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QUASILINEARIZATION METHOD FOR FINITE SYSTEMS OF NONLINEAR RL FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper the quasilinearization method is extended to finite systems of Riemann–Liouville fractional differential equations of order $0 < q < 1$. Existence and comparison results of the linear Riemann–Liouville fractional differential systems are recalled and modified where necessary. Using upper and lower solutions, sequences are constructed that are monotonic such that the weighted sequences converge uniformly and quadratically to the unique solution of the system. A numerical example illustrating the main result is given.

Keywords: fractional differential systems, lower and upper solutions, quasilinearization method.

Mathematics Subject Classification: 34A08, 34A34, 34A45.

1. INTRODUCTION

Fractional differential equations have various applications in widespread fields of science, such as in engineering [17], chemistry [19, 29, 30], physics [7, 10, 20], and others [21, 22] (we will detail some of the motivating applications at the end of this section). Initially in this paper we will recall existence results via the lower and upper solution method, which will be useful to developing our main results. Despite there being a number of existence theorems for nonlinear fractional differential equations, much as in the integer order case, this does not necessarily imply that calculating a solution explicitly will be routine, or even possible. Therefore, in this paper we construct an iterative numerical method to approximate the needed solution.

Specifically, we will construct an extension of the quasilinearization method to systems of nonlinear Riemann–Liouville (RL) fractional differential equations of order q, where $0 < q < 1$. The quasilinearization method was first developed in [4, 5, 27], but the method we construct is more closely related to those found in [26], which is a method via lower and upper solutions. This method is very similar to the monotone method in that we construct monotone sequences from linear equations based on upper and lower solutions, which converge monotonically and uniformly. However, for the quasilinearization method we require the nonlinear forcing function to be convex in x, or at least can be made convex by the addition of a function. In the process, we are guaranteed that the constructed sequences converge quadradically to the unique solution.

There are notable complications that arise when developing the quasilinearization method for RL fractional differential equations. First of all, the iterates of the constructed sequences are solutions to the linear fractional differential equation with variable coefficients. The solution of this equation is quite unwieldy, therefore we will recall existence, comparison, and inequality results for this case, including a generalized Gronwall type inequality, which will be paramount to our main result. Another complication that stems from using the RL derivative is that, in general, the sequences we construct, $\{\alpha_n\}, \{\beta_n\}$ do not converge uniformly to the unique solution, but the weighted sequences $\{t^{1-q}\alpha_n\}$, $\{t^{1-q}\beta_n\}$ converge uniformly and quadratically to $t^{1-q}x$, where x is the unique solution of the original equation. Various quasilinearization techniques have been established for the scalar nonlinear RL fractional differential equation in [12, 13, 16].

In this paper we extend the method to finite n-systems, and as such we will present necessary preliminary results for RL fractional systems where needed. We will also finish with a constructed example detailing the main results of the paper. Unfortunately we are limited in constructing such examples due to the nature of the foundations of fractional calculus. We will detail this more explicitly in the final section. However, we briefly describe an application that acts as motivation for considering this method. In control theory, fractional models have found use in the design of state estimators, specifically in nonlinear observer-based control [6, 28]. In [6] Boroujeni and Momeni considered a nonlinear fractional order system with a nonlinear fractional order observer leading to the following observer error dynamic equation:

$$D^q\tilde{x} = (A - LC)\tilde{x} + \phi(x, u) - \phi(\hat{x}, u),$$

where x is the state, $\hat{x}$ is the state estimation, u is the input, L is the proportional observer gain, ϕ is a nonlinear Lipshitz function with $\phi(0, u) = 0$, and $\tilde{x} = x - \hat{x}$ is the state estimation error. We note that this system is a special case of the general system we develop our iterative technique for, since the above system could be generalized to:

$$D^q\tilde{x} = f(u, \tilde{x}).$$

Another possible motivating application can be found in viscoelasticity models with the stress-strain relationship of energy passing through a medium is presented as a linear system [11]. RL differential equations have been found to produce convenient models for viscoelasticity [18, 23], and in [31] the following fractional order model was developed

$$\sigma(t) = E_0 D^q \varepsilon(t), \quad 0 < q < 1,$$

where σ is the strain, ε is the stress, and E_0 is a constant all depending on the nature of the material. Then in [1–3, 24] this model was extended to consider properties of multiple viscoelastic materials with the following system

$$\sigma(t) = E_0\varepsilon(t) + E_1 D^q \varepsilon(t).$$

We posit that some materials may elicit a nonlinear representation:

$$\sigma(t) = F(t,\varepsilon) + E_1 D^q \varepsilon(t),$$

and if so, the quasilinearization method, or other generalizations of the monotone method, could prove helpful in approximating solutions. We hope that the method developed herein will lead the way to approximating solutions of models such as those discussed above, or others similar to these motivating applications.

2. PRELIMINARY RESULTS

In this section we will recall definitions and results that will be used in our main methods. Let $T > 0$, we will be developing our results on the half open interval $J = (0, T]$. We will let J_0 be the closure of J. We will also be focusing on the RL derivative of order q, $0 < q < 1$, further, let $p = 1 - q$. When solving RL differential equations of this order we will be looking in the following space of functions.

Definition 2.1. A function $f \in C(J, \mathbb{R})$ is C_p continuous if $t^p f(t) \in C(J_0, \mathbb{R})$. We will use $C_p(J, \mathbb{R})$ to denote all C_p continuous functions over J. For simplicity we will sometimes use the notation f_p to denote the weighted function $t^p f$.

In most cases the functions that naturally occur as solutions in RL differential equations are C_p in that there is a weak singularity at the left-most endpoint. Thus many of our results involve using t^p as a factor to use properties regarding compact intervals. Further, if a function is C_p then the qth order RL derivative exists, see [21]. We give the definition of the RL derivative and integral below.

