{ii_1j} d_{e_1} x^j,$$

where $q_{ii_1j} \in \mathbf{R}$ are the numbers. Then by the invariance of C with respect to $(\frac{1}{\tau^1}x^1, \dots, \frac{1}{\tau^m}x^m, y^1, \dots, y^n)$ we get $\tau^1\tau^2\tau^3\frac{1}{\tau^i}\frac{1}{\tau^{i_1}}q_{ii_1j} = \tau^j q_{ii_1j}$. Then $q_{ii_1j} = 0$ if $\{i, i_1, j\} \neq \{1,2,3\}$. Further, there exists a 0-preserving embedding $\varphi : \mathbf{R} \to \mathbf{R}$ sending (the germ at 0 of) $\frac{\partial}{\partial x}$ into $\frac{\partial}{\partial x} + x\frac{\partial}{\partial x}$. Then, by the invariance of C with respect to $(x^1, x^2, \varphi(x^3), \dots, x^m, y^1, \dots, y^n)$, from

$$C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0\Big)_{e_1} = 0$$

we get

$$C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1} \oplus 0, \Big(\frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^3}\Big) \oplus 0\Big)_{e_1} = 0,$$

and then

$$C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1} \oplus 0, x^3\frac{\partial}{\partial x^3} \oplus 0\Big)_{e_1} = 0,$$

i.e. $q_{132} = 0$. Then using the invariance of C with respect to changing x^1 by x^2 (and vice-versa) we get that $q_{231} = -q_{132} = 0$ and $q_{321} = -q_{312}$ and $q_{123} = -q_{213}$. We put $\tilde{f} := q_{123}$ and $\tilde{g} := q_{321}$. The lemma is complete. □

Lemma 3.14. *Let $m \geq 3$. All values $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(x^3\frac{\partial}{\partial x^{i_1}} \oplus 0, \frac{\partial}{\partial x^i} \oplus 0)_{e_1}$ are equal to zero except (eventually) of $B^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(x^3\frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$ and $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(x^3\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0)_{e_1}$ and $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(x^3\frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0)_{e_1}$ and $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(x^3\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0)_{e_1}$. Moreover, we have*

$$\begin{aligned} C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(x^3\frac{\partial}{\partial x^2}\oplus 0, \frac{\partial}{\partial x^1}\oplus 0\Big)_{e_1} &= \tilde{h} d_{e_1} x^3, \\ C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(x^3\frac{\partial}{\partial x^1}\oplus 0, \frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1} &= -\tilde{h} d_{e_1} x^3, \\ C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(x^3\frac{\partial}{\partial x^2}\oplus 0, \frac{\partial}{\partial x^3}\oplus 0\Big)_{e_1} &= \tilde{k} d_{e_1} x^1, \\ C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(x^3\frac{\partial}{\partial x^1}\oplus 0, \frac{\partial}{\partial x^3}\oplus 0\Big)_{e_1} &= -\tilde{k} d_{e_1} x^2, \end{aligned} \tag{3.16}$$

where $\tilde{h}$ and $\tilde{k}$ are the real numbers (determined by the operator C).

Proof. The proof is almost the same as the one of Lemma 3.13. In fact, this lemma is Lemma 3.13 for C^{op} instead of C. □

Lemma 3.15. *Let $m \geq 3$. We have*

$$\tilde{f} = \tilde{a} + \tilde{c} \tag{3.17}$$

where $\tilde{a}$ is the real number from Lemma 3.9 and $\tilde{c}$ is the real number from Lemma 3.11 and $\tilde{f}$ is the number from Lemma 3.13.

