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MAXIMUM PACKINGS OF THE λ-FOLD COMPLETE 3-UNIFORM HYPERGRAPH WITH LOOSE 3-CYCLES

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Abstract. It is known that the 3-uniform loose 3-cycle decomposes the complete 3-uniform hypergraph of order v if and only if $v \equiv 0$, 1, or 2 (mod 9). For all positive integers λ and v, we find a maximum packing with loose 3-cycles of the λ-fold complete 3-uniform hypergraph of order v. We show that, if $v \geq 6$, such a packing has a leave of two or fewer edges.

Keywords: maximum packing, λ-fold complete 3-uniform hypergraph, loose 3-cycle.

Mathematics Subject Classification: 05C65, 05C85.

1. INTRODUCTION

A *hypergraph* H consists of a finite nonempty set V of *vertices* and a finite collection $E = \{e_1, e_2, \ldots, e_m\}$ of nonempty subsets of V called *hyperedges* or simply *edges*. For a given hypergraph H, we use $V(H)$ and $E(H)$ to denote the vertex set and the edge set (or multiset) of H, respectively. We call $|V(H)|$ and $|E(H)|$ the *order* and *size* of H, respectively. The *degree* of a vertex $v \in V(H)$ is the number of edges in $E(H)$ that contain v. A hypergraph H is *simple* if no edge appears more than once in $E(H)$. If for each $e \in E(H)$ we have $|e| = t$, then H is said to be *t-uniform*. Thus t-uniform hypergraphs are generalizations of the concept of a graph (where $t = 2$). Graphs with repeated edges are often called *multigraphs*. If H is a simple hypergraph and λ is a positive integer, then *λ-fold H*, denoted ${}^{\lambda}H$, is the multi-hypergraph obtained from H by repeating each edge exactly λ times. The hypergraph with vertex set V and edge set the set of all t-element subsets of V is called the *complete t-uniform hypergraph on V* and is denoted by $K_V^{(t)}$. If $v = |V|$, then ${}^{\lambda}K_v^{(t)}$ is called the *λ-fold complete t-uniform hypergraph of order v* and is used to denote any hypergraph isomorphic to ${}^{\lambda}K_V^{(t)}$. When $t = 2$, we will use ${}^{\lambda}K_v$ in place of ${}^{\lambda}K_v^{(2)}$. Similarly, if $\lambda = 1$, then we will use $K_v^{(t)}$ in place of ${}^{1}K_v^{(t)}$. If H' is a subhypergraph of H, then $H \setminus H'$ denotes the hypergraph obtained from H by deleting the edges of H'. We may refer to $H \setminus H'$ as the hypergraph H with a *hole* H'. The vertices in H' may be referred to as the vertices in the hole.

A commonly studied problem in combinatorics concerns decompositions of graphs or multigraphs into edge-disjoint subgraphs. A *decomposition* of a multigraph M is a set $\Delta = \{G_1, G_2, \ldots, G_s\}$ of subgraphs of M such that $\{E(G_1), E(G_2), \ldots, E(G_s)\}$ is a partition of $E(M)$. If each element of Δ is isomorphic to a fixed graph G, then Δ is called a *G-decomposition* of M. If L is a subgraph of M and Δ is a G-decomposition of $M \setminus L$, then Δ is called a *G-packing* of M with *leave* L. Such a G-packing is *maximum* if no other possible G-packing of M has a leave of a smaller size than that of L. Clearly, if $|E(L)| < |E(G)|$, then the G-packing is maximum. Moreover, a G-decomposition of M can be viewed as a maximum G-packing with an empty leave.

A G-decomposition of ${}^{\lambda}K_v$ is also known as a *G-design of order v and index λ*. A K_k-design of order v and index λ is usually known as a 2-(v, k, λ) *design* or as a *balanced incomplete block design of index λ* or a (v, k, λ)*-BIBD*. The problem of determining all v for which there exists a G-design of order v is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to hypergraphs. A *decomposition* of a hypergraph M is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of subhypergraphs of M such that $\{E(H_1), E(H_2), \ldots, E(H_s)\}$ is a partition of $E(M)$. Any element of Δ isomorphic to a fixed hypergraph H is called an *H-block*. If all elements of Δ are H-blocks, then Δ is called an *H-decomposition* of M. If L is a subgraph of M and Δ is an H-decomposition of $M \setminus L$, then Δ is called an *H-packing* of M with *leave* L, where we again define such a packing to be *maximum* if L has the fewest edges possible. An H-decomposition of ${}^{\lambda}K_v^{(t)}$ is called an *H-design of order v and index λ*. The problem of determining all v for which there exists an H-design of order v and index λ is called the *λ-fold spectrum problem for H-designs.*

A $K_k^{(t)}$-design of order v and index λ is a generalization of 2-(v, k, λ) designs and is known as a t-(v, k, λ) *design* or simply as a *t-design.* A summary of results on t-designs appears in [15]. A t-$(v, k, 1)$ design is also known as a *Steiner system* and is denoted by $S(t, v, k)$ (see [8] for a summary of results on Steiner systems). Keevash [14] has recently shown that for all t and k the obvious necessary conditions for the existence of an $S(t, k, v)$-design are sufficient for sufficiently large values of v. Similar results were obtained by Glock, Kühn, Lo, and Osthus [9, 10] and extended to include the corresponding asymptotic results for H-designs of order v for all uniform hypergraphs H. These results for t-uniform hypergraphs mirror the celebrated results of Wilson [23] for graphs. Although these asymptotic results assure the existence of H-designs for sufficiently large values of v for any uniform hypergraph H, the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

In the study of graph decompositions, a fair amount of the focus has been on G-decompositions of K_v where G is a graph with a relatively small number of edges (see [1] and [5] for known results). Some authors have investigated the corresponding problem for 3-uniform hypergraphs. For example, in [4], the 1-fold spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the 1-fold spectrum problem was settled in [6] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [6], they also settle the 1-fold spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered H-designs where H is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let T, O, and I denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph T is the same as $K_4^{(3)}$, and its spectrum was settled in 1960 by Hanani [11]. In another paper [12], Hanani settled the spectrum problem for O-designs and gave necessary conditions for the existence of I-designs. The 1-fold spectrum problem is also settled for a type of 3-uniform hyperstars which is part of a larger class of hypergraphs known as delta-systems. For a positive integer m, let $S_m^{(3)}$ denote the 3-uniform hypergraph of size m which consists of one vertex of degree m and $2m$ vertices of degree one. Necessary and sufficient conditions for the existence of $S_m^{(3)}$-decompositions of $K_v^{(3)}$ are given in [21] for $m \in [4, 6]$ and for all m in [18]. Some results on maximum $S_m^{(3)}$-packings of $K_v^{(3)}$ are given in [19]. Perhaps the best known general result on decompositions of complete t-uniform hypergraphs is Baranyai's result [3] on the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers m. There are, however, several articles on decompositions of complete t-uniform hypergraphs (see [2] and [20]) and of t-uniform t-partite hypergraphs (see [16] and [22]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [13] and [17]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

