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Dedicated to Prof. Dušan D. Repovš
on the occasion of his 65th birthday

ON THE REGULARITY OF SOLUTION TO THE TIME-DEPENDENT p-STOKES SYSTEM

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Abstract. In this paper we consider the time evolutionary p-Stokes problem in a smooth and bounded domain. This system models the unsteady motion or certain non-Newtonian incompressible fluids in the regime of slow motions, when the convective term is negligible. We prove results of space/time regularity, showing that first-order time-derivatives and second-order space-derivatives of the velocity and first-order space-derivatives of the pressure belong to rather natural Lebesgue spaces.

Keywords: regularity, evolution problem, p-Stokes.

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1. INTRODUCTION

We consider the time-dependent p-Stokes system

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{u}) + \nabla \pi = \mathbf{f} \qquad & \text{in } I \times \Omega, \\
\operatorname{div} \mathbf{u} = 0 \qquad & \text{in } I \times \Omega, \\
\mathbf{u} = \mathbf{0} \qquad & \text{on } I \times \partial\Omega, \\
\mathbf{u}(0, \cdot) = \mathbf{u}_0 \qquad & \text{in } \Omega,
\end{aligned} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$ and $I := [0, T]$, for some $T > 0$. The unknowns are the velocity $\mathbf{u} : I \times \Omega \to \mathbb{R}^3$ and the pressure $\pi : I \times \Omega \to \mathbb{R}$. The stress tensor $\mathbf{S}$ has (p, δ)-structure, for $1 < p \leq 2$ and some $\delta \geq 0$, see Assumption 2.1 for the precise definition. The system (1.1) can be used to model certain non-Newtonian fluids in the case in which the velocity is small enough such that the convective term can be disregarded. For the system (1.1), since the principal part is nonlinear (and in the equations there is not a term corresponding to the linear elliptic part, that is a term as $-\nu \operatorname{div} \mathbf{Du}$), the proofs of the various results cannot be obtained as a perturbation of the ones known for the linear Stokes system. Nevertheless, the presence of a nonlinear convective term can be handled in a rather standard way by means of a linearization argument, once the regularity for the p-Stokes has been established and once the range of p has been adequately restricted.

The analysis of this problem has a long history and several results are concerned with interior regularity or with the space-periodic setting. We also observe that many results focus on the presence of the convective term, which enforces some limitations to both the technique (passage to the limit) to be used to construct weak solutions as well as to the range of allowed exponents. We refer to [4] for the analysis in the space-periodic case, but also the interior case can be treated similarly.

The analysis in a bounded domain with Dirichlet conditions requires a more technical local argument, as that employed for $p \geq 2$ in [15], taking into account of the divergence free constraint. Here, we follow the same approach and we adapt the techniques used for the steady problem and $1 < p < 2$ in [5].

We also wish to mention that the 2D case can be handled with different techniques as in Kaplický, Málek, and Stará [14] and also that the shear thinning case $p > 2$ requires a different treatment, see also [2].

We also wish to mention the results of Bothe and Prüss [7], where local existence and uniqueness results of rather smooth solutions is proved under the condition $\delta > 0$. Here we are considering the case in which the data are not so regular and also include in our treatment the degenerate case $\delta = 0$ (for which some of our results are valid).

We wish also to mention that a similar approach has been also recently used by the same authors in [6] to prove optimal regularity for solutions of the (technically simpler by the absence of the pressure) initial boundary value problem for a parabolic system

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \operatorname{div} \mathbf{S}(\mathbf{Du}) &= \mathbf{f} && \text{in } I \times \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } I \times \partial\Omega, \\ \mathbf{u}(0,\cdot) &= \mathbf{u}_0 && \text{in } \Omega, \end{aligned} \tag{1.2}$$

with a tensor $\mathbf{S}$ satisfying Assumption 2.1. The technical novelty here is the derivation of the appropriate estimates for π.

Our results are expressed in terms of the quantity

$$\mathbf{F}(\mathbf{A}) := \big(\delta + |\mathbf{A}^{\mathrm{sym}}|\big)^{\frac{p-2}{2}} \mathbf{A}^{\mathrm{sym}}, \tag{1.3}$$

since its space and time derivatives represent the natural quantity to be controlled. Bounds on $\partial_t \mathbf{F}(\mathbf{Du}), \nabla \mathbf{F}(\mathbf{Du})$ (as the quasi-norm in Barrett and Liu [1]) allow to prove error estimates for the numerical discretization; the results imply also certain regularity for the usual partial derivatives. We will study in a forthcoming paper the bounds on the numerical error and the dependence on the regularity of the continuous solution.

