

ON THE ASYMPTOTIC BEHAVIOR
OF NONOSCILLATORY SOLUTIONS
OF CERTAIN FRACTIONAL DIFFERENTIAL EQUATIONS
WITH POSITIVE AND NEGATIVE TERMS

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Abstract. This paper is concerned with the asymptotic behavior of the nonoscillatory solutions of the forced fractional differential equation with positive and negative terms of the form

$${}^C D_c^\alpha y(t) + f(t, x(t)) = e(t) + k(t)x^\eta(t) + h(t, x(t)),$$

where $t \geq c \geq 1$, $\alpha \in (0, 1)$, $\eta \geq 1$ is the ratio of positive odd integers, and ${}^C D_c^\alpha y$ denotes the Caputo fractional derivative of y of order α . The cases

$$y(t) = (a(t) (x'(t))^\eta)' \quad \text{and} \quad y(t) = a(t) (x'(t))^\eta$$

are considered. The approach taken here can be applied to other related fractional differential equations. Examples are provided to illustrate the relevance of the results obtained.

Keywords: integro-differential equations, fractional differential equations, nonoscillatory solutions, boundedness, Caputo derivative.

Mathematics Subject Classification: 34E10, 34A34.

1. INTRODUCTION

We consider the forced fractional differential equation with positive and negative terms

$${}^C D_c^\alpha y(t) + f(t, x(t)) = e(t) + k(t)x^\eta(t) + h(t, x(t)), \quad (1.1)$$

where $t \geq c > 1$, $\alpha \in (0, 1)$, $\eta \geq 1$ is the ratio of positive odd integers, and ${}^C D_c^\alpha y$ denotes the Caputo fractional derivative of y of order α as defined by

$${}^C D_c^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds, \quad \alpha \in (n-1, n), n \in \mathbb{N}.$$

If $\alpha \in (0, 1)$, this definition was given by Caputo in [4]; for the definition of the Caputo derivative of order $\alpha \in (n - 1, n)$, $n \geq 1$, see [1, 5, 6]. We will just consider the case $n = 1$, i.e., $\alpha \in (0, 1)$, here, but our results can easily be generalized to values of n greater than 1 (see Section 3). We will be considering the following choices for the function y , namely,

$$y(t) = (a(t) (x'(t))^\eta)' \quad (1.2)$$

and

$$y(t) = a(t) (x'(t))^\eta. \quad (1.3)$$

Throughout the paper, we assume that:

- (i) $a, k : [c, \infty) \rightarrow (0, \infty)$ and $e : [c, \infty) \rightarrow \mathbb{R}$ are continuous functions;
- (ii) $f, h : [c, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ are real-valued continuous functions and there exist continuous functions $b, m : [c, \infty) \rightarrow (0, \infty)$ and positive real numbers λ and γ with $\lambda > \gamma$ such that

$$xf(t, x) \geq b(t) |x|^{\lambda+1} \quad \text{and} \quad 0 \leq xh(t, x) \leq m(t) |x|^{\gamma+1} \quad \text{for} \quad x \neq 0 \quad \text{and} \quad t \geq c.$$

A function $y : [c, \infty) \rightarrow \mathbb{R}$ is a *solution* of equation (1.1) and either (1.2) or (1.3) if $x \in C^1([c, \infty), \mathbb{R})$ and y satisfies (1.1). Only those solutions that are continuable and nontrivial in any neighborhood of ∞ are under consideration here. Such a solution is said to be *oscillatory* if there exists a sequence $\{t_n\} \subseteq [c, \infty)$ with $t_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $x(t_n) = 0$, and it is *nonoscillatory* otherwise.

In recent years, integro-differential and fractional differential equations have gained considerably more attention due to their applicability to problems in engineering and other scientific disciplines. For example, they arise as mathematical models for systems and processes in physics, mechanics, chemistry, aerodynamics, electrodynamics, and more recently in social networking (see [13]). Additional examples can be found in [1, 15–17, 22–25].

