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ASYMPTOTIC EXPANSION OF LARGE EIGENVALUES FOR A CLASS OF UNBOUNDED JACOBI MATRICES

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Abstract. We investigate a class of infinite tridiagonal matrices which define unbounded self-adjoint operators with discrete spectrum. Our purpose is to establish the asymptotic expansion of large eigenvalues and to compute two correction terms explicitly.

Keywords: tridiagonal matrix, band matrix, unbounded self-adjoint operator, discrete spectrum, large eigenvalues, asymptotics.

Mathematics Subject Classification: 47B25, 47B36, 15A18.

1. INTRODUCTION

1.1. GENERAL REMARKS

Infinite Jacobi matrices have appeared in many recent papers related to various questions of pure and applied mathematics (see [7, 9, 14, 19, 23]). In this paper we consider a Hermitian tridiagonal matrix

$$\begin{pmatrix} d(1) & \overline{a(1)} & 0 & 0 & \cdots \\ a(1) & d(2) & \overline{a(2)} & 0 & \cdots \\ 0 & a(2) & d(3) & \overline{a(3)} & \cdots \\ 0 & 0 & a(3) & d(4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.1}$$

such that $(d(k))_{k=1}^{\infty}$ is an increasing sequence going to infinity and $(a(k))_{k=1}^{\infty}$ is a complex valued sequence dominated by $(d(k))_{k=1}^{\infty}$. Then (1.1) defines in ℓ^2 a self-adjoint operator J with discrete spectrum (see [8]). Moreover, J is bounded from below and *the eigenvalue sequence of* J is defined as the non-decreasing sequence

$(\lambda_n(J))_{n=1}^{\infty}$ such that $Jv_n = \lambda_n(J)v_n$ for $n \in \mathbb{N}^*$ and $(v_n)_{n=1}^{\infty}$ is an orthonormal basis of ℓ^2.

We begin the discussion of known results by the fundamental paper of J. Janas, S. Naboko [14]. In this paper the authors describe a method of approximative diagonalization and its application to the analysis of large eigenvalues in certain quantum models. The paper [14] gives also explanations why this type of analysis is important in Quantum Physics.

Concerning non self-adjoint problems, we refer to [10] and [16]. Concerning the self-adjoint problem, we remark that the papers [1–5, 7, 14, 15]

use the following hypothesis.

Hypothesis. There exist $\mu, \rho, C, c, k_0 \in (0; \infty)$ such that for $k \geq k_0$ one has

(H1) $ck^{\mu} \leq d(k) \leq Ck^{\mu}$,
(H2) $d(k+1) - d(k) \geq ck^{\mu-1}$,
(H3) $|a(k)| \leq Ck^{\mu-\rho}$.

It turns out that asymptotic estimates crucially depend on whether $\rho > 1$ or not.

Case $\rho > 1$. The papers [14] and [7] treat a type of coefficients satisfying (H1)–(H3) with $\mu = 2$ and $\rho = \frac{3}{2}$. An asymptotic expansion of large eigenvalues is constructed in [15] using (H1)–(H3) with $\mu \geq 1$, $\rho > 1$ and in [1] using (H1)–(H3) with $\mu > 0$, $\rho > 1$.

Case $\rho \leq 1$. It appears that $\rho = \frac{1}{2}$ is the most important value for the Quantum Optics and the asymptotic behaviour of large eigenvalues for models considered in [1–5, 20–22] turned out to have a quite special form. However, it is an open problem to describe eigenvalue asymptotics without additional regularity of entries. Other special models are investigated in papers [11, 13] and [17].

In this paper we construct an asymptotic expansion of large eigenvalues under additional regularity conditions imposed on the entries and we give explicit expressions for correction terms in the asymptotic formula with error $O(n^{\mu-6\rho})$. In Theorem 1.1 we assume that the entries have a classical expansion at infinity. The case $\rho = \frac{1}{2}$ gives a nice surprise by ensuring a classical expansion of eigenvalues. A similar construction still works if entries satisfy regularity conditions of symbol type (see Section 1.3). The restriction to tridiagonal matrices is not essential in our approach and we can work all the time with band matrices, but tridiagonal matrices make formulas and calculus more simple. We mention as well that this paper can be viewed as a development of [6] (see also [15]).

1.2. ASYMPTOTIC EXPANSION OF LARGE EIGENVALUES

We denote by ℓ^2 the Hilbert space of square summable complex valued sequences $x\colon \mathbb{N}^* \to \mathbb{C}$ with the norm

$$\|x\| = \Big(\sum_{k=1}^{\infty} |x(k)|^2 \Big)^{1/2}$$

and the scalar product $\langle x,y\rangle = \sum_{k=1}^{\infty} \overline{x(k)}y(k)$. For any $\theta > 0$ we denote

$$\ell^{2,\theta} := \Big\{ x \in \ell^2 \,:\, \sum_{k=1}^{\infty} |k^{\theta}x(k)|^2 < \infty \Big\}.$$

Let $(d(k))_{k=1}^{\infty}$, $(a(k))_{k=1}^{\infty}$ satisfy the hypothesis (H1)–(H3) for a fixed $\mu > 0$ and $\rho > 0$. For $x \in \ell^{2,\mu}$ we define $Jx \in \ell^2$ by the formula

$$Jx(k) = d(k)x(k) + \bar{a}(k)x(k+1) + a(k-1)x(k-1) \text{ for } k \in \mathbb{N}^*, \tag{1.2}$$

where we assume $x(0) = 0$ and $a(0) = 0$. The formula (1.2) defines the operator J on the domain of definition $\mathcal{D}(J) = \ell^{2,\mu}$. Then J is self-adjoint in ℓ^2, bounded from below and has compact resolvent (see [8]). We will prove the following result.

Theorem 1.1. *Let $\mu > 0$, $\rho > 0$ be fixed and assume*

$$d(k) \sim k^{\mu} \sum_{i=0}^{\infty} \frac{\delta_i}{k^i} \qquad \text{as } k \to \infty, \tag{1.3}$$

$$a(k) \sim k^{\mu-\rho} \sum_{i=0}^{\infty} \frac{\alpha_i}{k^i} \qquad \text{as } k \to \infty, \tag{1.4}$$

where $\alpha_i \in \mathbb{C}$, $\delta_i \in \mathbb{R}$ for $i \in \mathbb{N}$ and $\delta_0 > 0$. Then (H1)–(H3) *hold and* (1.2) *defines in ℓ^2 the self-adjoint operator J on the domain $\mathcal{D}(J) = \ell^{2,\mu}$. If $(\lambda_n(J))_{n=1}^{\infty}$ is the eigenvalue sequence of the operator J, then*

$$\lambda_n(J) = d(n) + \mathfrak{r}(n),$$

where $\mathfrak{r}(n)$ obeys the asymptotic expansion of the form

$$\mathfrak{r}(n) \sim n^{\mu-2\rho} \sum_{i,j=0}^{\infty} \frac{c_{i,j}}{n^{i+2j\rho}} \qquad \text{as } n \to \infty \tag{1.5}$$

and $c_{i,j}$ are real coefficients obtained by the induction scheme in Section 5. The explicit values of $c_{0,0}$, $c_{1,0}$, $c_{2,0}$ are computed in Section 6.4 and the value of $c_{0,1}$ in Section 6.5 (see also (1.11) *and* (1.14)–(1.15)*).*

The operator J is a relatively compact perturbation of the diagonal operator $\operatorname{diag}(d(n))_{n=1}^{\infty}$ and the special form of this perturbation allows one to deduce

$$\lambda_n(J) = d(n) + O(n^{\mu-\rho}) \tag{1.6}$$

from the min-max principle (see Corollary 3.3). It appears that the assumptions of Theorem 1.1 allow one to replace (1.6) by the stronger estimate

$$\lambda_n(J) = d(n) + O(n^{\mu-2\rho})$$

and the remainder $\mathfrak{r}(n) = \lambda_n(J) - d(n)$ obeys the asymptotic formula

$$\mathfrak{r}(n) = \mathfrak{r}_1(n) + O(n^{\mu-4\rho}),$$

where $\mathfrak{r}_1(n)$ is given by (1.9). The quantity $\mathfrak{r}_1(n)$ is of order $n^{\mu-2\rho}$ and we call it *the first correction term.* We mention that the expression (1.9) for the first correction term was obtained in [15] under the assumptions (H1)–(H3) with $\mu \geq 1$, $\rho > 1$ (and with weaker remainder estimates).

The next step of precision is attained in Theorem 6.2 which describes the asymptotic behaviour of $\mathfrak{r}(n)$ modulo $O(n^{\mu-6\rho})$. We show the formula (1.12) and give explicit expressions (1.13)–(1.15) for the leading coefficient of *the second correction term*, which is of order $O(n^{\mu-4\rho})$.

In order to obtain a complete asymptotic expansion, we use a method of approximative diagonalization. The idea of our approach is similar to the method presented in [14], but we do not need the assumptions $\mu \geq 1$ and $\rho > 1$ used in [14].

At the end of this discussion we remark that the result of Theorem 1.1 is inspired by the paper [18]. The main differences between [18] and our work are the following:

– we consider the Hilbert space $\ell^2 = \ell^2(\mathbb{N}^*)$ instead of $\ell^2(\mathbb{Z})$,
– we give a simple algorithm of computing the coefficients $c_{i,j}$ adopted to our problem,
– our approach allows us to see that (1.5) contains only even powers of $n^{-\rho}$.

1.3. FURTHER RESULTS

We construct an asymptotic expansion for large eigenvalues under assumptions slightly weaker than used in Theorem 1.1.

Notation 1.2. For any $f : \mathbb{N}^* \to \mathbb{C}$ we write $\partial^0 f(k) := f(k)$,

$$\partial f(k) = f(k+1) - f(k),$$

and, by using induction, we define $\partial^{m+1} f := \partial(\partial^m f)$ for $m = 0, 1, 2, \ldots$.

If $\nu \in \mathbb{R}$ then S_1^ν denotes the set of $p : \mathbb{N}^* \to \mathbb{C}$ such that the estimate

$$\partial^m p(k) = \mathrm{O}(k^{\nu-m})$$

holds for every $m \in \mathbb{N}$.

Throughout the paper $\mu > 0$, $\rho > 0$ are fixed and we centre our analysis on the Jacobi operators (1.2) such that

$$d \in \mathrm{S}_1^\mu, \tag{1.7}$$

$$a \in \mathrm{S}_1^{\mu-\rho} \tag{1.8}$$

and (H1)–(H2) hold. These conditions could be weakened if one wants to obtain the remainder estimate $O(n^{-\eta})$ for a fixed value of the exponent η (see Theorem 4.1), but for simplicity we use (1.7)–(1.8) throughout Section 5 and 6.

As before, J is the self-adjoint operator in ℓ^2 with $\mathcal{D}(J) = \ell^{2,\mu}$ and $(\lambda_n(J))_{n=1}^\infty$ denotes its non-decreasing eigenvalue sequence (counted with multiplicities). We consider the asymptotic formula for $(\lambda_n(J))_{n=1}^\infty$ with three different degrees of precision.

(i) *The asymptotic formula with one correction term.* We show that

$$\lambda_n(J) = d(n) + \mathfrak{r}_1(n) + O(n^{\mu-4\rho})$$

holds with

$$\mathfrak{r}_1(n) := \frac{|a(n-1)|^2}{d(n) - d(n-1)} - \frac{|a(n)|^2}{d(n+1) - d(n)}. \tag{1.9}$$

If (1.3)–(1.4) hold, then (1.7)–(1.8) hold as well. Under the assumptions (1.3)–(1.4), it is easy to see that the quantity $\mathfrak{r}_1(n)$ defined by (1.9) satisfies

$$\mathfrak{r}_1(n) \sim n^{\mu-2\rho} \sum_{i=0}^{\infty} \frac{c_{i,0}}{n^i} \quad \text{as } n \to \infty. \tag{1.10}$$

In Section 6.4 we show that in the case $\delta_0 = 1$ one has (1.10) with

$$\begin{cases} c_{0,0} = -\frac{\mu-2\rho+1}{\mu}|\alpha_0^2|, \\ c_{1,0} = \frac{\mu-2\rho}{\mu}\Big(\big(\mu-\rho+\delta_1\big(1-\frac{1}{\mu}\big)\big)|\alpha_0^2| - 2\mathrm{Re}(\alpha_0\overline{\alpha}_1)\Big). \end{cases} \tag{1.11}$$

It is clear that the general case can be reduced to the case $\delta_0 = 1$ by means of the multiplication of J by a suitable constant.

(ii) *The asymptotic formula with two correction terms.* In Theorem 6.2 we prove

$$\lambda_n(J) = d(n) + \mathfrak{r}_1(n) + \mathfrak{r}_2(n) + O(n^{\mu-6\rho}), \tag{1.12}$$

where $\mathfrak{r}_1$ is given by (1.9) and $\mathfrak{r}_2 \in \mathrm{S}_1^{\mu-4\rho}$.