In a future work we will consider the numerical analysis of the problem also with convective term, for which existence of weak solutions is known for $p > \frac{6}{5}$ (at least in the periodic case strong solutions are known to exists locally in time for $p > \frac{7}{5}$). Thus, from now on we suppose

$$p \geq \frac{6}{5},$$

even if we consider the problem without convection. Another technical reasons is that $p = \frac{6}{5}$ is also the critical exponent in $\mathbb{R}^3$ to have an evolution triple, in order to properly formulate the variational problem. This choice avoids the use of more technical definitions of weak solutions, as done in [13]. In particular in the case $1 < p < \frac{6}{5}$ the natural spaces $W^{1,p}_{0,\sigma}(\Omega)$, $L^2_\sigma(\Omega)$, and $(W^{1,p}_{0,\sigma}(\Omega))^*$ cannot be used to define a Gelfand evolution triple (in three space dimensions). Another reason for this restriction is a critical result for the regularity of the pressure, cf. Theorem 2.10.

The main result we prove is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$. Let the stress tensor $\mathbf{S}$ be with (p,δ)-structure, for $\frac{6}{5} \leq p \leq 2$ and some $\delta > 0$. Let be given $\mathbf{u}_0 \in L^2(\Omega) \cap W^{1,p}_{0,\sigma}(\Omega)$ such that* $\operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{u}_0) \in L^2(\Omega)$ *and let the external force satisfy $\mathbf{f} \in W^{1,2}(I; L^2(\Omega)) \cap L^{p'}(I; L^{p'}(\Omega))$. Then, there exists a unique $\mathbf{u}$ solution of* (1.1) *such that*

$$\begin{aligned}
&\mathbf{u} \in W^{1,\infty}(I; L^2(\Omega)),\\
&\mathbf{S}(\mathbf{D}\mathbf{u}) \in L^{p'}(I; L^{p'}(\Omega)),\\
&\nabla \mathbf{F}(\mathbf{D}\mathbf{u}) \in L^p(I \times \Omega),\\
&\mathbf{F}(\mathbf{D}\mathbf{u}) \in W^{1,2}(I; L^2(\Omega))\\
&\pi \in L^{p'}(I \times \Omega).
\end{aligned}$$

In addition, we have the following interior estimates

$$\begin{aligned}
&\nabla \mathbf{u} \in L^\infty(I; L^2_{loc}(\Omega)),\\
&\mathbf{F}(\mathbf{D}\mathbf{u}) \in L^2(I; W^{1,2}_{loc}(\Omega)),\\
&\nabla \pi \in L^\infty(I; L^2_{loc}(\Omega)),
\end{aligned}$$

and the following estimates valid up to the boundary

$$\begin{aligned}
&\xi \partial_\tau \mathbf{u} \in L^\infty(I; L^2(\Omega)),\\
&\xi \partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u}) \in L^2(I; L^2(\Omega)),\\
&\xi \partial_\tau \pi \in L^\infty(I; L^2(\Omega)),
\end{aligned}$$

where the localization function and the notion of tangential derivative ∂_τ are defined in detail in Section 2.3.

2. PRELIMINARIES AND FUNCTION SPACES

Let us collect some preliminary notation and definitions, together with the proof of the regularity with respect to the time variable, which can be obtained directly by the energy method. These results will allow us to prove the regularity for the pressure needed in the Section 3 for the treatment of the regularity with respect to the spatial variables.

2.1. FUNCTION SPACES

We use c, C to denote generic constants, which may change from line to line, but are not depending on the crucial quantities. Moreover we write $f \sim g$ if and only if there exist constants $c, C > 0$ such that $c\, f \leq g \leq C\, f$.

We will use the customary Lebesgue spaces $(L^p(\Omega), \|\cdot\|_p)$ and Sobolev spaces $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$, $k \in \mathbb{N}$. We do not distinguish between scalar, vector-valued or tensor-valued function spaces in the notation, if there is no danger of confusion. However, we denote scalar functions by roman letters, vector-valued functions by small boldfaced letters, and tensor-valued functions by capital boldfaced letters. We equip $W_0^{1,p}(\Omega)$ (based on the Poincaré Lemma) with the gradient norm $\|\nabla\, .\, \|_p$. We denote by $|M|$ the n-dimensional Lebesgue measure of a measurable $M \subset \mathbb{R}^3$. Since we consider divergence-free solutions, we denote by $L^p_\sigma(\Omega) \subset L^p(\Omega)$ the closed subspace of divergence-free vector fields, tangential to the boundary, while $W^{1,p}_{0,\sigma}(\Omega) \subset W_0^{1,p}(\Omega)$ is the counterpart in $W_0^{1,p}(\Omega)$ and observe that if the domain is smooth these spaces coincide with the closure of smooth and compactly supported divergence-free functions with respect to the norm of $L^p(\Omega)$ and of $W^{1,p}(\Omega)$, respectively.