In this paper we are interested in maximum H-packings of ${}^{\lambda}K_v^{(3)}$, where H is a 3-uniform loose 3-cycle. For integer $m \geq 3$, a *3-uniform loose m-cycle*, denoted $LC_m^{(3)}$, is a 3-uniform hypergraph with vertex set $\{v_1, v_2, \ldots, v_{2m}\}$ and edge set $\big\{\{v_{2i-1}, v_{2i}, v_{2i+1}\} : 1 \leq i \leq m-1\big\} \cup \{v_{2m-1}, v_{2m}, v_1\}$. Thus $LC_3^{(3)}$ has vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\big\{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_5, v_6, v_1\}\big\}$ for which we use $H[v_1, v_2, v_3, v_4, v_5, v_6]$ to denote (see Figure 1).

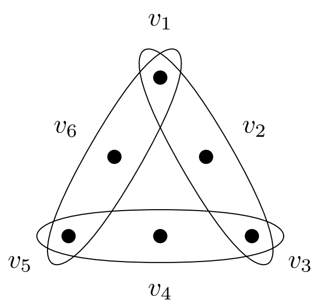

Fig. 1. The 3-uniform loose 3-cycle, $LC_3^{(3)}$, denoted by $H[v_1, v_2, v_3, v_4, v_5, v_6]$.

Since $LC_3^{(3)}$ has 3 edges and 6 vertices, it is one of the hypergraphs covered in the decomposition results by Bryant, Herke, Maenhaut, and Wannasit in [6]. It is shown in [6] that there exists an $LC_3^{(3)}$-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0$, 1, or 2 (mod 9). Similarly, it is shown in [7] that there exists an $LC_4^{(3)}$-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0$, 1, 2, 4, or 6 (mod 8) and $v \notin \{4, 6\}$. Here we focus on maximum $LC_3^{(3)}$-packings of ${}^{\lambda}K_v^{(3)}$ and show that if λ and $v \geq 6$ are positive integers, then there exists a maximum $LC_3^{(3)}$-packing of ${}^{\lambda}K_v^{(3)}$ where the leave has two or fewer edges.

1.1. ADDITIONAL NOTATION AND TERMINOLOGY

If a and b are integers with $a \leq b$, we define $[a, b]$ to be $\{a, a+1, \ldots, b\}$. We next define some notation for certain types of 3-uniform hypergraphs.

Let U_1, U_2, U_3 be pairwise disjoint sets. The hypergraph with vertex set $U_1 \cup U_2 \cup U_3$ and edge set consisting of all 3-element sets having exactly one vertex in each of U_1, U_2, U_3 is denoted by $K_{U_1,U_2,U_3}^{(3)}$. The hypergraph with vertex set $U_1 \cup U_2$ and edge set consisting of all 3-element sets having at most 2 vertices in each of U_1, U_2 is denoted by $L_{U_1,U_2}^{(3)}$. If $|U_i| = u_i$ for $i \in \{1, 2, 3\}$, we may use $K_{u_1,u_2,u_3}^{(3)}$ or $L_{u_1,u_2}^{(3)}$ to denote any hypergraph that is isomorphic to $K_{U_1,U_2,U_3}^{(3)}$ or $L_{U_1,U_2}^{(3)}$, respectively. From a hypergraph decomposition perspective, we note that if U_1, U_1', U_2, U_3 are pairwise vertex disjoint, then

$$E(K_{U_1 \cup U_1', U_2, U_3}^{(3)}) = E(K_{U_1,U_2,U_3}^{(3)}) \cup E(K_{U_1',U_2,U_3}^{(3)}).$$

Thus, for any positive integer x, it is simple to see that $K_{u_1,u_2,u_3}^{(3)}$ decomposes $K_{u_1 \cdot x, u_2, u_3}^{(3)}$ and, in general, $K_{u_1+x,u_2,u_3}^{(3)}$ decomposes into one copy of $K_{u_1,u_2,u_3}^{(3)}$ and one copy of $K_{x,u_2,u_3}^{(3)}$.

2. MAIN CONSTRUCTIONS

The constructions in this section are dependent on many small examples. These examples are given in the last section. Throughout, we will often identify a hypergraph (e.g., a leave in a packing) with its edge set only. Since the hypergraphs presented here do not contain isolated vertices, this will uniquely define them.

We begin by proving a lemma that is fundamental to our constructions.

Lemma 2.1. *Let $n \geq 1$, $x \geq 0$, and $r \geq 0$ be integers and let $v = nx + r$. There exists a decomposition of $K_v^{(3)}$ into:*

(i) 1 *copy of* $K_{n+r}^{(3)}$,
(ii) $x - 1$ *copies of* $K_{n+r}^{(3)} \setminus K_r^{(3)}$ (*these are isomorphic to* $K_{n+r}^{(3)}$ *if* $r \in [0, 2]$),
(iii) $\binom{x}{2}$ *copies of* $K_{r,n,n}^{(3)} \cup L_{n,n}^{(3)}$ (*here* $K_{r,n,n}^{(3)}$ *is empty if* $r = 0$), *and*
(iv) $\binom{x}{3}$ *copies of* $K_{n,n,n}^{(3)}$.

Proof. If $x \in \{0,1\}$, the decomposition is trivial. Thus we may assume that $x \geq 2$.