Definition 2.2. Let $f \in C_p(J, \mathbb{R})$, then the RL integral of order q is given by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

and the RL derivative of order q is given by

$$D^q f(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_0^t (t-s)^{-q} f(s) ds.$$

We refer the reader to [21] for more details.

In our main results we will be constructing a method to approximate solutions of finite n-systems of nonlinear RL initial value problems of the form,

$$D^q x = f(t,x), \quad \lim_{t\to 0^+} t^p x(t) = z, \tag{2.1}$$

where $x \in C_p(J_0, \mathbb{R}^n)$, $f \in C(J_0 \times \mathbb{R}^n, \mathbb{R}^n)$, and $z \in \mathbb{R}^n$. Thanks to our above simplified notation we can write the above initial value as $x_p(0) = z$, which we will do going forward. In [21] and [25] it was shown that this IVP is equivalent to the Volterra fractional integral equation, which we give specifically in the following theorem.

Theorem 2.3. *Let $f \in C(J_0 \times \mathbb{R}^n, \mathbb{R}^n)$, then $x \in C_p(J_0, \mathbb{R}^n)$ satisfies (2.1) if and only if it satisfies the Volterra fractional integral equation*

$$x(t) = zt^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x)ds. \tag{2.2}$$

The Mittag–Leffler function is paramount in the subject of RL Calculus since it behaves as a generalized exponential function. Here we give the definition of the Mittag–Leffler function.

Definition 2.4. The Mittag–Leffler function with parameters $a, b \in \mathbb{R}$ is given as

$$E_{a,b}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(ak+b)},$$

and is entire for $a, b > 0$.

The next result gives us that the q-th R-L integral of a C_p continuous function is also a C_p continuous function. This result will give us that the solutions of R-L differential equations are also C_p continuous.

Lemma 2.5. *Let $f \in C_p(J, \mathbb{R})$, then $I_t^q f(t) \in C_p(J, \mathbb{R})$, i.e. the q-th integral of a C_p continuous function is C_p continuous.*

Note the proof of this theorem for $q \in \mathbb{R}^+$ can be found in [15]. Now we consider results for the nonhomogeneous linear R-L differential equation. In our main results we need a matrix formulation of the weighted Mittag–Leffler function given in the next definition.

Definition 2.6. Let $A = [a_{ij}]$ be an $n \times n$ matrix where each $a_{ij} \in \mathbb{R}$, and $t > 0$, then the matrix q-weighted Mittag–Leffler function is given by

$$\mu(At) = t^{q-1} E_{q,q}(At^q) = \sum_{k=0}^{\infty} A^k \frac{t^{qk+q-1}}{\Gamma(qk+q)},$$

and converges uniformly on compacta of J. See [21] for more discussion on this function.

The importance of μ follows from the fact that

$$D^q \mu(At) = A\mu(At), \tag{2.3}$$

which leads us to the result of the following theorem.

Theorem 2.7. *Consider the linear RL fractional system given by*

$$D^q x(t) = Ax(t) + f(t), \quad \lim_{t \to 0^+} t^p x(t) = z, \tag{2.4}$$

where A is a fixed real $n \times n$ matrix, $z \in \mathbb{R}^n$, and $x, f \in C_p(J, \mathbb{R}^n)$. Then system (2.4) *has a unique solution given as*

$$x(t) = z\mu(At) + \int_0^t \mu\big(A(t-s)\big) f(s) ds. \tag{2.5}$$

The solution to system (2.4) was developed and presented in [8, 9] where x, f were measurable functions over J_0. We note that C_p functions can easily be extended to measurable functions over J_0 by setting the extensions $\tilde{x}(t), \tilde{f}(t) = x(t), f(t)$ for $t > 0$, and $\tilde{x}(0) = \tilde{f}(0) = 0$. Therefore Theorem 2.7 is a special case. Similarly, the case with variable coefficients

$$D^q x(t) = A(t)x(t) + f(t), \quad x_p(0) = z, \tag{2.6}$$

where $A(t) = [a_{ij}(t)]$ is a continuous $n \times n$ matrix function over J_0, has a unique solution on J. See [21] for more details.

The quasilinearization method is an iterative technique that is generated from lower and upper solutions. The remainder of this section will focus on definitions and theorems pertinent to the utilization of lower and upper solutions of (2.1).

Definition 2.8. $\alpha, \beta \in C_p(J_0, \mathbb{R}^n)$ are lower and upper solutions of (2.1) respectively if

$$\begin{aligned} D^q \alpha \le f(t, \alpha), \quad \alpha_p(0) = z_\alpha \le z, \\ D^q \beta \ge f(t, \beta), \quad \beta_p(0) = z_\beta \ge z. \end{aligned}$$

Counter intuitively, lower solutions are not guaranteed to live below upper solutions. In fact, the following theorem gives us a condition that ensures this desired result. While this theorem is a result for lower and upper solutions, it is essential for proving the sequences we construct are monotonic. The first condition required for this result is a generalization of function monotonicity that we now define.

Definition 2.9. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be quasimonotone increasing if for each i, $y \le x$ and $y_i = x_i$ implies $f_i(y) \le f_i(x)$. Naturally, f is quasimonotone decreasing if we reverse the inequalities.

Now if f also possesses a one-sided Lipschitz condition we can employ the following comparsion theorem.

Theorem 2.10 (Comparison Theorem). *Let $f \in C(J_0 \times \mathbb{R}^n, \mathbb{R}^n)$ and let $\alpha, \beta \in C_p(J_0, \mathbb{R}^n)$ be lower and upper solutions of* (2.1)*, then if f is quasimonotone increasing in x and satisfies the following one-sided Lipschitz condition*

$$f_i(t,x) - f_i(t,y) \le L_i \sum_{k=1}^{n} (x_k - y_k),$$

when $y \le x$, then $\alpha \le \beta$ on J.