Let us assume that (1.3) holds and (1.4) holds with $\delta_0 = 1$. Then (1.10)–(1.11) are still true and in Section 6.5 we show that

$$\mathfrak{r}_2(n) = c_{0,1} n^{\mu-4\rho} + O(n^{\mu-4\rho-1}) \tag{1.13}$$

holds with

$$c_{0,1} = c_1 + c_2 + c_3, \tag{1.14}$$

where

$$\begin{cases} c_1 = -(\mu-2\rho)(\mu+1-2\rho)(\mu+1-4\rho)\frac{|\alpha_0^4|}{2\mu^3}, \\ c_2 = (\mu-1)(\mu+2-4\rho)(\mu+1-4\rho)\frac{|\alpha_0^4|}{4\mu^3}, \\ c_3 = -(\mu-1)^2(\mu+1-4\rho)\frac{|\alpha_0^4|}{4\mu^3}. \end{cases} \tag{1.15}$$

(iii) *A complete asymptotic expansion.* In Section 5 we construct $\mathfrak{r}_l \in S_1^{\mu-2^l\rho}$, where $l = 1, 2, \ldots$ and for every $m \in \mathbb{N}^*$ one has

$$\lambda_n(J) = d(n) + \sum_{l=1}^{m} \mathfrak{r}_l(n) + O(n^{\mu-2^{m+1}\rho}).$$

This construction is used to deduce Theorem 1.1.

2. PRELIMINARIES

2.1. BASIC NOTATIONS AND PROPERTIES

Throughout the paper $\{e_k\}_{k\in\mathbb{Z}}$ denotes the canonical basis in ℓ^2 (i.e. $e_k(j) = \delta_{k,j}$) and the shift operator is denoted by S (i.e. S is the linear bounded operator in ℓ^2 satisfying $Se_k = e_{k+1}$ for $k \in \mathbb{N}^*$). We denote by Λ the self-adjoint operator in ℓ^2 defined by the formula

$$(\Lambda x)(k) = kx(k)$$

on the domain of definition $\mathcal{D}(\Lambda) = \ell^{2,1}$. Using the functional calculus, we can define $f(\Lambda)$ as the closed linear operator satisfying $f(\Lambda)e_k = f(k)e_k$ for all $k \in \mathbb{N}^*$. These notations allow us to write down the Jacobi operator (1.2) in the form

$$J = d(\Lambda) + Sa(\Lambda) + \bar{a}(\Lambda)S^*,$$

where S^* is the adjoint of S.

Below we introduce more notations.

Notation 2.1. Let $i \in \mathbb{Z}$. We write $S^{(i)} = S^i$ if $i \in \mathbb{N}$ and $S^{(i)} = S^{*|i|}$ if $i \in \mathbb{Z} \setminus \mathbb{N}$. According to this notation, the operator $S^{(i)}$ is the linear mapping on ℓ^2 satisfying

(i) $S^{(i)}e_k = e_{k+i}$ if $k + i \geq 1$,
(ii) $S^{(i)}e_k = 0$ if $k + i \leq 0$.

Let $i \in \mathbb{Z}$ and $f : \mathbb{N}^* \to \mathbb{C}$. Then $\tau_i f$ denotes the sequence $\mathbb{N}^* \to \mathbb{C}$ satisfying

(i) $\tau_i f(k) = f(k + i)$ if $k + i \geq 1$,
(ii) $\tau_i f(k) = 0$ if $k + i \leq 0$.

We denote by $\partial_i f$ the sequence $\mathbb{N}^* \to \mathbb{C}$ given by the formula

$$\partial_i f(k) := \tau_i f(k) - f(k).$$

In particular $\partial_0 f = 0$ and $\partial_1 f(k) = \partial f(k) = f(k+1) - f(k)$ is the derivative introduced in Section 1. We observe that for any fixed $i \in \mathbb{Z}$,

$$\partial f(k) = O(k^\nu) \implies \partial_i f(k) = O(k^\nu). \tag{2.1}$$

Remark 2.2. The above notations allow us to write $S^{(i)}x = \tau_{-i}x$ and

$$f(\Lambda)S^{(i)} = S^{(i)}(\tau_i f)(\Lambda), \tag{2.2}$$

where $(\tau_i f)(\Lambda)$ is the shifted diagonal operator satisfying $(\tau_i f)(\Lambda)e_k = \tau_i f(k)e_k$.

Notation 2.3. If $\nu \in \mathbb{R}$ and $N \in \mathbb{N}$, then $\mathrm{S}_1^\nu(N)$ denotes the set of all $p : \mathbb{N}^* \to \mathbb{C}$ such that $\partial^m p(k) = O(k^{\nu-m})$ holds for $m \in \{0, \dots, N\}$.

Lemma 2.4.

(i) *If* $N \geq 1$, $i \in \mathbb{Z}$ *and* $p \in \mathrm{S}_1^\nu(N)$ *then* $\partial_i p \in \mathrm{S}_1^{\nu-1}(N-1)$.
(ii) *If* $p \in \mathrm{S}_1^\nu(N)$ *and* $q \in \mathrm{S}_1^\eta(N)$ *then* $pq \in \mathrm{S}_1^{\nu+\eta}(N)$.
(iii) *Assume that* $p \in \mathrm{S}_1^\nu(N)$ *and there exists* $c > 0$ *such that* $|p(k)| \geq ck^\nu$ *for all* $k \in \mathbb{N}^*$.

Then $1/p \in \mathrm{S}_1^{-\nu}(N)$.

Proof. (i) It suffices to use (2.1) and the definition of $\mathrm{S}_1^\nu(N)$.

(ii) We obtain

$$\partial^m(pq)(k) = \mathrm{O}(k^{\nu+\eta-m})$$

by induction with respect to m using

$$\partial(fg) = f\partial g + g\partial f + \partial f \partial g.$$

(iii) We obtain

$$\partial^m\big(1/p\big)(k) = \mathrm{O}(k^{-\nu-m})$$

by induction with respect to m using

$$\partial\big(1/p\big)(k) = -\frac{\partial p(k)}{p(k)p(k+1)}.$$

□

2.2. OPERATOR VALUED ERRORS

In this section we introduce notations used throughout the paper to control errors of large eigenvalues. As before, S denotes the shift operator in ℓ^2, i.e. $Se_k = e_{k+1}$. If $\theta \geq 0$, then Λ^θ denotes the self-adjoint operator in ℓ^2 such that $\Lambda^\theta e_k = k^\theta e_k$ and $\mathcal{D}(\Lambda^\theta) = \ell^{2,\theta}$. If $\theta \leq 0$, then Λ^θ is the inverse of $\Lambda^{|\theta|}$. In order to deal with these operators, we introduce the subspace of fast decaying sequences

$$\mathrm{S}^{-\infty} = \bigcap_{\theta>0} \ell^{2,\theta} = \{x \in \ell^2 : x(k) = O(k^{-N}) \text{ for any } N \geq 0\}. \tag{2.3}$$

The space $\mathrm{S}^{-\infty}$ defined by (2.3) will be assumed to be invariant for all operators considered further on and we use the fact that this type of operators form an algebra. Clearly $\mathrm{S}^{-\infty}$ is invariant for the shifts S, S^* and for $f(\Lambda)$ if $f : \mathbb{N}^* \to \mathbb{C}$ is polynomially bounded.

We adopt the following convention: for a linear mapping $P : \mathcal{D}(P) \to \ell^2$ we write $P \in \mathcal{B}(\ell^2)$ if and only if the closure of P is a bounded operator $\ell^2 \to \ell^2$.

Let $\eta \in \mathbb{R}$. We introduce the notation

$$P = O(\Lambda^\eta) \quad \Longleftrightarrow \quad \forall\theta \in \mathbb{R}, \quad \Lambda^\theta P \Lambda^{-\theta-\eta} \in \mathcal{B}(\ell^2). \tag{2.4}$$

Then (2.4) immediately ensures the properties

$$P = O(\Lambda^\eta) \quad \Longrightarrow \quad \mathrm{S}^{-\infty} \text{ is an invariant subspace of } P,$$

$$f(k) = O(k^\eta) \quad \Longrightarrow \quad f(\Lambda) = O(\Lambda^\eta),$$

$$P = O(\Lambda^\eta), \quad Q = O(\Lambda^{\widetilde{\eta}}) \quad \Longrightarrow \quad PQ = O(\Lambda^{\eta+\widetilde{\eta}}). \tag{2.5}$$

The notation $P=O(\Lambda^{-\infty})$ means that $P=O(\Lambda^{-\theta})$ holds for all $\theta \geq 0$ and the notation

$$P = Q + O(\Lambda^\eta)$$

means that $P - Q = O(\Lambda^\eta)$ and $\mathrm{S}^{-\infty}$ is invariant for both operators P, Q.

2.3. FINITE DIFFERENCE OPERATORS.

Any complex matrix $(P(i,j))_{i,j=1}^\infty$ can be written in the form

$$\begin{pmatrix} p_0(1) & p_{-1}(2) & p_{-2}(3) & p_{-3}(4) & \dots \\ p_1(1) & p_0(2) & p_{-1}(3) & p_{-2}(4) & \dots \\ p_2(1) & p_1(2) & p_0(3) & p_{-1}(4) & \dots \\ p_3(1) & p_2(2) & p_1(3) & p_0(4) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.6}$$

where $(p_i(k))_{k=1}^\infty$ is a complex valued sequence (for any $i \in \mathbb{Z}$).

Definition 2.5. We say that P is *a band matrix* if and only if P has the form (2.6) and there is $i_0 \in \mathbb{N}$ such that $|i| > i_0$ implies $p_i(k) = 0$ for all $k \in \mathbb{N}^*$.

Definition 2.6. We say that P is *off-diagonal* if and only if $p_0(k) = 0$ holds for all $k \in \mathbb{N}^*$. If P is given by (2.6), then $p_0(\Lambda)$ is called *the diagonal part of P*.

Lemma 2.7. *We fix $\nu \in \mathbb{R}$ and $i_0 \in \mathbb{N}$. Assume that $p_i(k) = O(k^\nu)$ for all $i \in \mathbb{Z}$ and $|i| > i_0$ implies $p_i(k) = 0$ for all $k \in \mathbb{N}^*$. Then the band matrix P given by* (2.6) *can be identified with the linear mapping acting on $\mathrm{S}^{-\infty}$ according to the formula*

$$(Px)(k) = \sum_{i=-i_0}^{\min\{i_0,k-1\}} p_i(k-i)x(k-i). \tag{2.7}$$

Moreover $P = O(\Lambda^\nu)$.

Proof. We observe that (2.7) can be written in the form

$$Px = \sum_{-i_0 \leq i \leq i_0} S^{(i)} p_i(\Lambda)x \text{ for } x \in \mathrm{S}^{-\infty}. \tag{2.8}$$

The assumptions $p_i(k) = O(k^\nu)$ ensure $p_i(\Lambda) = O(\Lambda^\nu)$ and due to (2.5) it remains to show that $S = O(\Lambda^0)$ and $S^* = O(\Lambda^0)$. In order to show $\Lambda^\theta S \Lambda^{-\theta} \in \mathcal{B}(\ell^2)$ for any $\theta \in \mathbb{R}$, we use (2.2) to write $\Lambda^\theta S \Lambda^{-\theta} = S(\Lambda+1)^\theta \Lambda^{-\theta}$ and observe that $(\Lambda+1)^\theta\Lambda^{-\theta} \in \mathcal{B}(\ell^2)$. Taking the adjoint, we get $\Lambda^{-\theta} S^* \Lambda^\theta \in \mathcal{B}(\ell^2)$. $\square$

2.4. CLASS OF OPERATORS $\mathrm{FDO}_1^{\nu}(N)$

Notation 2.8. If $\nu \in \mathbb{R}$ and $N \in \mathbb{N}$, then $\mathrm{FDO}_1^{\mu}(N)$ denotes the set of all band matrices P given by (2.6), where $p_i \in \mathrm{S}_1^{\nu}(N)$ for all i. Further on the elements of $\mathrm{FDO}_1^{\nu}(N)$ are always identified with linear mappings given by (2.8) where $p_i \in \mathrm{S}_1^{\nu}(N)$.

Lemma 2.9. *If $P \in \mathrm{FDO}_1^{\nu}(N+1)$ and $Q \in \mathrm{FDO}_1^{\eta}(N+1)$, then the commutator*

$$[P,Q] := PQ - QP \in \mathrm{FDO}_1^{\nu+\eta-1}(N).$$

Proof. It suffices to prove the formula

$$B_{i,j} := [S^{(i)}p_i(\Lambda), S^{(j)}q_j(\Lambda)] = S^{(i+j)}(q_j\partial_j p_i - p_i\partial_i q_j)(\Lambda). \tag{2.9}$$

Indeed, if $p_i \in \mathrm{S}_1^{\nu}(N+1)$ and $q_j \in \mathrm{S}_1^{\mu}(N+1)$, then

$$b_{i,j} := q_j\partial_j p_i - p_i\partial_i q_j \in \mathrm{S}_1^{\nu+\mu-1}(N),$$

which ensures that the right-hand side of (2.9) belongs to $\mathrm{FDO}_1^{\nu+\mu-1}(N)$.

