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Dedicated to Prof. Dušan D. Repovš
on the occasion of his 65th birthday

GLOBAL EXISTENCE AND BLOW UP OF SOLUTION FOR SEMI-LINEAR HYPERBOLIC EQUATION WITH THE PRODUCT OF LOGARITHMIC AND POWER-TYPE NONLINEARITY

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Abstract. In this paper we consider the semilinear wave equation with the multiplication of logarithmic and polynomial nonlinearities. We establish the global existence and finite time blow up of solutions at three different energy levels ($E(0) < d$, $E(0) = d$ and $E(0) > 0$) using potential well method. The results in this article shed some light on using potential wells to classify the solutions of the semilinear wave equation with the product of polynomial and logarithmic nonlinearity.

Keywords: global existence, blow-up, logarithmic and polynomial nonlinearity, potential well.

Mathematics Subject Classification: 35L71, 35L20, 35L05.

1. INTRODUCTION

In this contribution, we would like to study the initial boundary value problem of a semilinear wave equation with polynomial nonlinearity of the factor of logarithmic term

$$\begin{cases} u_{tt} - \triangle u = |u|^p \ln|u|, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $1 < p < \infty$ if $n = 1, 2$ and $2 < p < \frac{4}{n-2}$ $(3 \le n \le 5)$ are constants, $u_0(x)$ and $u_1(x)$ are given initial data. One of the most important nonlinear evolution equations are the semilinear hyperbolic equations in the field of mathematical physics and engineering. This type of nonlinearities appear naturally in inflation cosmology and supersymmetric field theory (see [3, 18]). Furthermore, there are applications in many branches of physics such as nuclear physics, optics and geophysics (see [6, 14, 22]). For the problem under consideration, according to available literature, some special analytical solutions can be obtained in the logarithmic quantum mechanics (see [5, 23]). For instance, this model has a large set of oscillating localized solutions.

We start the literature review with the pioneer work of Sattinger [26] in which potential well W was first introduced to study the following initial boundary value problem of semilinear wave equation with polynomial nonlinearity

$$\begin{cases} u_{tt} - \triangle u = f(u), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.2}$$

and showed $u(t) \in W$ for every t when u_0 inside the potential well W and $E(0) < d$, where $E(0)$ is the initial energy and d is the depth of potential well. The class of initial data was precisely specified for which global existence and finite time blow up of the solutions were investigated. Payne and Sattinger [24] treated the case while u_0 lies outside the potential well W and proved that the solutions of problem (1.2) blows up in finite time. Illustrative explanations of the potential well W was given by certain differential-integral inequalities and found the existence of saddle point of the potential energy functionals J. In [15] the technique was first introduced to prove global nonexistence of solution for an abstract problem which includes (1.2). In [2] a stronger result was obtained for (1.2), namely pointwise blow up in finite time. The case of definitely positive initial energy was considered in [16] and [27] by proving a blowup (global nonexistence) that depends on the condition $(u_0, u_1) \geq 0$. Liu Yacheng [19] improved previous results proposing a new method that is the so-called family of potential wells which includes single potential well W as a particular case. All solutions of problem (1.2) with the typical form of the source term $f(u) = |u|^{p-1}u$ were proved that the solutions can be only either inside of a smaller ball or outside of some bigger balls of space $H_0^1(\Omega)$ under low initial energy, i.e., $E(0) < d$ and can never be located in the vacuum isolating region. Liu Yacheng, Zhao Junsheng [21] proved the threshold result of global existence and non-existence by family of potential wells for problem (1.2) with critical conditions $I(u_0) \geq 0$, $E(0) = d$. The authors [20], for the first time, deal problem (1.2) with combined power type nonlinearities of different sign and investigated global existence of solutions under critical initial conditions $I(u_0) \geq 0,\ E(0) = d$. Xu [29] handled the case considering typical form of source term $f(u) = |u|^{p-1}u$ for critical initial data $I(u_0) < 0$, $E(0) = d$ adding $(u_0, u_1) \geq 0$, and showed problem (1.2) does not have any global solution, i.e., the nonexistence of the solutions. Filippo Gazzola and Marco Squassina [9] studied the following damped semilinear wave equations

$$\begin{cases} u_{tt} - \triangle u - \omega\triangle u_t + \mu u_t = |u|^{p-2}u & \text{in } [0,T]\times\Omega, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } [0,T]\times\partial\Omega. \end{cases} \tag{1.3}$$

When $\omega = 0,\ \mu = 0$, then this problem turns into very classical undamped problem (1.2) which was studied by many authors [19, 21, 24, 29]. When $\omega = 0,\ \mu \geq 0$, the finite time blow up result was acquired for this problem with arbitrarily high initial energy in an open bounded Lipschitz subset of $\mathbb{R}^n$(see [9], Theorem 3.12, Theorem 3.13). Their results also hold for the case $\omega = 0,\ \mu = 0$. Yanjin Wang [28] extended the result of blow up in finite time to the whole space $\mathbb{R}^n$ at arbitrarily positive initial energy for the nonlinear Klein-Gordon equation of the form

$$u_{tt} - \triangle u + m^2 u = f(u), \quad (t,x) \in [0,T)\times\mathbb{R}^n.$$

There are lots of investigation at high initial energy level (see, e.g., [8, 25, 30–32]). All the study above was on polynomial nonlinearity.

