MAXIMUM PACKINGS
OF THE $\lambda$-FOLD COMPLETE 3-UNIFORM
HYPERGRAPH
WITH LOOSE 3-CYCLES

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Abstract. It is known that the 3-uniform loose 3-cycle decomposes the complete 3-uniform hypergraph of order $v$ if and only if $v \equiv 0, 1, \text{ or } 2 \pmod{9}$. For all positive integers $\lambda$ and $v$, we find a maximum packing with loose 3-cycles of the $\lambda$-fold complete 3-uniform hypergraph of order $v$. We show that, if $v \geq 6$, such a packing has a leave of two or fewer edges.

Keywords: maximum packing, $\lambda$-fold complete 3-uniform hypergraph, loose 3-cycle.

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1. INTRODUCTION

A hypergraph $H$ consists of a finite nonempty set $V$ of vertices and a finite collection $E = \{e_1, e_2, \ldots, e_n\}$ of nonempty subsets of $V$ called hyperedges or simply edges. For a given hypergraph $H$, we use $V(H)$ and $E(H)$ to denote the vertex set and the edge set (or multiset) of $H$, respectively. We call $|V(H)|$ and $|E(H)|$ the order and size of $H$, respectively. The degree of a vertex $v \in V(H)$ is the number of edges in $E(H)$ that contain $v$. A hypergraph $H$ is simple if no edge appears more than once in $E(H)$. If for each $e \in E(H)$ we have $|e| = t$, then $H$ is said to be $t$-uniform. Thus $t$-uniform hypergraphs are generalizations of the concept of a graph (where $t = 2$). Graphs with repeated edges are often called multigraphs. If $H$ is a simple hypergraph and $\lambda$ is a positive integer, then $\lambda$-fold $H$, denoted $\lambda H$, is the multi-hypergraph obtained from $H$ by repeating each edge exactly $\lambda$ times. The hypergraph with vertex set $V$ and edge set the set of all $t$-element subsets of $V$ is called the complete $t$-uniform hypergraph on $V$ and is denoted by $K^t_v$. If $v = |V|$, then $\lambda K^t_v$ is called the $\lambda$-fold complete $t$-uniform hypergraph of order $v$ and is used to denote any hypergraph isomorphic.

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to $\lambda K_v^{(t)}$. When $t = 2$, we will use $\lambda K_v^{(2)}$ in place of $\lambda K_v^{(2)}$. Similarly, if $\lambda = 1$, then we will use $K_v^{(t)}$ in place of $\lambda K_v^{(t)}$. If $H'$ is a subhypergraph of $H$, then $H \setminus H'$ denotes the hypergraph obtained from $H$ by deleting the edges of $H'$. We may refer to $H \setminus H'$ as the hypergraph $H$ with a hole $H'$. The vertices in $H'$ may be referred to as the vertices in the hole.

A commonly studied problem in combinatorics concerns decompositions of graphs or multigraphs into edge-disjoint subgraphs. A decomposition of a multigraph $M$ is a set $\Delta = \{G_1, G_2, \ldots, G_s\}$ of subgraphs of $M$ such that $\{E(G_1), E(G_2), \ldots, E(G_s)\}$ is a partition of $E(M)$. If each element of $\Delta$ is isomorphic to a fixed graph $G$, then $\Delta$ is called a $G$-decomposition of $M$. If $L$ is a subgraph of $M$ and $\Delta$ is a $G$-decomposition of $M \setminus L$, then $\Delta$ is called a $G$-packing of $M$ with leave $L$. Such a $G$-packing is maximum if no other possible $G$-packing of $M$ has a leave of a smaller size than that of $L$. Clearly, if $|E(L)| < |E(G)|$, then the $G$-packing is maximum. Moreover, a $G$-decomposition of $M$ can be viewed as a maximum $G$-packing with an empty leave.

A $G$-decomposition of $\lambda K_v$ is also known as a $G$-design of order $v$ and index $\lambda$. A $K_k$-design of order $v$ and index $\lambda$ is usually known as a $2-(v, k, \lambda)$ design or as a balanced incomplete block design of index $\lambda$ or a $(v, k, \lambda)$-BIBD. The problem of determining all $v$ for which there exists a $G$-design of order $v$ is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to hypergraphs. A decomposition of a hypergraph $M$ is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of subhypergraphs of $M$ such that $\{E(H_1), E(H_2), \ldots, E(H_s)\}$ is a partition of $E(M)$. Any element of $\Delta$ isomorphic to a fixed hypergraph $H$ is called an $H$-block. If all elements of $\Delta$ are $H$-blocks, then $\Delta$ is called an $H$-decomposition of $M$. If $L$ is a subgraph of $M$ and $\Delta$ is an $H$-decomposition of $M \setminus L$, then $\Delta$ is called an $H$-packing of $M$ with leave $L$, where we again define such a packing to be maximum if $L$ has the fewest edges possible. An $H$-decomposition of $\lambda K_v^{(t)}$ is called an $H$-design of order $v$ and index $\lambda$. The problem of determining all $v$ for which there exists an $H$-design of order $v$ and index $\lambda$ is called the $\lambda$-fold spectrum problem for $H$-designs.

A $K_k^{(t)}$-design of order $v$ and index $\lambda$ is a generalization of $2-(v, k, \lambda)$ designs and is known as a $t-(v, k, \lambda)$ design or simply as a $t$-design. A summary of results on $t$-designs appears in [15]. A $t-(v, k, 1)$ design is also known as a Steiner system and is denoted by $S(t, v, k)$ (see [8] for a summary of results on Steiner systems). Keevash [14] has recently shown that for all $t$ and $k$ the obvious necessary conditions for the existence of an $S(t, k, v)$-design are sufficient for sufficiently large values of $v$. Similar results were obtained by Glock, Kühn, Lo, and Osthus [9, 10] and extended to include the corresponding asymptotic results for $H$-designs of order $v$ for all uniform hypergraphs $H$. These results for $t$-uniform hypergraphs mirror the celebrated results of Wilson [23] for graphs. Although these asymptotic results assure the existence of $H$-designs for sufficiently large values of $v$ for any uniform hypergraph $H$, the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