This comparison theorem is a special case of the one found in [14], and yields a Gronwall-type inequality given in the following corollary.

Corollary 2.11. *If A is a fixed real $n \times n$ matrix, $\alpha, f \in C_p(J_0, \mathbb{R}^n)$, and if*

$$D^q \alpha(t) \le A\alpha(t) + f(t), \quad \alpha_p(0) \le z,$$

then

$$\alpha(t) \le z\mu(At) + \int_0^t \mu\big(A(t-s)\big) f(s) ds.$$

Now, we will recall a result that gives us existence of a solution to (2.1) via lower and upper solutions.

Theorem 2.12. *Let $\alpha, \beta \in C_p(J, \mathbb{R}^n)$ be lower and upper solutions of* (2.1) *such that $\alpha(t) \le \beta(t)$ on J and let $f \in C(\Omega, \mathbb{R}^n)$, where Ω is defined as*

$$\Omega = \{(t,y) \, : \, \alpha(t) \le y \le \beta(t), t \in J\}.$$

Then there exists a solution $x \in C_p(J, \mathbb{R}^n)$ of (2.1) *such that $\alpha(t) \le x(t) \le \beta(t)$ on J.*

The proof of this Theorem can be found in [15] with minor additions needed to generalize it to systems. From here we have the necessary results to construct our main result which we will do in the next section.

3. QUASILINEARIZATION METHOD

In this section we will construct the quasilinearization method for (2.1). The basis of this method is built from the lower and upper solutions of (2.1). Here we assume f is twice differentiable in x and that there exists a function g such that that $f_{xx} + g_{xx} \ge 0$. So we do not require f to be convex, but instead can be made that way by another function. We note that such a function always exists, since we are assuming that f_{xx} is continuous over a compact set we can choose M such that $|f_{xx}| \le M$. Then letting $g = Mx^2$ will yield $f_{xx} + g_{xx} = f_{xx} + 2M > 0$.

The proof that the constructed sequences converge quadradically will utilize a generalized Mean Value Theorem. In order to simplify this proof we present this specific Mean Value Theorem in the following remark. Then we will give our main theorem.

Remark 3.1. In our main result we will be using a generalized Mean Value Theorem. Specifically, if $f \in C^{0,2}(J \times \mathbb{R}^n, \mathbb{R}^n)$, $f_{xx} \geq 0$, and $u \geq v$, then

$$f(t,u) - f(t,v) = \left[\int_0^1 f_x(t, su + (1-s)v) ds \right] \cdot (u - v) \leq [f_x(t,u)](u - v).$$

Here, and going forward, $[f_x]$ is the Jacobian matrix of f. Further, we will need another application of this Mean Value Theorem involving $[f_x(u) - f_x(v)]$, specifically,

$$[f_x(u) - f_x(v)](u - v) = (u - v)^T \left[\int_0^1 f_{xx}(t, su + (1-s)v) ds \right] (u - v),$$

where T represents the transpose of a matrix and $[f_{xx}]$ is the Hessian of f, which in this case is an n-array of Hessian matrices $(Hf_1, Hf_2, Hf_3, \dots, Hf_n)$. Further, given f and a vector c, the Hessian expression above can be expressed in terms of a sum of Hessian matrices:

$$c^T [f_{xx}] c = \sum_{i=1}^n c^T [Hf_i] c,$$

which is a formulation we will utilize in our main result which is given below.

Theorem 3.2. *Suppose the following hypotheses hold.*

(H_1) $\alpha_0, \beta_0 \in C_p(J, \mathbb{R})$ *are lower and upper solutions of* (2.1) *such that* $\alpha_0 \leq \beta_0$ *on* J.

(H_2) $f, g \in C^{0,2}(\Omega, \mathbb{R}^n)$, f, g *are quasimonotone increasing in* x *for* $t \in J$, *and* $f_{xx} + g_{xx} \geq 0$ *and* $g_{xx} \geq 0$ *on* Ω, *where*

$$\Omega = \{(t,u) : \alpha_0 \leq u \leq \beta_0, t \in J\}.$$

(H_3) $a_{ij}(t, \alpha_0, \beta_0) \geq 0$ *for* $i \neq j$ *where* $A(t, \alpha_0, \beta_0) = [a_{ij}(t, \alpha_0, \beta_0)]$ *is an* $n \times n$ *matrix given by*

$$A(t, \alpha_0, \beta_0) = f_x(t, \alpha_0) + g_x(t, \alpha_0) - g_x(t, \beta_0).$$

Then there exist monotone sequences $\{\alpha_k\}$, $\{\beta_k\}$ *where the* p*-weighted sequences* $\{t^p \alpha_k\}$, $\{t^p \beta_k\}$ *converge uniformly to* $t^p x$, *where* x *is the unique solution of* (2.1).

Proof. To begin, note that by (H_2) we have that x exists and is unique, further (H_2) also implies that f has the one-sided Lipschitz condition from the Comparison Theorem. Applying (H_2) and the Mean Value Theorem for any $u, v \in \Omega$, $u \geq v$ we have

$$\begin{aligned} f(t,u) &\geq f(t,v) + \big[f_x(t,v) + g_x(t,v)\big](u - v) - \big[g(t,u) - g(t,v)\big] \\ &\geq f(t,v) + \big[f_x(t,v) + g_x(t,v) - g_x(t,u)\big](u - v) \\ &= f(t,v) + [A(t,u,v)](u - v). \end{aligned} \tag{3.1}$$

Further $A(t, \xi, \eta)$ has a mixed monotone property, in that A is increasing in ξ and decreasing in η. This is proven directly from the fact that (H_2) implies $f_x + g_x$ is increasing and $-g_x$ is decreasing. This implies that for u, v defined above that

$$a_{ij}(t, u, v) \geq a_{ij}(t, \alpha_0, \beta_0) \geq 0$$

for $i \neq j$. Further, we can finally refine (3.1) to yield

$$f(t, u) \geq f(t, v) + [A(t, v, u)](u - v), \tag{3.2}$$

for $\alpha_0 \leq v \leq u \leq \beta_0$.