Let $V_0, V_1, \ldots, V_x$ be pairwise disjoint sets of vertices with $|V_0| = r$, $|V_1| = |V_2| = \ldots = |V_x| = n$ and let $V = V_0 \cup V_1 \cup \ldots \cup V_x$. Then, $K_V^{(3)}$ can be viewed as the (edge-disjoint) union

$$K^{(3)}_{V_1 \cup V_0} \cup \bigcup_{2 \leq i \leq x} \left(K^{(3)}_{V_i \cup V_0} \backslash K^{(3)}_{V_0} \right) \cup \bigcup_{1 \leq i < j \leq x} \left(K^{(3)}_{V_0, V_i, V_j} \cup L^{(3)}_{V_i, V_j} \right) \cup \bigcup_{1 \leq i < j < k \leq x} \left(K^{(3)}_{V_i, V_j, V_k} \right)$$

Thus the result follows. □

If $LC_3^{(3)}$ decomposes $K_v^{(3)}$, then we must have $3 \mid \binom{v}{3}$ and hence $18 \mid v(v-1)(v-2)$. Therefore we have $v \equiv 0$, 1, or 2 (mod 9). In [6], it is shown that these necessary conditions are sufficient. Although a proof of Theorem 2.2 is given in [6], we include a proof here for the sake of completeness.

Theorem 2.2. *There exists an $LC_3^{(3)}$-decomposition of K_v if and only if $v \equiv 0$, 1, or* 2 (mod 9)*.*

Proof. Let $v = 9x + r$, where $r \in [0,2]$. If $x = 0$, the result is vacuously true. If $x = 1$, we give $LC_3^{(3)}$-decompositions of $K_9^{(3)}$ in Example 3.1, of $K_{10}^{(3)}$ in Example 3.2, and of $K_{11}^{(3)}$ in Example 3.3. Thus we may assume that $x \geq 2$. By Lemma 2.1, it suffices to give $LC_3^{(3)}$-decompositions of $K_{9+r}^{(3)}$, of $K_{9+r}^{(3)} \setminus K_r^{(3)}$ which is isomorphic to $K_{9+r}^{(3)}$ since $r \in [0,2]$, of $K_{r,9,9}^{(3)} \cup L_{9,9}^{(3)}$, and of $K_{9,9,9}^{(3)}$. A decomposition of $K_{1,9,9}^{(3)} \cup L_{9,9}^{(3)}$ is given in Example 3.7, and a decomposition of $L_{9,9}^{(3)}$ is given in Example 3.6. Since $K_{2,3,3}^{(3)}$ decomposes $K_{2,9,9}^{(3)}$ and $K_{3,3,3}^{(3)}$ decomposes $K_{9,9,9}^{(3)}$ and since $LC_3^{(3)}$-decompositions of $K_{2,3,3}^{(3)}$ of $K_{3,3,3}^{(3)}$ are given in Examples 3.4 and 3.5, we have that $LC_3^{(3)}$ decomposes both $K_{2,9,9}^{(3)}$ and $K_{9,9,9}^{(3)}$. Thus the result follows. □

Next, we give our main result on maximum $LC_3^{(3)}$-packings of $K_v^{(3)}$.

Theorem 2.3. *If $v \geq 6$ is an integer, then there exists a maximum $LC_3^{(3)}$-packing of $K_v^{(3)}$ where the leave has two or fewer edges.*

Proof. If $v \equiv 0$, 1, or 2 (mod 9), then the result follows from the $LC_3^{(3)}$-decomposition result in Theorem 2.2, which translates to a maximum $LC_3^{(3)}$-packing with an empty leave. If $v \in [6,8]$, a maximum $LC_3^{(3)}$-packing of $K_v^{(3)}$ with a two edge leave is given in Examples 3.13–3.15. Hence, we need only consider when $v = 9x + r$ where $x \geq 1$ and $r \in [3,8]$. By Lemma 2.1 it suffices to find

(i) a maximum $LC_3^{(3)}$-packing of $K_{9+r}^{(3)}$ with a leave consisting of two or fewer edges and
(ii) $LC_3^{(3)}$-decompositions of $K_{9+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,9,9}^{(3)} \cup L_{9,9}^{(3)}$, and $K_{9,9,9}^{(3)}$.

We note that an $LC_3^{(3)}$-decomposition of $K_{12}^{(3)} \setminus K_3^{(3)}$ is equivalent an $LC_3^{(3)}$-packing of $K_{12}^{(3)}$ with a leave consisting of the single edge in the hole, which is given in Example 3.16. Also, for $r \geq 3$, it is simple to see that $K_{r,9,9}^{(3)}$ is decomposable into copies of $K_{2,3,3}^{(3)}$ and $K_{3,3,3}^{(3)}$. Maximum $LC_3^{(3)}$-packings (with leaves of two or fewer edges) of $K_{9+r}^{(3)}$, for $r \in [3,8]$, are given in Examples 3.16–3.21. Similarly, $LC_3^{(3)}$-decompositions of $K_{9+r}^{(3)} \setminus K_r^{(3)}$, for $r \in [4,8]$, are given in Examples 3.8–3.12. Finally, an $LC_3^{(3)}$-decomposition of $L_{9,9}^{(3)}$ is given in Example 3.6, and $LC_3^{(3)}$-decompositions of $K_{2,3,3}^{(3)}$ and of $K_{3,3,3}^{(3)}$ are given in Examples 3.4 and 3.5, respectively. $\square$

Next, we give a lemma on maximum $LC_3^{(3)}$-packings of ${}^2K_v^{(3)}$ for $v \in [6,17]$.

Lemma 2.4. *If $v \in [6,17]$, then there exists a maximum $LC_3^{(3)}$-packing of ${}^2K_v^{(3)}$ where the leave has two or fewer edges.*

Proof. Let $V\left({}^2K_v^{(3)}\right) = \mathbb{Z}_v$. If $v \in [9,11]$, there exists an $LC_3^{(3)}$-decomposition of $K_v^{(3)}$ and hence of ${}^2K_v^{(3)}$. Next, if $v \in [6,8] \cup [15,17]$, let Δ_1 be a maximum $LC_3^{(3)}$-packing of $K_v^{(3)}$ where the leave has edge set $\big\{\{0,1,2\},\{2,3,4\}\big\}$ (which exists by Examples 3.13–3.15 and Examples 3.19–3.21) and let Δ_2 be another maximum $LC_3^{(3)}$-packing of $K_v^{(3)}$ where the leave has edge set $\big\{\{4,5,0\},\{0,1,2\}\big\}$. Then $\Delta_1 \cup \Delta_2 \cup \big\{H[0,1,2,3,4,5]\big\}$ is a maximum $LC_3^{(3)}$-packing of ${}^2K_v^{(3)}$ where $\{0,1,2\}$ is the only edge in the leave. Finally, if $v \in [12,14]$, let Δ_1 be a maximum packing of $K_v^{(3)}$ where $\{0,1,2\}$ is the only edge in the leave (which exists by Examples 3.16–3.18) and let Δ_2 be a maximum $LC_3^{(3)}$-packing of $K_v^{(3)}$ where $\{2,3,4\}$ is the only edge in the leave. Then $\Delta_1 \cup \Delta_2$ is a maximum $LC_3^{(3)}$-packing of ${}^2K_v^{(3)}$ where the leave has edge set $\big\{\{0,1,2\},\{2,3,4\}\big\}$. $\square$

Now we extend our results to maximum $LC_3^{(3)}$-packings of ${}^2K_v^{(3)}$ in general.