Proof. Given a positive number τ, $\psi_\tau := (x^1, \frac{x^2}{1+\tau x^3}, x^3, \dots, x^m, y^1, \dots, y^n)$ preserves e_1 and $\frac{\partial}{\partial x^1}$ and sends $dy^1 \wedge dx^1 \wedge dx^2$ into $dy^1 \wedge dx^1 \wedge d(x^2 + \tau x^2 x^3)$ and $\frac{\partial}{\partial x^2}$ into $\frac{1}{1+\tau x^3}\frac{\partial}{\partial x^2}$. Moreover, by the invariance of C with respect to the base homotheties $(\frac{1}{t}x^1, \dots, \frac{1}{t}x^m, y^1, \dots, y^n)$, we can easily see that

$$C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0, \frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1} = 0.$$

Then, by the invariance of C with respect to ψ_τ, we get

$$C^{2,1}_{dy^1\wedge dx^1\wedge d(x^2+\tau x^2 x^3)}\Big(\frac{\partial}{\partial x^1}\oplus 0, \frac{1}{1+\tau x^3}\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1} = 0.$$

Then by the order argument, we get

$$C^{2,1}_{dy^1\wedge dx^1\wedge d(x^2+\tau x^2 x^3)}\Big(\frac{\partial}{\partial x^1}\oplus 0, (1-\tau x^3)\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1} = 0.$$

Then, comparing the coefficients on τ of both sides of this equality, we easily get

$$C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0, x^3\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1} = C^{2,1}_{dy^1\wedge dx^1\wedge d(x^2 x^3)}\Big(\frac{\partial}{\partial x^1}\oplus 0, \frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}. \tag{3.18}$$

Further, by the invariance of C with respect to $(x^1,\dots,x^m,\frac{1}{1+x^3}y^1,y^2,\dots,y^n)$, from

$$C_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=0\oplus\tilde{a}d_{e_1}y^1,$$

we get

$$C_{d(y^1+x^3y^1)\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=0\oplus(\tilde{a}d_{e_1}y^1+\tilde{a}d_{e_1}x^3),$$

and then

$$C^{2,1}_{d(x^3y^1)\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=\tilde{a}d_{e_1}x^3,\tag{3.19}$$

where $\tilde{a}$ is the number from Lemma 3.9.

Further, by the invariance of C with respect to $(x^1,\frac{1}{t}x^2,x^3,\dots,x^m,y^1,\dots,y^n)$, we can easily see that

$$C_{dy^1\wedge dx^1\wedge dx^3}(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0)_{e_1}=0\oplus 0.$$

Then, by the invariance of C with respect to $(x^1,\dots,x^m,\frac{1}{1+\tau x^2}y^1,y^2,\dots,y^n)$, we get

$$C_{d(y^1+\tau x^2y^1)\wedge dx^1\wedge dx^3}\Big(\frac{\partial}{\partial x^1}\oplus 0,\Big(\frac{\partial}{\partial x^2}-\frac{\tau}{1+\tau x^2}y^1\frac{\partial}{\partial y^1}\Big)\oplus 0\Big)_{e_1}=0\oplus 0,$$

and then (by the order argument and comparing the coefficients on τ) we get

$$C^{2,1}_{d(x^2y^1)\wedge dx^1\wedge dx^3}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=C^{2,1}_{dy^1\wedge dx^1\wedge dx^3}\Big(\frac{\partial}{\partial x^1}\oplus 0,y^1\frac{\partial}{\partial y^1}\oplus 0\Big)_{e_1}.$$

Further, by the invariance of C with respect to $(x^1,x^2+x^3,x^3,\dots,x^m,y^1,\dots,y^n)$, from the first equality of (3.13), we get

$$C^{2,1}_{dy^1\wedge dx^1\wedge(dx^2-dx^3)}\Big(\frac{\partial}{\partial x^1}\oplus 0,y^1\frac{\partial}{\partial y^1}\oplus 0\Big)_{e_1}=\tilde{c}d_{e_1}(x^2-x^3),$$

and then

$$C^{2,1}_{dy^1\wedge dx^1\wedge dx^3}\Big(\frac{\partial}{\partial x^1}\oplus 0,y^1\frac{\partial}{\partial y^1}\oplus 0\Big)_{e_1}=\tilde{c}d_{e_1}x^3,$$

where $\tilde{c}$ is the number from Lemma 3.11. Then

$$C^{2,1}_{d(x^2y^1)\wedge dx^1\wedge dx^3}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=\tilde{c}d_{e_1}x^3.\tag{3.20}$$