From here the sequences $\{\alpha_k\}$, $\{\beta_k\}$ will be defined as the solutions to the linear RL fractional systems of the form

$$D^q\alpha_{k+1} = f(t, \alpha_k) + [A(t, \alpha_k, \beta_k)](\alpha_{k+1} - \alpha_k), \qquad \alpha^p_{k+1}(0) = z, \tag{3.3}$$

$$D^q\beta_{k+1} = f(t, \beta_k) + [A(t, \alpha_k, \beta_k)](\beta_{k+1} - \beta_k), \qquad \beta^p_{k+1}(0) = z, \tag{3.4}$$

for $k \geq 0$, where α_0, β_0 are the lower and upper solutions defined in the hypotheses. We note that the solutions of these systems exist and are unique since they are the form of (2.6).

First we will consider α_1, defined as the solution to the system

$$D^q\alpha_1 = f(t, \alpha_0) + [A(t, \alpha_0, \beta_0)](\alpha_1 - \alpha_0). \tag{3.5}$$

Now note that by definition

$$D^q\alpha_0 \leq f(t, \alpha_0) + [A(t, \alpha_0, \beta_0](\alpha_0 - \alpha_0),$$

and by applying (3.2) we obtain

$$D^q\beta_0 \geq f(t, \alpha_0) + [A(t, \alpha_0, \beta_0)](\beta_0 - \alpha_0).$$

Thus, by the Comparison Theorem, $\alpha_0 \leq \alpha_1 \leq \beta_0$ on J. Similarly we can prove that $\alpha_0 \leq \beta_1 \leq \beta_0$ on J.

Now, using (3.1) and the monotone property of A we have

$$D^q x \geq f(t, \alpha_0) + [A(t, \alpha_0, x)](x - \alpha_0) \geq f(t, \alpha_0) + [A(t, \alpha_0, \beta_0)](x - \alpha_0).$$

By the Comparison Theorem we have $\alpha_1 \leq x$ on J. Similarly we can prove that $x \leq \beta_1$ on J. So we have that

$$\alpha_0 \leq \alpha_1 \leq x \leq \beta_1 \leq \beta_0$$

on J. Using this as our basis step we can inductively prove that

$$\alpha_{k-1} \leq \alpha_k \leq x \leq \beta_k \leq \beta_{k-1}$$

on J for all $k \geq 1$. Each step of this induction follows in an analogous manner to what was done previously. Specifically, using the mixed monotone property of A along with the Comparison Theorem. This gives us that the sequences $\{\alpha_k\}$, $\{\beta_k\}$ are monotone.

Now we will use the Arzelá–Ascoli Theorem to prove that the weighted sequences $\{t^p\alpha_k\}$, $\{t^p\beta_k\}$ converge uniformly. To do so we note that both weighted sequences are uniformly bounded since

$$|t^p\alpha_{ki}| \le |t^p(\alpha_{ki} - \alpha_{0i})| + |t^p\alpha_{0i}| \le |t^p(\beta_{0i} - \alpha_{0i})| + |t^p\alpha_{0i}|,$$
$$|t^p\beta_{ki}| \le |t^p(\beta_{ki} - \alpha_{0i})| + |t^p\alpha_{0i}| \le |t^p(\beta_{0i} - \alpha_{0i})| + |t^p\alpha_{0i}|,$$

for each $k \in \mathbb{N}$ and for each $i \in \{1, 2, 3, \dots, n\}$. From here we need to show that the weighted sequences are equicontinuous. We will do so with a technique similar to [14]. For simplicity we introduce the notation

$$F(t, \alpha_{k+1}) = f(t, \alpha_k) + [A(t, \alpha_k, \beta_k)](\alpha_{k+1} - \alpha_k)$$

for each $k \ge 0$. Note that since f and A are continuous over J_0 and each α_k is C_p, and since the weighted sequences are uniformly bounded, there exists a bound $M \in \mathbb{R}^n$ such that

$$\left|t^p F(t, \alpha_{k+1})\right| \le M$$

for $t \in J_0$ and for each $k \ge 0$. For simplicity we are using the absolute value bars to mean

$$|x| = (|x_1|, |x_2|, |x_3|, \dots, |x_n|).$$

Further in our result we will need a property regarding the function ϕ defined as

$$\phi(t) = \tfrac{1}{\Gamma(q)} t^p (t - s)^{q-1}.$$

We note that ϕ is decreasing in t, provided $t > s \ge 0$. To show this note that the derivative

$$\tfrac{d}{dt}\phi(t) = -\frac{s}{\Gamma(q)}(1 - q)t^{-q}(t - s)^{q-2} \le 0.$$

For our first case consider, without loss of generality, that $t \ge \tau > 0$. Then applying Theorem 2.3 we obtain

$$\begin{aligned} |t^p\alpha_{k+1}(t) - \tau^p\alpha_{k+1}(\tau)| &= \left| \int_0^t \phi(t)F(s, \alpha_{k+1})ds - \int_0^\tau \phi(\tau)F(s, \alpha_{k+1})ds \right| \\ &\le \int_\tau^t \phi(t)\left|F(s, \alpha_{k+1})\right|ds + \int_0^\tau |\phi(t) - \phi(\tau)|\left|F(s, \alpha_{k+1})\right|ds \\ &\le M \int_\tau^t \phi(t)s^{q-1}ds + M \int_0^\tau [\phi(\tau) - \phi(t)]s^{q-1}ds. \end{aligned} \tag{3.6}$$