Theorem 2.5. *If $v \geq 6$, then there exists a maximum $LC_3^{(3)}$-packing of ${}^2K_v^{(3)}$ where the leave has two or fewer edges.*

Proof. If $v \equiv 0$, 1, or 2 (mod 9), then the result follows from Theorem 2.2, which translates to a maximum $LC_3^{(3)}$-packing with an empty leave. If $v \in [6,8]$, a maximum $LC_3^{(3)}$-packing of ${}^2K_v^{(3)}$ with a one edge leave is given in Lemma 2.4. Hence, we need only consider when $v \equiv r \pmod 9, r \geq 3, v \geq 12$. Let $v = 9x + r$ where $x \geq 1$ and $r \in [3,8]$. By Lemma 2.1 it suffices to find

(i) a maximum $LC_3^{(3)}$-packing of ${}^2K_{9+r}^{(3)}$ with a leave consisting of two or fewer edges and
(ii) $LC_3^{(3)}$-decompositions of ${}^2K_{9+r}^{(3)} \setminus {}^2K_r^{(3)}$, ${}^2K_{r,9,9}^{(3)} \cup {}^2L_{9,9}^{(3)}$, and ${}^2K_{9,9,9}^{(3)}$.

But since $LC_3^{(3)}$ decomposes $K_{9+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,9,9}^{(3)} \cup L_{9,9}^{(3)}$, and $K_{9,9,9}^{(3)}$ (see argument in proof of Theorem 2.3), $LC_3^{(3)}$ decomposes the 2-fold versions of these hypergraphs. Maximum $LC_3^{(3)}$-packings (with leaves of two or fewer edges) of ${}^2K_{9+r}^{(3)}$, for $r \in [3,8]$, are given in Lemma 2.4. The result now follows. $\square$

Next, we give a lemma on $LC_3^{(3)}$-decompositions of ${}^3K_v^{(3)}$ for $v \in [6,17]$.

Lemma 2.6. *If $v \in [6,17]$, then there exists an $LC_3^{(3)}$-decomposition of ${}^3K_v^{(3)}$.*

Proof. Let $V\left({}^3K_v^{(3)}\right) = \mathbb{Z}_v$. If $v \in [9,11]$, there exists an $LC_3^{(3)}$-decomposition of $K_v^{(3)}$ and hence of ${}^3K_v^{(3)}$. Next, if $v \in [6,8] \cup [15,17]$, let Δ_1 be a maximum packing of $K_v^{(3)}$ where the leave has edge set $\{\{2,3,4\},\{4,5,0\}\}$ (which exists by Examples 3.13–3.15 and Examples 3.19–3.21) and let Δ_2 be a maximum $LC_3^{(3)}$-packing of ${}^2K_v^{(3)}$ where $\{0,1,2\}$ is the only edge in the leave (which exists by Lemma 2.4). Then $\Delta_1 \cup \Delta_2 \cup \{H[0,1,2,3,4,5]\}$ is an $LC_3^{(3)}$-decomposition of ${}^3K_v^{(3)}$. Finally, if $v \in [12,14]$, let Δ_1 be a maximum packing of $K_v^{(3)}$ where $\{4,5,0\}$ is the only edge in the leave (which exists by Examples 3.16–3.18) and let Δ_2 be a maximum $LC_3^{(3)}$-packing of ${}^2K_v^{(3)}$ where the leave has edge set $\{\{0,1,2\},\{2,3,4\}\}$ (which exists by Lemma 2.4). Then $\Delta_1 \cup \Delta_2 \cup \{H[0,1,2,3,4,5]\}$ is an $LC_3^{(3)}$-decomposition of ${}^3K_v^{(3)}$. □

It is simple to see that if there is an $LC_3^{(3)}$-decomposition of ${}^\lambda K_v^{(3)}$, then we must have $v \notin [3,5]$ and either $\lambda \equiv 0 \pmod 3$ or $v \equiv 0,1,2 \pmod 9$. Thus, in light of Theorem 2.2 and Lemmas 2.1 and 2.6 and because ${}^3K_v^{(3)}$ decomposes ${}^{3k}K_v^{(3)}$ for all positive integers k, we have the following obvious corollary.

Corollary 2.7. *Let λ and $v \notin [3,5]$ be positive integers. There exists an $LC_3^{(3)}$-decomposition of ${}^\lambda K_v^{(3)}$ if and only if $\lambda \equiv 0 \pmod 3$ or $v \equiv 0$, 1, or 2 $\pmod 9$.*

Finally we give our general main result.

Theorem 2.8. *If λ and $v \notin [3,5]$ are positive integers, then there exists a maximum $LC_3^{(3)}$-packing of ${}^\lambda K_v^{(3)}$ where the leave has two or fewer edges.*

Proof. If $\lambda \in \{1,2\}$, the result follows from Theorems 2.3 and 2.5. If $\lambda \equiv 0 \pmod 3$, the result follows from Corollary 2.7. Suppose $\lambda \geq 4$ and let $\lambda = 3b + r$ for integers $b \geq 1$ and $r \in \{1,2\}$. We can view ${}^\lambda K_v^{(3)}$ as the edge disjoint union of ${}^{3b}K_v^{(3)}$ and ${}^rK_v^{(3)}$. An $LC_3^{(3)}$-decomposition of ${}^{3b}K_v^{(3)}$ exists by Corollary 2.7 and a maximum $LC_3^{(3)}$-packing of ${}^rK_v^{(3)}$ where the leave has two or fewer edges follows from Theorems 2.3 and 2.5. Thus the result follows. □

3. SMALL EXAMPLES

Example 3.1. Let

$$V\left(K_9^{(3)}\right) = \mathbb{Z}_7 \cup \{\infty_1,\infty_2\}$$

and let

$$B = \{H[0,1,2,4,6,3], H[\infty_1,0,1,\infty_2,2,4], H[0,1,3,\infty_2,6,4], H[\infty_1,0,3,1,\infty_2,2]\}.$$

Then an $LC_3^{(3)}$-decomposition of $K_9^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 7$.