Our aim here is to obtain some new results on the asymptotic behavior of nonoscillatory solutions of equation (1.1). We note that this equation is equivalent to the Volterra type equation

$$y(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} [e(s) + k(s)x^\eta(s) + h(s, x(s)) - f(s, x(s))] ds, \quad (1.4)$$

where $c \geq 1$, $\alpha \in (0, 1)$, $c_0 = \frac{y(c)}{\Gamma(1)} = y(c)$, and c_0 is a real constant.

Oscillation and asymptotic properties of solutions of integro-differential equations and fractional differential equations are relatively scarce in the literature; some results can be found in [2, 7, 9, 10, 12, 17–21]. The only results to date for forced fractional differential equations with positive and negative terms of the type (1.1) appear to be those in [11] where sufficient conditions for boundedness of nonoscillatory solutions are obtained. In that paper, $\eta = 1$ and, as remarked below, additional conditions were needed on the function $a(t)$. As a consequence, in the present paper, we are able to obtain growth estimates on the nonoscillatory solutions of (1.1) (see (2.6) and (2.23) below).

The idea to transform a fractional differential equation into a Volterra integral equation is not new. For example, Medved' did this for a much simpler equation in [20, Lemma 2.5]; in this regard also see [21, Lemma 1]. We make use of Young's inequality which is not the case for example in [20] and [21]. It is not difficult to see that the approach we use in this paper can be useful in the study of other types of fractional differential equations as well.

2. MAIN RESULTS

We will make use of the following two lemmas in the proofs of our main results.

Lemma 2.1 ([3]). *Let α and p be positive constants such that $p(\alpha - 1) + 1 > 0$. Then*

$$\int_0^t (t - s)^{p(\alpha-1)} e^{ps} ds \leq Q e^{pt}, \quad t \geq 0,$$

where

$$Q = \frac{\Gamma(1 + p(\alpha - 1))}{p^{1+p(\alpha-1)}},$$

and

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds, \quad x > 0,$$

is the usual Euler-Gamma function.

Lemma 2.2 (Young's inequality, [14]). *If X and Y are nonnegative, $\delta > 1$, and $1/\delta + 1/\beta = 1$, then*

$$XY \leq \frac{1}{\delta} X^\delta + \frac{1}{\beta} Y^\beta, \tag{2.1}$$

where equality holds if and only if $Y = X^{\delta-1}$.

For notational purpose, we put

$$g(t) = \left(\frac{\lambda - \gamma}{\gamma}\right) \left(\frac{\gamma}{\lambda} m(t)\right)^{\lambda/(\lambda-\gamma)} (b(t))^{\gamma/(\gamma-\lambda)},$$

$$R(t, c) = \int_c^t a^{-1/\eta}(s) ds,$$

and we assume that

$$R(t, c) \rightarrow \infty \quad \text{as } t \rightarrow \infty. \tag{2.2}$$

We now give sufficient conditions under which any nonoscillatory solution x of equation (1.1) with (1.2) satisfies

$$|x(t)| = O\left(t^{1/\eta} e^{t/\eta} R(t, c)\right) \quad \text{as } t \rightarrow \infty.$$

Theorem 2.3. *Let conditions (i)–(ii) and (2.2) hold and assume that there exist real numbers $p > 1$ and $0 < \alpha < 1$ such that $p(\alpha - 1) + 1 > 0$. If*

$$\int_c^\infty k^q(s) s^q R^{\eta q}(s, c) ds < \infty, \quad \text{where } q = \frac{p}{p-1}, \quad (2.3)$$

$$\lim_{t \rightarrow \infty} \int_c^t (t-s)^{\alpha-1} |e(s)| ds < \infty, \quad (2.4)$$

and

$$\lim_{t \rightarrow \infty} \int_c^t (t-s)^{\alpha-1} g(s) ds < \infty, \quad (2.5)$$

then every nonoscillatory solution $x(t)$ of equation (1.1) with (1.2) satisfies

$$\limsup_{t \rightarrow \infty} \frac{|x(t)|}{t^{1/\eta} e^{t/\eta} R(t, c)} < \infty. \quad (2.6)$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) with (1.2), say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. Setting $F(t) = h(t, x(t)) - f(t, x(t))$, it follows from (i)–(ii) and (1.1) that

$$\begin{aligned} (a(t)(x'(t))^\eta)' &\leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)| ds + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} [m(s)x^\gamma(s) - b(s)x^\lambda(s)] ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} k(s) |x^\eta(s)| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} k(s) x^\eta(s) ds. \end{aligned} \quad (2.7)$$