Let us go to view some work with logarithmic nonlinearity which gives impetus to study problem (1.1). In [7], Cazenave and Haraux first dealt with the Cauchy problem (1.1) with logarithmic nonlinearity in $\mathbb{R}^3$ and found the existence and uniqueness of the solutions. The Cauchy problem (1.1) with u plus restricting logarithmic term was treated by Bartkowski and Gorka [4] where they obtained the existence of classical solutions. Gorka [10] inquired the following initial boundary value problem

$$\begin{cases} u_{tt} - \triangle u = -u + \varepsilon u \log|u|^2, & x\in\Omega,\ t\in(0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x\in\Omega, \\ u(x,t) = 0, & x\in\partial\Omega,\ t\in(0,T), \end{cases} \tag{1.4}$$

where Ω is a finite interval $\Omega = [a,b]$ and parameter $\varepsilon \in [0,1]$ fixed. The existence of weak solution was proved for all $u_0 \in H_0^1(\Omega),\ u_1 \in L^2(\Omega)$ by using compactness method. Hiramatus et.al [12] introduced the following equation

$$u_{tt} - \triangle u + u + u_t + u|u|^2 = u\ln|u|, \quad x\in\Omega,\ t>0, \tag{1.5}$$

for studying the dynamics of Q-ball in theoretical physics. A numerical research was given in that work. For the initial boundary value problem of (1.5), Han [11] obtained the global existence of weak solution in $\mathbb{R}^3$, and Zhang *et al.* [33] proved the decay estimate of energy for this problem in finite dimensional case. In [13], the authors studied logarithmic Boussinesq-type equation, and got the global existence and exponential growth of the solution in the potential well under sub-critical initial energy ($E(0) < d$). Recently in [1], the authors treated the following problem

$$\begin{cases} u_{tt} - \triangle^2 u + u + h(u_t) = ku\log|u|, & x\in\Omega,\ t>0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x\in\Omega, \\ u(x,t) = \frac{\partial u}{\partial v}(x,t) = 0, & x\in\partial\Omega,\ t>0, \end{cases}$$

and found the global existence and decay rate of the solution using the multiplier method in $\mathbb{R}^2$. From the above literature survey we were first motivated to work with problem (1.1) besides the power-type term $|u|^{p-1}$ and got some interesting result about global existence and blowup time of the solutions (see [17]). Although, above studies are pioneer about consistence of the wave function with either polynomial or logarithmic nonlinearity, there are no investigation considering the polynomial nonlinearity with the factor of logarithmic term. All the investigations above, however, motivate us to consider such fundamental model of wave equations in the present paper to see what kind of conclusions we can have for problem (1.1) with the product of logarithmic and polynomial nonlinearity. Moreover, we investigate the problem (1.1) using so-called potential well method which has been one of the most important and sophisticated methods for studying nonlinear evolution equations. Finally, for the first time, we go to search global existence and blowup time of solutions of (1.1) at three different energy level cases, i.e., ($E(0) < d$, $E(0) = d$ and $E(0) > 0$). We can summarize our main results in Table 1 in which "✓" will represent successful investigation and "?" for open problem.

Table 1. Essence

Initial energy level	Global existence	Finite time blow up
$0 < E(0) < d$	✓	✓
$E(0) = d$	✓	✓
$E(0) > 0$	?	✓

This paper is organized as follows. In Section 2 some preliminaries and necessary lemmas are included. Moreover, potential wells, their properties are also described here. Section 3 summarises the key result under the condition of $E(0) < d$. The result under the condition $E(0) = d$ is demonstrated in Section 4. Furthermore, in Section 5 main result and proofs are given under $E(0) > 0$.

2. PRELIMINARIES

We commence this section by introducing the norms $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ and the inner product $(u, v) = \int_\Omega uvdx$.

A weak solution $u(x,t)$ of problem (1.1) on $\Omega \times [0,T)$ by which we mean

$$u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right), \quad u_{tt} \in \left([0,T]; H^{-1}(\Omega)\right)$$

such that $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$, $u_t(x,0) = u_1(x)$ in $L^2(\Omega)$ and there holds

$$(u_t, v) + \int_0^t (\nabla u, \nabla v) d\tau = \int_0^t (|u|^p \ln|u|, v) d\tau + (u_1, v) \tag{2.1}$$

for any $v \in H_0^1(\Omega)$, $t \in [0,T)$ and u holds the following energy inequality

$$E(t) \le E(0) \text{ for every } t \in [0,T), \tag{2.2}$$

where

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}\ln|u|dx + \frac{1}{(p+1)^2}\|u\|_{p+1}^{p+1}$$

and

$$E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{p+1}\int_\Omega |u_0|^{p+1}\ln|u_0|dx + \frac{1}{(p+1)^2}\|u_0\|_{p+1}^{p+1}.$$

2.1. POTENTIAL WELLS

In this section, we shall set up the corresponding method of potential wells and series of their properties, which will be used to prove the theorems in all the sections.

First of all, we define two C^1 functionals on $H_0^1(\Omega)$, known as potential energy functional and Nehari functional respectively as follows

$$J(u) = \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}\ln|u|dx + \frac{1}{(p+1)^2}\|u\|_{p+1}^{p+1} \tag{2.3}$$

and

$$I(u) = \|\nabla u\|^2 - \int_\Omega |u|^{p+1}\ln|u|dx. \tag{2.4}$$

Then it is obvious that

$$J(u) = \frac{p-1}{2(p+1)}\|\nabla u\|^2 + \frac{1}{p+1}I(u) + \frac{1}{(p+1)^2}\|u\|_{p+1}^{p+1} \tag{2.5}$$

and

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + J(u) \\ &= \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|\nabla u\|^2 + \frac{1}{p+1}I(u) + \frac{1}{(p+1)^2}\|u\|_{p+1}^{p+1}. \end{aligned} \tag{2.6}$$

We also define Nehari manifold as

$$\mathcal{N}(u) = \{u \in H_0^1(\Omega)|\ I(u) = 0,\ \|\nabla u\|^2 \neq 0\}$$

and the depth of the potential well or the mountain pass level

$$d = \inf_{u\in\mathcal{N}} J(u), \tag{2.7}$$

which will be figured out to be positive later.

Now, we define the potential well

$$W = \{u \in H_0^1(\Omega)\,|\,I(u) > 0,\ J(u) < d\} \cup \{0\}$$

and the outer of the potential well

$$V = \{u \in H_0^1(\Omega) \mid I(u) < 0,\ J(u) < d\}.$$

Next, we try to extend the above single potential well to the family of potential wells by extending the above functional to following ones for $\delta > 0$

$$J_\delta(u) = \frac{\delta}{2}\|\nabla u\|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}\ln|u|dx + \frac{1}{(p+1)^2}\|u\|_{p+1}^{p+1}$$

and

$$I_\delta(u) = \delta\|\nabla u\|^2 - \int_\Omega |u|^{p+1}\ln|u|dx.$$

Also corresponding Nehari manifolds

$$\mathcal{N}_\delta(u) = \{u \in H_0^1(\Omega) \mid I_\delta(u) = 0,\ \|\nabla u\|^2 \neq 0\}$$

and depth of family of potential wells

$$d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u). \tag{2.8}$$

With the aid of the above functionals we introduce the family of potential wells

$$W_\delta = \{u \in H_0^1(\Omega) \mid I_\delta(u) > 0,\ J(u) < d(\delta)\} \cup \{0\}$$

and the outer of the family of potential wells

$$V_\delta = \{u \in H_0^1(\Omega) \mid I_\delta(u) < 0,\ J(u) < d(\delta)\}.$$

To study problem (1.1) in critical case we need to define the following set

$$V' = \{u \in H_0^1(\Omega) \mid I(u) < 0\}.$$

The lemma stated below informs that the functional $J(\lambda u)$ has a unique positive critical point $\lambda = \lambda^*$.