We will consider the integrals in the final step above separately. For the first integral we note that $s \geq \tau$ implies that

$$\begin{aligned} M\int_{\tau}^{t}\phi(t)s^{q-1}ds &= M\frac{t^p}{\Gamma(q)}\int_{\tau}^{t}(t-s)^{q-1}s^{q-1}ds \leq M\frac{t^p\tau^{q-1}}{\Gamma(q)}\int_{\tau}^{t}(t-s)^{q-1}ds \\ &= \frac{M}{\Gamma(q+1)}\left[\frac{t}{\tau}\right]^p(t-\tau)^q \end{aligned} \tag{3.7}$$

Now, for simplicity name the second integral from (3.6) $\mathcal{I}$, we will employ the RL-integral power rule along with the Beta Function

$$B(q,q) = \int_0^1 (1-u)^{q-1}u^{q-1}du = \frac{\Gamma(q)\Gamma(q)}{\Gamma(2q)}.$$

From this and using the transformation $tu = s$, $\mathcal{I}$ becomes

$$\begin{aligned} \mathcal{I} &= \frac{M}{\Gamma(q)}\left[\tau^p\int_0^{\tau}(\tau-s)^{q-1}s^{q-1}ds - t^p\int_0^{\tau}(t-s)^{q-1}s^{q-1}ds\right] \\ &= \frac{M}{\Gamma(q)}\left[B(q,q)\tau^q - t^q\int_0^{\tau/t}(1-u)^{q-1}u^{q-1}du\right]. \end{aligned}$$

From here we will add and subtract $t^pB(q,q)$, and use the fact that $u \geq \tau/t$ to obtain

$$\begin{aligned} \mathcal{I} &= \frac{M}{\Gamma(q)}\left[B(q,q)\tau^q - t^pB(q,q) + t^q\int_{\tau/t}^{1}(1-u)^{q-1}u^{q-1}du\right] \\ &\leq \frac{M}{\Gamma(q)}t^p(\tau/t)^{q-1}\int_{\tau/t}^{1}(1-u)^{q-1}du = \frac{M}{\Gamma(q+1)}\left[\frac{t}{\tau}\right]^p(t-\tau)^q. \end{aligned}$$

Now, in the proof of equicontinuity, we can freely choose a δ small enough where $0 < t-\tau < \delta$ and $1 \leq (t/\tau)^p \leq 2^p$. Giving us finally that

$$|t^p\alpha_{k+1}(t) - \tau^p\alpha_{k+1}(\tau)| \leq \frac{2^{p+1}M}{\Gamma(q+1)}(t-\tau)^q.$$

For our second case consider when $\tau = 0$. Here we get

$$|t^p\alpha_{k+1}(t) - z| \leq M\frac{t^p}{\Gamma(q)}\int_0^t(t-s)^{q-1}s^{q-1}ds = M\frac{\Gamma(q)}{\Gamma(2q)}t^q.$$

So finally, letting $\mathcal{M} = \max\left\{\frac{2^{p+1}M}{\Gamma(q+1)}, \frac{M\Gamma(q)}{\Gamma(2q)}\right\}$ we get

$$|t^p\alpha_{k+1}(t) - \tau^p\alpha_{k+1}(\tau)| \leq \mathcal{M}|t-\tau|^q$$

for all $t \geq \tau \geq 0$. Since this expression does not depend on k, the remainder of proof of equicontinuity for $\{t^p\alpha_k\}$ is routine. Equicontinuity for $\{t^p\beta_k\}$ follows in an analogous manner. So by the Arzelá–Ascoli Theorem we have that both weighted sequences converge uniformly on J_0. Suppose that $t^p\alpha, t^p\beta$ are the limits of the weighted sequences respectively. This also gives us that $\alpha_k \to \alpha$, $\beta_k \to \beta$ pointwise on J. Then due to these convergence properties, the continuity properties of f and A, and Theorem 2.2 we have that

$$\begin{aligned} t^p\alpha &= \lim\left(z + \frac{t^p}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[f(s,\alpha_k) + [A(s,\alpha_k,\beta_k)](\alpha_{k+1}-\alpha_k)\big]ds\right) \\ &= z + \frac{t^p}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[f(s,\alpha)\big]ds. \end{aligned}$$

Implying that α satisfies (2.1) and therefore $\alpha = x$, and similarly $\beta = x$, since x is the unique solution of (2.1).

Now we will prove that the weighted sequences converge quadradically. To do so, consider the sequence $\gamma_k = \beta_k - \alpha_k$. Further note, since f and g are continuous over J_0 and x is C_p there exists a function $\mathcal{F}$ such that

$$\mathcal{F}(t, t^px) = f(t,x) + g(t,x),$$

implying that

$$f_{xx}(t,x) + g_{xx}(t,x) = t^{2p}\mathcal{F}_{xx}(t,t^px).$$

Using the Mean Value Theorem, i.e. Remark 3.1, and the monotone properties of f_x, we get

$$\begin{aligned} D^q\gamma_{k+1} &\leq f_x(t,\beta_k)\gamma_k + [A(t,\alpha_k,\beta_k)](\gamma_{k+1}-\gamma_k) \\ &\leq [f_x(t,\beta_k) - f_x(t,\alpha) + g_x(t,\beta_k) - g_x(t,\alpha_k)]\gamma_k + [A(t,\alpha_k,\beta_k)]\gamma_{k+1} \\ &\leq \gamma_k^T\left[\int_0^1 (f+g)_{xx}\big(t, c\beta_k + (1-c)\alpha_k\big)dc\right]\gamma_k + N\gamma_{k+1} \\ &= \gamma_k^T\left[\int_0^1 t^{2p}\mathcal{F}_{xx}\big(t, ct^p\beta_k + (1-c)t^p\alpha_k\big)dc\right]\gamma_k + N\gamma_{k+1} \\ &\leq Qt^{2p}(\gamma_k\cdot\gamma_k) + N\gamma_{k+1}. \end{aligned}$$