Applying (2.1) to $[m(t)x^\gamma(t) - b(t)x^\lambda(t)]$ with

$$\delta = \frac{\lambda}{\gamma}, \quad X = x^\gamma(t), \quad Y = \frac{\gamma m(t)}{\lambda b(t)}, \quad \text{and } \beta = \frac{\lambda}{\lambda - \gamma},$$

we obtain

$$\begin{aligned}
 m(t)x^\gamma(t) - b(t)x^\lambda(t) &= \frac{\lambda}{\gamma}b(t) \left[x^\gamma(t) \frac{\gamma}{\lambda} \frac{m(t)}{b(t)} - \frac{\gamma}{\lambda} (x^\gamma(t))^{\lambda/\gamma} \right] \\
 &= \frac{\lambda}{\gamma}b(t) \left[XY - \frac{1}{\delta} X^\delta \right] \leq \frac{\lambda}{\gamma}b(t) \left(\frac{1}{\beta} Y^\beta \right) \\
 &= \left(\frac{\lambda - \gamma}{\gamma} \right) \left[\frac{\gamma}{\lambda} m(t) \right]^{\lambda/(\lambda - \gamma)} (b(t))^{\gamma/(\gamma - \lambda)} := g(t).
 \end{aligned}
 \tag{2.8}$$

Using (2.8) in (2.7) gives

$$\begin{aligned}
 (a(t) (x'(t))^\eta)' &\leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t_1 - s)^{\alpha - 1} |F(s)| ds + \frac{1}{\Gamma(\alpha)} \int_c^t (t - s)^{\alpha - 1} |e(s)| ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t - s)^{\alpha - 1} g(s) ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t_1 - s)^{\alpha - 1} k(s) |x^\eta(s)| ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t - s)^{\alpha - 1} k(s) x^\eta(s) ds.
 \end{aligned}
 \tag{2.9}$$

In view of (2.4) and (2.5), it follows from (2.9) that

$$(a(t) (x'(t))^\eta)' \leq C_1 + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t - s)^{\alpha - 1} k(s) x^\eta(s) ds,
 \tag{2.10}$$

for some constant $C_1 > 0$. Integrating (2.10) from t_1 to t gives

$$a(t) (x'(t))^\eta \leq a(t_1) |(x'(t_1))^\eta| + C_1(t - t_1) + \frac{1}{\alpha \Gamma(\alpha)} \int_{t_1}^t (t - s)^\alpha k(s) x^\eta(s) ds := w(t),
 \tag{2.11}$$

which can be written as

$$x'(t) \leq \left(\frac{w(t)}{a(t)} \right)^{1/\eta}.
 \tag{2.12}$$

Noting that $w(t)$ is an increasing function, it follows from (2.12) that

$$\begin{aligned} x(t) &\leq x(t_1) + w^{1/\eta}(t) \int_{t_1}^t a^{-1/\eta}(s) ds \\ &= x(t_1) + w^{1/\eta}(t) R(t, t_1) \\ &= \left[\frac{x(t_1)}{R(t, t_1)} + w^{1/\eta}(t) \right] R(t, t_1) \\ &\leq \left[\frac{x(t_1)}{R(t_2, t_1)} + w^{1/\eta}(t) \right] R(t, t_1) \end{aligned} \quad (2.13)$$

for $t \geq t_2$ and any fixed $t_2 > t_1$. From (2.13), we see that

$$\frac{x(t)}{R(t, t_1)} \leq C_2 + w^{1/\eta}(t) \quad \text{for } t \geq t_2, \quad (2.14)$$

where $C_2 = x(t_1)/R(t_2, t_1) > 0$. Applying the elementary inequality

$$(A + B)^\mu \leq 2^{\mu-1}(A^\mu + B^\mu), \quad A, B \geq 0 \quad \text{and} \quad \mu \geq 1, \quad (2.15)$$

to (2.14) gives

$$\left(\frac{x(t)}{R(t, t_1)} \right)^\eta \leq 2^{\eta-1}(C_2)^\eta + 2^{\eta-1}w(t) \quad \text{for } t \geq t_2. \quad (2.16)$$