Example 3.2. Let
$$V\left(K_{10}^{(3)}\right) = \mathbb{Z}_{10}$$
and let
$$B = \big\{H[0,2,1,3,4,9], H[0,7,1,2,4,5], H[0,4,2,5,7,6], H[0,6,2,4,7,3]\big\}.$$
Then an $LC_3^{(3)}$-decomposition of $K_{10}^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $j \mapsto j+1 \pmod{10}$.

Example 3.3. Let
$$V\left(K_{11}^{(3)}\right) = \mathbb{Z}_{11}$$
and let
$$B = \big\{H[0,8,2,6,9,1], H[0,8,1,4,7,2], H[1,0,5,3,8,9], H[0,7,1,10,5,6], H[0,3,1,8,10,9]\big\}.$$
Then an $LC_3^{(3)}$-decomposition of $K_{11}^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $j \mapsto j+1 \pmod{11}$.

Example 3.4. Let
$$V\left(K_{2,3,3}^{(3)}\right) = \mathbb{Z}_6 \cup \{\infty_1, \infty_2\}$$
with the vertex partition $\big\{\{\infty_1,\infty_2\},\{0,2,4\},\{1,3,5\}\big\}$ and let
$$B = \big\{H[\infty_1,0,1,\infty_2,2,5], H[5,\infty_2,0,1,\infty_1,2,5]\big\}.$$
Then an $LC_3^{(3)}$-decomposition of $K_{2,3,3}^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+2 \pmod 6$.

Example 3.5. Let
$$V\left(K_{3,3,3}^{(3)}\right) = \mathbb{Z}_9$$
with vertex partition $\big\{\{0,3,6\},\{1,4,7\},\{2,5,8\}\big\}$ and let
$$B = \big\{H[7,3,2,1,0,5]\big\}.$$
Then an $LC_3^{(3)}$-decomposition of $K_{3,3,3}^{(3)}$ consists of the $LC_3^{(3)}$-block in B under the action of the map $j \mapsto j+1 \pmod 9$.

Example 3.6. Let
$$V\left(L_{9,9}^{(3)}\right) = \mathbb{Z}_{18}$$
with vertex partition $\big\{\{0,2,\ldots,16\},\{1,3,\ldots,17\}\big\}$, and let
$$B = \big\{H[0,16,1,4,15,2], H[14,0,1,11,16,9], H[4,0,9,1,14,3], H[9,0,1,6,17,2], H[0,5,1,12,15,14], H[0,1,2,5,3,15], H[0,1,15,10,4,13], H[1,5,12,7,0,13], H[0,1,12,3,9,16], H[10,2,17,8,1,0], H[1,0,8,2,17,4], H[1,0,7,12,6,16]\big\}.$$

Then an $LC_3^{(3)}$-decomposition of $L_{9,9}^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $j \mapsto j+1 \pmod{18}$.

Example 3.7. Let
$$V\left(L_{9,9}^{(3)} \cup K_{1,9,9}^{(3)}\right) = \mathbb{Z}_{18} \cup \{\infty\}$$
with vertex partition $\big\{\{\infty\}, \{0,2,\ldots,16\}, \{1,3,\ldots,17\}\big\}$, and let
$$\begin{aligned} B = \big\{&H[0,16,1,4,15,2], H[14,0,1,11,16,9], H[4,0,9,1,14,3], H[9,0,1,6,17,2],\\ &H[0,5,1,12,15,14], H[0,1,2,\infty,3,15], H[0,1,15,10,4,13], H[1,5,12,7,0,13],\\ &H[0,\infty,3,12,9,16], H[14,7,17,8,1,\infty], H[1,\infty,8,2,17,4], H[1,0,7,12,6,16]\big\}, \end{aligned}$$
$$\begin{aligned} B' = \big\{&H[0,\infty,9,10,1,3], H[1,\infty,10,11,2,4], H[2,\infty,11,12,3,5],\\ &H[3,\infty,12,13,4,6], H[4,\infty,13,14,5,7], H[5,\infty,14,15,6,8],\\ &H[6,\infty,15,16,7,9], H[7,\infty,16,17,8,10], H[8,\infty,17,0,9,11],\\ &H[10,0,1,2,9,3], H[11,1,2,3,10,4], H[12,2,3,4,11,5],\\ &H[13,3,4,5,12,6], H[14,4,5,6,13,7], H[15,5,6,7,14,8],\\ &H[16,6,7,8,15,9], H[17,7,8,9,16,10], H[0,8,9,10,17,11],\\ &H[12,9,10,11,0,1], H[13,10,11,12,1,2], H[14,11,12,13,2,3],\\ &H[15,12,13,14,3,4], H[16,13,14,15,4,5], H[17,14,15,16,5,6],\\ &H[0,15,16,17,6,7], H[1,16,17,0,7,8], H[2,17,0,1,8,9]\big\}. \end{aligned}$$
Then an $LC_3^{(3)}$-decomposition of $L_{9,9}^{(3)} \cup K_{1,9,9}^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{18}$, along with the $LC_3^{(3)}$-blocks in B'.