In the above steps N and Q are $n\times n$ matrices such that N is invertible, and an upper bound of $[A(t,\alpha_k,\beta_k)]$, and $\sum_{i=1}^n (H\mathcal{F}_i) \leq Q$. Then by Corollary 2.11 we have that

$$t^p\gamma_{k+1} \leq t^p\int_0^t \mu\big(N(t-s)\big)Qs^{2p}(\gamma_k\cdot\gamma_k)ds,$$

which implies that

$$||t^p\gamma_{k+1}|| \le Q||t^p\gamma_k||^2 t^p \int_0^t \mu\big(N(t-s)\big)ds = Q||t^p\gamma_k||^2 t^p \int_0^t \sum_{k=0}^{\infty} \frac{N^k(t-s)^{kq+q-1}}{\Gamma(qk+q)}$$
$$= Q||t^p\gamma_k||^2 t^p \sum_{k=0}^{\infty} \frac{N^k t^{q(k+1)}}{\Gamma(q(k+1)+1)} \le N^{-1}T^p E_{q,1}(NT^q)Q||t^p\gamma_k||^2.$$

The above norm is $||t^p\gamma|| = \max_{J_0}|t^p\gamma|$, where $|\cdot|$ is as used previously. Further, $\|t^p\gamma\|^2 = (\max_{J_0}|t^p\gamma_i|^2)_{i=1}^n$. Finally this gives that $\{t^p\gamma_n\}$ converges quadratically to zero and thus completes the proof. □

We note that this acts as a generalization to the standard quasilinearization method, since if f is convex then we can simply choose $g=0$. Further, even though convergence is quadratic, which is stronger than convergence in the monotone method, the computation of each iterate is far more computer intensive than iterates from the monotone method. We will exemplify this conundrum with the following illustrative example.

Example 3.3. Let $J=(0,1]$ and consider the system

$$D^{1/2}x_1 = \tfrac{1}{2} + \tfrac{5}{8}t + \tfrac{1}{16}x_2^2, \quad \lim_{t\to 0^+} t^{1/2}x_1(t) = 0,$$
$$D^{1/2}x_2 = \tfrac{1}{10} + \tfrac{1}{6}t + \tfrac{1}{20}x_1^3, \quad \lim_{t\to 0^+} t^{1/2}x_2(t) = 0.$$

Then

$$f(t,x_1,x_2) = \begin{bmatrix} \frac{1}{2} + \frac{5}{8}t + \frac{1}{16}x_2^2 \\ \\ \frac{1}{10} + \frac{1}{6}t + \frac{1}{20}x_1^3 \end{bmatrix},$$

where f is quasimonotone increasing in $x=(x_1,x_2)$ for $t\in J$, and $g\equiv 0$. Also, $f_{xx}+g_{xx}\ge 0$ and (H_2) is satsified.

Now we will denote

$$f_1(t,x_1,x_2) = \tfrac{1}{2} + \tfrac{5}{8}t + \tfrac{1}{16}x_2^2$$

and

$$f_2(t,x_1,x_2) = \tfrac{1}{10} + \tfrac{1}{6}t + \tfrac{1}{20}x_1^3,$$

so $(f_1)_{x_1}=0$, $(f_1)_{x_2}=\frac{1}{8}x_2$, $(f_2)_{x_1}=\frac{3}{20}x_1^2$, and $(f_2)_{x_2}=0$.

Consider the functions $\alpha_0=(\alpha_{0,1},\alpha_{0,2})$ and $\beta_0=(\beta_{0,1},\beta_{0,2})$, where $\alpha_{0,1}=\frac{\sqrt{t}}{2}$, $\alpha_{0,2}=0$, $\beta_{0,1}=3$, and $\beta_{0,2}=3-\frac{t}{16}$. Then

$$\lim_{t\to 0^+} t^{1/2}\alpha_0(t) \le 0, \quad \lim_{t\to 0^+} t^{1/2}\beta_0(t) \ge 0,$$

and for $t\in J$,

$$\tfrac{\sqrt{\pi}}{4} = D^{1/2}\alpha_{0,1} \le f_1(t,\alpha_{0,1},\alpha_{0,2}) = \frac{1}{2} + \frac{5}{8}t,$$
$$0 = D^{1/2}\alpha_{0,2} \le f_2(t,\alpha_{0,1},\alpha_{0,2}) = \frac{1}{10} + \frac{1}{6}t + \frac{t^{3/2}}{160},$$

and

$$\frac{3}{\sqrt{\pi t}} = D^{1/2}\beta_{0,1} \geq f_1(t,\beta_1,\beta_2) = \frac{1}{2} + \frac{5}{8}t + \frac{1}{16}\Big(3 - \frac{1}{16}t\Big)^2,$$
$$\frac{24-t}{8\sqrt{\pi t}} = D^{1/2}\beta_{0,2} \geq f_2(t,\beta_1,\beta_2) = \frac{1}{10} + \frac{1}{6}t + \frac{27}{20},$$

that is, α_0, β_0 are lower and upper solutions for the system with $\alpha_0 \leq \beta_0$ which satisfies (H_1).