Lemma 2.1. *For any $u \in H_0^1(\Omega)$, $\|u\| \neq 0$ and let $g(\lambda) = J(\lambda u)$, then the following assertions hold:*

(i) $\lim_{\lambda\to 0} J(\lambda u) = 0$, $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$,

(ii) *in the interval $0 < \lambda < +\infty$ there exists a unique $\lambda^* = \lambda^*(u)$ such that*

$$\frac{d}{d\lambda}J(\lambda u)\mid_{\lambda=\lambda^*} = 0,$$

(iii) *$J(\lambda u)$ is increasing on $0 \le \lambda \le \lambda^*$, decreasing on $\lambda^* \le \lambda < +\infty$ and takes the maximum at $\lambda = \lambda^*$,*

(iv) *in the other words, $I(\lambda^* u) = 0$ and $I(\lambda u) = \lambda\frac{d}{d\lambda}J(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$.*

Proof. (i) We know that

$$\begin{aligned} g(\lambda) :=& J(\lambda u) \\ =& \frac{1}{2}\lambda^2\|\nabla u\|^2 - \frac{\lambda^{p+1}}{p+1}\int\limits_\Omega |u|^{p+1}\ln|u|dx - \frac{\lambda^{p+1}}{p+1}\ln|\lambda|\|u\|_{p+1}^{p+1} \\ &+ \frac{\lambda^{p+1}}{(p+1)^2}\|u\|_{p+1}^{p+1}. \end{aligned}$$

Since $\|u\| \neq 0$, then clearly $g(0) = 0,\ g(+\infty) = -\infty$.

(ii) Taking derivative of $g(\lambda)$ and making equals zero, we obtain

$$g'(\lambda) = \frac{d}{d\lambda}J(\lambda u) = \lambda\|\nabla u\|^2 - \lambda^p\int\limits_\Omega |u|^{p+1}\ln|u|dx - \lambda^p\ln|\lambda|\|u\|_{p+1}^{p+1} = 0,$$

which is equivalent to

$$\|\nabla u\|^2 = \lambda^{p-1}\int\limits_\Omega |u|^{p+1}\ln|u|dx + \lambda^{p-1}\ln|\lambda|\|u\|_{p+1}^{p+1}. \tag{2.9}$$

Let

$$l(\lambda) := \lambda^{p-1}\int\limits_\Omega |u|^{p+1}\ln|u|dx + \lambda^{p-1}\ln|\lambda|\|u\|_{p+1}^{p+1}.$$

We can clearly perceive that $l(\lambda)$ is increasing on $0 < \lambda < \infty$. Again, we have

$$\lim_{\lambda\to 0} l(\lambda) = -\infty,\ \lim_{\lambda\to\infty} l(\lambda) = \infty.$$

Therefore, there exists a unique λ_0 such that $l(\lambda_0) = 0,\ l(\lambda) < 0$ for $0 < \lambda < \lambda_0$ and $l(\lambda) > 0$ for $\lambda_0 < \lambda < \infty$. Hence, for any $\|\nabla u\| > 0$ there exists a unique $\lambda^* > \lambda_0$ such that (2.9) holds.

(iii) We have

$$\frac{d}{d\lambda}J(\lambda u) = \lambda\left(\|\nabla u\|^2 - l(\lambda)\right).$$

From the proof of (ii) it implies that if $0 < \lambda \leq \lambda_0$, then $l(\lambda) \leq 0$; if $\lambda_0 < \lambda < \lambda^*$, then $0 < l(\lambda) < \|\nabla u\|^2$; if $\lambda^* < \lambda < \infty$, then $l(\lambda) > \|\nabla u\|^2$. Hence, we arrive at $\frac{d}{d\lambda}J(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $\frac{d}{d\lambda}J(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$. From this, the conclusion of (iii) follows.

(iv) The conclusion follows from the proof of (iii) and

$$I(\lambda u) = \lambda^2\|\nabla u\|^2 - \lambda^{p+1}\int\limits_\Omega |u|^{p+1}\ln|\lambda u|dx = \lambda\frac{d}{d\lambda}J(\lambda u).$$

$\square$

The above lemma also tells $\mathcal{N} \neq \emptyset$. The following lemma provides some crucial features of the Nehari functional $I_\delta(u)$.

Lemma 2.2. *Suppose $\delta > 0$. We have the following statements:*

(i) *if $0 < \|\nabla u\| \leq r(\delta)$, then $I_\delta(u) > 0$,*
(ii) *if $I_\delta(u) < 0$, then $\|\nabla u\| > r(\delta)$,*
(iii) *if $I_\delta(u) = 0$, then $\|\nabla u\| > r(\delta)$ or $\|\nabla u\| = 0$,*

where $r(\delta)$ is the unique real root of equation $\phi(r) = \delta$,

$$\phi(r) = C^{p+2} r^p, \; C = \sup_{u \in H_0^1(\Omega)} \frac{\|u\|_{p+2}}{\|\nabla u\|}.$$

Proof. (i) From $0 < \|\nabla u\| \leq r(\delta)$ we have $\|u\|_{p+2} > 0$ and by

$$\int_\Omega |u|^{p+1} \ln |u| dx < \|u\|_{p+2}^{p+2} \leq C^{p+2} \|\nabla u\|^{p+2} = \phi(\|\nabla u\|)\|\nabla u\|^2 \leq \delta \|\nabla u\|^2$$

we obtain $I_\delta(u) > 0$.