In order to prove (2.9), we observe that (2.2) allows us to express

$$\begin{aligned} B_{i,j} &= S^{(i)}p_i(\Lambda)S^{(j)}q_j(\Lambda) - S^{(j)}q_j(\Lambda)S^{(i)}p_i(\Lambda) \\ &= S^{(i)}S^{(j)}(\tau_j p_i)(\Lambda)q_j(\Lambda) - S^{(j)}S^{(i)}(\tau_i q_j)(\Lambda)p_i(\Lambda). \end{aligned}$$

Since $b_{i,j} = q_j\tau_j p_i - p_i\tau_i q_j$, it remains to check the equalities

$$S^{(i)}S^{(j)}(\tau_j p_i)(\Lambda)q_j(\Lambda) = S^{(i+j)}(\tau_j p_i)(\Lambda)q_j(\Lambda), \tag{2.10}$$

$$S^{(j)}S^{(i)}(\tau_i q_j)(\Lambda)p_i(\Lambda) = S^{(i+j)}(\tau_i q_j)(\Lambda)p_i(\Lambda). \tag{2.11}$$

Case $j \geq 0$. In this case $S^{(i)}S^{(j)} = S^{(i+j)}$ holds for any $i \in \mathbb{Z}$.

Case $j < 0$, $i < 0$. In this case $S^{(i)}S^{(j)} = S^{(i+j)}$ holds as well.

Case $j < 0$, $i \geq 0$. In this case $S^{(i)}S^{(j)}e_k = S^{(i+j)}e_k$ holds if $k > -j$. Assume now that $k \leq -j$. Then $S^{(i)}S^{(j)}e_k = 0$ and $(\tau_j p_i)(\Lambda)e_k = \tau_j p_i(k)e_k = 0$, hence (2.10) holds. Similarly we check (2.11). □

3. MAIN INGREDIENTS

3.1. CONSEQUENCES OF THE MIN-MAX PRINCIPLE

In this section we prove Proposition 3.2 which is the first fundamental tool of our approach. Its purpose is to detect perturbations which give small errors for large eigenvalues. The importance of Proposition 3.2 lies in the fact that it is used as the ingredient of our approach replacing Lemma 2.1 of J. Janas, S. Naboko [14].

We notice that Lemma 2.1 from [14] is used in [7, 15, 16, 22], but we prefer using Proposition 3.2 for several reasons. One of them is the fact that Lemma 2.1 from [14] needs the additional assumption $\mu \geq 1$ in the hypothesis (H1).

Notation 3.1. If L is a self-adjoint, bounded from below operator with compact resolvent in ℓ^2, then $\lambda_1(L) \leq \ldots \leq \lambda_n(L) \leq \lambda_{n+1}(L) \leq \ldots$ are eigenvalues of L, enumerated in non-decreasing order, counting multiplicities.

Proposition 3.2. *We fix μ, ρ, $\eta \in (0;\infty)$. Let $D = d(\Lambda)$ be the diagonal operator with d satisfying* (H1). *Let A be a symmetric operator such that $\mathcal{D}(A) \supset \ell^{2,\mu}$ and*

$$A = O(\Lambda^{\mu-\rho}). \tag{3.1}$$

Let J be the operator defined by

$$J = D + A \tag{3.2}$$

and $\mathcal{D}(J) = \ell^{2,\mu}$. Then J is self-adjoint, bounded from below and has compact resolvent in ℓ^2. If R is a symmetric operator such that $\mathcal{D}(R) \supset \ell^{2,\mu}$ and

$$R = O(\Lambda^{\mu-\eta}),$$

then the operator $J + R$ is self-adjoint in ℓ^2 on the domain $\mathcal{D}(J + R) = \ell^{2,\mu}$. Moreover, $J + R$ is a bounded from below operator with compact resolvent and one has

$$\lambda_n(J + R) = \lambda_n(J) + O(n^{\mu-\eta}). \tag{3.3}$$

Proof. Let $\tilde{C} > 0$ and denote $\tilde{D} := D + \tilde{C}I$. Assume that $\tilde{C}$ is fixed large enough. Then (H1) ensures $C^{-1}\Lambda^\mu \leq \tilde{D} \leq C\Lambda^\mu$ with $C > 0$, hence $\tilde{D}^{-1}$ is compact. Since A has zero relative bound with respect to $\tilde{D}$ (due to (3.1)), the operaror

$$\tilde{J} := J + \tilde{C}I = \tilde{D} + A$$

is self-adjoint, bounded from below, has compact resolvent and

$$(2C)^{-1}\Lambda^\mu \leq \tilde{J} \leq 2C\Lambda^\mu \tag{3.4}$$

holds if $\tilde{C}$ is large enough. Since $\lambda_n(\tilde{J}) = \lambda_n(J) + \tilde{C}$ and $\lambda_n(\tilde{J}+R) = \lambda_n(J+R) + \tilde{C}$, it remains to prove

$$\lambda_n(\tilde{J}+R) = \lambda_n(\tilde{J}) + O(n^{\mu-\eta}). \tag{3.5}$$

Step 1. We prove that (3.5) holds if $\eta \leq 2\mu$.

Since $\lambda_n((2C)^{\pm 1}\Lambda^\mu) = (2C)^{\pm 1}n^\mu$, the min-max principle used in (3.4) gives

$$(2C)^{-1}n^\mu \leq \lambda_n(\tilde{J}) \leq 2Cn^\mu. \tag{3.6}$$

The well known result concerning the powers of positive operators ensures that

$$(2C)^{-|s|}\Lambda^{\mu s} \leq \tilde{J}^s \leq (2C)^{|s|}\Lambda^{\mu s} \tag{3.7}$$

follows from (3.4) if $-1 \leq s \leq 1$. Since $0 \leq \eta \leq 2\mu$, we can use (3.7) with $s = 1 - \eta/\mu$ ensuring existence of positive constants C_1, C_2 such that

$$\pm R \leq C_1\Lambda^{\mu-\eta} \leq C_2\tilde{J}^{1-\eta/\mu}. \tag{3.8}$$

Let $f_\pm : (0;\infty) \to \mathbb{R}$ be given by the formula $f_\pm(t) := t \pm C_2 t^{1-\eta/\mu}$. Then (3.8) ensures

$$f_-(\tilde{J}) \le \tilde{J} + R \le f_+(\tilde{J})$$

and the min-max principle ensures

$$\lambda_n(f_-(\tilde{J})) \le \lambda_n(\tilde{J} + R) \le \lambda_n(f_+(\tilde{J})). \tag{3.9}$$

Since $f_\pm$ is increasing on $(t_0;\infty)$ if t_0 is fixed large enough, there is $n_0 \in \mathbb{N}$ such that

$$n \ge n_0 \Longrightarrow \lambda_n(f_\pm(\tilde{J})) = f_\pm(\lambda_n(\tilde{J})). \tag{3.10}$$

Combining (3.9) and (3.10), we find that for $n \ge n_0$ one has

$$\lambda_n(\tilde{J}) - C_2\lambda_n(\tilde{J})^{1-\eta/\mu} \le \lambda_n(\tilde{J} + R) \le \lambda_n(\tilde{J}) + C_2\lambda_n(\tilde{J})^{1-\eta/\mu}. \tag{3.11}$$

Using (3.6), we conclude that (3.11) ensures

$$|\lambda_n(\tilde{J} + R) - \lambda_n(\tilde{J})| \le C_2\lambda_n(\tilde{J})^{1-\eta/\mu} = O(n^{\mu-\eta}).$$

Step 2. Let $k \in \mathbb{N}^*$. We claim that the assertion of Proposition 3.2 holds if $\eta \le 2^k\mu$.

Indeed, the case $k = 1$ was proved in Step 1 and using induction with respect to $k \in \mathbb{N}^*$, we fix $k \in \mathbb{N}^*$ and assume that the assertion of Proposition 3.2 holds if $\eta \le 2^k\mu$.

As before $\tilde{C} > 0$ is fixed large enough, $\tilde{D} = d(\Lambda) + \tilde{C}I$ and d satisfies (H1) for a given $\mu > 0$. Let $\tilde{J} = \tilde{D} + A$ where $A = O(\Lambda^{\mu-\rho})$ and $\rho > 0$. Then

$$\tilde{J}^2 = \tilde{D}^2 + A' \text{ holds with } A' = \tilde{D}A + A\tilde{D} + A^2 = O(\Lambda^{2\mu-\rho}),$$

i.e. $J' := \tilde{J}^2$ satisfies the hypotheses of Proposition 3.2 with $\mu' := 2\mu$ instead of μ. Assume now that $R = O(\Lambda^{\mu-\eta})$ and $0 < \eta \le 2^{k+1}\mu = 2^k\mu'$. Then

$$(\tilde{J} + R)^2 = \tilde{J}^2 + R' \text{ holds with } R' = \tilde{J}R + R\tilde{J} + R^2 = O(\Lambda^{2\mu-\eta})$$

and by using Proposition 3.2 with $J' := \tilde{J}^2$, $\mu' := 2\mu$ instead of J, μ, we obtain

$$\lambda_n((\tilde{J} + R)^2) = \lambda_n(\tilde{J}^2) + O(n^{2\mu-\eta}). \tag{3.12}$$

Let $\tilde{C}$ be large enough to ensure $\tilde{J} + R \ge 0$. Then (3.12) implies

$$\lambda_n(\tilde{J} + R)^2 = \lambda_n(\tilde{J})^2 + O(n^{2\mu-\eta}) = \lambda_n(\tilde{J})^2\big(1 + O(n^{-\eta})\big),$$

where the last estimate follows from (3.6). Thus

$$\lambda_n(\tilde{J} + R) = \lambda_n(\tilde{J})\big(1 + O(n^{-\eta})\big)^{1/2} = \lambda_n(\tilde{J})\big(1 + O(n^{-\eta})\big)$$

and using (3.6) in the last estimate, we deduce that (3.5) holds with $\eta \le 2^{k+1}\mu$. □

Corollary 3.3. *Let $\mu > 0$ and $\rho > 0$. Assume that* (H1) *and* (H3) *hold and there exists k_0 such that $d(k+1) \ge d(k)$ for $k \ge k_0$. If the operator J is given by* (1.2)*, then*

$$\lambda_n(J) = d(n) + O(n^{\mu-\rho}).$$

Proof. Denote $D = d(\Lambda)$ and $R = Sa(\Lambda) + \bar{a}(\Lambda)S^*$. Since (H3) ensures $R = O(\Lambda^{\mu-\rho})$, we can write (3.3) with D instead of J and find $\lambda_n(D + R) = \lambda_n(D) + O(n^{\mu-\rho})$. To complete the proof, we observe that $\lambda_n(d(\Lambda)) = d(n)$ for $n \ge n_0$. □

3.2. SPECIAL SELF-ADJOINT OPERATORS

Lemma 3.4. *Let P be a symmetric operator on* $\mathrm{S}^{-\infty}$ *given by the formula*

$$Px = \sum_{1\le i\le i_0} \big(S^i p_i(\Lambda) + \overline{p_i}(\Lambda)S^{(-i)}\big)x \text{ for } x \in \mathrm{S}^{-\infty}. \tag{3.13}$$

If $p_i(k) = O(k)$ for $i = 1, \ldots, i_0$, then P is essentially self-adjoint on $\mathrm{S}^{-\infty}$.