In the study of graph decompositions, a fair amount of the focus has been on $G$-decompositions of $K_v$, where $G$ is a graph with a relatively small number of edges (see [1] and [5] for known results). Some authors have investigated the corresponding
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problem for 3-uniform hypergraphs. For example, in [4], the 1-fold spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the 1-fold spectrum problem was settled in [6] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [6], they also settle the 1-fold spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered $H$-designs where $H$ is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let $T$, $O$, and $I$ denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph $T$ is the same as $K^{(3)}_4$, and its spectrum was settled in 1960 by Hanani [11]. In another paper [12], Hanani settled the spectrum problem for $O$-designs and gave necessary conditions for the existence of $I$-designs. The 1-fold spectrum problem is also settled for a type of 3-uniform hyperstars which is part of a larger class of hypergraphs known as delta-systems. For a positive integer $m$, let $S^{(3)}_m$ denote the 3-uniform hypergraph of size $m$ which consists of one vertex of degree $m$ and $2m$ vertices of degree one. Necessary and sufficient conditions for the existence of $S^{(3)}_m$-decompositions of $K^{(3)}_v$ are given in [21] for $m \in [4, 6]$ and for all $m$ in [18]. Some results on maximum $S^{(3)}_m$-packings of $K^{(3)}_v$ are given in [19]. Perhaps the best known general result on decompositions of complete $t$-uniform hypergraphs is Baranyai’s result [3] on the existence of $1$-factorizations of $K^{(t)}_m$ for all positive integers $m$. There are, however, several articles on decompositions of complete $t$-uniform hypergraphs (see [2] and [20]) and of $t$-uniform $t$-partite hypergraphs (see [16] and [22]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [13] and [17]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

In this paper we are interested in maximum $H$-packings of $\lambda K^{(3)}_v$, where $H$ is a 3-uniform loose 3-cycle. For integer $m \geq 3$, a 3-uniform loose $m$-cycle, denoted $LC^{(3)}_m$, is a 3-uniform hypergraph with vertex set $\{v_1, v_2, \ldots, v_{2m}\}$ and edge set $\{\{v_{2i-1}, v_{2i}, v_{2i+1}\} : 1 \leq i \leq m - 1\} \cup \{v_{2m-1}, v_{2m}, v_1\}$. Thus $LC^{(3)}_3$ has vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}, \{v_5, v_6, v_1\}\}$ for which we use $H[v_1, v_2, v_3, v_4, v_5, v_6]$ to denote (see Figure 1).

![Fig. 1. The 3-uniform loose 3-cycle, $LC^{(3)}_3$, denoted by $H[v_1, v_2, v_3, v_4, v_5, v_6]$.](image-url)
Since $LC_3^{(3)}$ has 3 edges and 6 vertices, it is one of the hypergraphs covered in the decomposition results by Bryant, Herke, Maenhaut, and Wannasit in [6]. It is shown in [6] that there exists an $LC_3^{(3)}$-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, \text{ or } 2 \pmod{9}$. Similarly, it is shown in [7] that there exists an $LC_3^{(3)}$-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$ and $v \notin \{4, 6\}$. Here we focus on maximum $LC_3^{(3)}$-packings of $\lambda K_v^{(3)}$ and show that if $\lambda$ and $v \geq 6$ are positive integers, then there exists a maximum $LC_3^{(3)}$-packing of $\lambda K_v^{(3)}$ where the leave has two or fewer edges.

1.1. ADDITIONAL NOTATION AND TERMINOLOGY

If $a$ and $b$ are integers with $a \leq b$, we define $[a, b]$ to be $\{a, a + 1, \ldots, b\}$. We next define some notation for certain types of 3-uniform hypergraphs.

Let $U_1, U_2, U_3$ be pairwise disjoint sets. The hypergraph with vertex set $U_1 \cup U_2 \cup U_3$ and edge set consisting of all 3-element sets having exactly one vertex in each of $U_1, U_2, U_3$ is denoted by $K^{(3)}_{U_1, U_2, U_3}$. The hypergraph with vertex set $U_1 \cup U_2$ and edge set consisting of all 3-element sets having at most 2 vertices in each of $U_1, U_2$ is denoted by $L^{(3)}_{U_1, U_2}$. If $|U_i| = u_i$ for $i \in \{1, 2, 3\}$, we may use $K^{(3)}_{u_1, u_2, u_3}$ or $L^{(3)}_{u_1, u_2}$ to denote any hypergraph that is isomorphic to $K^{(3)}_{U_1, U_2, U_3}$ or $L^{(3)}_{U_1, U_2}$, respectively. From a hypergraph decomposition perspective, we note that if $U_1, U_1', U_2, U_2', U_3$ are pairwise vertex disjoint, then

$$E(K^{(3)}_{U_1 \cup U_1', U_2, U_3}) = E(K^{(3)}_{U_1, U_2, U_3}) \cup E(K^{(3)}_{U_1', U_2, U_3}).$$

Thus, for any positive integer $x$, it is simple to see that $K^{(3)}_{u_1, u_2, u_3}$ decomposes $K^{(3)}_{u_1 + x, u_2, u_3}$ and, in general, $K^{(3)}_{u_1 + x, u_2, u_3}$ decomposes into one copy of $K^{(3)}_{u_1, u_2, u_3}$ and one copy of $K^{(3)}_{x, u_2, u_3}$.

2. MAIN CONSTRUCTIONS

The constructions in this section are dependent on many small examples. These examples are given in the last section. Throughout, we will often identify a hypergraph (e.g., a leave in a packing) with its edge set only. Since the hypergraphs presented here do not contain isolated vertices, this will uniquely define them.

We begin by proving a lemma that is fundamental to our constructions.

**Lemma 2.1.** Let $n \geq 1$, $x \geq 0$, and $r \geq 0$ be integers and let $v = nx + r$. There exists a decomposition of $K_v^{(3)}$ into:

1. 1 copy of $K_n^{(3)}$,
2. $x - 1$ copies of $K_n^{(3)} \setminus K_r^{(3)}$ (these are isomorphic to $K_{n + r}^{(3)}$ if $r \in [0, 2]$),
3. $\binom{x}{3}$ copies of $K_{n,n,n}^{(3)} \cup L_{n,n}^{(3)}$ (here $K_{r,n,n}$ is empty if $r = 0$), and
4. $\binom{x - 1}{3}$ copies of $K_{n,n,n}^{(3)}$. 


Proof. If \( x \in \{0, 1\} \), the decomposition is trivial. Thus we may assume that \( x \geq 2 \).