(ii) $I_\delta(u) < 0$ gives

$$\delta \|\nabla u\|^2 < \int_\Omega |u|^{p+1} \ln |u| dx < \|u\|_{p+2}^{p+2} \leq \phi(\|\nabla u\|)\|\nabla u\|^2. \tag{2.10}$$

From (2.10) we get $\|\nabla u\| > r(\delta)$.

(iii) If $\|\nabla u\| = 0$, then $I_\delta(u) = 0$. If $I_\delta(u) = 0$ and $\|\nabla u\| \neq 0$, then from

$$\delta \|\nabla u\|^2 = \int_\Omega |u|^{p+1} \ln |u| dx < \|u\|_{p+2}^{p+2} \leq \phi(\|\nabla u\|)\|\nabla u\|^2$$

we have $\|\nabla u\| > r(\delta)$. □

The lemma below illustrates the depth of the potential well or the mountain pass level.

Lemma 2.3. *For $d(\delta)$ in (2.8) we have the following properties:*

(i) *$d(\delta) = a(\delta) r^2(\delta) > 0$ for $0 < \delta < \frac{p+1}{2}$, $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}$,*
(ii) *there exists a unique $\delta_0 > \frac{p+1}{2}$ such that $d(\delta_0) = 0$, and $d(\delta) > 0$ for $0 < \delta < \delta_0$,*
(iii) *$d(\delta)$ is strictly increasing on $0 < \delta \leq 1$, decreasing on $1 \leq \delta \leq \delta_0$ and maximum $d = d(1)$ at $\delta = 1$.*

Proof. (i) If $I_\delta(u) = 0$ and $\|\nabla u\| \neq 0$, then by Lemma 2.2(iii) we have $\|\nabla u\| > r(\delta)$ and using this, we get

$$\begin{aligned} J(u) =& \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}\ln|u|dx + \frac{1}{(p+1)^2}\|u\|_{p+1}^{p+1} \\ =& \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\nabla u\|^2 + \frac{1}{p+1}I_\delta(u) + \frac{1}{(p+1)^2}\|u\|_{p+1}^{p+1} \\ >& \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\nabla u\|^2 + \frac{1}{p+1}I_\delta(u) \\ >& a(\delta)r^2(\delta) > 0. \end{aligned}$$

(ii) For any $u \in H_0^1(\Omega),\ \|\nabla u\| \neq 0$ and $\delta > 0$, we define $\lambda = \lambda(\delta)$ such that

$$\delta\lambda^2\|\nabla u\|^2 = \lambda^{p+1}\int_\Omega |u|^{p+1}\ln|\lambda u|dx. \tag{2.11}$$

Then $I_\delta(\lambda u) = 0$ and

$$\delta\|\nabla u\|^2 = \lambda^{p-1}\int_\Omega |u|^{p+1}\ln|u|dx + \lambda^{p-1}\ln|\lambda|\|u\|_{p+1}^{p+1}. \tag{2.12}$$

As Lemma 2.1 says, $J(\lambda u)$ is increasing on $(0, \lambda^*)$, decreasing on $[\lambda^*, +\infty)$, and (i) of this lemma gives $d(\delta) > 0$ on $(0, \frac{p+1}{2})$. According to that $\lambda(\delta)$ is increasing on $(0, +\infty)$, we know that for some a (in the next part $a = 1$ will be proved) $d(\delta)$ is increasing on $(0, a)$ and decreasing on $[a, 0)$ and hence hits the δ-axis at some point δ_0. Since (i) of this lemma says, $d(\delta) > 0$ on $(0, \frac{p+1}{2})$, thus $\delta_0 > \frac{p+1}{2}$.

(iii) We prove that $d(\delta') < d(\delta'')$ for any $0 < \delta' < \delta'' < 1$ or $1 < \delta'' < \delta' < \delta_0$. Clearly it is sufficient to prove that for any $0 < \delta' < \delta'' < 1$ or $1 < \delta'' < \delta' < \delta_0$ and any $u \in H_0^1(\Omega),\ I_{\delta''}(u) = 0$ and $\|\nabla u\| \neq 0$ there exists a $v \in H_0^1(\Omega)$ and a constant $\varepsilon(\delta', \delta'') > 0$ such that $I_{\delta'}(v) = 0,\ \|\nabla v\| \neq 0$ and $J(v) < J(u) - \varepsilon(\delta', \delta'')$. Actually, for above u we can define $\lambda(\delta)$ by (2.11) such that $I_\delta(\lambda(\delta)u) = 0,\ \lambda(\delta'') = 1$ and (2.12) holds. Let $g(\lambda) = J(\lambda u)$. Then

$$\begin{aligned} \frac{d}{d\lambda}J(\lambda u) &= \frac{1}{\lambda}I(\lambda u) \\ &= \frac{1}{\lambda}\left((1-\delta)\|\nabla(\lambda u)\|^2 + I_\delta(\lambda u)\right) \\ &= (1-\delta)\lambda\|\nabla u\|^2. \end{aligned}$$

Taking $v = \lambda(\delta')u$, then $I_{\delta'}(v) = 0$ and $\|\nabla v\| \neq 0$.

If $0 < \delta' < \delta'' < 1$, then since $\lambda(\delta)$ is increasing in δ,

$$\begin{aligned} J(u) - J(v) &= g(1) - g(\lambda(\delta')) \\ &= g(\lambda(\delta'')) - g(\lambda(\delta')) \\ &= (\lambda(\delta'') - \lambda(\delta'))\, g'(\lambda) \\ &= (1-\delta)\lambda(1-\lambda(\delta'))\|\nabla u\|^2 \\ &> (1-\delta'')\lambda(\delta')r^2(\delta'')(1-\lambda(\delta')) \\ &\equiv \varepsilon(\delta', \delta'') > 0. \end{aligned}$$

If $1 < \delta'' < \delta' < \delta_0$, then

$$J(u) - J(v) = g(1) - g(\lambda(\delta')) > (\delta'' - 1)\lambda(\delta'')r^2(\delta'')(\lambda(\delta') - 1) \equiv \varepsilon(\delta', \delta'') > 0. \quad \square$$

2.2. INVARIANT SETS

To obtain the invariant sets, the lemma below will be used.