Proof. Let $\theta \ge 0$. Then $[\mathrm{i}P, \Lambda^{2\theta}]$ is symmetric on $\mathrm{S}^{-\infty}$ and we claim that

$$[\mathrm{i}P, \Lambda^{2\theta}] = O(\Lambda^{2\theta}). \tag{3.14}$$

Indeed, if $f_\theta(k) = k^{2\theta}$, then $\partial_i f_\theta(k) = O(k^{2\theta-1})$, hence (2.2) ensures

$$[P, \Lambda^{2\theta}] = \sum_{1\le i\le i_0} \Big(S^i(p_i\partial_i f_\theta)(\Lambda) + \mathrm{hc}\Big) \text{ with } (p_i\partial_i f_\theta)(k) = O(k^{2\theta}).$$

By (3.14) we can find a constant $C_\theta > 0$ such that

$$\pm\langle y, \Lambda^{-\theta}[\mathrm{i}P, \Lambda^{2\theta}]\Lambda^{-\theta}y\rangle \le C_\theta\|y\|^2 \text{ for } y \in \ell^2.$$

Writing $y = \Lambda^\theta x$, we get

$$\pm\langle x, [\mathrm{i}P, \Lambda^{2\theta}]x\rangle \le C_\theta\|\Lambda^\theta x\|^2 \text{ for } x \in \ell^{2,\theta}. \tag{3.15}$$

Using (3.15) with $\theta = 1/2$ and the fact that

$$\|Px\| \le C\|\Lambda x\| \text{ for } x \in \mathrm{S}^{-\infty}, \tag{3.16}$$

we deduce the assertion by Nelson's Commutator Theorem. Indeed, the properties (3.16) and (3.15) with $\theta = \frac{1}{2}$ allow us to apply Corollary 1.1 in [12] (using P, Λ and $\mathrm{S}^{-\infty}$ instead of H, N and $\mathcal{C}$ in [12]). □

Notation 3.5. If $P \in \mathrm{FDO}_1^\nu(N)$ is given by (2.8), then the adjoint matrix $(P^\dagger(i,j))_{i,j=1}^\infty = (\overline{P(j,i)})_{i,j=1}^\infty$ satisfies

$$P^\dagger x = \sum_{-i_0\le i\le i_0} \overline{p_i}(\Lambda)S^{(-i)}x = \sum_{-i_0\le i\le i_0} S^{(-i)}(\tau_{-i}\overline{p_i})(\Lambda)x \quad \text{for } x \in \mathrm{S}^{-\infty}$$

and it is clear that $P^\dagger \in \mathrm{FDO}_1^\nu(N)$. We will use the notation

$$P + \mathrm{hc} := P + P^\dagger.$$

If moreover $p_i(k) = O(k)$ for $i = 1, \ldots, i_0$, then

$$P = \sum_{1\le i\le i_0} \big(S^i p_i(\Lambda) + \mathrm{hc}\big) \tag{3.17}$$

will always denote the self-adjoint operator satisfying (3.13).

Proposition 3.6. *Assume that $p_i(k) = O(k)$ for $i = 1, \dots, i_0$ and P is the self-adjoint operator given by* (3.17)*. If $\theta \geq 0$ then $\ell^{2,\theta}$ is an invariant subspace of $\mathrm{e}^{\mathrm{i}tP}$ for any $t \in \mathbb{R}$. Moreover there is a constant C_θ such that*

$$\|\Lambda^\theta \mathrm{e}^{\mathrm{i}tP} x\| \leq \mathrm{e}^{C_\theta |t|} \|\Lambda^\theta x\| \tag{3.18}$$

holds for all $x \in \mathrm{S}^{-\infty}$ and $t \in \mathbb{R}$.

Proof. Let $\mathcal{D}(P)$ denote the domain of the self-adjoint operator P equipped with its graph norm. Let $\varepsilon > 0$ and denote

$$\Lambda_\varepsilon^{(\theta)} := \Lambda^{2\theta}(\varepsilon\Lambda^{2\theta} + I)^{-1} = \varepsilon^{-1} I - \varepsilon^{-1}(\varepsilon\Lambda^{2\theta} + I)^{-1}$$

and we define the quadratic form

$$\mathrm{q}_\varepsilon^{(\theta)}(x) = \langle \mathrm{i}Px, \Lambda_\varepsilon^{(\theta)} x\rangle + \langle \Lambda_\varepsilon^{(\theta)} x, \mathrm{i}Px\rangle \text{ for } x \in \mathcal{D}(P).$$

We claim that there is a constant $C_\theta > 0$ such that

$$\pm \mathrm{q}_\varepsilon^{(\theta)}(x) \leq C_\theta \|\Lambda^\theta(\varepsilon\Lambda^{2\theta} + I)^{-1/2} x\|^2 = C_\theta \langle x, \Lambda_\varepsilon^{(\theta)} x\rangle \tag{3.19}$$

holds for all $x \in \mathcal{D}(P)$. Indeed, if $x \in \mathrm{S}^{-\infty}$ then we compute

$$\mathrm{q}_\varepsilon^{(\theta)}(x) = \langle x, [\Lambda_\varepsilon^{(\theta)}, \mathrm{i}P]x\rangle = \langle (\varepsilon\Lambda^{2\theta} + I)^{-1} x, [\mathrm{i}P, \Lambda^{2\theta}](\varepsilon\Lambda^{2\theta} + I)^{-1} x\rangle$$

and by using $(\varepsilon\Lambda^{2\theta}+I)^{-1}x$ instead of x in (3.15), we find that (3.19) holds for $x \in \mathrm{S}^{-\infty}$. Since $\mathrm{S}^{-\infty}$ is dense in $\mathcal{D}(P)$, the estimate (3.19) holds for $x \in \mathcal{D}(P)$ as well.

Let $x \in \mathcal{D}(P)$ and denote $x_t := \mathrm{e}^{\mathrm{i}tP} x$ for $t \in \mathbb{R}$. By using $\frac{\mathrm{d}}{\mathrm{d}t}\langle x_t, \Lambda_\varepsilon^{(\theta)} x_t\rangle = \mathrm{q}_\varepsilon^{(\theta)}(x_t)$ and (3.19), we obtain

$$\pm \frac{\mathrm{d}}{\mathrm{d}t} \langle x_t, \Lambda_\varepsilon^{(\theta)} x_t\rangle \leq C_\theta \langle x_t, \Lambda_\varepsilon^{(\theta)} x_t\rangle$$

and Gronwall's inequality ensures

$$\langle x_t, \Lambda_\varepsilon^{(\theta)} x_t\rangle \leq \mathrm{e}^{C_\theta |t|} \langle x, \Lambda_\varepsilon^{(\theta)} x\rangle. \tag{3.20}$$

If $x \in \mathrm{S}^{-\infty}$, then the limit $\varepsilon \to 0$ in (3.20) gives $x_t \in \ell^{2,\theta}$ and (3.18) holds. Since $\mathrm{S}^{-\infty}$ is dense in $\ell^{2,\theta}$, we still have (3.18) for all $x \in \ell^{2,\theta}$ and $x_t \in \ell^{2,\theta}$ if $x \in \ell^{2,\theta}$. □

Corollary 3.7. *Let P be as in Lemma 3.4. Then*

(i) $\mathrm{S}^{-\infty}$ *is an invariant subspace of* $\mathrm{e}^{\mathrm{i}tP}$ *for any* $t \in \mathbb{R}$,

(ii) $t \to \mathrm{e}^{\mathrm{i}tP} x$ *is a smooth function* $\mathbb{R} \to \ell^{2,\theta}$ *for any* $\theta \geq 0$ *if* $x \in \mathrm{S}^{-\infty}$.

Proof. (ii) Let $x \in \mathrm{S}^{-\infty}$, $\varepsilon \geq 0$ and denote

$$x_t^{\varepsilon,\theta} := \Lambda^\theta(\varepsilon\Lambda^\theta + I)^{-1} \mathrm{e}^{\mathrm{i}tP} x.$$

It is clear that $t \to x_t^{\varepsilon,\theta}$ is smooth $\mathbb{R} \to \ell^2$ if $\varepsilon > 0$. Let us fix $N \in \mathbb{N}^*$ and $t_0 > 0$. Then the family $(x_t^{\varepsilon,\theta})_{0<\varepsilon\leq 1}$ is bounded in $C^N([-t_0, t_0]; \ell^2)$ due to (3.18), hence using the Ascoli Theorem we can find a sequence $(\varepsilon_n)_{n=0}^\infty$ which converges to 0 and $(x_t^{\varepsilon_n,\theta})_{n=0}^\infty$ is convergent in $C^{N-1}([-t_0, t_0]; \ell^2)$. We conclude that the pointwise limit $t \to \Lambda^\theta \mathrm{e}^{\mathrm{i}tP} x$ is a function belonging to $C^{N-1}([-t_0, t_0]; \ell^2)$. □

3.3. APPLICATION OF THE TAYLOR FORMULA

Let P be the operator satisfying the assumptions of Lemma 3.4 and assume $Q = O(\Lambda^\mu)$. We will denote

$$F_{tP}(Q) := \mathrm{e}^{-\mathrm{i}tP} Q \mathrm{e}^{\mathrm{i}tP} \quad \text{for } t \in \mathbb{R}.$$

Let $x \in \mathrm{S}^{-\infty}$ and $\eta \in \mathbb{R}$. Then the function $t \to \Lambda^\eta F_{tP}(Q)x$ is smooth $\mathbb{R} \to \ell^2$ due to Corollary 3.7. Denoting

$$\mathrm{ad}^1_{\mathrm{i}P} Q = \mathrm{ad}_{\mathrm{i}P} Q := [Q, \mathrm{i}P] \quad \text{and} \quad \mathrm{ad}^{m+1}_{\mathrm{i}P} Q = [\mathrm{ad}^m_{\mathrm{i}P} Q, \mathrm{i}P],$$

we can express the m-th derivative with respect to $t \in \mathbb{R}$ in the form

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} F_{tP}(Q)x = F_{tP}(\mathrm{ad}^m_{\mathrm{i}P} Q)x \quad \text{for } x \in \mathrm{S}^{-\infty}$$

and by using the Taylor formula in $t = 1$, we obtain on $\mathrm{S}^{-\infty}$ the equality

$$\mathrm{e}^{-\mathrm{i}P} Q \mathrm{e}^{\mathrm{i}P} = Q + \sum_{m=1}^{N-1} \frac{1}{m!} \mathrm{ad}^m_{\mathrm{i}P} Q + R_N(\mathrm{ad}^N_{\mathrm{i}P} Q), \tag{3.21}$$

where

$$R_N(T) := \frac{1}{(N-1)!} \int_0^1 \mathrm{e}^{-\mathrm{i}tP} T \mathrm{e}^{\mathrm{i}tP} (1-t)^{N-1} \mathrm{d}t.$$

It is clear that Proposition 3.6 ensures

$$T = O(\Lambda^\nu) \implies R_N(T) = O(\Lambda^\nu). \tag{3.22}$$

4. ASYMPTOTIC FORMULA WITH ONE CORRECTION TERM FOR BAND MATRICES

4.1. STATEMENT OF THE RESULT

In this section we will prove the following result.

Theorem 4.1. *Let $\mu > 0$ and $\rho > 0$ be fixed. Assume that $d \in \mathrm{S}^\mu_1(2)$ satisfies* (H1) *and* (H2)*. Assume that $a_i \in \mathrm{S}^{\mu-\rho}_1(1)$ for $i = 1, \ldots, i_0$, and define the operator*

$$A = d(\Lambda) + \sum_{1 \le i \le i_0} \big(S^i a_i(\Lambda) + \mathrm{hc} \big) \tag{4.1}$$

on the domain $\mathcal{D}(A) = \ell^{2,\mu}$. Then A is a self-adjoint and bounded from below operator with compact resolvent in ℓ^2. Moreover,

(i) *the eigenvalue sequence of A satisfies*

$$\lambda_n(A) = d(n) + O(n^{\mu - 2\rho}),$$

(ii) *if* $d \in \mathrm{S}_1^{\mu}(3)$ *and* $a_i \in \mathrm{S}_1^{\mu-\rho}(2)$ *for* $i = 1, \dots, i_0$, *then*

$$\lambda_n(A) = d(n) + \mathfrak{r}_1(n) + O(n^{\mu-3\rho})$$

holds with

$$\mathfrak{r}_1(n) := \sum_{i=1}^{i_0} \left(\frac{|a_i(n-i)|^2}{d(n) - d(n-i)} - \frac{|a_i(n)|^2}{d(n+i) - d(n)} \right). \tag{4.2}$$

4.2. SOLUTION OF AN AUXILIARY MATRIX EQUATION

We observe that the assumptions (H1) and (H2) concern the sequence $(d(k))_{k=k_0}^{\infty}$, where $k_0 \in \mathbb{N}$ is fixed. Since any modification of $\{d(k) : k < k_0\}$ can be viewed as a perturbation R satisfying $R = \mathrm{O}(\Lambda^{-\infty})$, the corresponding error of the n-th eigenvalue is $O(n^{-\infty})$ due to Proposition 3.2. Neglecting such errors we may assume that (H2) holds with $k_0 = 1$, i.e.

$$\exists c > 0 \, \forall k \in \mathbb{N}^* \quad d(k+1) - d(k) \geq ck^{\mu-1}. \tag{4.3}$$

Lemma 4.2. *Let* $\mu > 0$, $\rho > 0$ *and consider a symmetric operator on* $\mathrm{S}^{-\infty}$ *given by*

$$A^0 = \sum_{1 \leq i \leq i_0} \big(S^i a_i(\Lambda) + \mathrm{hc} \big). \tag{4.4}$$

Assume that $a_i(k) = O(k^{\mu-\rho})$ *for* $i = 1, \dots, i_0$, $(d(k))_{k=1}^{\infty}$ *satisfies* (4.3) *and denote*

$$p_i(k) = \mathrm{i} \frac{a_i(k)}{d(k+i) - d(k)} = \mathrm{i} \frac{a_i(k)}{\partial_i d(k)}. \tag{4.5}$$

(i) *If* P *is expressed by* (3.13) *with* p_i *given by* (4.5), *then*

$$[D, \mathrm{i}P] + A^0 = 0. \tag{4.6}$$

(ii) *If* $d \in \mathrm{S}_1^{\mu}(N+1)$ *and* $a_i \in \mathrm{S}_1^{\mu-\rho}(N)$, *then* $p_i \in \mathrm{S}_1^{1-\rho}(N)$.