Let \( V_0, V_1, \ldots, V_x \) be pairwise disjoint sets of vertices with \( |V_0| = r, |V_1| = |V_2| = \ldots = |V_x| = n \) and let \( V = V_0 \cup V_1 \cup \ldots \cup V_x \). Then, \( K^{(3)}_V \) can be viewed as the (edge-disjoint) union

\[
K^{(3)}_{V_1 \cup V_0} \cup \bigcup_{2 \leq i \leq x} (K^{(3)}_{V_i \cup V_0} \setminus K^{(3)}_{V_0}) \cup \bigcup_{1 \leq i < j \leq x} (K^{(3)}_{V_i, V_j, V_0} \cup L^{(3)}_{V_i, V_j}) \cup \bigcup_{1 \leq i < j < k \leq x} (K^{(3)}_{V_i, V_j, V_k})
\]

Thus the result follows. \( \square \)

If \( LC_3^{(3)} \) decomposes \( K^{(3)}_9 \), then we must have \( 3 | \binom{9}{3} \) and hence \( 18 | v(v-1)(v-2) \). Therefore we have \( v \equiv 0, 1, \) or \( 2 \) (mod 9). In [6], it is shown that these necessary conditions are sufficient. Although a proof of Theorem 2.2 is given in [6], we include a proof here for the sake of completeness.

Theorem 2.2. There exists an \( LC_3^{(3)} \)-decomposition of \( K_v \) if and only if \( v \equiv 0, 1, \) or \( 2 \) (mod 9).

Proof. Let \( v = 9x + r \), where \( r \in [0, 2] \). If \( x = 0 \), the result is vacuously true. If \( x = 1 \), we give \( LC_3^{(3)} \)-decompositions of \( K^{(3)}_9 \) in Example 3.1, of \( K^{(3)}_{10} \) in Example 3.2, and of \( K^{(3)}_{11} \) in Example 3.3. Thus we may assume that \( x \geq 2 \). By Lemma 2.1, it suffices to give \( LC_3^{(3)} \)-decompositions of \( K^{(3)}_9 \) of \( K^{(3)}_{9+r} \setminus K^{(3)}_9 \) which is isomorphic to \( K^{(3)}_{9+r} \) since \( r \in [0, 2] \), of \( K^{(3)}_{r,9} \cup L^{(3)}_9 \), and of \( K^{(3)}_{9,9,9} \). A decomposition of \( K^{(3)}_{1,9,9} \cup L^{(3)}_9 \) is given in Example 3.7, and a decomposition of \( L^{(3)}_9 \) is given in Example 3.6. Since \( K^{(3)}_{2,3,3} \) decomposes \( K^{(3)}_{2,9,9} \) and \( K^{(3)}_{9,9} \) decomposes \( K^{(3)}_{9,9,9} \), and since \( LC_3^{(3)} \)-decompositions of \( K^{(3)}_{2,3,3} \) and \( K^{(3)}_{3,3,3} \) are given in Examples 3.4 and 3.5, we have that \( LC_3^{(3)} \) decomposes both \( K^{(3)}_{1,9,9} \) and \( K^{(3)}_{9,9,9} \). Thus the result follows. \( \square \)

Next, we give our main result on maximum \( LC_3^{(3)} \)-packings of \( K^{(3)}_9 \).

Theorem 2.3. If \( v \geq 6 \) is an integer, then there exists a maximum \( LC_3^{(3)} \)-packing of \( K^{(3)}_9 \) where the leave has two or fewer edges.

Proof. If \( v \equiv 0, 1, \) or \( 2 \) (mod 9), then the result follows from the \( LC_3^{(3)} \)-decomposition result in Theorem 2.2, which translates to a maximum \( LC_3^{(3)} \)-packing with an empty leave. If \( v \in [6, 8] \), a maximum \( LC_3^{(3)} \)-packing of \( K^{(3)}_9 \) with a two edge leave is given in Examples 3.13–3.15. Hence, we need only consider when \( v = 9x + r \) where \( x \geq 1 \) and \( r \in [3, 8] \). By Lemma 2.1 it suffices to find

(i) a maximum \( LC_3^{(3)} \)-packing of \( K^{(3)}_{3+r} \) with a leave consisting of two or fewer edges and
(ii) \( LC_3^{(3)} \)-decompositions of \( K^{(3)}_{9+r} \setminus K^{(3)}_9 \), \( K^{(3)}_{r,9,9} \cup L^{(3)}_9 \), and \( K^{(3)}_{9,9,9} \).

We note that an \( LC_3^{(3)} \)-decomposition of \( K^{(3)}_{12} \setminus K^{(3)}_3 \) is equivalent an \( LC_3^{(3)} \)-packing of \( K^{(3)}_{12} \) with a leave consisting of the single edge in the hole, which is given...
in Example 3.16. Also, for \( r \geq 3 \), it is simple to see that \( K_{r,9}^{(3)} \) is decomposable into copies of \( K_{2,3,3}^{(3)} \) and \( K_{3,3,3}^{(3)} \). Maximum \( LC_{3}^{(3)} \)-packings (with leaves of two or fewer edges) of \( K_{9+r}^{(3)} \), for \( r \in [3,8] \), are given in Examples 3.16–3.21. Similarly, \( LC_{3}^{(3)} \)-decompositions of \( K_{9+r}^{(3)} \backslash K_{r}^{(3)} \), for \( r \in [4,8] \), are given in Examples 3.8–3.12. Finally, an \( LC_{3}^{(3)} \)-decomposition of \( L_{9,9}^{(3)} \) is given in Example 3.6, and \( LC_{3}^{(3)} \)-decompositions of \( K_{2,3,3}^{(3)} \) and of \( K_{3,3,3}^{(3)} \) are given in Examples 3.4 and 3.5, respectively.

Next, we give a lemma on maximum \( LC_{3}^{(3)} \)-packings of \( 2K_{v}^{(3)} \) for \( v \in [6,17] \).

**Lemma 2.4.** If \( v \in [6,17] \), then there exists a maximum \( LC_{3}^{(3)} \)-packing of \( 2K_{v}^{(3)} \) where the leave has two or fewer edges.