Lemma 2.4. *Let $0 < J(u) < d$ for some $u \in H_0^1(\Omega)$ and $\delta_1 < 1 < \delta_2$ are the two roots of the equation $d(\delta) = J(u)$, then the sign of $I_\delta(u)$ are unchangeable for $\delta_1 < \delta < \delta_2$.*

Proof. Firstly, $J(u) > 0$ implies that $\|\nabla u\| \neq 0$. Arguing by contradiction, we suppose that the sign of $I_\delta(u)$ are changeable for $\delta_1 < \delta < \delta_2$, then there exists a $\bar{\delta} \in (\delta_1, \delta_2)$ such that $I_{\bar{\delta}}(u) = 0$. Hence, by the definition of $d(\delta)$, we have $J(u) \geq d(\bar{\delta})$ which contradicts

$$J(u) = d(\delta_1) = d(\delta_2) < d(\bar{\delta}).$$

$\square$

Theorem 2.5 (Invariant sets). *Let $u_0 \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Assume that $0 < e < d$, $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = e$, then*

(i) *all solutions of problem* (1.1) *with $0 < E(0) \leq e$ belong to W_δ for $\delta_1 < \delta < \delta_2$, provided $I(u_0) > 0$ or $\|\nabla u_0\| = 0$,*
(ii) *all solutions of problem* (1.1) *with $0 < E(0) \leq e$ belong to V_δ for $\delta_1 < \delta < \delta_2$, provided $I(u_0) < 0$.*

Proof. (i) Let $u(t)$ be any solution of problem (1.1) with $E(0) = e$ and $I(u_0) > 0$ or $\|\nabla u_0\| = 0$, T be the existence time of $u(t)$. If $\|\nabla u_0\| = 0$, then clearly $u_0(x) \in W_\delta$ for $0 < \delta < \delta_0$. Since $I(u_0) > 0$ and by Lemma 2.4 the sign of $I_\delta(u)$ is unchangeable for $\delta_1 < \delta < \delta_2$, so we have $I_\delta(u_0) > 0$ for $\delta \in (\delta_1, \delta_2)$. From the energy equality

$$\frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) \leq d(\delta_1) = d(\delta_2) < d(\delta), \quad \delta \in (\delta_1, \delta_2) \tag{2.13}$$

we have $J(u_0) < d(\delta)$, i.e., $u_0(x) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. Next, we prove $u(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$, where T is the maximal existence time of $u(t)$. Arguing by contradiction, we suppose that there must exist a $t_0 \in (0, T)$ such that $u(t_0) \in \partial W_\delta$

for some $\delta \in (\delta_1, \delta_2)$, i.e., $I_\delta(u(t_0)) = 0,\ \|\nabla u(t_0)\| \neq 0$ or $J(u(t_0)) = d(\delta)$. From the energy inequality (2.2)

$$\frac{1}{2}\|u_t\|^2 + J(u) \leq E(0) < d(\delta), \quad t \in (0,T),\ \delta \in (\delta_1, \delta_2), \tag{2.14}$$

we see that $J(u(t_0)) = d(\delta)$ is impossible. On the other hand, if $I_\delta(u(t_0)) = 0$ and $\|\nabla u(t_0)\| \neq 0$, then by the definition of $d(\delta)$ we have $J(u(t_0)) \geq d(\delta)$ which contradicts (2.14).

(ii) The proof is similar to (i) of this theorem. □

In fact, we have the following result.

Theorem 2.6. *All nontrivial solutions of problem* (1.1) *with* $E(0) = 0$ *belong to*

$$B_{r_0}^c = \left\{ u \in H_0^1(\Omega) \;\middle|\; \|\nabla u\| \geq r_0 := \left(\frac{1}{C^{p+2}}\right)^{\frac{1}{p}} \right\}.$$

Proof. Let $u(t)$ be the any solution of problem (1.1) with $E(0) = 0$, T be the existence time of $u(t)$. From the energy inequality (2.2) we have

$$\frac{1}{2}\|u_t\|^2 + J(u) \leq E(0) = 0,$$

which means that $J(u) \leq 0$ for $0 \leq t < T$. Hence by (2.5) we have

$$\frac{p-1}{2(p+1)}\|\nabla u\|^2 + \frac{1}{p+1}I(u) + \frac{1}{(p+1)^2}\|u\|_{p+1}^{p+1} \leq 0,$$

which implies $I(u) \leq 0$. So by definition of $I(u)$ we have

$$\|\nabla u\|^2 \leq \int_\Omega |u|^{p+1} \ln |u| dx \leq \|u\|_{p+2}^{p+2} \leq C^{p+2}\|\nabla u\|^p \|\nabla u\|^2, \quad 0 \leq t < T.$$

From this we must have either $\|\nabla u\| = 0$ or $\|\nabla u\| \geq r_0$. If $\|\nabla u_0\| = 0$, then $\|\nabla u\| \equiv 0$ for $0 \leq t < T$. Otherwise there exists a $t_0 \in (0,T)$ such that $0 < \|\nabla u(t_0)\| < r_0$. By similar logics we can prove that if $\|\nabla u_0\| \geq r_0$, then $\|\nabla u\| \geq r_0$ for $0 < t < T$. □

Theorem 2.7. *Let* $u_0(x) \in H_0^1(\Omega)$ *and* $u_1(x) \in L^2(\Omega)$*. Assume that* $E(0) < 0$ *or* $E(0) = 0,\ \|\nabla u_0\| \neq 0$*, then all solutions of problem* (1.1) *belong to* V_δ *for* $0 < \delta < \frac{p+1}{2}$.