Proof. (i) By (2.2), we have

$$[D, \mathrm{i}P] = \sum_{1 \leq i \leq i_0} \Big([d(\Lambda), \mathrm{i}S^i p_i(\Lambda)] + \mathrm{hc} \Big) = \sum_{1 \leq i \leq i_0} \Big(S^i \partial_i d(\Lambda) \mathrm{i} p_i(\Lambda) + \mathrm{hc} \Big)$$

and (4.6) holds if and only if $\partial_i d(k) \mathrm{i} p_i(k) + a_i(k) = 0$ for all $k \in \mathbb{N}^*$.

(ii) It suffices to use Lemma 2.4. □

4.3. PROOF OF THEOREM 4.1(i)

Step 1. Let A be given by (4.1). Then

$$A = D + A^0$$

holds with $D = d(\Lambda)$ and A^0 given by (4.4). Let $P \in \mathrm{FDO}_1^{1-\rho}(1)$ be as in Section 4.2. According to Section 3.2 the operator P is self-adjoint and we claim that

$$\mathrm{e}^{-\mathrm{i}P} A^0 \mathrm{e}^{\mathrm{i}P} = A^0 + O(\Lambda^{\mu-2\rho}). \tag{4.7}$$

Indeed, $A^0 \in \mathrm{FDO}_1^{\mu-\rho}(1)$ and $P \in \mathrm{FDO}_1^{1-\rho}(1)$ ensure $[A^0, P] \in \mathrm{FDO}_1^{\mu-2\rho}(0)$ due to Lemma 2.9, hence $[A^0, P] = O(\Lambda^{\mu-2\rho})$ and the Taylor formula gives

$$\mathrm{e}^{-\mathrm{i}P} A^0 \mathrm{e}^{\mathrm{i}P} = A^0 + R_1([A^0, \mathrm{i}P]),$$

where $R_1([A^0, \mathrm{i}P]) = O(\Lambda^{\mu-2\rho})$ follows from $[A^0, \mathrm{i}P] = O(\Lambda^{\mu-2\rho})$ due to (3.22).

Step 2. We claim that

$$\mathrm{e}^{-\mathrm{i}P} D \mathrm{e}^{\mathrm{i}P} = D - A^0 + O(\Lambda^{\mu-2\rho}). \tag{4.8}$$

Indeed, by using $[D, \mathrm{i}P] = -A^0$ in the Taylor formula

$$\mathrm{e}^{-\mathrm{i}P} D \mathrm{e}^{\mathrm{i}P} = D + [D, \mathrm{i}P] + R_2([[D, \mathrm{i}P], \mathrm{i}P]),$$

we obtain

$$\mathrm{e}^{-\mathrm{i}P} D \mathrm{e}^{\mathrm{i}P} = D - A^0 - R_2([A^0, \mathrm{i}P])$$

and $R_2([A^0, \mathrm{i}P]) = \mathrm{O}(\Lambda^{\mu-2\rho})$ follows from $[A^0, \mathrm{i}P] = O(\Lambda^{\mu-2\rho})$ due to (3.22).

Step 3. Denote $\widehat{A} := \mathrm{e}^{-\mathrm{i}P} A \mathrm{e}^{\mathrm{i}P}$. Summing up (4.7) and (4.8), we obtain

$$\widehat{A} = \mathrm{e}^{-\mathrm{i}P}(D + A^0)\mathrm{e}^{\mathrm{i}P} = D + O(\Lambda^{\mu-2\rho}). \tag{4.9}$$

Using (4.9) and Proposition 3.2, we obtain

$$\lambda_n(A) = \lambda_n(\widehat{A}) = \lambda_n(D) + O(n^{\mu-2\rho}) = d(n) + \mathrm{O}(n^{\mu-2\rho}).$$

4.4. PROOF OF THEOREM 4.1(ii)

Step 1. As before we write $A = D + A^0$ and P is defined as in Section 4.2. Then

$$A^0 \in \mathrm{FDO}_1^{\mu-\rho}(2), \quad P \in \mathrm{FDO}_1^{1-\rho}(2) \quad \Rightarrow [A^0, P] \in \mathrm{FDO}_1^{\mu-2\rho}(1)$$

follows from Lemma 2.9 and we claim that the diagonal part of $[A^0, \mathrm{i}P]$ satisfies

$$[A^0, \mathrm{i}P](k,k) = 2\mathfrak{r}_1(k),$$

where $\mathfrak{r}_1(k)$ is given by (4.2). Indeed, for $i = 1, \ldots, i_0$, we have

$$[S^i a_i(\Lambda), \mathrm{i}\bar{p}_i(\Lambda)(S^i)^*] = S^i(\mathrm{i}a_i\bar{p}_i)(\Lambda)(S^i)^* - (\mathrm{i}a_i\bar{p}_i)(\Lambda) = \partial_{-i}(\mathrm{i}a_i\bar{p}_i)(\Lambda)$$

and the diagonal part of $[A^0, \mathrm{i}P]$ equals

$$\sum_{1 \le i \le i_0} \Big([S^i a_i(\Lambda), \mathrm{i}\bar{p}_i(\Lambda)(S^i)^*] + \mathrm{hc} \Big) = \sum_{1 \le i \le i_0} q_i(\Lambda),$$

where $q_i(\Lambda) = \partial_{-i}(\mathrm{i}a_i\bar{p}_i)(\Lambda) + \mathrm{hc}$, i.e.

$$q_i = 2\mathrm{Re}\partial_{-i}(\mathrm{i}a_i\bar{p}_i) = -2\partial_{-i}\mathrm{Im}(a_i\bar{p}_i)$$

and

$$[A^0, \mathrm{i}P](k,k) = \sum_{1\le i\le i_0} q_i(k) = \sum_{1\le i\le i_0} 2\partial_{-i}\left(\frac{|a_i(k)|^2}{\partial_i d(k)}\right) = 2\mathfrak{r}_1(k).$$

Step 2. We claim that

$$\mathrm{e}^{-\mathrm{i}P}A^0\mathrm{e}^{\mathrm{i}P} = A^0 + [A^0, \mathrm{i}P] + O(\Lambda^{\mu-3\rho}). \tag{4.10}$$

Indeed, Lemma 2.9 ensures

$$[A^0, \mathrm{i}P] \in \mathrm{FDO}_1^{\mu-2\rho}(1), \quad P \in \mathrm{FDO}_1^{1-\rho}(2) \quad \Longrightarrow \quad \mathrm{ad}^2_{\mathrm{i}P}A^0 \in \mathrm{FDO}_1^{\mu-3\rho}(0),$$

hence $\mathrm{ad}^2_{\mathrm{i}P}A^0 = O(\Lambda^{\mu-3\rho})$ and the Taylor formula gives

$$\mathrm{e}^{-\mathrm{i}P}A^0\mathrm{e}^{\mathrm{i}P} = A^0 + [A^0, \mathrm{i}P] + R_2(\mathrm{ad}^2_{\mathrm{i}P}A^0),$$

where $R_2(\mathrm{ad}^2_{\mathrm{i}P}A^0) = O(\Lambda^{\mu-3\rho})$ follows from $\mathrm{ad}^2_{\mathrm{i}P}A^0 = O(\Lambda^{\mu-3\rho})$ and (3.22).

Step 3. We claim that

$$\mathrm{e}^{-\mathrm{i}P}D\mathrm{e}^{\mathrm{i}P} = D - A^0 - \tfrac{1}{2}[A^0, \mathrm{i}P] + O(\Lambda^{\mu-3\rho}). \tag{4.11}$$

Indeed, it suffices to use $[D, \mathrm{i}P] = -A^0$ in the Taylor formula

$$\mathrm{e}^{-\mathrm{i}P}D\mathrm{e}^{\mathrm{i}P} = D + [D, \mathrm{i}P] + \tfrac{1}{2}[[D, \mathrm{i}P], \mathrm{i}P] + R_3(\mathrm{ad}^3_{\mathrm{i}P}D)$$

and $R_3(\mathrm{ad}^3_{\mathrm{i}P}D) = O(\Lambda^{\mu-3\rho})$ follows from $\mathrm{ad}^3_{\mathrm{i}P}D = -\mathrm{ad}^2_{\mathrm{i}P}A^0 = O(\Lambda^{\mu-3\rho})$.

Step 4. Denote $\widehat{A} := \mathrm{e}^{-\mathrm{i}P}A\mathrm{e}^{\mathrm{i}P}$. Summing up (4.10) and (4.11), we obtain

$$\widehat{A} = \mathrm{e}^{-\mathrm{i}P}(D + A^0)\mathrm{e}^{\mathrm{i}P} = D + \tfrac{1}{2}[A^0, \mathrm{i}P] + O(\Lambda^{\mu-3\rho}) \tag{4.12}$$

and Proposition 3.2 ensures

$$\lambda_n(A) = \lambda_n(\widehat{A}) = \lambda_n(A_1) + O(n^{\mu-3\rho}), \tag{4.13}$$

where we denoted

$$A_1 := D + \tfrac{1}{2}[A^0, \mathrm{i}P].$$

Let

$$d_1(k) := d(k) + \frac{1}{2}[A^0, \mathrm{i}P](k,k) = d(k) + \mathfrak{r}_1(k).$$

Then $D_1 = d_1(\Lambda)$ is the diagonal part of A_1 and we can write

$$A_1 = d_1(\Lambda) + A_1^0,$$

where $A_1^0 \in \mathrm{FDO}_1^{\mu-2\rho}(1)$ is off-diagonal. Thus Theorem 4.1(i) can be applied to A_1 with $d_1(k) = d(k) + \mathfrak{r}_1(k)$ instead of $d(k)$ and 2ρ instead of ρ. We obtain

$$\lambda_n(A_1) = d_1(n) + O(n^{\mu-4\rho}),$$

which completes the proof due to (4.13).

5. A METHOD OF APPROXIMATIVE DIAGONALIZATION

5.1. INTRODUCTION

In this section J satisfies (H1)–(H3) and assuming (1.7)–(1.8), we describe a special sequence of operators $(J_N^{(l)})_{l=0}^{\infty}$ such that $J_N^{(l)}$ is self-adjoint in ℓ^2 on the domain $\mathcal{D}(J_N^{(l)}) = \ell^{2,\mu}$ for any l, $N \in \mathbb{N}$. Moreover $J_N^{(l)}$ is a bounded from below operator with compact resolvent and the eigenvalue sequence $(\lambda_n(J_N^{(l)}))_{n=1}^{\infty}$ satisfies

$$\lambda_n(J) = \lambda_n(J_N^{(l)}) + O(n^{\mu-\rho N}). \tag{5.1}$$

Besides (5.1), the operators $(J_N^{(l)})_{l=0}^{\infty}$ have a special structure of band matrices (see (5.3), (5.4)) and in Section 5.2 we describe its construction. Using $(J_N^{(l)})_{l=0}^{\infty}$ and suitable classes of operators defined in Section 5.3, we complete the proof of Theorem 1.1 in Section 5.4.

Notation 5.1. If $\nu \in \mathbb{R}$ then FDO_1^{ν} denotes the set of all linear operators on $\mathrm{S}^{-\infty}$ of the form

$$P = \sum_{-i_0 \le i \le i_0} S^{(i)} p_i(\Lambda) \ \text{ with } p_i \in \mathrm{S}_1^{\nu}$$
$$\text{for } i \in [-i_0, i_0] \cap \mathbb{Z}.$$

Clearly Lemma 2.4 holds with S_1^{ν} instead of $\mathrm{S}_1^{\nu}(N)$ and the proof of Lemma 2.9 ensures

$$P \in \mathrm{FDO}_1^{\nu}, \quad Q \in \mathrm{FDO}_1^{\eta} \implies [P, Q] \in \mathrm{FDO}_1^{\nu+\eta-1}. \tag{5.2}$$

The structure of $(J_N^{(l)})_{l=0}^{\infty}$ is the following. We write $\rho_l := 2^l \rho$ and construct $J_N^{(l)}$ such that $J_N^{(0)} = J$ and

$$J_N^{(l)} = d_N^{(l)}(\Lambda) + A_N^{(l)} \text{ with } A_N^{(l)} \in \mathrm{FDO}_1^{\mu-\rho_l}, \tag{5.3}$$

where $A_N^{(l)}$ is off-diagonal and

$$d_N^{(l)}(k) = d(k) + \sum_{1 \le m \le l} \mathfrak{r}_{m,N}(k) \text{ with } \mathfrak{r}_{m,N} \in \mathrm{S}_1^{\mu-\rho_m}. \tag{5.4}$$

Remark 5.2. Since (5.4) ensures $d_N^{(l)} - d \in \mathrm{S}_1^{\mu-2\rho}$, (H1) and (H2) ensure

$$d_N^{(l)}(k) \ge \tfrac{1}{2} c k^{\mu} \text{ for } k \ge k_N^{(l)}, \tag{5.5}$$

$$\partial d_N^{(l)}(k) \ge \tfrac{1}{2} c k^{\mu-1} \text{ for } k \ge k_N^{(l)}, \tag{5.6}$$

where $k_N^{(l)} \in \mathbb{N}^*$ is fixed large enough.