We can now compute A and $\mu(A(t))$. In fact

$$A(t) = \begin{bmatrix} (f_1)_{x_1}(t,\alpha_{0,1},\alpha_{0,2}) & (f_1)_{x_2}(t,\alpha_{0,1},\alpha_{0,2}) \\ (f_2)_{x_1}(t,\alpha_{0,1},\alpha_{0,2}) & (f_2)_{x_2}(t,\alpha_{0,1},\alpha_{0,2}) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{8}\alpha_{0,2} \\ \frac{3}{20}\alpha_{0,1}^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{3}{80}t & 0 \end{bmatrix}.$$

Also,

$$\mu\left(A(t)\right) = t^{q-1}E_{q,q}\left(A(t^q)\right) = \sum_{k=0}^{\infty} A(t)^k \frac{t^{qk+q-1}}{\Gamma(qk+q)}.$$

Observe that $A(t)^k = 0$ for $k \geq 2$, since $q = \frac{1}{2}$, then

$$\mu\left(A(t)\right) = \frac{t^{-1/2}}{\Gamma(\frac{1}{2})} + A(t^{1/2})\frac{1}{\Gamma(1)} = \begin{bmatrix} \frac{t^{-1/2}}{\sqrt{\pi}} & 0 \\ \frac{3}{80}t^{1/2} & \frac{t^{-1/2}}{\sqrt{\pi}} \end{bmatrix}.$$

Finally, the solution of

$$D^{1/2}x(t) = A(t)x(t) + f(t), \quad \lim_{t\to 0^+} t^{1/2}x(t) = 0,$$

is given by

$$x(t) = \int_0^t \begin{bmatrix} \frac{(t-s)^{-1/2}}{\sqrt{\pi}} & 0 \\ \frac{3}{80}(t-s)^{1/2} & \frac{(t-s)^{-1/2}}{\sqrt{\pi}} \end{bmatrix} f(s)ds.$$

We now use this formula to compute the first iterates α_1 and β_1.

Let us compute α_1 first,

$$D^{1/2}\alpha_1 = \begin{bmatrix} \frac{1}{2} + \frac{5}{8}t \\ \frac{1}{10} + \frac{1}{6}t - \frac{t^{3/2}}{80} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{3}{80}t & 0 \end{bmatrix}\begin{bmatrix} \alpha_{1,1} \\ \alpha_{1,2} \end{bmatrix}.$$

Then, from (2.5),

$$\alpha_1 = \int_0^t \begin{bmatrix} \frac{(t-s)^{-1/2}}{\sqrt{\pi}} & 0 \\ \frac{3}{80}(t-s)^{1/2} & \frac{(t-s)^{-1/2}}{\sqrt{\pi}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} + \frac{5}{8}s \\ \frac{1}{10} + \frac{1}{6}s - \frac{1}{80}s^{3/2} \end{bmatrix} ds$$
$$= \begin{bmatrix} \frac{\sqrt{t}(6+5t)}{6\sqrt{\pi}} \\ \frac{\sqrt{t}\left(t\left(36\sqrt{\pi}(t+2) - 27\pi\sqrt{t} + 1280\right) + 1152\right)}{5760\sqrt{\pi}} \end{bmatrix}.$$

Similarly, we get from (2.5) that

$$\beta_1 = \int_0^t \begin{bmatrix} \frac{(t-s)^{-1/2}}{\sqrt{\pi}} & 0 \\ \frac{3}{80}(t-s)^{1/2} & \frac{(t-s)^{-1/2}}{\sqrt{\pi}} \end{bmatrix} \begin{bmatrix} \frac{17}{16} + \frac{77}{128}s + \frac{s^2}{4096} \\ \frac{29}{20} + \frac{1}{6}s \end{bmatrix} ds$$

$$= \begin{bmatrix} \frac{\sqrt{t}(t(t+3080)+8160)}{3840\sqrt{\pi}} \\ \frac{\left(t^2+4928t+26112\right)t^2}{1310720} + \frac{(13t+348)\sqrt{t}}{120\sqrt{\pi}} \end{bmatrix}.$$

In Figures 1 and 2 we have shown two sets of the four iterates graphically.

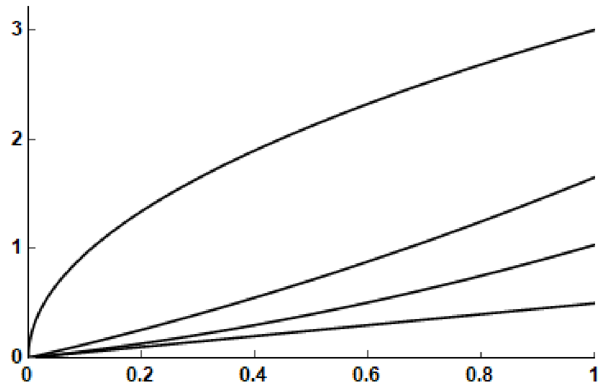

Fig. 1. $\sqrt{t}\alpha_{0,1}(t) \leq \sqrt{t}\alpha_{1,1}(t) \leq \sqrt{t}\beta_{1,1}(t) \leq \sqrt{t}\beta_{0,1}(t)$

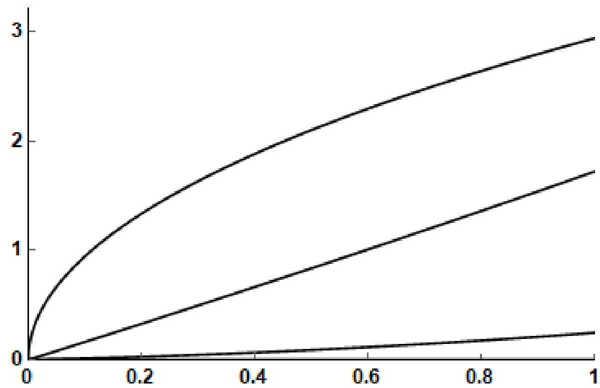

Fig. 2. $\sqrt{t}\alpha_{0,2}(t) \leq \sqrt{t}\alpha_{1,2}(t) \leq \sqrt{t}\beta_{1,2}(t) \leq \sqrt{t}\beta_{0,2}(t)$

In order to find α_2, β_2 we would need to take the derivative of α_1 and then compute $A(t)$ which will involve the weighted Mittag–Leffler function. Integrating this expression is significantly more challenging. One of the future plans is to find an approximation for the matrix q-weighted Mittag–Leffler function so we can to obtain higher order iterates. However, even with this limitation we can see the convergence developing graphically.