**Proof.** Let \( V(2K_{v}^{(3)}) = Z_v \). If \( v \in [9,11] \), there exists an \( LC_{3}^{(3)} \)-decomposition of \( K_{v}^{(3)} \) and hence of \( 2K_{v}^{(3)} \). Next, if \( v \in [6,8] \cup [15,17] \), let \( \Delta_1 \) be a maximum \( LC_{3}^{(3)} \)-packing of \( K_{v}^{(3)} \) where the leave has edge set \( \{0,1,2\},\{2,3,4\} \) (which exists by Examples 3.13–3.15 and Examples 3.19–3.21) and let \( \Delta_2 \) be another maximum \( LC_{3}^{(3)} \)-packing of \( K_{v}^{(3)} \) where the leave has edge set \( \{4,5,0\},\{0,1,2\} \). Then \( \Delta_1 \cup \Delta_2 \cup \{H[0,1,2,3,4,5]\} \) is a maximum \( LC_{3}^{(3)} \)-packing of \( 2K_{v}^{(3)} \) where \( 0,1,2 \) is the only edge in the leave. Finally, if \( v \in [12,14] \), let \( \Delta_1 \) be a maximum packing of \( K_{v}^{(3)} \) where \( 0,1,2 \) is the only edge in the leave (which exists by Examples 3.16–3.18) and let \( \Delta_2 \) be a maximum \( LC_{3}^{(3)} \)-packing of \( K_{v}^{(3)} \) where \( 2,3,4 \) is the only edge in the leave. Then \( \Delta_1 \cup \Delta_2 \) is a maximum \( LC_{3}^{(3)} \)-packing of \( 2K_{v}^{(3)} \) where the leave has edge set \( \{0,1,2\},\{2,3,4\} \).

Now we extend our results to maximum \( LC_{3}^{(3)} \)-packings of \( 2K_{v}^{(3)} \) in general.

**Theorem 2.5.** If \( v \geq 6 \), then there exists a maximum \( LC_{3}^{(3)} \)-packing of \( 2K_{v}^{(3)} \) where the leave has two or fewer edges.

**Proof.** If \( v \equiv 0,1, \text{ or } 2 \pmod{9} \), then the result follows from Theorem 2.2, which translates to a maximum \( LC_{3}^{(3)} \)-packing with an empty leave. If \( v \in [6,8] \), a maximum \( LC_{3}^{(3)} \)-packing of \( 2K_{v}^{(3)} \) with a one edge leave is given in Lemma 2.4. Hence, we need only consider when \( v \equiv r \pmod{9} \), \( r \geq 3, v \geq 12 \). Let \( v = 9x + r \) where \( x \geq 1 \) and \( r \in [3,8] \). By Lemma 2.1 it suffices to find

(i) a maximum \( LC_{3}^{(3)} \)-packing of \( 2K_{9+r}^{(3)} \) with a leaf consisting of two or fewer edges and

(ii) \( LC_{3}^{(3)} \)-decompositions of \( 2K_{9+r}^{(3)} \backslash 2K_{r}^{(3)}, 2K_{r,9}^{(3)} \cup L_{9,9}^{(3)}, \) and \( 2K_{9,9,9}^{(3)} \).

But since \( LC_{3}^{(3)} \) decomposes \( K_{r+9}^{(3)} \backslash K_{r}^{(3)}, K_{r,9}^{(3)} \cup L_{9,9}^{(3)}, \) and \( K_{9,9,9}^{(3)} \) (see argument in proof of Theorem 2.3), \( LC_{3}^{(3)} \) decomposes the 2-fold versions of these hypergraphs. Maximum \( LC_{3}^{(3)} \)-packings (with leaves of two or fewer edges) of \( 2K_{9+r}^{(3)} \), for \( r \in [3,8] \), are given in Lemma 2.4. The result now follows.
Next, we give a lemma on $LC^3(3)$ decompositions of $3K_v(3)$ for $v \in [6, 17]$.

**Lemma 2.6.** If $v \in [6, 17]$, then there exists an $LC^3(3)$-decomposition of $3K_v(3)$.

**Proof.** Let $V(3K_v(3)) = \mathbb{Z}_v$. If $v \in [9, 11]$, there exists an $LC^3(3)$-decomposition of $K_v(3)$ and hence of $3K_v(3)$. Next, if $v \in [6, 8] \cup [15, 17]$, let $\Delta_1$ be a maximum packing of $K_v(3)$ where the leaves have edge set $\{\{2, 3, 4\}, \{4, 5, 0\}\}$ (which exists by Examples 3.13–3.15 and Examples 3.19–3.21) and let $\Delta_2$ be a maximum $LC^3(3)$-packing of $2K_v(3)$ where $\{0, 1, 2\}$ is the only edge in the leave (which exists by Lemma 2.4). Then $\Delta_1 \cup \Delta_2 \cup \{H[0, 1, 2, 3, 4, 5]\}$ is an $LC^3(3)$-decomposition of $3K_v(3)$. Finally, if $v \in [12, 14]$, let $\Delta_1$ be a maximum packing of $K_v(3)$ where $\{4, 5, 0\}$ is the only edge in the leave (which exists by Examples 3.16–3.18) and let $\Delta_2$ be a maximum $LC^3(3)$-packing of $2K_v(3)$ where the leave has edge set $\{\{0, 1, 2\}, \{2, 3, 4\}\}$ (which exists by Lemma 2.4). Then $\Delta_1 \cup \Delta_2 \cup \{H[0, 1, 2, 3, 4, 5]\}$ is an $LC^3(3)$-decomposition of $3K_v(3)$.

It is simple to see that if there is an $LC^3(3)$-decomposition of $\lambda K_v(3)$, then we must have $v \not\in [3, 5]$ and either $\lambda \equiv 0 \pmod{3}$ or $v \equiv 0, 1, 2 \pmod{9}$. Thus, in light of Theorem 2.2 and Lemmas 2.1 and 2.6 and because $3K_v(3)$ decomposes $3K_v(3)$ for all positive integers $k$, we have the following obvious corollary.

**Corollary 2.7.** Let $\lambda$ and $v \not\in [3, 5]$ be positive integers. There exists an $LC^3(3)$-decomposition of $\lambda K_v(3)$ if and only if $\lambda \equiv 0 \pmod{3}$ or $v \equiv 0, 1, 2 \pmod{9}$.

Finally we give our general main result.

**Theorem 2.8.** If $\lambda$ and $v \not\in [3, 5]$ are positive integers, then there exists a maximum $LC^3(3)$-packing of $\lambda K_v(3)$ where the leave has two or fewer edges.

**Proof.** If $\lambda \in \{1, 2\}$, the result follows from Theorems 2.3 and 2.5. If $\lambda \equiv 0 \pmod{3}$, the result follows from Corollary 2.7. Suppose $\lambda \geq 4$ and let $\lambda = 3(b + r)$ for integers $b \geq 1$ and $r \in \{1, 2\}$. We can view $3K_v(3)$ as the edge disjoint union of $3bK_v(3)$ and $bK_v(3)$. An $LC^3(3)$-decomposition of $3bK_v(3)$ exists by Corollary 2.7 and a maximum $LC^3(3)$-packing of $bK_v(3)$ where the leave has two or fewer edges follows from Theorems 2.3 and 2.5. Thus the result follows.