Remark 5.3. It is clear that (5.3) implies

$$J_N^{(l)} = d_N^{(l)}(\Lambda) + O(\Lambda^{\mu-\rho_l}). \tag{5.7}$$

Thus the operator $J_N^{(l)}$ has the domain $\ell^{2,\mu}$ and is a bounded from below self-adjoint operator with compact resolvent in ℓ^2 due to (5.4)–(5.5).

Remark 5.4. The estimate (5.1) ensures

$$\lambda_n(J) = d_N^{(l)}(n) + O(n^{\mu-N\rho}) \text{ if } l \geq \log_2 N. \tag{5.8}$$

Indeed, using (5.7) and Proposition 3.2, we obtain

$$\lambda_n(J_N^{(l)}) = \lambda_n(d_N^{(l)}(\Lambda)) + O(n^{\mu-\rho_l}). \tag{5.9}$$

Combining (5.9), (5.1) with $\rho_l = 2^l\rho \geq N\rho$, we obtain

$$\lambda_n(J) = \lambda_n(d_N^{(l)}(\Lambda)) + O(n^{\mu-N\rho}) \text{ if } l \geq \log_2 N. \tag{5.10}$$

To deduce (5.8) from (5.10), we observe that due to (5.5) and (5.6) we can find $n_N^{(l)} \in \mathbb{N}^*$ large enough to ensure $\lambda_n(d_N^{(l)}(\Lambda))) = d_N^{(l)}(n)$ for $n \geq n_N^{(l)}$.

5.2. CONSTRUCTION OF THE SEQUENCE $(J_N^{(l)})_{l=0}^{\infty}$

We fix N and skip this index writing $J^{(l)}$ instead of $J_N^{(l)}$. If $l = 0$, then $J^{(0)} = J$ and (5.3) takes the form

$$J^{(0)} = d(\Lambda) + A^{(0)} \text{ with } A^{(0)} = Sa(\Lambda) + \text{hc},$$

i.e. $A^{(0)} \in \text{FDO}_1^{\mu-\rho_0}$ holds with $\rho_0 = \rho$.

Next we fix $l \in \mathbb{N}$ and make the induction hypothesis that $J^{(l)}$ has the properties given in Section 5.1. Our purpose is to construct $J^{(l+1)}$ having analogical properties for $l+1$ instead of l. At the beginning we assume that $A^{(l)} \in \text{FDO}_1^{\mu-\rho_l}$ is off-diagonal, i.e.

$$A^{(l)} = \sum_{1\leq i\leq i_l} \big(S^i a_i^{(l)}(\Lambda) + \text{hc}\big) \text{ with } a_i^{(l)} \in \text{S}_1^{\mu-\rho_l}.$$

Let $k^{(l)} \in \mathbb{N}^*$ be fixed large enough. Then $\partial d^{(l)}(k) \geq \frac{1}{2}ck^{\mu-1}$ holds for $k \geq k^{(l)}$ (see (5.6)), hence we can define $p_i^{(l)} \in \text{S}_1^{1-\rho_l}$ by the formula

$$p_i^{(l)}(k) = \mathrm{i}\frac{a_i^{(l)}(k)}{d^{(l)}(k+i) - d^{(l)}(k)} \text{ for } k \geq k^{(l)}$$

and $p_i^{(l)}(k) = 0$ if $k < k^{(l)}$. Similarly as in Section 4.2 we find that the operator

$$P^{(l)} = \sum_{1\leq i\leq i_l} \big(S^i p_i^{(l)}(\Lambda) + \text{hc}\,\big)$$

satisfies

$$[d^{(l)}(\Lambda), \mathrm{i}P^{(l)}] + A^{(l)} = O(\Lambda^{-\infty}). \tag{5.11}$$

We observe that (5.2) ensures

$$P^{(l)} \in \mathrm{FDO}_1^{1-\rho_l}, \quad A^{(l)} \in \mathrm{FDO}_1^{\mu-\rho_l} \implies [P^{(l)}, A^{(l)}] \in \mathrm{FDO}_1^{\mu-2\rho_l}$$

and it is easy to see that using induction, we obtain

$$\mathrm{ad}_{\mathrm{i}P^{(l)}}^{m-1} A^{(l)} \in \mathrm{FDO}_1^{\mu-m\rho_l} \text{ holds for } m \geq 2. \tag{5.12}$$

Next we choose $m_l \in \mathbb{N}^*$ large enough and write the Taylor formula

$$\mathrm{e}^{-\mathrm{i}P^{(l)}} D^{(l)} \mathrm{e}^{\mathrm{i}P^{(l)}} = D^{(l)} + \sum_{m=1}^{m_l-1} \frac{1}{m!} \mathrm{ad}_{\mathrm{i}P^{(l)}}^{m} D^{(l)} + R_{m_l}(\mathrm{ad}_{\mathrm{i}P^{(l)}}^{m_l} D^{(l)}), \tag{5.13}$$

where $D^{(l)} := d^{(l)}(\Lambda)$. Since $\mathrm{ad}_{\mathrm{i}P^{(l)}}^{m} D^{(l)} = -\mathrm{ad}_{\mathrm{i}P^{(l)}}^{m-1} A^{(l)} + O(\Lambda^{-\infty})$ holds due to (5.11), we can rewrite (5.13) in the form

$$\mathrm{e}^{-\mathrm{i}P^{(l)}} D^{(l)} \mathrm{e}^{\mathrm{i}P^{(l)}} = D^{(l)} - A^{(l)} - \sum_{m=2}^{m_l-2} \frac{1}{m!} \mathrm{ad}_{\mathrm{i}P^{(l)}}^{m-1} A^{(l)} + O(\Lambda^{\mu-\rho_l m_l}), \tag{5.14}$$

where the estimate of last term follows from $\mathrm{ad}_{\mathrm{i}P^{(l)}}^{m_l-1} A^{(l)} = O(\Lambda^{\mu-\rho_l m_l})$ and (3.22). Writing $m-1$ instead of m in the Taylor's formula (3.21), we get

$$\mathrm{e}^{-\mathrm{i}P^{(l)}} A^{(l)} \mathrm{e}^{\mathrm{i}P^{(l)}} = A^{(l)} + \sum_{m=2}^{m_l-2} \frac{1}{(m-1)!} \mathrm{ad}_{\mathrm{i}P^{(l)}}^{m-1} A^{(l)} + O(\Lambda^{\mu-\rho_l m_l}). \tag{5.15}$$

Combining (5.15) with (5.14), we obtain

$$\mathrm{e}^{-\mathrm{i}P^{(l)}} (D^{(l)} + A^{(l)}) \mathrm{e}^{\mathrm{i}P^{(l)}} = D^{(l)} + \tilde{A}^{(l)} + O(\Lambda^{\mu-\rho_l m_l}), \tag{5.16}$$

where

$$\tilde{A}^{(l)} := \sum_{m=2}^{m_l-2} \Big(\frac{1}{(m-1)!} - \frac{1}{m!} \Big) \mathrm{ad}_{\mathrm{i}P^{(l)}}^{m-1} A^{(l)} \in \mathrm{FDO}_1^{\mu-2\rho_l}. \tag{5.17}$$

Let us denote

$$J^{(l+1)} := D^{(l)} + \tilde{A}^{(l)}. \tag{5.18}$$

Using (5.18) and $J^{(l)} = D^{(l)} + A^{(l)}$, we find that (5.16) gives

$$\mathrm{e}^{-\mathrm{i}P^{(l)}} J^{(l)} \mathrm{e}^{\mathrm{i}P^{(l)}} = J^{(l+1)} + O(\Lambda^{\mu-m_l\rho_l}). \tag{5.19}$$

Applying Proposition 3.2, we find that (5.19) ensures

$$\lambda_n(J^{(l)}) = \lambda_n(\mathrm{e}^{-\mathrm{i}P^{(l)}} J^{(l)} \mathrm{e}^{\mathrm{i}P^{(l)}}) = \lambda_n(J^{(l+1)}) + O(n^{\mu-m_l\rho_l}). \tag{5.20}$$

Let $A^{(l+1)}$ be the off-diagonal part of $\tilde{A}^{(l)}$ and let $\mathfrak{r}_{l+1}(\Lambda)$ be its diagonal part. Then

$$\tilde{A}^{(l)} = \mathfrak{r}_{l+1}(\Lambda) + A^{(l+1)},$$

where $A^{(l+1)} \in \mathrm{FDO}_1^{\mu-\rho_{l+1}}$, $\mathfrak{r}_{l+1} \in \mathrm{S}_1^{\mu-\rho_{l+1}}$ due to (5.17) and $\rho_{l+1} = 2\rho_l$. Thus setting

$$d^{(l+1)} := d^{(l)} + \mathfrak{r}_{l+1}$$

we find

$$J^{(l+1)} = d^{(l)}(\Lambda) + \mathfrak{r}_{l+1}(\Lambda) + A^{(l+1)} = d^{(l+1)}(\Lambda) + A^{(l+1)},$$

i.e. (5.3), (5.4) hold with $l+1$ instead of l. Finally (5.1) and (5.20) ensure

$$\lambda_n(J) = \lambda_n(J^{(l+1)}) + O(n^{\mu - m_l\rho_l}) + O(n^{\mu - N\rho}),$$

hence (5.1) holds with $l+1$ instead of l if $m_l\rho_l \geq N\rho$, i.e. if we choose $m_l \geq 2^{-l}N$.

5.3. AUXILIARY CLASSES OF SYMBOLS AND OPERATORS

Notation 5.5. If $\nu \in \mathbb{R}$, $\rho > 0$, then $\mathrm{S}^{\nu}_{[1,2\rho]}$ denotes the set of $p : \mathbb{N}^* \to \mathbb{C}$ such that

$$p(k) \sim \sum_{i,j=0}^{\infty} \gamma_{i,j} k^{\nu - 2\rho j - i} \tag{5.21}$$

holds for a certain complex valued sequence $(\gamma_{i,j})_{i,j=0}^{\infty}$. The formula (5.21) means that

$$p(k) = \sum_{i,j=0}^{N-1} \gamma_{i,j} k^{\nu-2\rho j - i} + O(k^{\nu - N\min\{2\rho,1\}})$$

holds for any $N \in \mathbb{N}^*$. One checks easily the following properties

$$\mathrm{S}^{\nu-2k\rho-m}_{[1,2\rho]} \subset \mathrm{S}^{\nu}_{[1,2\rho]} \text{ if } k, m \in \mathbb{N}, \tag{5.22}$$

$$p \in \mathrm{S}^{\nu}_{[1,2\rho]} \quad \Longrightarrow \quad \tau_i p \in \mathrm{S}^{\nu}_{[1,2\rho]}, \quad \partial_i p \in \mathrm{S}^{\nu-1}_{[1,2\rho]},$$

$$p \in \mathrm{S}^{\nu}_{[1,2\rho]}, \quad q \in \mathrm{S}^{\eta}_{[1,2\rho]} \quad \Longrightarrow \quad pq \in \mathrm{S}^{\nu+\eta}_{[1,2\rho]}.$$

If there exists $c > 0$ such that $|p(k)| \geq ck^{\nu}$ for all $k \in \mathbb{N}^*$, then

$$p \in \mathrm{S}^{\nu}_{[1,2\rho]} \quad \Longrightarrow \quad 1/p \in \mathrm{S}^{-\nu}_{[1,2\rho]}.$$

Notation 5.6. We denote by $\mathrm{FDO}^{\nu}_{[1,2\rho]}$ the set of all linear operators on $\mathrm{S}^{-\infty}$ of the form

$$P = \sum_{-i_0 \leq i \leq i_0} S^{(i)} p_i(\Lambda) \text{ with } p_i \in \mathrm{S}^{\nu-\rho|i|}_{[1,2\rho]} \text{ for } i \in [-i_0, i_0] \cap \mathbb{Z}.$$

Lemma 5.7. *If $P \in \mathrm{FDO}^{\nu}_{[1,2\rho]}$ and $Q \in \mathrm{FDO}^{\eta}_{[1,2\rho]}$ then $[P,Q] \in \mathrm{FDO}^{\nu+\eta-1}_{[1,2\rho]}$.*

Proof. It suffices to check that the right hand side of (2.9) belongs to $\mathrm{FDO}^{\nu+\mu-1}_{[1,2\rho]}$, i.e.

$$q_j \partial_j p_i - p_i \partial_i q_j \in \mathrm{S}^{\nu+\eta-1-\rho|i+j|}_{[1,2\rho]}. \tag{5.23}$$