Proof. Let $u(t)$ be the any solution of problem (1.1) with $E(0) = 0$, T be the existence time of $u(t)$. The energy equality gives

$$\frac{1}{2}\|u_t\|^2 + a(\delta)\|\nabla u\|^2 + \frac{1}{p+1}I_\delta(u) \leq \frac{1}{2}\|u_t\|^2 + J(u) = E(0), \quad 0 < \delta < \frac{p+1}{2}. \tag{2.15}$$

From (2.15) it implies that if $E(0) < 0$, then $I_\delta(u) < 0,\ J(u) < 0 < d(\delta)$ since $d(\delta) > 0$ by Lemma 2.3 for $0 < \delta < \frac{p+1}{2}$; if $E(0) = 0,\ \|\nabla u_0\| \neq 0$, then by Theorem 2.6 we have $\|\nabla u_0\| \geq r_0$ for $0 \leq t < T$. Again by (2.15) we get $I_\delta(u) < 0,\ J(u) < 0 < d(\delta)$ for $0 < \delta < \frac{p+1}{2}$. Thus for above two cases we always have $u(t) \in V_\delta$ for $0 < \delta < \frac{p+1}{2}$, $0 \leq t < T$. □

3. GLOBAL EXISTENCE AND FINITE TIME BLOWUP AT $E(0) < d$

Here, we shall prove the global existence and blow up property in finite time of the solutions for problem (1.1) by using potential wells introduced above.

Theorem 3.1 (Global existence for $E(0) < d$)**.** *Let $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Assume that $0 < E(0) < d$ and $I(u_0) > 0$ or $\|\nabla u_0\| = 0$, then problem (1.1) admits a global weak solution $u(t) \in L^\infty\left(0,\infty; H_0^1(\Omega)\right)$ with $u_t(t) \in L^\infty\left(0,\infty; L^2(\Omega)\right)$ and $u(t) \in W$ for $0 \le t < \infty$.*

Proof. Construct approximate solutions $u_m(x,t)$ of problem (1.1) as did in [19]. Then by the same logics used in the proof of Theorem 3.2 in [19] we can get

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) = E_m(0) < d, \quad 0 \le t < \infty \tag{3.1}$$

and $u_m(t) \in W$ for sufficiently large m and $0 \le t < \infty$. From (3.1) and

$$\begin{aligned} J(u_m) &= \frac{1}{2}\|\nabla u_m\|^2 - \frac{1}{p+1}\int_\Omega |u_m|^{p+1}\ln|u_m|dx + \frac{1}{(p+1)^2}\|u_m\|_{p+1}^{p+1} \\ &\ge \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\nabla u_m\|^2 + \frac{1}{p+1}I(u_m) \\ &\ge \frac{p-1}{2(p+1)}\|\nabla u_m\|^2 \end{aligned}$$

we can write

$$\frac{1}{2}\|u_{mt}\|^2 + \frac{p-1}{2(p+1)}\|\nabla u_m\|^2 < d, \quad 0 \le t < \infty,$$

which implies that

$$\|\nabla u_m\|^2 < \frac{2(p+1)}{p-1}d, \quad 0 \le t < \infty, \tag{3.2}$$

$$\|u_m\|_{p+1}^2 \le C^2\|\nabla u_m\|^2 < C^2\frac{2(p+1)}{p-1}d, \quad 0 \le t < \infty, \tag{3.3}$$

$$\begin{aligned} \int_\Omega |u_m|^{p+1}\ln|u_m|dx &< \|u\|_{p+2}^{p+2} \le C^{p+2}\|\nabla u_m\|^{p+2} \\ &< C^{p+2}\left(\frac{2(p+1)d}{p-1}\right)^{\frac{p+2}{2}}, \quad 0 \le t < \infty, \end{aligned} \tag{3.4}$$

and

$$\|u_{mt}\|^2 < 2d, \quad 0 \le t < \infty. \tag{3.5}$$

From (3.2)–(3.5) and compactness method it follows that problem (1.1) admits a global weak solution $u(t) \in L^\infty\left(0,\infty; H_0^1(\Omega)\right)$ with $u_t(t) \in L^\infty\left(0,\infty; L^2(\Omega)\right)$. Ultimately, by Theorem 2.5, we have $u(t) \in W$ for $0 \le t < \infty$. □

Theorem 3.2 (Finite time blow up for $E(0) < d$)**.** *Let $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Assume that $E(0) < d,\ I(u_0) < 0$ and $(u_0, u_1) \geq 0$, then the weak solution of problem* (1.1) *blows up in finite time such that*

$$\lim_{t \to T^-} \|u(\cdot, t)\| = +\infty.$$

Proof. Suppose $u(x,t)$ be any solution of problem (1.1) with $E(0) < d$ and $I(u_0) < 0$. Let us consider the function $L(t) : [0, +\infty) \to \mathbb{R}^+$ defined by

$$L(t) := \|u\|^2. \tag{3.6}$$

Differentiating this we have

$$L'(t) = 2(u, u_t) \tag{3.7}$$

and

$$\begin{aligned} L''(t) &= 2\|u_t\|^2 + 2(u, u_{tt}) \\ &= 2\|u_t\|^2 - 2\left(\|\nabla u\|^2 - \int_\Omega |u|^{p+1} \ln |u| dx\right) \qquad (3.8) \\ &= 2\|u_t\|^2 - 2I(u). \qquad (3.9) \end{aligned}$$

From (2.2) we have

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} \ln |u| dx + \frac{1}{(p+1)^2}\|u\|_{p+1}^{p+1} \leq E(0).$$

From this we can write

$$\begin{aligned} 2\int_\Omega |u|^{p+1} \ln |u| dx \geq &(p+1)\|u_t\|^2 + (p+1)\|\nabla u\|^2 + \frac{2}{(p+1)}\|u\|_{p+1}^{p+1} \\ &- 2(p+1)E(0) \\ \geq &(p+1)\|u_t\|^2 + (p+1)\|\nabla u\|^2 - 2(p+1)E(0). \end{aligned} \tag{3.10}$$

From (3.8) and (3.10) we obtain

$$\begin{aligned} L''(t) &\geq 2\|u_t\|^2 - 2\|\nabla u\|^2 + (p+1)\|u_t\|^2 + (p+1)\|\nabla u\|^2 - 2(p+1)E(0) \\ &= (p+3)\|u_t\|^2 + (p-1)\|\nabla u\|^2 - 2(p+1)E(0) \\ &\geq (p+3)\|u_t\|^2 + (p-1)\lambda_1 L(t) - 2(p+1)E(0), \end{aligned} \tag{3.11}$$

where $\lambda_1 > 0$ is the first eigenvalue of the problem $\Delta\varphi + \lambda\varphi = 0,\ \phi|_{\partial\Omega} = 0$.