3. SMALL EXAMPLES

**Example 3.1.** Let

$$V(K_v(3)) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$$

and let

$$B = \{H[0, 1, 2, 4, 6, 3], H[\infty_1, 0, 1, \infty_2, 2, 4], H[0, 1, 3, \infty_1, 2, 4], H[\infty_1, 0, 3, 1, \infty_2, 2]\}.$$ 

Then an $LC^3(3)$-decomposition of $K_v(3)$ consists of the $LC^3(3)$-blocks in $B$ under the action of the map $\infty_1 \mapsto \infty_1$ and $j \mapsto j + 1 \pmod{7}$. 


Example 3.2. Let
\[ V(K_{10}^{(3)}) = \mathbb{Z}_{10} \]
and let
\[ B = \{ H[0, 2, 1, 3, 4, 9], H[0, 7, 1, 2, 4, 5], H[0, 4, 2, 5, 7, 6], H[0, 6, 2, 4, 7, 3] \}. \]
Then an \( LC_3^{(3)} \)-decomposition of \( K_{10}^{(3)} \) consists of the \( LC_3^{(3)} \)-blocks in \( B \) under the action of the map \( j \mapsto j + 1 \) (mod 10).

Example 3.3. Let
\[ V(K_{11}^{(3)}) = \mathbb{Z}_{11} \]
and let
\[ B = \{ H[0, 8, 2, 6, 9, 1], H[0, 8, 1, 4, 7, 2], H[1, 0, 5, 3, 8, 9], H[0, 7, 1, 10, 5, 6], H[0, 3, 1, 8, 10, 9] \}. \]
Then an \( LC_3^{(3)} \)-decomposition of \( K_{11}^{(3)} \) consists of the \( LC_3^{(3)} \)-blocks in \( B \) under the action of the map \( j \mapsto j + 1 \) (mod 11).

Example 3.4. Let
\[ V(K_{2,3,3}^{(3)}) = \mathbb{Z}_6 \cup \{ \infty_1, \infty_2 \} \]
with the vertex partition \( \{ \{ \infty_1, \infty_2 \}, \{ 0, 2, 4 \}, \{ 1, 3, 5 \} \} \) and let
\[ B = \{ H[\infty_1, 0, 1, \infty_2, 2, 5], H[5, \infty_2, 0, 1, \infty_1, 2, 5] \}. \]
Then an \( LC_3^{(3)} \)-decomposition of \( K_{2,3,3}^{(3)} \) consists of the \( LC_3^{(3)} \)-blocks in \( B \) under the action of the map \( \infty_1 \mapsto \infty_1 \) and \( j \mapsto j + 2 \) (mod 6).

Example 3.5. Let
\[ V(K_{3,3,3}^{(3)}) = \mathbb{Z}_9 \]
with vertex partition \( \{ \{ 0, 3, 6 \}, \{ 1, 4, 7 \}, \{ 2, 5, 8 \} \} \) and let
\[ B = \{ H[7, 3, 2, 1, 0, 5] \}. \]
Then an \( LC_3^{(3)} \)-decomposition of \( K_{3,3,3}^{(3)} \) consists of the \( LC_3^{(3)} \)-block in \( B \) under the action of the map \( j \mapsto j + 1 \) (mod 9).

Example 3.6. Let
\[ V(L_{9,9}^{(3)}) = \mathbb{Z}_{18} \]
with vertex partition \( \{ \{ 0, 2, \ldots, 16 \}, \{ 1, 3, \ldots, 17 \} \} \), and let
\[ B = \{ H[0, 16, 1, 4, 15, 2], H[14, 0, 1, 11, 16, 9], H[4, 0, 9, 1, 14, 3], H[9, 0, 1, 6, 17, 2], H[0, 5, 1, 12, 15, 14], H[0, 1, 2, 5, 3, 15], H[0, 1, 15, 10, 4, 13], H[1, 5, 12, 7, 0, 13], H[0, 1, 12, 3, 9, 16], H[10, 2, 17, 8, 1, 0], H[1, 0, 8, 2, 17, 4], H[1, 0, 7, 12, 6, 16] \}. \]
Then an $LC_{9,9}^{(3)}$-decomposition of $L_{9,9}^{(3)}$ consists of the $LC_{9,9}^{(3)}$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{18}$.

**Example 3.7.** Let
\[ V \left( L_{9,9}^{(3)} \cup K_{1,9,9}^{(3)} \right) = Z_{18} \cup \{ \infty \} \]
with vertex partition $\{ \{ \infty \}, \{ 0, 2, \ldots, 16 \}, \{ 1, 3, \ldots, 17 \} \}$, and let
\[ B = \{ \{ 0, 16, 1, 4, 15, 2 \}, \{ 5, 1, 12, 15, 14 \}, \{ 0, 1, 2, \infty, 3, 15 \}, \{ 0, 1, 15, 10, 4, 13 \}, \{ 1, 5, 12, 7, 0, 13 \}, \{ 0, \infty, 3, 12, 9, 16 \}, \{ 14, 7, 17, 8, 1, \infty \}, \{ 1, \infty, 8, 2, 17, 4 \}, \{ 1, 0, 7, 12, 6, 16 \} \}, \]
\[ B' = \{ \{ 0, \infty, 9, 10, 1, 3 \}, \{ 1, \infty, 10, 11, 2, 4 \}, \{ 2, \infty, 11, 12, 3, 5 \}, \{ 3, \infty, 12, 13, 4, 6 \}, \{ 4, \infty, 13, 14, 5, 7 \}, \{ 5, \infty, 14, 15, 6, 8 \}, \{ 6, \infty, 15, 16, 7, 9 \}, \{ 7, \infty, 16, 17, 8, 10 \}, \{ 8, \infty, 17, 0, 9, 11 \}, \{ 10, 0, 1, 2, 9, 3 \}, \{ 11, 1, 2, 3, 10, 4 \}, \{ 12, 2, 3, 4, 11, 5 \}, \{ 13, 3, 4, 5, 12, 6 \}, \{ 14, 4, 5, 6, 13, 7 \}, \{ 15, 5, 6, 7, 14, 8 \}, \{ 16, 6, 7, 8, 15, 9 \}, \{ 17, 7, 8, 9, 16, 10 \}, \{ 0, 8, 9, 10, 17, 11 \}, \{ 12, 9, 10, 11, 0, 1 \}, \{ 13, 10, 11, 12, 1, 2 \}, \{ 14, 11, 12, 13, 2, 3 \}, \{ 15, 12, 13, 14, 3, 4 \}, \{ 16, 13, 14, 15, 4, 5 \}, \{ 17, 14, 15, 16, 5, 6 \}, \{ 0, 15, 16, 17, 6, 7 \}, \{ 1, 16, 17, 0, 7, 8 \}, \{ 2, 17, 0, 1, 8, 9 \} \}. \]

Then an $LC_{9}^{(3)}$-decomposition of $L_{9}^{(3)} \cup K_{1,9,9}^{(3)}$ consists of the $LC_{9}^{(3)}$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{18}$, along with the $LC_{9}^{(3)}$-blocks in $B'$.