Now, from (3.15), (3.18), (3.19) and (3.20), since

$$d(x^3y^1)\wedge dx^1\wedge dx^2+d(x^2y^1)\wedge dx^1\wedge dx^3=dy^1\wedge dx^1\wedge d(x^2x^3),$$

we get

$$\tilde{f}d_{e_1}x^3=C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,x^3\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=(\tilde{a}+\tilde{c})d_{e_1}x^3,$$

as well. The lemma is complete. □

Lemma 3.16. *Let $m \geq 3$. We have*

$$\tilde{h} = -\tilde{a} + \tilde{e}, \tag{3.21}$$

where $\tilde{a}$ is the real number from Lemma 3.9 and $\tilde{e}$ is the real number from Lemma 3.12 and $\tilde{h}$ is the number from Lemma 3.14.

Proof. In fact, this lemma is Lemma 3.15 for C^{op} instead of C. So, the proof is almost the same as the one of Lemma 3.15. □

Lemma 3.17. *Let $m \geq 3$. We have*

$$\tilde{f} + \tilde{g} + \tilde{k} + \tilde{h} = 0, \tag{3.22}$$

where $\tilde{f}$ and $\tilde{g}$ are the numbers from Lemma 3.13 and $\tilde{h}$ and $\tilde{k}$ are the numbers from Lemma 3.14.

Proof. By the invariance of C with respect to $(x^1 + \tau x^3, x^2, \dots, x^m, y^1, \dots, y^n)$, from the third equality of (3.15) we get

$$C^{2,1}_{dy^1\wedge d(x^1-\tau x^3)\wedge dx^2}\Big(\Big(\frac{\partial}{\partial x^3} + \tau\frac{\partial}{\partial x^1}\Big)\oplus 0, x^3\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1} = \tilde{g}d_{e_1}(x^1 - \tau x^3),$$

and then considering the coefficients on τ and using the first equation of (3.15) we obtain

$$-C^{2,1}_{dy^1\wedge dx^3\wedge dx^2}\Big(\frac{\partial}{\partial x^3}\oplus 0, x^3\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1} + \tilde{f}d_{e_1}x^3 = -\tilde{g}d_{e_1}x^3.$$

Then using (in particular) the invariance of C with replacing x^3 by x^1 (and vice-versa) we get

$$C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0, x^1\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1} = (\tilde{g} + \tilde{f})d_{e_1}x^1. \tag{3.23}$$

Quite similarly, using (3.16) instead of (3.15) we get

$$C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(x^1\frac{\partial}{\partial x^2}\oplus 0, \frac{\partial}{\partial x^1}\oplus 0\Big)_{e_1} = (\tilde{k} + \tilde{h})d_{e_1}x^1\ . \tag{3.24}$$

(In fact, the equality (3.24) is the equality (3.23) for C^{op} instead of C.)

Next, by invariance of C with respect to $(x^1, x^2 + \tau(x^1)^2, x^3, \dots, x^m, y^1, \dots, y^n)$, from $C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}(\frac{\partial}{\partial x^1}\oplus 0, \frac{\partial}{\partial x^1}\oplus 0)_{e_1} = 0$, we get

$$C^{2,1}_{dy^1\wedge dx^1\wedge d(x^2-\tau(x^1)^2)}\Big(\Big(\frac{\partial}{\partial x^1} + 2\tau x^1\frac{\partial}{\partial x^2}\Big)\oplus 0, \Big(\frac{\partial}{\partial x^1} + 2\tau x^1\frac{\partial}{\partial x^2}\Big)\oplus 0\Big)_{e_1} = 0,$$

and then considering the coefficients on τ we get

$$\begin{aligned} &C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0, x^1\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1} \\ &+ C^{2,1}_{dy^1\wedge dx^1\wedge dx^2}\Big(x^1\frac{\partial}{\partial x^2}\oplus 0, \frac{\partial}{\partial x^1}\oplus 0\Big)_{e_1} = 0. \end{aligned} \tag{3.25}$$