In view of the definition of $w(t)$, it follows from (2.16) that

$$\begin{aligned} \left(\frac{x(t)}{R(t, t_1)} \right)^\eta &\leq 2^{\eta-1}(C_2)^\eta + 2^{\eta-1}a(t_1)|(x'(t_1))^\eta| + 2^{\eta-1}C_1(t - t_1) \\ &\quad + \frac{2^{\eta-1}}{\alpha\Gamma(\alpha)} \int_{t_1}^t (t-s)^\alpha k(s)x^\eta(s) ds \\ &\leq 2^{\eta-1}(C_2)^\eta + 2^{\eta-1}a(t_1)|(x'(t_1))^\eta| + 2^{\eta-1}C_1t \\ &\quad + \frac{2^{\eta-1}t}{\alpha\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} k(s)x^\eta(s) ds. \end{aligned} \quad (2.17)$$

From (2.17), we arrive at

$$\left(\frac{x(t)}{t^{1/\eta}R(t, t_1)} \right)^\eta \leq C_3 + \frac{2^{\eta-1}}{\alpha\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} k(s)x^\eta(s) ds \quad (2.18)$$

for some constant $C_3 > 0$. Applying Hölder's inequality and Lemma 2.1 to the integral on the right in (2.18) yields

$$\begin{aligned}
 & \int_{t_1}^t [(t-s)^{\alpha-1} e^s] [e^{-s} k(s)x^\eta(s)] ds \\
 & \leq \left(\int_{t_1}^t (t-s)^{p(\alpha-1)} e^{ps} ds \right)^{1/p} \left(\int_{t_1}^t e^{-qs} k^q(s)x^{\eta q}(s) ds \right)^{1/q} \\
 & \leq \left(\int_0^t (t-s)^{p(\alpha-1)} e^{ps} ds \right)^{1/p} \left(\int_{t_1}^t e^{-qs} k^q(s)x^{\eta q}(s) ds \right)^{1/q} \tag{2.19} \\
 & \leq (Qe^{pt})^{1/p} \left(\int_{t_1}^t e^{-qs} k^q(s)x^{\eta q}(s) ds \right)^{1/q} \\
 & = Q^{1/p} e^t \left(\int_{t_1}^t e^{-qs} k^q(s)x^{\eta q}(s) ds \right)^{1/q}.
 \end{aligned}$$

Using (2.19) in (2.18), we obtain

$$z(t) := \left(\frac{x(t)}{t^{1/\eta} e^{t/\eta} R(t, t_1)} \right)^\eta \leq 1 + C_4 + K \left(\int_{t_1}^t e^{-qs} k^q(s)x^{\eta q}(s) ds \right)^{1/q}, \tag{2.20}$$

where C_4 is an upper bound for $C_3 e^{-t}$ and $K = 2^{\eta-1} Q^{1/p} / \alpha \Gamma(\alpha)$. Employing again inequality (2.15), we obtain from (2.20) that

$$z^q(t) \leq 2^{q-1} (1 + C_4)^q + 2^{q-1} K^q \int_{t_1}^t e^{-qs} k^q(s)x^{\eta q}(s) ds,$$

which can be written as

$$z^q(t) \leq 2^{q-1} (1 + C_4)^q + 2^{q-1} K^q \int_{t_1}^t k^q(s) s^q R^{\eta q}(s, t_1) z^q(s) ds. \tag{2.21}$$

Setting $P_1 = 2^{q-1} (1 + C_4)^q$, $Q_1 = 2^{q-1} K^q$, and $w(t) = z^q(t)$ so that $z(t) = w^{1/q}(t)$, inequality (2.21) becomes

$$w(t) \leq P_1 + Q_1 \int_{t_1}^t k^q(s) s^q R^{\eta q}(s, t_1) w(s) ds.$$

The conclusion then follows from Gronwall's inequality and condition (2.3), that is,

$$\limsup_{t \rightarrow \infty} \frac{x(t)}{t^{1/\eta} e^{t/\eta} R(t, t_1)} < \infty.$$