**Example 3.8.** Let
\[ V \left( K_{13}^{(3)} \setminus K_{4}^{(3)} \right) = Z_{9} \cup \{ \infty_{1}, \infty_{2}, \infty_{3}, \infty_{4} \} \]
with $\infty_{1}, \ldots, \infty_{4}$ being the vertices in the hole and let
\[ B = \{ \{ \infty_{1}, 0, \infty_{2}, 1, 4, 8 \}, \{ \infty_{2}, 0, \infty_{3}, 1, 4, 8 \}, \{ \infty_{3}, 0, \infty_{4}, 1, 4, 8 \}, \{ \infty_{4}, 0, \infty_{1}, 1, 4, 8 \}, \{ \infty_{1}, 0, \infty_{2}, 1, 2, 3 \}, \{ \infty_{2}, 0, \infty_{3}, 1, 2, 3 \}, \{ \infty_{3}, 0, \infty_{4}, 1, 2, 3 \}, \{ \infty_{4}, 0, \infty_{1}, 2, 6, 3 \}, \{ 0, 1, 4, 8, 3, \infty_{4} \}, \{ 0, 2, 4, 6, 1, 8 \} \}, \]
\[ B' = \{ \{ 0, 2, 8, 5, 3, 6 \}, \{ 1, 3, 0, 6, 4, 7 \}, \{ 2, 4, 1, 7, 5, 8 \}, \{ 5, 2, 3, 6, 4, 7 \}, \{ 8, 5, 6, 0, 7, 1 \}, \{ 6, 2, 8, 4, 1, 3 \}, \{ 7, 3, 0, 5, 2, 4 \}, \{ 6, 1, 0, 4, 3, 7 \}, \{ 7, 2, 1, 5, 4, 8 \}, \{ 8, 3, 2, 6, 5, 0 \}, \{ 0, 7, 1, 8, 2, 3 \}, \{ 3, 1, 4, 2, 5, 6 \}, \{ 6, 4, 7, 5, 8, 0 \} \}. \]

Then an $LC_{9}^{(3)}$-decomposition of $K_{13}^{(3)} \setminus K_{4}^{(3)}$ consists of the $LC_{9}^{(3)}$-blocks in $B$ under the action of the map $\infty_{i} \mapsto \infty_{i}$ and $j \mapsto j + 1 \pmod{9}$ along with the $LC_{9}^{(3)}$-blocks in $B'$. 
Example 3.9. Let 
\[ V\left(K_{14}^{(3)} \setminus K_5^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \]
with \(\infty_1, \ldots, \infty_5\) being the vertices in the hole and let

\[ B = \{H[\infty_1, 0, 0, 2, 1, 6], H[\infty_2, 0, 0, 3, 1, 2, 6], H[\infty_3, 0, 0, 4, 1, 2, 6], \]
\[ H[\infty_4, 0, 0, 5, 1, 3, 6], H[\infty_5, 0, 0, 1, 3, 6], \]
\[ H[\infty_6, 0, 0, 2, 1, 3, 6], H[0, 2, 4, 6, 1, 8], H[1, 0, 4, 5, 2, 6]\}, \]
\[ B' = \{H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7], \]
\[ H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3], \]
\[ H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0]\} \].

Then an \(\text{LC}_{3}^{(3)}\)-decomposition of \(K_{14}^{(3)} \setminus K_5^{(3)}\) consists of the \(\text{LC}_{3}^{(3)}\)-blocks in \(B\) under the action of the map \(\infty_i \mapsto \infty_i\) and \(j \mapsto j + 1 \pmod{9}\) along with the \(\text{LC}_{3}^{(3)}\)-blocks in \(B'\).

Example 3.10. Let 
\[ V\left(K_{15}^{(3)} \setminus K_6^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\} \]
with \(\infty_1, \ldots, \infty_6\) being the vertices in the hole and let

\[ B = \{H[\infty_1, 0, 0, 2, 1, 3, 2], H[\infty_2, 0, 0, 4, 1, 3, 2], H[\infty_3, 0, 0, 5, 1, 3, 2], \]
\[ H[\infty_4, 0, 0, 6, 1, 1, 2], H[0, 2, 0, 4, 1, 6, 2], H[0, 2, 0, 4, 1, 6, 2], \]
\[ H[0, 2, 0, 4, 1, 6, 2], H[0, 2, 0, 4, 1, 6, 2], \}
\[ B' = \{H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7], \]
\[ H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3], \]
\[ H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0]\} \].

Then an \(\text{LC}_{3}^{(3)}\)-decomposition of \(K_{15}^{(3)} \setminus K_6^{(3)}\) consists of the \(\text{LC}_{3}^{(3)}\)-blocks in \(B\) under the action of the map \(\infty_1 \mapsto \infty_1\) and \(j \mapsto j + 1 \pmod{9}\) along with the \(\text{LC}_{3}^{(3)}\)-blocks in \(B'\).

Example 3.11. Let 
\[ V\left(K_{16}^{(3)} \setminus K_7^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\} \]
with \(\infty_1, \ldots, \infty_7\) being the vertices in the hole and let
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Let

$B = \{H[\infty_1, 0, \infty_2, 1, \infty_4, 2], H[\infty_2, 0, \infty_3, 1, \infty_5, 2], H[\infty_3, 0, \infty_4, 1, \infty_6, 2], H[\infty_4, 0, \infty_5, 1, \infty_7, 2], H[\infty_5, 0, \infty_6, 1, \infty_1, 2], H[\infty_6, 0, \infty_7, 1, \infty_2, 2], H[\infty_7, 0, \infty_8, 1, \infty_3, 2], H[\infty_1, 0, 1, \infty_2, 3, 7], H[\infty_2, 0, 1, \infty_3, 3, 7], H[\infty_3, 0, 1, \infty_4, 3, 7], H[\infty_4, 0, 1, \infty_5, 3, 7], H[\infty_5, 0, 1, \infty_6, 3, 7], H[\infty_6, 0, 1, \infty_7, 3, 7], H[\infty_7, 0, 1, \infty_1, 3, 7], \}$

$H[0, 0, 1, 3, \infty_2, 6, \infty_3], H[0, \infty_4, 3, \infty_5, 6, \infty_6], H[0, 1, 4, 8, 3, \infty_7], H[0, 2, 4, 6, 1, 8]\}

Then an $LC^{(3)}_{\lambda}-$decomposition of $K^{(3)}_{16} \setminus K^{(3)}_{7}$ consists of the $LC^{(3)}_{\lambda}$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{9}$ along with the $LC^{(3)}_{\lambda}$-blocks in $B'$.