Example 3.8. Let
$$V\left(K_{13}^{(3)} \setminus K_4^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$$
with $\infty_1, \ldots, \infty_4$ being the vertices in the hole and let
$$\begin{aligned} B = \big\{&H[\infty_1,0,\infty_2,1,4,8], H[\infty_2,0,\infty_3,1,4,8], H[\infty_3,0,\infty_4,1,4,8],\\ &H[\infty_4,0,\infty_1,1,4,8], H[\infty_1,0,\infty_3,1,2,3], H[\infty_2,0,\infty_4,1,2,3],\\ &H[0,\infty_1,3,\infty_2,6,\infty_3], H[0,1,4,8,3,\infty_4], H[0,2,4,6,1,8]\big\}, \end{aligned}$$
$$\begin{aligned} B' = \big\{&H[0,2,8,5,3,6], H[1,3,0,6,4,7], H[2,4,1,7,5,8], H[5,2,3,6,4,7],\\ &H[8,5,6,0,7,1], H[6,2,8,4,1,3], H[7,3,0,5,2,4], H[6,1,0,4,3,7],\\ &H[7,2,1,5,4,8], H[8,3,2,6,5,0], H[0,7,1,8,2,3], H[3,1,4,2,5,6],\\ &H[6,4,7,5,8,0]\big\}. \end{aligned}$$
Then an $LC_3^{(3)}$-decomposition of $K_{13}^{(3)} \setminus K_4^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 9$ along with the $LC_3^{(3)}$-blocks in B'.

Example 3.9. Let

$$V\left(K_{14}^{(3)} \setminus K_5^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$$

with $\infty_1, \ldots, \infty_5$ being the vertices in the hole and let

$$\begin{aligned} B = \big\{ & H[\infty_1, 0, \infty_2, 1, 2, 6], H[\infty_2, 0, \infty_3, 1, 2, 6], H[\infty_3, 0, \infty_4, 1, 2, 6], \\ & H[\infty_4, 0, \infty_5, 1, 2, 6], H[\infty_5, 0, \infty_1, 1, 2, 6], H[\infty_1, 0, \infty_3, 1, 3, 6], \\ & H[\infty_2, 0, \infty_4, 1, 3, 6], H[\infty_3, 0, \infty_5, 1, 3, 6], H[\infty_4, 0, \infty_1, 1, 3, 6], \\ & H[\infty_5, 0, \infty_2, 1, 3, 6], H[0, 2, 4, 6, 1, 8], H[1, 0, 4, 5, 2, 6]\big\}, \\ B' = \big\{ & H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7], \\ & H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3], \\ & H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0]\big\}. \end{aligned}$$

Then an $LC_3^{(3)}$-decomposition of $K_{14}^{(3)} \setminus K_5^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod 9$ along with the $LC_3^{(3)}$-blocks in B'.

Example 3.10. Let

$$V\left(K_{15}^{(3)} \setminus K_6^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$$

with $\infty_1, \ldots, \infty_6$ being the vertices in the hole and let

$$\begin{aligned} B = \big\{ & H[\infty_1, 0, \infty_2, 1, \infty_3, 2], H[\infty_3, 0, \infty_4, 1, \infty_5, 2], H[\infty_5, 0, \infty_6, 1, \infty_1, 2], \\ & H[\infty_2, 0, \infty_4, 1, \infty_6, 2], H[\infty_1, 0, 4, \infty_2, 3, 6], H[\infty_2, 0, 4, \infty_3, 3, 6], \\ & H[\infty_3, 0, 4, \infty_4, 3, 6], H[\infty_4, 0, 4, \infty_5, 3, 6], H[\infty_5, 0, 4, \infty_6, 3, 6], \\ & H[\infty_6, 0, 4, \infty_1, 3, 6], H[0, 2, \infty_1, 1, \infty_4, 7], H[0, 2, \infty_2, 1, \infty_5, 7], \\ & H[0, 2, \infty_3, 1, \infty_6, 7], H[0, 2, 4, 6, 1, 8], H[1, 0, 4, 5, 2, 6]\big\}, \\ B' = \big\{ & H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7], \\ & H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3], \\ & H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0]\big\}. \end{aligned}$$

Then an $LC_3^{(3)}$-decomposition of $K_{15}^{(3)} \setminus K_6^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod 9$ along with the $LC_3^{(3)}$-blocks in B'.

Example 3.11. Let

$$V\left(K_{16}^{(3)} \setminus K_7^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$$

with $\infty_1, \ldots, \infty_7$ being the vertices in the hole and let

$$\begin{aligned} B = \{ & H[\infty_1, 0, \infty_2, 1, \infty_4, 2], H[\infty_2, 0, \infty_3, 1, \infty_5, 2], H[\infty_3, 0, \infty_4, 1, \infty_6, 2],\\ & H[\infty_4, 0, \infty_5, 1, \infty_7, 2], H[\infty_5, 0, \infty_6, 1, \infty_1, 2], H[\infty_6, 0, \infty_7, 1, \infty_2, 2],\\ & H[\infty_7, 0, \infty_1, 1, \infty_3, 2], H[\infty_1, 0, 1, \infty_2, 3, 7], H[\infty_2, 0, 1, \infty_3, 3, 7],\\ & H[\infty_3, 0, 1, \infty_4, 3, 7], H[\infty_4, 0, 1, \infty_5, 3, 7], H[\infty_5, 0, 1, \infty_6, 3, 7],\\ & H[\infty_6, 0, 1, \infty_7, 3, 7], H[\infty_7, 0, 1, \infty_1, 3, 7], H[0, \infty_1, 3, \infty_2, 6, \infty_3],\\ & H[0, \infty_4, 3, \infty_5, 6, \infty_6], H[0, 1, 4, 8, 3, \infty_7], H[0, 2, 4, 6, 1, 8]\},\end{aligned}$$

$$\begin{aligned} B' = \{ & H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7],\\ & H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[6, 1, 0, 4, 3, 7],\\ & H[7, 2, 1, 5, 4, 8], H[8, 3, 2, 6, 5, 0], H[0, 7, 1, 8, 2, 3], H[3, 1, 4, 2, 5, 6],\\ & H[6, 4, 7, 5, 8, 0]\}.\end{aligned}$$

Then an $LC_3^{(3)}$-decomposition of $K_{16}^{(3)} \setminus K_7^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 9$ along with the $LC_3^{(3)}$-blocks in B'.