From (3.25), (3.24) and (3.23) we obtain (3.22), as well. The lemma is complete. □

We continue the proof of Theorem 3.5. By Lemmas 3.6–3.17, any $\mathcal{VB}_{m,n}$-gauge-natural operator C with $C_0 = 0$ is uniquely determined by the corresponding 5-tuple $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{g}, \tilde{k})$. Further, one can easily directly compute the corresponding 5-tuples of $\mathcal{VB}_{m,n}$-gauge natural operators $0 \oplus i_{X^1} i_{X^2} H$ and $0 \oplus i_L i_{X^1} i_{X^2} dH$ and $0 \oplus i_L i_{X^2} d i_{X^1} H$ and $0 \oplus i_L i_{X^1} d i_{X^2} H$ and $0 \oplus i_L d i_{X^2} i_{X^1} H$. They are equal to $(-1, -1, 1, 0, 0)$ and $(0, -1, 0, 0, 0)$ and $(0, -1, 0, 0, 1)$ and $(0, 1, -1, 1, 0)$ and $(0, 1, -1, 0, 0)$, respectively. The determinant of the matrix of the above vectors is 1. So, the dimension argument complete Theorem 3.5. □

4. THE GENERALIZED TWISTED DORFMAN–COURANT BRACKETS WITH THE JACOBI IDENTITY IN LEIBNIZ FORM

Definition 4.1. Let C be a $\mathcal{VB}_{m,n}$-gauge-natural operator in the sense of Definition 3.1. We say that C is a generalized twisted Dorfman–Courant bracket if C_0 is the (usual) Dorfman–Courant bracket.

Corollary 4.2. *Let $m \geq 3$ and $n \geq 1$. Any generalized twisted Dorfman–Courant bracket C is of the form*

$$\begin{aligned} C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [X^1, X^2] \oplus \{&\mathcal{L}_{X^1}\omega^2 - i_{X^2} d\omega^1 \\ &+ c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH + c_3 i_L i_{X^2} d i_{X^1} H \\ &+ c_4 i_L i_{X^1} d i_{X^2} H + c_5 i_L d i_{X^2} i_{X^1} H\} \end{aligned} \tag{4.1}$$

for any $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_E(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$-object E, where c_1, c_2, c_3, c_4, c_5 are (uniquely determined by C) real numbers.

Proof. It is a immediate consequence of Theorem 3.5. □

Definition 4.3. We say that a generalized twisted Dorfman–Courant bracket C satisfies the Jacobi identity in Leibniz form for closed linear 3-forms if

$$C_H(\rho^1, C_H(\rho^2, \rho^3)) = C_H(C_H(\rho^1, \rho^2), \rho^3) + C_H(\rho^2, C_H(\rho^1, \rho^3)) \tag{4.2}$$

for all closed linear 3-forms $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ and all linear sections $\rho^i = X^i \oplus \omega^i \in \Gamma^l_E(TE \oplus T^*E)$ for $i = 1, 2, 3$ and all $\mathcal{VB}_{m,n}$-objects E.

Remark 4.4. It is well-known that the twisted Dorfman-Courant bracket (i.e. the generalized one satisfying (4.1) with $(c_1, c_2, c_3, c_4, c_5) = (1, 0, 0, 0, 0)$) satisfies the Jacobi identity in Leibniz form for closed linear 3-forms.

Lemma 4.5. *Let C be a generalized twisted Dorfman–Courant bracket of the form* (4.1). *If C satisfies the Jacobi identity in Leibniz form for closed linear* 3*-forms, then*