**Example 3.12.** Let

$V(K^{(3)}_{17} \setminus K^{(3)}_{8}) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8\}$

with $\infty_1, \ldots, \infty_8$ being the vertices in the hole and let

$B = \{H[\infty_1, 0, \infty_2, 1, \infty_4, 2], H[\infty_2, 0, \infty_3, 1, \infty_5, 2], H[\infty_3, 0, \infty_4, 1, \infty_6, 2], H[\infty_4, 0, \infty_5, 1, \infty_7, 2], H[\infty_5, 0, \infty_6, 1, \infty_8, 2], H[\infty_6, 0, \infty_7, 1, \infty_1, 2], H[\infty_7, 0, \infty_8, 1, \infty_2, 2], H[\infty_8, 0, \infty_1, 1, \infty_3, 2], H[\infty_1, 0, 4, \infty_2, 3, 6], H[\infty_2, 0, 4, \infty_3, 3, 6], H[\infty_3, 0, 4, \infty_4, 3, 6], H[\infty_4, 0, 4, \infty_5, 3, 6], H[\infty_5, 0, 4, \infty_6, 3, 6], H[\infty_6, 0, 4, \infty_7, 3, 6], H[\infty_7, 0, 4, \infty_8, 3, 6], H[\infty_8, 0, 4, \infty_1, 3, 6], H[\infty_1, 0, 2, 4, \infty_5, 1], H[\infty_2, 0, 2, 4, \infty_6, 1], H[\infty_3, 0, 2, 4, \infty_7, 1], H[\infty_4, 0, 2, 4, \infty_8, 1], H[0, 0, 1, 4, 8, 3, 6], H[0, 2, 4, 6, 1, 8], H[1, 0, 1, 4, 5, 2, 6]\}.$

Then an $LC^{(3)}_{\lambda}-$decomposition of $K^{(3)}_{17} \setminus K^{(3)}_{8}$ consists of the $LC^{(3)}_{\lambda}$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{9}$ along with the $LC^{(3)}_{\lambda}$-blocks in $B'$.

**Example 3.13.** Let

$V(K^{(3)}_{6}) = \mathbb{Z}_6$

and let

$B = \{H[4, 0, 1, 5, 3, 2], H[5, 1, 2, 0, 4, 3], H[4, 5, 0, 3, 2, 1], H[5, 0, 1, 4, 3, 2], H[0, 1, 2, 5, 4, 3], H[1, 2, 3, 0, 5, 4]\}.$
Then $B$ is a maximum $LC_{3}^{(3)}$-packing of $K_{6}^{(3)}$, where the leave has edge set $\{\{0,1,3\}, \{1,2,5\}\}$. Note that by renaming the vertices in this packing, any two hyperedges in $K_{6}^{(3)}$ that intersect in a single vertex can be made into the edge set of the leave of a maximum $LC_{3}^{(3)}$-packing of $K_{6}^{(3)}$.

**Example 3.14.** Let

$$V(K_{7}^{(3)}) = \mathbb{Z}_7$$

and let

$$B = \{H[0, 1, 2, 4, 6, 3]\},$$

$$B' = \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 0, 5, 1], H[1, 6, 2, 0, 3, 4], H[4, 2, 5, 3, 6, 0]\}.$$ 

Then a maximum $LC_{3}^{(3)}$-packing of $K_{7}^{(3)}$, where the leave has edge set $\{\{0,1,5\}, \{0,2,6\}\}$, consists of the $LC_{3}^{(3)}$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{7}$ along with the $LC_{3}^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{7}^{(3)}$ that intersect in a single vertex can be made into the edge set of a maximum $LC_{3}^{(3)}$-packing of $K_{7}^{(3)}$.

**Example 3.15.** Let

$$V(K_{8}^{(3)}) = \mathbb{Z}_8$$

and let

$$B = \{H[6, 0, 7, 2, 3, 1], H[0, 2, 6, 7, 4, 1]\},$$

$$B' = \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 0]\}.$$ 

Then a maximum $LC_{3}^{(3)}$-packing of $K_{8}^{(3)}$, where the leave has edge set $\{\{1,6,7\}, \{0,2,7\}\}$, consists of the $LC_{3}^{(3)}$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{8}$ along with the $LC_{3}^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{8}^{(3)}$ that intersect in a single vertex can be made into the edge set of a maximum $LC_{3}^{(3)}$-packing of $K_{8}^{(3)}$.

**Example 3.16.** Let

$$V(K_{12}^{(3)}) = \mathbb{Z}_{11} \cup \{\infty\}$$

and let

$$B = \{H[0, \infty, 1, 3, 2, 4], H[0, \infty, 2, 8, 5, 10], H[0, \infty, 3, 7, 4, 8], H[0, \infty, 4, 9, 5, 2],$$

$$H[0, \infty, 5, 10, 9, 6], H[0, 6, 2, 9, 4, 3]\},$$

$$B' = \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 0], H[1, 10, 2, 0, 3, 4],$$

$$H[4, 2, 5, 3, 6, 7], H[8, 6, 9, 1, 10, 0], H[9, 7, 10, 2, 0, 1]\}.$$
Then a maximum $L C_3^{(3)}$-packing of $K_{12}^{(3)}$, where the leave is the single edge $\{5, 7, 8\}$, consists of the $L C_3^{(3)}$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{11}$ along with the $L C_3^{(3)}$-blocks in $B'$. Note that by renaming the vertices in this packing, any edge in $K_{12}^{(3)}$ can be made into the leave of a maximum $L C_3^{(3)}$-packing of $K_{12}^{(3)}$.

Example 3.17. Let

$$V(K_{13}^{(3)}) = \mathbb{Z}_{13}$$

and let

$$B = \{ H[0, 3, 9, 12, 1, 11], H[0, 4, 8, 12, 1, 10], H[12, 4, 9, 0, 1, 7], H[12, 5, 8, 0, 1, 6], H[7, 10, 4, 11, 0, 1], H[6, 10, 5, 2, 0, 1], H[0, 2, 3, 5, 1, 0] \},$$

$$B' = \{ H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], H[12, 9, 10, 0, 11, 1] \}.$$

Then a maximum $L C_3^{(3)}$-packing of $K_{13}^{(3)}$, where the leave is the single edge $\{0, 2, 12\}$, consists of the $L C_3^{(3)}$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{13}$ along with the $L C_3^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any edge in $K_{13}^{(3)}$ can be made into the leave of a maximum $L C_3^{(3)}$-packing of $K_{13}^{(3)}$.