If $q_j \in \mathrm{S}^{\eta-\rho|j|}_{[1,2\rho]}$, $p_i \in \mathrm{S}^{\nu-\rho|i|}_{[1,2\rho]}$, then $\partial_i q_j \in \mathrm{S}^{\eta-1-\rho|j|}_{[1,2\rho]}$, $\partial_j p_i \in \mathrm{S}^{\nu-1-\rho|i|}_{[1,2\rho]}$ and

$$q_j \partial_j p_i - p_i \partial_i q_j \in \mathrm{S}^{\nu+\eta-1-\rho|i|-\rho|j|}_{[1,2\rho]}.$$

Hence, (5.23) follows if we know that

$$\mathrm{S}^{\nu+\eta-1-\rho|i|-\rho|j|}_{[1,2\rho]} \subset \mathrm{S}^{\nu+\eta-1-\rho|i+j|}_{[1,2\rho]}.$$

We deduce the last inclusion using (5.22) with $k = \frac{1}{2}(|i| + |j| - |i+j|)$, $m = 0$ and $\nu + \eta - 1$ instead of ν. Indeed, it is easy to see that $\frac{1}{2}(|i| + |j| - |i+j|) \in \mathbb{N}$ for every i, $j \in \mathbb{Z}$. □

5.4. PROOF OF THEOREM 1.1

The assertion of Theorem 1.1 follows from

Theorem 5.8. *Let $\mu > 0$, $\rho > 0$ and let J be given by* (1.2) *with entries satisfying* (H1)–(H3). *If $d \in \mathrm{S}^{\mu}_{[1,2\rho]}$ and $a \in \mathrm{S}^{\mu-\rho}_{[1,2\rho]}$, then $n \to \lambda_n(J)$ belongs to $\mathrm{S}^{\mu}_{[1,2\rho]}$.*

Proof. Let $N \in \mathbb{N}^*$ and as in Section 5.2 denote $J^{(l)}$ instead of $J^{(l)}_N$. We claim that

$$J^{(l)} \in \mathrm{FDO}^{\mu}_{[1,2\rho]} \tag{5.24}$$

holds for every $l \in \mathbb{N}$. Indeed, $J^{(0)} = d(\Lambda) + (Sa(\Lambda) + \mathrm{hc}) \in \mathrm{FDO}^{\mu}_{[1,2\rho]}$ follows from the assumptions $d \in \mathrm{S}^{\mu}_{[1,2\rho]}$ and $a \in \mathrm{S}^{\mu-\rho}_{[1,2\rho]}$. Assume now that (5.24) holds for a given $l \in \mathbb{N}$. Then $a^{(l)}_i \in \mathrm{S}^{\mu-\rho|i|}_{[1,2\rho]}$ and $\partial_k d^{(l)} \in \mathrm{S}^{\mu-1}_{[1,2\rho]}$ implies $p^{(l)}_i \in \mathrm{S}^{1-\rho|i|}_{[1,2\rho]}$, hence $P^{(l)} \in \mathrm{FDO}^{1}_{[1,2\rho]}$ and $[P^{(l)}, A^{(l)}] \in \mathrm{FDO}^{\mu}_{[1,2\rho]}$ due to Lemma 5.7.

We observe that $\mathrm{ad}^{m}_{\mathrm{i}P^{(l)}} A^{(l)} \in \mathrm{FDO}^{\mu}_{[1,2\rho]}$ follows by induction with respect to m, hence $\tilde{A}^{(l)} \in \mathrm{FDO}^{\mu}_{[1,2\rho]}$ and (5.24) still holds with $l+1$ instead of l.

In order to deduce that $n \to \lambda_n(J)$ belongs to $\mathrm{S}^{\mu}_{[1,2\rho]}$, it suffices to observe that for any $N \in \mathbb{N}^*$ one has the estimate (5.8) and $d^{(l)}_N \in \mathrm{S}^{\mu}_{[1,2\rho]}$ holds due to (5.24). □

6. EXPRESSION OF THE SECOND CORRECTION TERM

6.1. STATEMENT OF THE RESULT

Notation 6.1. If i, $j \in \mathbb{Z}$ and a, $b : \mathbb{N}^* \to \mathbb{C}$, then we denote

$$\{a, b\}_{i,j}(k) := b(k)\partial_j a(k) - a(k)\partial_i b(k),$$

$$\{a, b\}(k) := \{a, b\}_{1,1}(k) = b(k)\partial a(k) - a(k)\partial b(k).$$

Theorem 6.2. *Let $\mu > 0$ and $\rho > 0$. Let $a \in \mathrm{S}_1^{\mu-\rho}$, let $d \in \mathrm{S}_1^{\mu}$ be such that* (H1), (H2) *hold for all $k \in \mathbb{N}^*$ and define*

$$p := \mathrm{i}\frac{a}{\partial d}, \tag{6.1}$$

$$\mathfrak{r}_1 := -\partial_{-1}\mathrm{Im}(a\bar{p}) = \partial_{-1}\Big(\frac{|a|^2}{\partial d}\Big). \tag{6.2}$$

If J is given by (1.2)*, then its eigenvalue sequence $(\lambda_n(J))_{n=1}^{\infty}$ satisfies*

$$\lambda_n(J) = d(n) + \mathfrak{r}_1(n) + \mathfrak{r}_2(n) + O(n^{\mu-6\rho}),$$

where $\mathfrak{r}_1$ is given by (6.2) *and*

$$\mathfrak{r}_2 = \mathfrak{r}_{2,1} + \mathfrak{r}_{2,2} + \mathfrak{r}_{2,3} \tag{6.3}$$

holds with

$$\mathfrak{r}_{2,1} := -\tfrac{1}{2}\partial_{-1}(|p|^2\partial\mathfrak{r}_1), \tag{6.4}$$

$$\mathfrak{r}_{2,2} := \tfrac{1}{4}\partial_{-1}\mathrm{Im}(\bar{p}\{\{a,p\},\tau_{-1}\bar{p}\}_{2,-1}), \tag{6.5}$$

$$\mathfrak{r}_{2,3} := \tfrac{1}{4}\partial_{-2}\Big(\frac{|\{a,p\}|^2}{\partial_2 d}\Big). \tag{6.6}$$

6.2. ANALYSIS OF COMMUTATORS

As in Section 4 we assume that $(d(k))_{k=1}^{\infty}$ is strictly increasing. We also assume that p is given by (6.1), $\mathfrak{r}_1$ is given by (6.2) and the operators $A^{(0)} \in \mathrm{FDO}_1^{\mu-\rho}$, $P \in \mathrm{FDO}_1^{1-\rho}$ are defined by

$$A^{(0)} = Sa(\Lambda) + \mathrm{hc}, \tag{6.7}$$

$$P = Sp(\Lambda) + \mathrm{hc}. \tag{6.8}$$

Lemma 6.3. *If $A^{(0)}$ and P are given by* (6.7) *and* (6.8)*, then*

$$\mathrm{ad}_{\mathrm{i}P}A^{(0)} = q_0(\Lambda) + (S^2q_2(\Lambda) + \mathrm{hc}), \tag{6.9}$$

$$\mathrm{ad}_{\mathrm{i}P}^2A^{(0)} = (Sq_1'(\Lambda) + S^3q_3'(\Lambda)) + \mathrm{hc}, \tag{6.10}$$

$$\mathrm{ad}_{\mathrm{i}P}^3A^{(0)} = q_0''(\Lambda) + (S^2q_2''(\Lambda) + \mathrm{hc}) + (S^4q_4''(\Lambda) + \mathrm{hc}), \tag{6.11}$$

$$\mathrm{ad}_{\mathrm{i}P}^4A^{(0)} = (Sq_1'''(\Lambda) + S^3q_3'''(\Lambda) + S^5q_5'''(\Lambda)) + \mathrm{hc} \tag{6.12}$$

hold with

$$q_0 = 2\mathfrak{r}_1, \tag{6.13}$$

$$q_2 = \{a, \mathrm{i}p\}, \tag{6.14}$$

$$q_1' := 2\mathrm{i}p\partial\mathfrak{r}_1 - \{\{a,p\},\tau_{-1}\bar{p}\}_{2,-1}, \tag{6.15}$$

$$q_0'' = -2\partial_{-1}\mathrm{Im}(q_1'\bar{p}) \tag{6.16}$$

and

$$q_i \in \mathrm{S}_1^{\mu-2\rho}, \quad q_i' \in \mathrm{S}_1^{\mu-3\rho}, \quad q_i'' \in \mathrm{S}_1^{\mu-4\rho}, \quad q_i''' \in \mathrm{S}_1^{\mu-5\rho} \tag{6.17}$$

hold for any i.

Proof. In this proof we deal with operators of the form $S^{(i)}p_i(\Lambda)$, which are written in the shortened way as $S^{(i)}p_i$. In particular, the formula (2.9) expressed in the shortened way, takes the form

$$[S^{(i)}p_i, S^{(j)}q_j] = S^{(i+j)}\{p_i, q_j\}_{i,j}.$$

Step 1. Analysis of $Q := \mathrm{ad}_{\mathrm{i}P}A^{(0)}$. The diagonal part of $Q = [Sa + \mathrm{hc}, \mathrm{i}P]$ is

$$[Sa, \mathrm{i}\bar{p}S^*] + \mathrm{hc} = \partial_{-1}(\mathrm{i}a\bar{p}) + \mathrm{hc} = 2\partial_{-1}\mathrm{Re}(\mathrm{i}a\bar{p}) = -2\partial_{-1}\mathrm{Im}(a\bar{p}) = 2\mathfrak{r}_1 \tag{6.18}$$

and the off-diagonal part of $[Sa + \mathrm{hc}, \mathrm{i}P]$ is

$$[Sa, \mathrm{i}Sp] + \mathrm{hc} = S^2\{a, \mathrm{i}p\} + \mathrm{hc}.$$

Thus (6.9) holds with q_0, q_2 given by (6.13) and (6.14).
Step 2. Analysis of $Q' := \mathrm{ad}^2_{\mathrm{i}P}A^{(0)}$. Combining (6.9) with $Q' = [Q, \mathrm{i}P]$, we get

$$Q' = [q_0, \mathrm{i}P] + [S^2q_2 + \mathrm{hc}, \mathrm{i}S^*\tau_{-1}\bar{p}] + [S^2q_2 + \mathrm{hc}, \mathrm{i}Sp] = Q'_1 + \widetilde{Q}'_1 + Q'_3,$$

where

$$\begin{aligned} Q'_1 &= [q_0, \mathrm{i}P] = [q_0, \mathrm{i}Sp] + \mathrm{hc} = S\mathrm{i}p\partial q_0 + \mathrm{hc}, \\ \widetilde{Q}'_1 &= [S^2q_2, \mathrm{i}S^{(-1)}\tau_{-1}\bar{p}] + \mathrm{hc} = S\mathrm{i}\{q_2, \tau_{-1}\bar{p}\}_{2,-1} + \mathrm{hc}, \\ Q'_3 &= [S^2q_2, \mathrm{i}Sp] + \mathrm{hc} = S^3\mathrm{i}\{q_2, p\}_{2,1} + \mathrm{hc} = S^3q'_3 + \mathrm{hc}. \end{aligned}$$

Thus (6.10) holds with $q'_1 = \mathrm{i}p\partial q_0 + \mathrm{i}\{q_2, \tau_{-1}\bar{p}\}_{2,-1}$ and we get (6.15) from (6.13)–(6.14).
Step 3. Analysis of $Q'' := \mathrm{ad}^3_{\mathrm{i}P}A^{(0)}$. Using (6.10), (6.15), we find that the diagonal part of $Q'' = [Q', \mathrm{i}P]$ can be obtained, similarly as in (6.18), by taking q'_1 instead of a, i.e.

$$[Sq'_1, \mathrm{i}\bar{p}S^*] + \mathrm{hc} = 2\partial_{-1}\mathrm{Re}(\mathrm{i}q'_1\bar{p}) = -2\partial_{-1}\mathrm{Im}(q'_1\bar{p}) = q''_0.$$

The off-diagonal part of Q'' is the sum of

$$\begin{aligned} [Sq'_1, \mathrm{i}Sp] + \mathrm{hc} &= S^2\mathrm{i}\{q'_1, p\} + \mathrm{hc}, \\ [S^3q'_3, \mathrm{i}S^{(-1)}\tau_{-1}\bar{p}] + \mathrm{hc} &= S^2\mathrm{i}\{q'_3, \tau_{-1}\bar{p}\}_{3,-1} + \mathrm{hc}, \\ [S^3q'_3, \mathrm{i}Sp] + \mathrm{hc} &= S^4\mathrm{i}\{q'_3, p\}_{3,1} + \mathrm{hc}. \end{aligned}$$