Example 3.12. Let

$$V\left(K_{17}^{(3)} \setminus K_8^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8\}$$

with $\infty_1, \ldots, \infty_8$ being the vertices in the hole and let

$$\begin{aligned} B = \{ & H[\infty_1, 0, \infty_2, 1, \infty_4, 2], H[\infty_2, 0, \infty_3, 1, \infty_5, 2], H[\infty_3, 0, \infty_4, 1, \infty_6, 2],\\ & H[\infty_4, 0, \infty_5, 1, \infty_7, 2], H[\infty_5, 0, \infty_6, 1, \infty_8, 2], H[\infty_6, 0, \infty_7, 1, \infty_1, 2],\\ & H[\infty_7, 0, \infty_8, 1, \infty_2, 2], H[\infty_8, 0, \infty_1, 1, \infty_3, 2], H[\infty_1, 0, 4, \infty_2, 3, 6],\\ & H[\infty_2, 0, 4, \infty_3, 3, 6], H[\infty_3, 0, 4, \infty_4, 3, 6], H[\infty_4, 0, 4, \infty_5, 3, 6],\\ & H[\infty_5, 0, 4, \infty_6, 3, 6], H[\infty_6, 0, 4, \infty_7, 3, 6], H[\infty_7, 0, 4, \infty_8, 3, 6],\\ & H[\infty_8, 0, 4, \infty_1, 3, 6], H[\infty_1, 0, 2, 4, \infty_5, 1], H[\infty_2, 0, 2, 4, \infty_6, 1],\\ & H[\infty_3, 0, 2, 4, \infty_7, 1], H[\infty_4, 0, 2, 4, \infty_8, 1], H[0, 2, 4, 6, 1, 8], H[1, 0, 4, 5, 2, 6]\},\end{aligned}$$

$$\begin{aligned} B' = \{ & H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7],\\ & H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3],\\ & H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0]\}.\end{aligned}$$

Then an $LC_3^{(3)}$-decomposition of $K_{17}^{(3)} \setminus K_8^{(3)}$ consists of the $LC_3^{(3)}$-blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j+1 \pmod 9$ along with the $LC_3^{(3)}$-blocks in B'.

Example 3.13. Let

$$V\left(K_6^{(3)}\right) = \mathbb{Z}_6$$

and let

$$\begin{aligned} B = \{ & H[4, 0, 1, 5, 3, 2], H[5, 1, 2, 0, 4, 3], H[4, 5, 0, 3, 2, 1], H[5, 0, 1, 4, 3, 2],\\ & H[0, 1, 2, 5, 4, 3], H[1, 2, 3, 0, 5, 4]\}.\end{aligned}$$

Then B is a maximum $LC_3^{(3)}$-packing of $K_6^{(3)}$, where the leave has edge set $\{\{0,1,3\},\{1,2,5\}\}$. Note that by renaming the vertices in this packing, any two hyperedges in $K_6^{(3)}$ that intersect in a single vertex can be made into the edge set of the leave of a maximum $LC_3^{(3)}$-packing of $K_6^{(3)}$.

Example 3.14. Let
$$V\left(K_7^{(3)}\right) = \mathbb{Z}_7$$
and let
$$\begin{aligned} B &= \{H[0,1,2,4,6,3]\},\\ B' &= \{H[3,0,1,4,2,5], H[6,3,4,0,5,1], H[1,6,2,0,3,4], H[4,2,5,3,6,0]\}. \end{aligned}$$
Then a maximum $LC_3^{(3)}$-packing of $K_7^{(3)}$, where the leave has edge set $\{\{0,1,5\},\{0,2,6\}\}$, consists of the $LC_3^{(3)}$-blocks in B under the action of the map $j \mapsto j+1 \pmod 7$ along with the $LC_3^{(3)}$-blocks in B'. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_7^{(3)}$ that intersect in a single vertex can be made into the leave edge set of a maximum $LC_3^{(3)}$-packing of $K_7^{(3)}$.

Example 3.15. Let
$$V\left(K_8^{(3)}\right) = \mathbb{Z}_8$$
and let
$$\begin{aligned} B &= \{H[6,0,7,2,3,1], H[0,2,6,7,4,1]\},\\ B' &= \{H[3,0,1,4,2,5], H[6,3,4,7,5,0]\}. \end{aligned}$$
Then a maximum $LC_3^{(3)}$-packing of $K_8^{(3)}$, where the leave has edge set $\{\{1,6,7\},\{0,2,7\}\}$, consists of the $LC_3^{(3)}$-blocks in B under the action of the map $j \mapsto j+1 \pmod 8$ along with the $LC_3^{(3)}$-blocks in B'. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_8^{(3)}$ that intersect in a single vertex can be made into the leave edge set of a maximum $LC_3^{(3)}$-packing of $K_8^{(3)}$.

Example 3.16. Let
$$V\left(K_{12}^{(3)}\right) = \mathbb{Z}_{11} \cup \{\infty\}$$
and let
$$\begin{aligned} B = \{&H[0,\infty,1,3,2,4], H[0,\infty,2,8,5,10], H[0,\infty,3,7,4,8], H[0,\infty,4,9,5,2],\\ &H[0,\infty,5,10,9,6], H[0,6,2,9,4,3]\},\\ B' = \{&H[3,0,1,4,2,5], H[6,3,4,7,5,8], H[9,6,7,10,8,0], H[1,10,2,0,3,4],\\ &H[4,2,5,3,6,7], H[8,6,9,1,10,0], H[9,7,10,2,0,1]\}. \end{aligned}$$

Then a maximum $LC_3^{(3)}$-packing of $K_{12}^{(3)}$, where the leave is the single edge $\{5,7,8\}$, consists of the $LC_3^{(3)}$-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{11}$ along with the $LC_3^{(3)}$-blocks in B'. Note that by renaming the vertices in this packing, any edge in $K_{12}^{(3)}$ can be made into the leave of a maximum $LC_3^{(3)}$-packing of $K_{12}^{(3)}$.

Example 3.17. Let

$$V\left(K_{13}^{(3)}\right) = \mathbb{Z}_{13}$$

and let

$$\begin{aligned} B = \big\{&H[0,3,9,12,1,11], H[0,4,8,12,1,10], H[12,4,9,0,1,7], H[12,5,8,0,1,6],\\ &H[7,10,4,11,0,1], H[6,10,5,2,0,1], H[4,2,3,5,1,0]\big\},\\ B' = \big\{&H[3,0,1,4,2,5], H[6,3,4,7,5,8], H[9,6,7,10,8,11], H[12,9,10,0,11,1]\big\}. \end{aligned}$$

Then a maximum $LC_3^{(3)}$-packing of $K_{13}^{(3)}$, where the leave is the single edge $\big\{\{0,2,12\}\big\}$, consists of the $LC_3^{(3)}$-blocks in B under the action of the map $j \mapsto j+1 \pmod{13}$ along with the $LC_3^{(3)}$-blocks in B'. Again, we note that by renaming the vertices in this packing, any edge in $K_{13}^{(3)}$ can be made into the leave of a maximum $LC_3^{(3)}$-packing of $K_{13}^{(3)}$.