(i) If $E(0) \leq 0$, then (3.11) implies

$$L''(t) \geq (p+3)\|u_t\|^2. \tag{3.12}$$

(ii) If $0 < E(0) < d$, then from Theorem 2.5 it follows that $u(t) \in V_\delta$ for $1 < \delta < \delta_2$ and $t > 0$, where δ_2 is the same as that in Theorem 2.5. Thus $I_\delta(u) < 0$ and by Lemma 2.2(ii) $\|\nabla u\| > r(\delta)$ for $1 < \delta < \delta_2$ and $t > 0$. Therefore, we get $I_{\delta_2}(u) \leq 0$ and $\|\nabla u\| \geq r(\delta_2)$ for $t > 0$. Again, as $L'(0) = 2(u_0, u_1) \geq 0$, by (3.9) we obtain

$$L''(t) \geq 2(\delta_2 - 1)\|\nabla u\|^2 - 2I_{\delta_2}(u) \geq 2(\delta_2 - 1)r^2(\delta_2) > 0,$$

$$L'(t) \geq 2(\delta_2 - 1)r^2(\delta_2)t + L'(0) \geq 2(\delta_2 - 1)r^2(\delta_2)t,$$

and

$$L(t) \geq (\delta_2 - 1)r^2(\delta_2)t^2 + L(0) \geq (\delta_2 - 1)r^2(\delta_2)t^2.$$

Thus, for sufficiently large t we have $(p-1)\lambda_1 L(t) > 2(p+1)E(0)$. Using this into (3.11) we can achieve (3.12). Ultimately (3.12) gives

$$L(t)L''(t) - \frac{p+3}{4}\left(L'(t)\right)^2 \geq (p+3)\left(\|u\|^2\|u_t\| - (u, u_t)^2\right) \geq 0$$

and

$$\left(L^{-\alpha}(t)\right)'' = -\frac{\alpha}{L^{\alpha+2}(t)}\left(L(t)L''(t) - (\alpha+1)\left(L'(t)\right)^2\right) \leq 0, \quad \alpha = \frac{p-1}{4}.$$

Thus, the conclusion of this theorem follows for some $T > 0$. □

4. GLOBAL EXISTENCE AND FINITE TIME BLOWUP AT $E(0) = d$

In this section, we shall prove the global existence and blow up property in finite time of the solutions for problem (1.1) at critical energy level by using potential well method.

Theorem 4.1 (Global existence for $E(0) = d$)**.** *Let $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Assume that $E(0) = d$ and $I(u_0) \geq 0$, then problem* (1.1) *admits a global weak solution $u(t) \in L^\infty\left(0, \infty; H_0^1(\Omega)\right)$ with $u_t(t) \in L^\infty\left(0, \infty; L^2(\Omega)\right)$ and $u(t) \in W \cup \partial W$ for $0 \leq t < \infty$.*

Proof. We prove this theorem considering two cases (i) and (ii).

(i) $\|\nabla u_0\| \neq 0$.

Let $\lambda_m = 1 - \frac{1}{m}$ and $u_{0m} = \lambda_m u_0,\ m = 2, 3, \ldots$. Consider the initial conditions

$$u(x,0) = u_{0m}(x),\ u_t(x,0) = u_1(x)$$

and the corresponding problem

$$\begin{cases} u_{tt} - \triangle u = |u|^p \ln|u|, & x \in \Omega,\ t > 0, \\ u(x,0) = u_{0m}(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{4.1}$$

From $I(u_0) \geq 0$ and Lemma 2.1, we have $\lambda^* = \lambda^*(u_0) \geq 1$. Hence $I(u_{0m}) > 0$ and $J(u_{0m}) = J(\lambda_m u_0) < J(u_0)$. In addition

$$0 < E_m(0) \equiv \frac{1}{2}\|u_1\|^2 + J(u_{0m}) < \frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) = d.$$

Thus it follows from Theorem 3.1 that for each m, problem (4.1) admits a global solution $u_m(t) \in L^\infty\left(0, \infty; H_0^1(\Omega)\right)$ and $u_m(t) \in W$ for $0 \leq t < \infty$ satisfying

$$(u_{mt}, v) + \int_0^t (\nabla u_m, \nabla v) d\tau = \int_0^t (f(u_m), v) d\tau + (u_1, v) \tag{4.2}$$

for every $v \in H_0^1(\Omega),\ 0 \leq t < \infty$ and

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) = E_m(0) < d,\ 0 \leq t < \infty. \tag{4.3}$$

The rest of the proof is similar to Theorem 3.1

(ii) $\|\nabla u_0\| = 0$.

Note that $\|\nabla u_0\| = 0$ implies that $J(u_0) = 0$ and $\frac{1}{2}\|u_1\|^2 = E(0) = d$. Let $\lambda_m = 1 - \frac{1}{m}$ and $u_{1m}(x) = \lambda_m u_1,\ m = 2,\ 3, \ldots$. Consider the initial conditions

$$u(x, 0) = u_0(x),\ u_t(x, 0) = u_{1m}(x)$$

and the corresponding problem

$$\begin{cases} u_{tt} - \triangle u = |u|^p \ln |u|, & x \in \Omega,\ t > 0, \\ u(x, 0) = u_0(x),\ u_t(x, 0) = u_{1m}(x), & x \in \Omega, \\ u(x, t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{4.4}$$

From

$$\|\nabla u_0\| = 0,\ 0 < E_m(0) = \frac{1}{2}\|u_{1m}\|^2 + J(u_0) = \frac{1}{2}\|\lambda_m u_1\|^2 < E(0) = d$$

and Theorem 3.1 it follows that for each m, problem (4.4) admits a global solution $u_m(t) \in L^\infty\left(0, \infty; H_0^1(\Omega)\right)$ with $u_t(t) \in L^\infty\left(0, \infty; L^2(\Omega)\right)$ and $u_m(t) \in W$ for $0 \leq t < \infty$, satisfying (4.2) and (4.3). The remainder proof is similar as part (i) of this theorem. $\square$

To prove blow up of solution under critical energy condition, we need following lemma. For the proof of Lemma 4.2, we refer the readers to see Lemma 2.7 in [29].

Lemma 4.2 (Invariant set V')**.** *Let $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Assume that $E(0) = d$, $I(u_0) < 0$ and $(u_0, u_1) \geq 0$, then the set V' is invariant under the flow of* (1.1).