The proof in case $x(t)$ is an eventually negative solution is similar. This completes the proof of the theorem. \square

We next give sufficient conditions under which any nonoscillatory solution x of equation (1.1) with (1.3) satisfies

$$|x(t)| = O\left(e^{t/\eta} R(t, c)\right) \quad \text{as } t \rightarrow \infty.$$

Theorem 2.4. *Let conditions (i)–(ii) and (2.2) hold and assume that there exist real numbers $p > 1$ and $\alpha \in (0, 1)$ such that $p(\alpha - 1) + 1 > 0$. If*

$$\int_c^\infty k^q(s) R^{\eta q}(s, c) ds < \infty, \quad \text{where } q = \frac{p}{p-1}, \quad (2.22)$$

and conditions (2.4)–(2.5) hold, then every nonoscillatory solution $x(t)$ of equation (1.1) with (1.3) satisfies

$$\limsup_{t \rightarrow \infty} \frac{|x(t)|}{e^{t/\eta} R(t, c)} < \infty. \quad (2.23)$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) with (1.3), say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. As in the proof of Theorem 2.3, we again let

$$F(t) = h(t, x(t)) - f(t, x(t)).$$

Then, in view of (i)–(ii), it follows from equation (1.1) that

$$\begin{aligned} a(t) (x'(t))^\eta &\leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} |e(s)| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} [m(s)x^\gamma(s) - b(s)x^\lambda(s)] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} k(s) |x^\eta(s)| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} k(s) x^\eta(s) ds. \end{aligned} \quad (2.24)$$

Proceeding as in the proof of Theorem 2.3, it follows from (2.24) that

$$a(t) (x'(t))^\eta \leq C_5 + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} k(s)x^\eta(s)ds := \varphi(t), \tag{2.25}$$

for some constant $C_5 > 0$. Again as in Theorem 2.3, inequality (2.25) can be written as

$$x'(t) \leq \left(\frac{\varphi(t)}{a(t)} \right)^{1/\eta}. \tag{2.26}$$

The remainder of the proof is similar to that of Theorem 2.3 and so we omit the details. \square

We conclude this paper with two examples to illustrate our results.

Example 2.5. Consider the equation

$${}^C D_8^{1/2} (t(x'(t))^3)' + f(t, x(t)) = e^{-2t} \cos t + \frac{1}{1+t^4} x^3(t) + h(t, x(t)), \quad t \geq 8. \tag{2.27}$$

Here we have $y(t) = (t(x'(t))^3)'$, $\alpha = 1/2$, $c = 8$, $\eta = 3$, $a(t) = t$, $e(t) = e^{-2t} \cos t$, $k(t) = 1/(1+t^4)$, and

$$R(t, c) = R(t, 8) = \int_8^t s^{-1/3} ds = \frac{3}{2}(t^{2/3} - 4).$$

Then, it is easy to see that conditions (i) and (2.2) hold. Letting $p = 3/2$, we see that $q = 3$, and $p(\alpha - 1) + 1 = 1/4 > 0$. Letting $f(t, x(t)) = b(t) |x(t)|^{\lambda-1} x(t)$ and $h(t, x(t)) = m(t) |x(t)|^{\gamma-1} x(t)$ with $\lambda > \gamma$, and taking $b(t) = m(t) = e^{-t}$, we see that condition (ii) holds. Since

$$\int_c^\infty k^q(s) s^q R^{\eta q}(s, c) ds \leq \left(\frac{3}{2} \right)^9 \int_8^\infty \frac{s^9}{(1+s^4)^3} ds < \infty,$$

condition (2.3) holds. To see that (2.4) holds, note that letting $u = t - s + 8$, the integral becomes