$$\begin{aligned}
&c_1\mathcal{L}_{X^1}i_{X^2}i_{X^3}H + c_3\mathcal{L}_{X^1}i_Li_{X^3}di_{X^2}H \\
&+ c_4\mathcal{L}_{X^1}i_Li_{X^2}di_{X^3}H + c_5\mathcal{L}_{X^1}i_Ldi_{X^3}i_{X^2}H \\
&+ c_1i_{X^1}i_{[X^2,X^3]}H + c_3i_Li_{[X^2,X^3]}di_{X^1}H \\
&+ c_4i_Li_{X^1}di_{[X^2,X^3]}H + c_5i_Ldi_{[X^2,X^3]}i_{X^1}H \\
&= -c_1i_{X^3}di_{X^1}i_{X^2}H - c_3i_{X^3}di_Li_{X^2}di_{X^1}H \\
&\quad - c_4i_{X^3}di_Li_{X^1}di_{X^2}H - c_5i_{X^3}di_Ldi_{X^2}i_{X^1}H \\
&\quad + c_1i_{[X^1,X^2]}i_{X^3}H + c_3i_Li_{X^3}di_{[X^1,X^2]}H \\
&\quad + c_4i_Li_{[X^1,X^2]}di_{X^3}H + c_5i_Ldi_{X^3}di_{[X^1,X^2]}H \\
&\quad + c_1\mathcal{L}_{X^2}i_{X^1}i_{X^3}H + c_3\mathcal{L}_{X^2}i_Li_{X^3}di_{X^1}H \\
&\quad + c_4\mathcal{L}_{X^2}i_Li_{X^1}di_{X^3}H + c_5\mathcal{L}_{X^2}i_Ldi_{X^3}i_{X^1}H \\
&\quad + c_1i_{X^2}i_{[X^1,X^3]}H + c_3i_Li_{[X^1,X^3]}di_{X^2}H \\
&\quad + c_4i_Li_{X^2}di_{[X^1,X^3]}H + c_5i_Ldi_{[X^1,X^3]}i_{X^2}H
\end{aligned} \tag{4.3}$$

for any linear vector fields X^1, X^2, X^3 and any closed linear 3*-form H on E.*

Proof. For any linear vector fields X^1, X^2, X^3 on E and any closed linear 3-form H on E, we can write

$$\begin{aligned}
C_H(X^1\oplus 0, C_H(X^2\oplus 0, X^3\oplus 0)) &= [X^1,[X^2,X^3]]\oplus\Omega, \\
C_H(C_H(X^1\oplus 0, X^2\oplus 0), X^3\oplus 0) &= [[X^1,X^2],X^3]\oplus\Theta, \\
C_H(X^2\oplus 0, C_H(X^1\oplus 0, X^3\oplus 0)) &= [X^2,[X^1,X^3]]\oplus\mathcal{T},
\end{aligned}$$

where

$$\begin{aligned}
\Omega = {} & c_1\mathcal{L}_{X^1}i_{X^2}i_{X^3}H + c_3\mathcal{L}_{X^1}i_Li_{X^3}di_{X^2}H \\
&+ c_4\mathcal{L}_{X^1}i_Li_{X^2}di_{X^3}H + c_5\mathcal{L}_{X^1}i_Ldi_{X^3}i_{X^2}H \\
&+ c_1i_{X^1}i_{[X^2,X^3]}H + c_3i_Li_{[X^2,X^3]}di_{X^1}H \\
&+ c_4i_Li_{X^1}di_{[X^2,X^3]}H + c_5i_Ldi_{[X^2,X^3]}i_{X^1}H,
\end{aligned}$$

$$\begin{aligned}
\Theta = {} & -c_1i_{X^3}di_{X^1}i_{X^2}H - c_3i_{X^3}di_Li_{X^2}di_{X^1}H \\
&- c_4i_{X^3}di_Li_{X^1}di_{X^2}H - c_5i_{X^3}di_Ldi_{X^2}i_{X^1}H \\
&+ c_1i_{[X^1,X^2]}i_{X^3}H + c_3i_Li_{X^3}di_{[X^1,X^2]}H \\
&+ c_4i_Li_{[X^1,X^2]}di_{X^3}H + c_5i_Ldi_{X^3}di_{[X^1,X^2]}H,
\end{aligned}$$

$$\begin{aligned}
\mathcal{T} = {} & c_1\mathcal{L}_{X^2}i_{X^1}i_{X^3}H + c_3\mathcal{L}_{X^2}i_Li_{X^3}di_{X^1}H \\
&+ c_4\mathcal{L}_{X^2}i_Li_{X^1}di_{X^3}H + c_5\mathcal{L}_{X^2}i_Ldi_{X^3}i_{X^1}H \\
&+ c_1i_{X^2}i_{[X^1,X^3]}H + c_3i_Li_{[X^1,X^3]}di_{X^2}H \\
&+ c_4i_Li_{X^2}di_{[X^1,X^3]}H + c_5i_Ldi_{[X^1,X^3]}i_{X^2}H.
\end{aligned}$$