2.2. BASIC PROPERTIES OF THE ELLIPTIC OPERATOR

For a tensor $\mathbf{P} \in \mathbb{R}^{3\times 3}$ we denote its symmetric part by

$$\mathbf{P}^{\mathrm{sym}} := \frac{1}{2}(\mathbf{P} + \mathbf{P}^\top) \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} := \{\mathbf{A} \in \mathbb{R}^{3\times 3} \,|\, \mathbf{P} = \mathbf{P}^\top\}.$$

The scalar product between two tensors $\mathbf{P}, \mathbf{Q}$ is denoted by $\mathbf{P} \cdot \mathbf{Q}$, and we use the notation $|\mathbf{P}|^2 = \mathbf{P} \cdot \mathbf{P}$. We assume that the extra stress tensor $\mathbf{S}$ has (p, δ)-structure, which will be defined now. A detailed discussion and full proofs of the following results can be found in [8, 16]

Assumption 2.1. We assume that $\mathbf{S}\colon \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ belongs to $C^0(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \cap C^1(\mathbb{R}^{3\times 3} \setminus \{\mathbf{0}\}, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, satisfies $\mathbf{S}(\mathbf{P}) = \mathbf{S}\big(\mathbf{P}^{\mathrm{sym}}\big)$, and $\mathbf{S}(\mathbf{0}) = \mathbf{0}$. Moreover, we assume that $\mathbf{S}$ has (p, δ)-structure, i.e., there exist $p \in (1, \infty)$, $\delta \in [0, \infty)$, and constants $C_0, C_1 > 0$ such that

$$\sum\nolimits_{i,j,k,l=1}^{3} \partial_{kl} S_{ij}(\mathbf{P}) Q_{ij} Q_{kl} \geq C_0 \big(\delta + |\mathbf{P}^{\mathrm{sym}}|\big)^{p-2} |\mathbf{Q}^{\mathrm{sym}}|^2, \tag{2.1a}$$

$$\big|\partial_{kl} S_{ij}(\mathbf{P})\big| \leq C_1 \big(\delta + |\mathbf{P}^{\mathrm{sym}}|\big)^{p-2}, \tag{2.1b}$$

are satisfied for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}$ with $\mathbf{P}^{\mathrm{sym}} \neq \mathbf{0}$ and all $i, j, k, l = 1, \ldots, 3$. The constants C_0, C_1, and p are called the *characteristics* of $\mathbf{S}$.

Remark 2.2. We would like to emphasize that, if not otherwise stated, the constants in the paper depend only on the characteristics of $\mathbf{S}$, but are independent of $\delta \geq 0$.

Defining for $t \geq 0$ a special N-function φ by

$$\varphi(t) := \int_0^t \varphi'(s)\, ds \qquad \text{with} \quad \varphi'(t) := (\delta + t)^{p-2} t\,, \tag{2.2}$$

we can replace $C_i\big(\delta + |\mathbf{P}^{\mathrm{sym}}|\big)^{p-2}$ in the right-hand side of (2.1) by $\widetilde{C}_i\,\varphi''\big(|\mathbf{P}^{\mathrm{sym}}|\big)$, $i = 0, 1$. Next, the shifted functions are defined for $t \geq 0$ by

$$\varphi_a(t) := \int_0^t \varphi_a'(s)\,ds \qquad \text{with} \qquad \varphi_a'(t) := \varphi'(a+t)\frac{t}{a+t}.$$

In the following lemma we recall several useful results, which will be frequently used in the paper. The proofs of these results and more details can be found in [3, 8, 9, 16].

Proposition 2.3. *Let* $\mathbf{S}$ *satisfy Assumption 2.1, let* φ *be defined in* (2.2)*, and let* $\mathbf{F}$ *be defined in* (1.3)*.*

(i) *For all* $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}$

$$\begin{aligned}
\big(\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})\big) \cdot \big(\mathbf{P} - \mathbf{Q}\big) &\sim \big|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})\big|^2 \\
&\sim \varphi_{|\mathbf{P}^{\mathrm{sym}}|}(|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|) \\
&\sim \varphi''\big(|\mathbf{P}^{\mathrm{sym}}| + |\mathbf{Q}^{\mathrm{sym}}|\big)|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|^2, \\
\mathbf{S}(\mathbf{Q}) \cdot \mathbf{Q} &\sim |\mathbf{F}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}^{\mathrm{sym}}|), \\
|\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})| &\sim \varphi'_{|\mathbf{P}^{\mathrm{sym}}|}\big(|\mathbf{P}^{\mathrm{sym}} - \mathbf{Q}^{\mathrm{sym}}|\big).
\end{aligned}$$