Theorem 4.3 (Finite time blow up for $E(0) = d$). *Let $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. Assume that $E(0) = d$, $I(u_0) < 0$ and $(u_0, u_1) \geq 0$, then the weak solution of problem (1.1) blows up in finite time such that*

$$\lim_{t \to T^-} \|u(\cdot, t)\| = +\infty.$$

Proof. From (3.11) we have

$$\begin{aligned} L''(t) &\geq (p+3)\|u_t\|^2 + (p-1)\lambda_1 L(t) - 2(p+1)E(0) \\ &= (p+3)\|u_t\|^2 + (p-1)\lambda_1 L(t) - 2(p+1)d. \end{aligned} \tag{4.5}$$

Eq. (3.9) and Lemma 4.2 yield $L''(t) > 0$ for $0 \leq t < \infty$ which means $L'(t)$ is strictly increasing for $0 \leq t < \infty$. Since $L'(0) = 2(u_0, u_1) \geq 0$, for any $t_0 > 0$ we have

$$L'(t) \geq L'(t_0) > 0, \quad t \geq t_0$$

and

$$L(t) \geq L'(t_0)(t - t_0) + L(t_0) > L'(t_0)(t - t_0), \quad t \geq t_0.$$

So for sufficiently large t, we can obtain

$$(p-1)\lambda_1 L(t) > 2(p+1)d.$$

From this and (4.5) we get

$$L''(t) \geq (p+3)\|u_t\|^2.$$

Hence,

$$L(t)L''(t) - \frac{p+3}{4}\left(L'(t)\right)^2 \geq (p+3)\left(\|u\|^2\|u_t\| - (u, u_t)^2\right) \geq 0.$$

The rest of the proof is similar to Theorem 3.2. □

5. FINITE TIME BLOWUP AT $E(0) > 0$

In this section we shall prove the blowup result at high initial energy.

Theorem 5.1 (Finite time blow up for $E(0) > 0$). *If the initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfy*

(i) $E(0) > 0$,
(ii) $(u_0, u_1) > 0$,
(iii) $\|u_0\|^2 > \frac{2(p+1)}{\lambda_1(p-1)} E(0)$,
(iv) $I(u_0) < 0$,

then the solutions of problem (1.1) *blow up in finite time.*

Proof. We shall prove the result by following two steps:
Step 1. In this step, we prove that $I(u) < 0$ and $\|u(t)\|^2 > \frac{2(p+1)}{\lambda_1(p-1)}E(0)$ for every $t \in (0,T)$. For $I(u) < 0$, arguing by contradiction we suppose that there exists a first time $t_0 \in (0,T)$ such that $I(u(t_0)) = 0$ and $I(u) < 0$ for $t \in [0,t_0)$. Again, we consider the function $L(t)$ as before and its first and second derivative are as below

$$L'(t) = 2(u, u_t)$$

and

$$L''(t) = 2\|u_t\| - 2I(u).$$

Since $I(u) < 0$ for $t \in [0,t_0)$, we have $L''(t) > 0$ for any $t \in (0,t_0)$, which means that $L'(t)$ is increasing. Due to $L'(0) = 2(u_0,u_1) > 0$, we obtain $L'(t) > 0$ for every $t \in (0,t_0)$, which implies that $L(t)$ is strictly increasing. Thus, we reach

$$L(t) > \|u_0\|^2 > \frac{2(p+1)}{\lambda_1(p-1)}E(0) \text{ for any } t \in (0,t_0).$$

Consequently, we have

$$L(t_0) > \frac{2(p+1)}{\lambda_1(p-1)}E(0). \tag{5.1}$$

In the meantime, we know

$$J(u((t_0)) \leq E(t_0) \leq E(0)$$

that is

$$\begin{aligned}&\frac{1}{2}\|\nabla u(t_0)\|^2 - \frac{1}{p+1}\int\limits_\Omega |u(t_0)|^{p+1}\ln|u(t_0)|dx + \frac{1}{(p+1)^2}\|u(t_0)\|_{p+1}^{p+1}\\ \leq& E(t_0) \leq E(0).\end{aligned} \tag{5.2}$$

In addition, $I(u(t_0)) = 0$ implies

$$\|\nabla u(t_0)\|^2 = \int\limits_\Omega |u(t_0)|^{p+1}\ln|u(t_0)|dx.$$

Now, we can write (5.2) as follows

$$\begin{aligned}&\frac{1}{2}\|\nabla u(t_0)\|^2 - \frac{1}{p+1}\int\limits_\Omega |u(t_0)|^{p+1}\ln|u(t_0)|dx + \frac{1}{(p+1)^2}\|u(t_0)\|_{p+1}^{p+1}\\ =&\frac{1}{2}\|\nabla u(t_0)\|^2 - \frac{1}{p+1}\|\nabla u(t_0)\|^2 + \frac{1}{(p+1)^2}\|u(t_0)\|_{p+1}^{p+1}\\ \geq&\frac{p-1}{2(p+1)}\|\nabla u(t_0)\|^2\\ \geq&\frac{\lambda_1(p-1)}{2(p+1)}\|u(t_0)\|^2.\end{aligned} \tag{5.3}$$

From (5.2) and (5.3), we get

$$\frac{\lambda_1(p-1)}{2(p+1)}\|u(t_0)\|^2 \leq E(0)$$

that is

$$L(t_0) \leq \frac{2(p+1)}{\lambda_1(p-1)}E(0),$$

which contradicts (5.1). Thus, we have

$$I(u) < 0 \text{ for every } t \in (0,T)$$

and

$$L(t) > \frac{2(p+1)}{\lambda_1(p-1)}E(0) \text{ for every } t \in (0,T). \tag{5.4}$$

Step 2. Here, we prove the blowup result. From (3.11) and (5.4) we have

$$L''(t) \geq (p+3)\|u_t\|^2.$$

Hence, we obtain

$$L(t)L''(t) - \frac{p+3}{4}\left(L'(t)\right)^2 \geq (p+3)\left(\|u\|^2\|u_t\| - (u,u_t)^2\right) \geq 0.$$

The remainder proof is similar to Theorem 3.2. □

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