If C satisfies the Jacobi identity in Leibniz form of C for closed linear 3-forms, then $\Omega = \Theta + \mathcal{T}$, i.e. (4.3). The lemma is complete. □

Lemma 4.6. *Let C be a generalized twisted Dorfman–Courant bracket of the form* (4.1). *If C satisfies the Jacobi identity in Leibniz form for closed linear* 3*-forms, then*

$$\begin{aligned} & c_3\mathcal{L}_{X^1} i_L i_{X^3} d i_{X^2} H \\ & + c_4\mathcal{L}_{X^1} i_L i_{X^2} d i_{X^3} H + c_5\mathcal{L}_{X^1} i_L d i_{X^3} i_{X^2} H \\ & + c_3 i_L i_{[X^2,X^3]} d i_{X^1} H \\ & + c_4 i_L i_{X^1} d i_{[X^2,X^3]} H + c_5 i_L d i_{[X^2,X^3]} i_{X^1} H \\ & = -c_3 i_{X^3} d i_L i_{X^2} d i_{X^1} H + \\ & \quad - c_4 i_{X^3} d i_L i_{X^1} d i_{X^2} H - c_5 i_{X^3} d i_L d i_{X^2} i_{X^1} H \\ & \quad + c_3 i_L i_{X^3} d i_{[X^1,X^2]} H \\ & \quad + c_4 i_L i_{[X^1,X^2]} d i_{X^3} H + c_5 i_L d i_{X^3} d i_{[X^1,X^2]} H \\ & \quad + c_3 \mathcal{L}_{X^2} i_L i_{X^3} d i_{X^1} H \\ & \quad + c_4 \mathcal{L}_{X^2} i_L i_{X^1} d i_{X^3} H + c_5 \mathcal{L}_{X^2} i_L d i_{X^3} i_{X^1} H \\ & \quad + c_3 i_L i_{[X^1,X^3]} d i_{X^2} H \\ & \quad + c_4 i_L i_{X^2} d i_{[X^1,X^3]} H + c_5 i_L d i_{[X^1,X^3]} i_{X^2} H \end{aligned} \tag{4.4}$$

for any linear vector fields X^1, X^2, X^3 and any closed linear 3*-form H on $\mathbf{R}^{m,n}$.*

Proof. It is well-known that the (usual) twisted Dorfman–Courant bracket satisfies the Jacobi identity in Leibniz form for closed linear 3-forms. So, we have (4.3) in the case $c_3 = c_4 = c_5 = 0$. So, formula (4.3) is equivalent to (4.4) for all linear vector fields X^1, X^2, X^3, X^4 and all closed linear 3-forms H on E. □

Lemma 4.7. *Let C be a generalized twisted Dorfman–Courant bracket of the form* (4.1). *Assume $m \geq 4$ and $n \geq 1$. If C satisfies the Jacobi identity in Leibniz form for closed linear* 3*-forms, then $c_3 = c_4 = c_5 = 0$.*

Proof. Putting linear vector fields $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = \frac{\partial}{\partial x^2}$ and $X^3 = L$ and closed linear 3-form $H = x^1 dx^1 \wedge dx^2 \wedge dy^1$ into (4.4), we get

$$\begin{aligned} & c_3 \cdot 0 + c_4 \cdot (y^1 dx^1) + c_5 \cdot (y^1 dx^1) + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\ & = -c_3 \cdot y^1 dx^1 - c_4 \cdot 0 - c_5 \cdot (-y^1 dx^1) + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\ & \quad + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0. \end{aligned}$$

Hence $c_3 = -c_4$.