Example 3.18. Let

$$V\left(K_{14}^{(3)}\right) = \mathbb{Z}_{14}$$

and let

$$\begin{aligned} B = \big\{&H[13,0,8,3,5,12], H[0,10,1,3,5,12], H[0,8,2,3,4,13], H[0,1,11,5,6,3],\\ &H[0,3,8,2,9,4], H[0,2,9,13,7,3], H[0,3,10,5,8,4]H[0,4,1,7,12,8]\big\},\\ B' =\big\{&H[3,0,1,4,2,5], H[6,3,4,7,5,8], H[9,6,7,10,8,11], H[12,9,10,13,11,0],\\ &H[1,13,2,0,3,4], H[4,2,5,3,6,7], H[7,5,8,6,9,10], H[11,9,12,1,13,0],\\ &H[12,10,13,2,0,1]\big\}. \end{aligned}$$

Then a maximum $LC_3^{(3)}$-packing of $K_{14}^{(3)}$, where the leave is the single edge $\{8,10,11\}$, consists of the $LC_3^{(3)}$-blocks in B under the action of the map $j \mapsto j+1 \pmod{14}$ along with the $LC_3^{(3)}$-blocks in B'. Again, we note that by renaming the vertices in this packing, any edge in $K_{14}^{(3)}$ can be made into the leave of a maximum $LC_3^{(3)}$-packing of $K_{14}^{(3)}$.

Example 3.19. Let

$$V\left(K_{15}^{(3)}\right) = \mathbb{Z}_{13} \cup \{\infty_1, \infty_2\}$$

and let

$$\begin{aligned}B = \big\{&H[\infty_1, 0\infty_2, 1, 2, 3], H[0, \infty_1, 2\infty_2, 4, 8], H[0, \infty_1, 3\infty_2, 6, 12],\\ &H[0, \infty_1, 4\infty_2, 8, 3], H[0, \infty_1, 5\infty_2, 10, 7], H[0, \infty_1, 6\infty_2, 12, 11],\\ &H[0, 9, 11, 8, 4, 10], H[0, 5, 2, 4, 8, 6], H[0, 10, 2, 4, 11, 6],\\ &H[0, 1, 4, 3, 8, 7], H[0, 1, 10, 9, 5, 6]\big\},\\ B' = \big\{&H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], H[12, 9, 10, 0, 11, 1],\\ &H[1, 12, 2, 0, 3, 4], H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[10, 8, 11, 9, 12, 0]\big\}.\end{aligned}$$

Then a maximum $LC_3^{(3)}$-packing of $K_{15}^{(3)}$, where the leave has edge set $\big\{\{0, 2, 14\}, \{2, 4, 5\}\big\}$, consists of the $LC_3^{(3)}$-blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{13}$ along with the $LC_3^{(3)}$-blocks in B'. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{15}^{(3)}$ that intersect in a single vertex can be made into the edge set the leave of a maximum $LC_3^{(3)}$-packing of $K_{15}^{(3)}$.

Example 3.20. Let

$$V\left(K_{16}^{(3)}\right) = \mathbb{Z}_{16}$$

and let

$$\begin{aligned}B = \big\{&H[1, 0, 13, 10, 6, 4], H[0, 5, 10, 6, 11, 4], H[1, 14, 9, 3, 7, 13],\\ &H[0, 11, 1, 12, 3, 10], H[0, 10, 2, 14, 7, 6], H[13, 5, 12, 0, 4, 7],\\ &H[0, 3, 8, 15, 7, 14], H[15, 5, 14, 8, 6, 2], H[0, 13, 3, 8, 1, 6],\\ &H[1, 0, 4, 8, 3, 7], H[0, 14, 12, 10, 11, 13]\big\},\\ B' = \big\{&H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11],\\ &H[12, 9, 10, 13, 11, 14], H[15, 12, 13, 0, 14, 1], H[1, 15, 2, 0, 3, 4],\\ &H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[10, 8, 11, 9, 12, 13],\\ &H[13, 11, 14, 12, 15, 0]\big\}.\end{aligned}$$

Then a maximum $LC_3^{(3)}$-packing of $K_{16}^{(3)}$, where the leave has edge set $\big\{\{0, 1, 14\}, \{0, 2, 15\}\big\}$, consists of the $LC_3^{(3)}$-blocks in B under the action of the map $j \mapsto j + 1 \pmod{16}$ along with the $LC_3^{(3)}$-blocks in B'. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{16}^{(3)}$ that intersect in a single vertex can be made into the edge set of the leave of a maximum $LC_3^{(3)}$-packing of $K_{16}^{(3)}$.

Example 3.21. Let
$$V\left(K_{17}^{(3)}\right) = \mathbb{Z}_{17}$$
and let
$$\begin{aligned} B = \big\{ & H[0,15,1,16,13,3], H[0,14,1,13,5,11], H[2,13,0,4,11,16], \\ & H[11,16,1,13,0,3], H[0,12,1,11,5,10], H[0,11,1,16,10,3], \\ & H[0,10,1,16,9,4], H[14,6,0,9,1,10], H[0,8,1,6,15,7], \\ & H[15,5,1,7,0,6], H[0,6,1,4,15,5], H[0,15,4,8,5,1], \\ & H[0,15,3,5,4,2]\big\}, \\ B' = \big\{ & H[3,0,1,4,2,5], H[6,3,4,7,5,8], H[9,6,7,10,8,11], \\ & H[12,9,10,13,11,14], H[15,12,13,16,14,0]\big\}. \end{aligned}$$

Then a maximum $LC_3^{(3)}$-packing of $K_{17}^{(3)}$, where the leave has edge set $\big\{\{1,15,16\},\{0,2,16\}\big\}$, consists of the $LC_3^{(3)}$-blocks in B under the action of the map $j \mapsto j+1 \pmod{17}$ along with the $LC_3^{(3)}$-blocks in B'. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{17}^{(3)}$ that intersect in a single vertex can be made into the edge set of the leave of a maximum $LC_3^{(3)}$-packing of $K_{17}^{(3)}$.

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