$$\begin{aligned}
\int_8^t (t-s)^{-1/2} |e^{-2s} \cos s| ds &= - \int_t^8 (u-8)^{-1/2} e^{2u-2t-16} |\cos(t-u+8)| du \\
&\leq \frac{1}{e^{2t+16}} \int_8^t (u-8)^{-1/2} e^{2u} du \\
&= \frac{1}{e^{2t+16}} \left[\int_8^{16} (u-8)^{-1/2} e^{2u} du + \int_{16}^t (u-8)^{-1/2} e^{2u} du \right] \\
&= \frac{1}{e^{2t+16}} \left[\lim_{b \rightarrow 8^+} \int_b^{16} (u-8)^{-1/2} e^{2u} du \right] \\
&\quad + \frac{1}{e^{2t+16}} \left[\int_{16}^t (u-8)^{-1/2} e^{2u} du \right] \\
&= \frac{e^{32}}{e^{2t+16}} \lim_{b \rightarrow 8^+} \int_b^{16} (u-8)^{-1/2} du + \frac{(16-8)^{-1/2}}{e^{2t+16}} \int_{16}^t e^{2u} du \\
&= \frac{2^{5/2} e^{32}}{e^{2t+16}} + \frac{2^{-5/2}}{e^{2t+16}} (e^{2t} - e^{32}) < \infty \text{ as } t \rightarrow \infty.
\end{aligned}$$

Hence, (2.4) and similarly (2.5) hold. Since all conditions of Theorem 2.3 are satisfied, every nonoscillatory solution $x(t)$ of equation (2.27) satisfies (2.6), that is,

$$\limsup_{t \rightarrow \infty} \frac{|x(t)|}{\frac{3}{2} e^{t/3} (t - 4t^{1/3})} < \infty. \quad (2.28)$$

Example 2.6. Consider the equation

$${}^C D_4^{3/5} \left(t^{1/2} x'(t) \right) + f(t, x(t)) = e^{-3t} \sin t + \frac{1}{t^2} x(t) + h(t, x(t)), \quad t \geq 4. \quad (2.29)$$

Here we have $y(t) = t^{1/2} x'(t)$, $\alpha = 3/5$, $c = 4$, $\eta = 1$, $a(t) = t^{1/2}$, $e(t) = e^{-3t} \sin t$, $k(t) = 1/t^2$, and

$$R(t, c) = R(t, 4) = \int_4^t s^{-1/2} ds = 2t^{1/2} - 4.$$

Then, it is easy to see that conditions (i) and (2.2) hold. Letting $p = 2$, we see that $q = 2$, and $p(\alpha - 1) + 1 = 1/5 > 0$. Letting $f(t, x(t)) = b(t) |x(t)|^{\lambda-1} x(t)$ and

$h(t, x(t)) = m(t)|x(t)|^{\gamma-1}x(t)$ with $\lambda > \gamma$ and $b(t) = m(t) = e^{-3t}$, we see that (ii) holds. Since

$$\int_c^\infty k^q(s)R^{nq}(s, c)ds \leq 4 \int_4^\infty \frac{1}{s^3}ds < \infty,$$

condition (2.22) holds. As in Example 2.5, it is easy to see that conditions (2.4)–(2.5) hold. Since all conditions of Theorem 2.4 are fulfilled, we have that every nonoscillatory solution $x(t)$ of equation (2.29) satisfies (2.23), i.e.,

$$\limsup_{t \rightarrow \infty} \frac{|x(t)|}{2e^t(t^{1/2} - 2)} < \infty. \quad (2.30)$$

3. REMARKS AND CONCLUSIONS

The results in this paper can also be obtained for higher fractional differential equations of order $\alpha \in (n-1, n)$ with $n \geq 1$. For example, if we take

$$y(t) = (a(t)|x'(t)|^{\nu-1}x'(t))'$$

with $\nu \geq 1$, then our equation would include the equation considered in [8] as a special case. Another possibility is to replace the fractional derivative ${}^C D_c^\alpha y(t)$ by ${}^C D_c^r y(t)$ where $r = \alpha + n - 1$ with $\alpha \in (0, 1)$ which would then include the problem considered by Medved [20, Eq. (24)]. These would make for interesting problems for future research.

We would also like to point out that if $\eta \equiv 1$ in (1.1), then the results in this paper generalize those in [11] where an additional condition on the function a was needed (see (2.3) and (2.18) in [11]). Another possible direction for future investigations is to consider equation (1.1) with $\eta < 1$.

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