Similarly, putting linear vector fields $X^1 = x^2 \frac{\partial}{\partial x^1}$ and $X^2 = \frac{\partial}{\partial x^2}$ and $X^3 = L$ and closed linear 3-form $H = dx^1 \wedge dx^2 \wedge dy^1$ into (4.4), we get

$$\begin{aligned} & c_3 \cdot 0 + c_4 \cdot y^1 dx^2 + c_5 \cdot y^1 dx^2 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\ & = -c_3 \cdot 0 - c_4 \cdot 0 - c_5 \cdot (-y^1 dx^2) + c_3 \cdot 0 + c_4 \cdot y^1 dx^2 + c_5 \cdot 0 \\ & \quad + c_3 \cdot 0 + c_4 \cdot (-y^1 dx^2) + c_5 \cdot (-y^1 dx^2) + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0. \end{aligned}$$

Hence $c_4 = -c_5$.

Similarly, putting linear vector fields $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = x^1 \frac{\partial}{\partial x^2}$ and $X^3 = \frac{\partial}{\partial x^3}$ and closed linear 3-form $H = d(x^2 x^4) \wedge dx^3 \wedge dy^1$ into (4.4), we get

$$\begin{aligned} &c_3 \cdot y^1 dx^4 + c_4 \cdot 0 + c_5 \cdot (-y^1 dx^4) + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\ &= -c_3 \cdot 0 - c_4 \cdot (y^1 dx^4 + x^4 dy^1) - c_5 \cdot 0 + c_3 \cdot y^1 dx^4 + c_4 \cdot 0 + c_5 \cdot 0 \\ &\quad + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0. \end{aligned}$$

Hence $c_4 = 0$.

Consequently, $c_3 = c_4 = c_5 = 0$, as well. □

Thus we have the following result.

Theorem 4.8. *Let $m \geq 4$ and $n \geq 1$. Any generalized twisted Dorfman–Courant bracket C satisfying the Jacobi identity in Leibniz form for closed linear 3-forms is of the form*

$$\begin{aligned} C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [X^1, X^2] \oplus \{&\mathcal{L}_{X^1}\omega^2 - i_{X^2} d\omega^1 \\ &+ c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH\} \end{aligned} \tag{4.5}$$

for any $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_E(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$-object E, where c_1, c_2 are (uniquely determined by C) real numbers.

Moreover, given $c_1, c_2 \in \mathbf{R}$, the generalized twisted Dorfman–Courant bracket C of the form (4.5) satisfies the Jacobi identity in Leibniz form for closed linear 3-forms.

Proof. The last sentence is an easy consequence of the fact that the usual twisted Dorfman–Courant bracket satisfies the Jacobi identity in Leibniz form for closed linear 3-forms. □

From Theorem 4.8, we have the following interesting natural characterization of the (usual) twisted Dorfman–Courant bracket.

Corollary 4.9. *Let $m \geq 4$ and $n \geq 1$. Any generalized twisted Dorfman–Courant bracket C_H satisfying the Jacobi identity in Leibniz form for closed linear 3-forms satisfies*

$$C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_{cH} \tag{4.6}$$

for any closed linear 3-form $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_E(TE \oplus T^*E)$, where $[[-,-]]_H$ is the usual twisted (H-twisted) Dorfman–Courant bracket and c is an arbitrary (uniquely determined by C) real number.

Remark 4.10. Roughly speaking, the above corollary means that (for $m \geq 4$) the (usual) twisted Dorfman–Courant bracket $[[-,-]]_H$ (for closed linear 3-forms H) is the unique (up to multiplication of H by a real number c) $\mathcal{VB}_{m,n}$-gauge-natural extension of the (usual) Dorfman–Courant bracket $[[-,-]]_0$ (by means of closed linear 3-forms H) satisfying the Jacobi identity in Leibniz form.

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Received: November 12, 2020.
Accepted: January 27, 2021.