Step 4. End of the proof. Denote $Q''' := \mathrm{ad}^4_{\mathrm{i}P}A^{(0)}$. By using (6.11), we find easily that the commutator $Q''' = [Q'', \mathrm{i}P]$ has the form given in (6.12). To complete the proof, we observe that the properties (6.17) follow from (5.12). □

6.3. END OF THE PROOF OF THEOREM 6.2

Writing $J = J^{(0)} = D^{(0)} + A^{(0)} = d(\Lambda) + (Sa(\Lambda) + \mathrm{hc})$ in (5.16)–(5.17) with $l = 0$, $\rho_0 = \rho$ and $m_0 = 6$, we obtain

$$\mathrm{e}^{-\mathrm{i}P}J\mathrm{e}^{\mathrm{i}P} = J' + O(\Lambda^{\mu-6\rho}), \tag{6.19}$$

where

$$J' = d(\Lambda) + \tfrac{1}{2}\mathrm{ad}_{\mathrm{i}P}A^{(0)} + \tfrac{1}{3}\mathrm{ad}^2_{\mathrm{i}P}A^{(0)} + \tfrac{1}{8}\mathrm{ad}^3_{\mathrm{i}P}A^{(0)} + \tfrac{1}{30}\mathrm{ad}^4_{\mathrm{i}P}A^{(0)}. \tag{6.20}$$

Taking into account the assertion of Lemma 6.3, we find

$$J' = d'(\Lambda) + A', \tag{6.21}$$

where

$$d' = d + \tfrac{1}{2}q_0 + \tfrac{1}{8}q_0'' \tag{6.22}$$

and

$$A' = \sum_{1\le i\le 5} (S^i a_i(\Lambda) + \mathrm{hc}) \tag{6.23}$$

holds with

$$\begin{aligned} a_1 &= \tfrac{1}{3}q_1' + \tfrac{1}{30}q_1''', \\ a_2 &= \tfrac{1}{2}q_2 + \tfrac{1}{8}q_2'', \\ a_i &\in S_1^{\mu-3\rho} \text{ if } i \neq 2. \end{aligned} \tag{6.24}$$

We observe that by using the expressions for q_0, q_0'' given by (6.13), (6.16), we can write (6.22) in the form

$$d' = d + \mathfrak{r}_1 + \mathfrak{r}_{2,1} + \mathfrak{r}_{2,2}, \tag{6.25}$$

where $\mathfrak{r}_1$, $\mathfrak{r}_{2,1}$, $\mathfrak{r}_{2,2}$ are given by (6.2), (6.4), (6.5) respectively.

We observe that $A' \in \mathrm{FDO}_1^{\mu-2\rho}$ and applying Theorem 4.1 to d', A', 2ρ instead of d, A^0, ρ, we obtain

$$\lambda_n(J') = d'(n) + \mathfrak{r}_1'(n) + O(n^{\mu-6\rho}), \tag{6.26}$$

where

$$\mathfrak{r}_1' := \sum_{1\le i\le 5} \partial_{-i}\Big(\frac{|a_i|^2}{\partial_i d'}\Big) \in \mathrm{S}_1^{\mu-4\rho}. \tag{6.27}$$

Since (6.19) ensures

$$\lambda_n(J) = \lambda_n(J') + O(n^{\mu-6\rho}),$$

the assertion of Theorem 6.2 will follow from (6.25)–(6.26), if we check that the quantity $\mathfrak{r}_1'$ introduced in (6.27) satisfies the estimate

$$\mathfrak{r}_1'(n) = \mathfrak{r}_{2,3}(n) + O(n^{\mu-6\rho}), \tag{6.28}$$

where $\mathfrak{r}_{2,3}$ is given by (6.6). In order to check (6.28), we first observe that (6.24) ensures

$$\mathfrak{r}_1'(n) = \partial_{-2}\Big(\frac{|a_2(n)|^2}{\partial_2 d'(n)}\Big) + O(n^{\mu-6\rho}).$$

We observe that $\partial_2 d'(n) = \partial_2 d(n)(1 + O(n^{-2\rho}))$ ensures

$$\mathfrak{r}_1'(n) = \partial_{-2}\Big(\frac{|a_2(n)|^2}{\partial_2 d}\Big) + O(n^{\mu-6\rho}). \tag{6.29}$$

Using $a_2 - \frac{1}{2}q_2 \in S_1^{\mu-4\rho}$, it is easy to check that (6.29) still holds if a_2 is replaced by $\frac{1}{2}q_2 = \frac{\mathrm{i}}{2}\{a, p\}$, i.e. (6.28) holds with $\mathfrak{r}_{2,3}$ given by (6.6).

6.4. EXPRESSIONS FOR COEFFICIENTS $c_{0,0}$, $c_{1,0}$, $c_{2,0}$ IN (1.10)

Notation 6.4. We write $p(k) = q(k) + \mathrm{S}_1^\nu$ if and only if $p,\ q : \mathbb{N}^* \to \mathbb{C}$ satisfy $p - q \in \mathrm{S}_1^\nu$.

Step 1. Expansion of $\partial d(k)$. Assume that (1.3) holds. Then for any $N \in \mathbb{N}^*$ one has

$$\partial d(k) = \sum_{i=0}^{N-1} \delta_i k^{\mu-i}\big((1+k^{-1})^{\mu-i} - 1\big) + O(k^{\mu-N}).$$

Let us assume $\delta_0 = 1$. Then

$$\partial d(k) = \mu k^{\mu-1}(1 + \delta_1' k^{-1} + \delta_2' k^{-2}) + \mathrm{S}_1^{\mu-4}$$

holds with $\delta_1' := \frac{1}{\mu}\big(\binom{\mu}{2} + (\mu-1)\delta_1\big)$, $\quad \delta_2' := \frac{1}{\mu}\big(\binom{\mu}{3} + \binom{\mu-1}{2}\delta_1 + (\mu-2)\delta_2\big)$.

Step 2. Expansion of $|a(k)|^2$. We assume that (1.4) holds. Then

$$|a(k)|^2 = k^{2\mu-2\rho}(|\alpha_0|^2 + \beta_1 k^{-1} + \beta_2 k^{-2}) + \mathrm{S}_1^{2\mu-2\rho-3}$$

holds with $\beta_1 := 2\mathrm{Re}(\alpha_0\overline{\alpha}_1)$, $\beta_2 := 2\mathrm{Re}(\alpha_0\overline{\alpha}_2) + |\alpha_1|^2$.

Step 3. Expansion of $|a(k)|^2/\partial d(k)$. Since

$$\frac{|\alpha_0|^2 + \beta_1 k^{-1} + \beta_2 k^{-2}}{1 + \delta_1' k^{-1} + \delta_2' k^{-2}} = |\alpha_0|^2 + \gamma_1 k^{-2} + \gamma_2 k^{-2} + \mathrm{S}_1^{2\mu-2\rho-3}$$

holds with $\gamma_1 := \beta_1 - |\alpha_0|^2\delta_1'$, $\gamma_2 := \beta_2 - \beta_1\delta_1' + |\alpha_0|^2({\delta_1'}^2 - \delta_2')$, we find

$$\frac{|a(k)|^2}{\partial d(k)} = \frac{k^{\mu-2\rho+1}}{\mu}\big(|\alpha_0|^2 + \gamma_1 k^{-1} + \gamma_2 k^{-2}\big) + \mathrm{S}_1^{\mu-2\rho-2}.$$

Step 4. Expansion of $\mathfrak{r}_1(k)$. We conclude that

$$\mathfrak{r}_1(k) = \partial_{-1}\frac{|a|^2}{\partial d}(k) = k^{\mu-2\rho}(c_{0,0} + c_{1,0}k^{-1} + c_{2,0}k^{-2}) + \mathrm{S}_1^{\mu-2\rho-3} \tag{6.30}$$

holds with

$$\begin{cases} c_{0,0} := -\frac{1}{\mu}(\mu - 2\rho + 1)|\alpha_0|^2, \\ c_{1,0} := \frac{1}{\mu}\big(\binom{\mu-2\rho+1}{2}|\alpha_0|^2 - (\mu-2\rho)\gamma_1\big), \\ c_{2,0} := \frac{1}{\mu}\big(-\binom{\mu-2\rho+1}{3}|\alpha_0|^2 + \binom{\mu-2\rho}{2}\gamma_1 - (\mu-2\rho-1)\gamma_2\big). \end{cases} \tag{6.31}$$

We deduce (1.11) using r_1 expressed in (6.31) with γ_1 given in Step 3, β_1 given in Step 2 and δ_1' given in Step 1.

6.5. EXPRESSION OF $c_{0,1}$ IN (1.13)

Due to (6.3), it suffices to show that

$$\mathfrak{r}_{2,i}(k) = c_i k^{\mu-4\rho} + O(k^{\mu-4\rho-1}) \text{ for } i = 1,2,3, \tag{6.32}$$

holds with $\mathfrak{r}_{2,1}$, $\mathfrak{r}_{2,2}$, $\mathfrak{r}_{2,3}$ given by (6.4)–(6.6) and c_1, c_2, c_3 given by (1.15).

Case $i = 1$. By using (6.30)–(6.31), we obtain

$$\partial\mathfrak{r}_1(k) = -\frac{|\alpha_0|^2}{\mu}(\mu+1-2\rho)(\mu-2\rho)k^{\mu-2\rho-1} + \mathrm{S}_1^{\mu-2\rho-2}. \tag{6.33}$$

Then we observe that (6.33) and $p(k) = \mathrm{i}\frac{a}{\partial d}(k) = \mathrm{i}\frac{\alpha_0}{\mu}k^{1-\rho} + \mathrm{S}_1^{-\rho}$ imply

$$|p(k)|^2\partial\mathfrak{r}_1(k) = -\frac{|\alpha_0|^4}{\mu^3}(\mu+1-2\rho)(\mu-2\rho)k^{\mu-4\rho+1} + \mathrm{S}_1^{\mu-4\rho}. \tag{6.34}$$

In order to deduce (6.32) for $i = 1$, it remains to remark that (6.34) gives

$$\partial_{-1}(|p|^2\partial\mathfrak{r}_1)(k) = \frac{|\alpha_0|^4}{\mu^3}(\mu+1-2\rho)(\mu-2\rho)(\mu+1-4\rho)k^{\mu-4\rho} + O(k^{\mu-4\rho-1}).$$

Case $i = 2$. Denote $q_2 := \{a, \mathrm{i}p\}$. By using

$$q_2(k) = \Big\{\alpha_0 k^{\mu-\rho}, -\frac{\alpha_0}{\mu}k^{1-\rho}\Big\} + \mathrm{S}_1^{\mu-2\rho-1} = -\frac{\alpha_0^2}{\mu}(\mu-1)k^{\mu-2\rho} + \mathrm{S}_1^{\mu-2\rho-1} \tag{6.35}$$

and $\tau_{-1}\bar{p}(k) = \bar{p}(k) + \mathrm{S}_1^{-\rho} = -\mathrm{i}\frac{\overline{\alpha}_0}{\mu}k^{1-\rho} + \mathrm{S}_1^{-\rho}$, we find

$$\begin{aligned}\bar{p}(k)\{q_2, \tau_{-1}\bar{p}\}_{2,-1}(k) &= -\mathrm{i}\frac{\overline{\alpha}_0}{\mu}k^{1-\rho}\Big\{-\frac{\alpha_0^2}{\mu}(\mu-1)k^{\mu-2\rho}, -\mathrm{i}\frac{\overline{\alpha}_0}{\mu}k^{1-\rho}\Big\}_{2,-1} + \mathrm{S}_1^{\mu-3\rho}\\ &= \frac{|\alpha_0|^4}{\mu^3}(\mu-1)(\mu+2-4\rho)k^{\mu+1-4\rho} + \mathrm{S}_1^{\mu-4\rho}.\end{aligned}$$

In order to deduce (6.32) for $i = 2$, it remains to remark that the last estimate gives

$$\begin{aligned}&\partial_{-1}\mathrm{Im}(\bar{p}\{-\mathrm{i}q_2, \tau_{-1}\bar{p}\}_{2,-1})(k)\\ &= \frac{|\alpha_0|^4}{\mu^3}(\mu-1)(\mu+2-4\rho)(\mu+1-4\rho)k^{\mu-4\rho} + O(k^{\mu-4\rho-1})\end{aligned}$$

Case $i = 3$. As before $q_2 := \{a, \mathrm{i}p\}$. Since (6.35) provides

$$\frac{|q_2(k)|^2}{\partial_2 d(k)} = \frac{|\alpha_0|^4}{2\mu^3}(\mu-1)^2 k^{\mu+1-4\rho} + \mathrm{S}_1^{\mu-4\rho}, \tag{6.36}$$

in order to deduce (6.32) for $i = 3$, it remains to remark that (6.36) gives

$$\partial_{-2}\frac{|q_2|^2}{\partial_2 d}(k) = -\frac{|\alpha_0|^4}{\mu^3}(\mu-1)^2(\mu+1-4\rho)k^{\mu-4\rho} + O(k^{\mu-4\rho-1}).$$

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