Example 3.18. Let

$$V(K_{14}^{(3)}) = \mathbb{Z}_{14}$$

and let

$$B = \{ H[13, 0, 8, 3, 5, 12], H[0, 10, 1, 3, 5, 12], H[0, 8, 2, 3, 4, 13], H[0, 1, 11, 5, 6, 3], H[0, 3, 8, 2, 9, 4], H[0, 2, 9, 13, 7, 3], H[0, 3, 10, 5, 8, 4] \},$$

$$B' = \{ H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], H[12, 9, 10, 13, 11, 0], H[1, 13, 2, 0, 3, 4], H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[11, 9, 12, 1, 13, 0], H[12, 10, 13, 2, 0, 1] \}.$$

Then a maximum $L C_3^{(3)}$-packing of $K_{14}^{(3)}$, where the leave is the single edge $\{8, 10, 11\}$, consists of the $L C_3^{(3)}$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{14}$ along with the $L C_3^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any edge in $K_{14}^{(3)}$ can be made into the leave of a maximum $L C_3^{(3)}$-packing of $K_{14}^{(3)}$.

Example 3.19. Let

$$V(K_{15}^{(3)}) = \mathbb{Z}_{13} \cup \{\infty_1, \infty_2\}$$
and let

\[ B = \{ H[\infty_1, 0, \infty_2, 1, 2, 3], H[0, \infty_1, 2, \infty_2, 4, 8], H[0, \infty_1, 3, \infty_2, 6, 12], \]
\[ H[0, \infty_1, 4, \infty_2, 8, 3], H[0, \infty_1, 5, \infty_2, 10, 7], H[0, \infty_1, 6, \infty_2, 12, 11], \]
\[ H[0, 9, 11, 8, 4, 10], H[0, 5, 2, 4, 8, 6], H[0, 10, 2, 4, 11, 6], \]
\[ H[0, 1, 4, 3, 8, 7], H[0, 1, 10, 9, 5, 6] \}, \]
\[ B' = \{ H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], H[12, 9, 10, 0, 11, 1], \]
\[ H[1, 12, 2, 0, 3, 4], H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[10, 8, 11, 9, 12, 0] \}. \]

Then a maximum \( LC_{3}^{(3)} \)-packing of \( K_{15}^{(3)} \), where the leave has edge set \( \{ 0, 2, 14 \}, \{ 2, 4, 5 \} \), consists of the \( LC_{3}^{(3)} \)-blocks in \( B \) under the action of the map \( \infty_i \mapsto \infty_i \) and \( j \mapsto j + 1 \) (mod 13) along with the \( LC_{3}^{(3)} \)-blocks in \( B' \). Again, we note that by renaming the vertices in this packing, any two hyperedges in \( K_{15}^{(3)} \) that intersect in a single vertex can be made into the edge set the leave of a maximum \( LC_{3}^{(3)} \)-packing of \( K_{15}^{(3)} \).

**Example 3.20.** Let

\[ V \left( K_{16}^{(3)} \right) = \mathbb{Z}_{16} \]

and let

\[ B = \{ H[1, 0, 13, 10, 6, 4], H[0, 5, 10, 6, 11, 4], H[1, 14, 9, 3, 7, 13], \]
\[ H[0, 11, 1, 12, 3, 10], H[0, 10, 2, 14, 7, 6], H[13, 5, 12, 0, 4, 7], \]
\[ H[0, 3, 8, 15, 7, 14], H[15, 5, 14, 8, 6, 2], H[0, 13, 3, 8, 1, 6], \]
\[ H[1, 0, 4, 8, 3, 7], H[0, 14, 12, 10, 11, 13] \}, \]
\[ B' = \{ H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], \]
\[ H[12, 9, 10, 13, 11, 14], H[15, 12, 13, 0, 14, 1], H[1, 15, 2, 0, 3, 4], \]
\[ H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[10, 8, 11, 9, 12, 13], \]
\[ H[13, 11, 14, 12, 15, 0] \}. \]

Then a maximum \( LC_{3}^{(3)} \)-packing of \( K_{16}^{(3)} \), where the leave has edge set \( \{ 0, 1, 14 \}, \{ 0, 2, 15 \} \), consists of the \( LC_{3}^{(3)} \)-blocks in \( B \) under the action of the map \( j \mapsto j + 1 \) (mod 16) along with the \( LC_{3}^{(3)} \)-blocks in \( B' \). Again, we note that by renaming the vertices in this packing, any two hyperedges in \( K_{16}^{(3)} \) that intersect in a single vertex can be made into the edge set the leave of a maximum \( LC_{3}^{(3)} \)-packing of \( K_{16}^{(3)} \).
Example 3.21. Let

\[ V(K_{17}^{(3)}) = \mathbb{Z}_{17} \]

and let

\[ B = \{ H[0, 15, 1, 16, 13, 3], H[0, 14, 1, 13, 5, 11], H[2, 13, 0, 4, 11, 16], \]
\[ H[11, 16, 1, 13, 0, 3], H[0, 12, 1, 11, 5, 10], H[0, 11, 1, 16, 10, 3], \]
\[ H[0, 10, 1, 16, 9, 4], H[14, 6, 0, 9, 1, 10], H[0, 8, 1, 6, 15, 7], \]
\[ H[15, 5, 1, 7, 0, 6], H[0, 6, 1, 4, 15, 5], H[0, 15, 4, 8, 5, 1], \]
\[ H[0, 15, 3, 5, 4, 2] \}, \]
\[ B' = \{ H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], \]
\[ H[12, 9, 10, 13, 11, 14], H[15, 12, 13, 16, 14, 0] \}. \]

Then a maximum $LC_{4}^{(3)}$-packing of $K_{17}^{(3)}$, where the leave has edge set \( \{ \{1, 15, 16\}, \{0, 2, 16\} \} \), consists of the $LC_{4}^{(3)}$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{17}$ along with the $LC_{3}^{(3)}$-blocks in $B'$. Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{17}^{(3)}$ that intersect in a single vertex can be made into the edge set of the leave of a maximum $LC_{3}^{(3)}$-packing of $K_{17}^{(3)}$.

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