REFERENCES

[1] R.L. Bagley, P.J. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, Journal of Rheology **27** (1983) 3, 201–210.

[2] R.L. Bagley, P.J. Torvik, *Fractional calculus in the transient analysis of viscoelastically damped structures*, AIAA Journal **23** (1985) 6, 918–925.

[3] R.L. Bagley, P.J. Torvik, *On the fractional calculus model of viscoelastic behavior*, Journal of Rheology **30** (1986) 1, 133–155.

[4] R. Bellman, *Methods of Nonlinear Analysis*, volume II, Academic Press, New York, 1973.

[5] R. Bellman, R. Kalaba, *Quasilinearization and Nonlinear Boundary Value Problems*, American Elsevier, New York, 1965.

[6] E.A. Boroujeni, H.R. Momeni, *Observer based control of a class of nonlinear fractional-order systems using lmi*, International Journal of Science and Engineering Investigations **1** (2012) 1, 48–52.

[7] M. Caputo, *Linear models of dissipation whose Q is almost independent, II*, Geophy. J. Roy. Astronom. **13** (1967), 529–539.

[8] A. Chikrii, S. Eidelman, *Generalized Mittag–Leffler matrix functions in game problems for evolutionary equations of fractional order*, Cybern. Syst. Analysis **36** (2000) 3, 315–338.

[9] A. Chikrii, I. Matichin, *Presentation of solutions of linear systems with fractional derivatives in the sense of Riemann–Liouville, Caputo, and Miller–Ross*, J. Autom. Inf. Sci. **40** (2008) 6, 1–11.

[10] A. Chowdhury, C.I. Christov, *Memory effects for the heat conductivity of random suspensions of spheres*, Proc. R. Soc. A **466** (2010), 3253–3273.

[11] L. Debnath, *Recent applications of fractional calculus to science and engineering*, International Journal of Mathematics and Mathematical Sciences **54** (2003), 3413–3442.

[12] Z. Denton, *Generalized extension of the quasilinearization for Riemann–Liouville fractional differential equations*, Dynamic Systems and Applications **23** (2014) 2 & 3, 333–350.

[13] Z. Denton, P.W. Ng, A.S. Vatsala, *Quasilinearization method via lower and upper solutions for Riemann–Liouville fractional differential equations*, Nonlinear Dynamics and Systems Theory **11** (2011) 3, 239–251.

[14] Z. Denton, J.D. Ramírez, *Monotone method for finite systems of nonlinear Riemann-Liouville fractional integro-differential equations*, Nonlinear Dynamics and Systems Theory **18** (2018) 2, 130–143.

[15] Z. Denton, A.S. Vatsala, *Monotone iterative technique for finite systems of nonlinear Riemann–Liouville fractional differential equations*, Opuscula Math. **31** (2011) 3, 327–339.

[16] Z. Denton, A.S. Vatsala, *Generalized quasilinearization method for RL fractional differential equations*, Nonlinear Studies **19** (2012) 4, 637–652.

[17] K. Diethelm, A.D. Freed, *On the solution of nonlinear fractional differential equations used in the modeling of viscoplasticity*, [in:] F. Keil, W. Mackens, H. Vob, J. Werther (eds), *Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties*, Heidelberg, Springer, 1999, 217–224.

[18] A.N. Gerasimov, *A generalization of linear laws of deformation and its application to problems of internal friction*, Akad. Nauk SSSR. Prikl. Mat. Meh. **12** (1948), 251–260.

[19] W.G. Glöckle, T.F. Nonnenmacher, *A fractional calculus approach to self similar protein dynamics*, Biophy. J. **68** (1995), 46–53.

[20] R. Hilfer (ed.), *Applications of Fractional Calculus in Physics*, World Scientific Publishing, Germany, 2000.

[21] A.A. Kilbas, H.M. Srivastava, J.J Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, North Holland, 2006.

[22] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Res. Notes Math. Ser., vol. 301, Longman-Wiley, New York, 1994.

[23] V. Kiryakova, *The multi-index Mittag–Leffler functions as an important class of special functions of fractional calculus*, Computers & Mathematics with Applications **59** (2010) 5, 1885–1895.

[24] R.C. Koeller, *Applications of fractional calculus to the theory of viscoelasticity*, J. Appl. Mech. **51** (1984), 229–307.

[25] V. Lakshmikantham, S. Leela, D.J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[26] V. Lakshmikantham, A.S. Vatsala, *Generalized Quasilinearization for Nonlinear Problems*, Kluwer Academic Publishers, Boston, 1998.

[27] E.S. Lee, *Quasilinearization and Invariant Imbedding*, Academic Press, New York, 1968.

[28] D. Matignon, *Stability results for fractional differential equations with applications to control processing*, [in:] *Computational Engineering in Systems Applications*, vol. 2, Lille, France, 1996, 963–968.

[29] R. Metzler, W. Schick, H.G. Kilian, T.F. Nonnenmacher, *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phy. **103** (1995), 7180–7186.

[30] B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, New York - London, 2002.

[31] G.W. Scott Blair, *The role of psychophysics in rheology*, Journal of Colloid Science **2** (1947) 1, 21–32.

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