The constants depend only on the characteristics of $\mathbf{S}$*.*

(ii) *For all* $\varepsilon > 0$*, there exist a constant* $c_\varepsilon > 0$ (*depending only on* $\varepsilon > 0$ *and on the characteristics of* $\mathbf{S}$) *such that for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W^{1,p}(\Omega)$

$$\big(\mathbf{S}(\mathbf{Du}) - \mathbf{S}(\mathbf{Dv}), \mathbf{Dw} - \mathbf{Dv}\big) \leq \varepsilon\, \|\mathbf{F}(\mathbf{Du}) - \mathbf{F}(\mathbf{Dv})\|_2^2 + c_\varepsilon\, \|\mathbf{F}(\mathbf{Dw}) - \mathbf{F}(\mathbf{Dv})\|_2^2\,,$$

and for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$*,* $t \geq 0$

$$\begin{aligned}
\varphi_{|\mathbf{Q}|}(t) &\leq c_\varepsilon\, \varphi_{|\mathbf{P}|}(t) + \varepsilon\, |\mathbf{F}(\mathbf{Q}) - \mathbf{F}(\mathbf{P})|^2, \\
(\varphi_{|\mathbf{Q}|})^*(t) &\leq c_\varepsilon\, (\varphi_{|\mathbf{P}|})^*(t) + \varepsilon\, |\mathbf{F}(\mathbf{Q}) - \mathbf{F}(\mathbf{P})|^2\,.
\end{aligned}$$

where the constants depend only on p*.*

2.3. DESCRIPTION AND PROPERTIES OF THE BOUNDARY

We assume that the boundary $\partial\Omega$ is of class $C^{2,1}$, that is for each point $P \in \partial\Omega$ there are local coordinates such that in these coordinates we have $P = 0$ and $\partial\Omega$ is locally described by a $C^{2,1}$-function, i.e., there exist R_P, $R_P' \in (0, \infty)$, $r_P \in (0, 1)$ and a $C^{2,1}$-function $a_P : B^2_{R_P}(0) \to B^1_{R_P'}(0)$ such that

(b1) $\mathbf{x} \in \partial\Omega \cap (B^2_{R_P}(0) \times B^1_{R_P'}(0)) \iff x_3 = a_P(x_1, x_2)$,

(b2) $\Omega_P := \{(x, x_3) \,\big|\, x = (x_1, x_2)^\top \in B^2_{R_P}(0),\ a_P(x) < x_3 < a_P(x) + R_P'\} \subset \Omega$,

(b3) $\nabla a_P(0) = \mathbf{0}$, and $\forall\, x = (x_1, x_2)^\top \in B^2_{R_P}(0) \quad |\nabla a_P(x)| < r_P$, where $B_r^k(0)$ denotes the k-dimensional open ball with center 0 and radius $r > 0$. Note that r_P can be made arbitrarily small if we make R_P small enough. In the sequel we will also use, for $0 < \lambda < 1$, the following scaled open sets, $\lambda\,\Omega_P \subset \Omega_P$ defined as follows

$$\lambda\,\Omega_P := \{(x, x_3)\,\big|\, x = (x_1, x_2)^\top \in B^2_{\lambda R_P}(0),\ a_P(x) < x_3 < a_P(x) + \lambda R'_P\}. \tag{2.3}$$

To localize near to $\partial\Omega \cap \partial\Omega_P$, for $P \in \partial\Omega$, we fix smooth functions $\xi_P : \mathbb{R}^3 \to \mathbb{R}$ such that

(ℓ1) $\chi_{\frac{1}{2}\Omega_P}(\mathbf{x}) \le \xi_P(\mathbf{x}) \le \chi_{\frac{3}{4}\Omega_P}(\mathbf{x})$,

where $\chi_A(\mathbf{x})$ is the indicator function of the measurable set A. For the remaining interior estimate we localize by a smooth function $0 \le \xi_{00} \le 1$ with $\operatorname{spt}\xi_{00} \subset \Omega_{00}$, where $\Omega_{00} \subset \Omega$ is an open set such that $\operatorname{dist}(\partial\Omega_{00}, \partial\Omega) > 0$. Since the boundary $\partial\Omega$ is compact, we can use an appropriate finite sub-covering which, together with the interior estimate, yields the global estimate.

In particular, in the interior we will use the well-known results linking difference quotients and derivatives. If $E \subset \mathbb{R}^n$, we denote

$$\begin{aligned} E \pm h\mathbf{e}^k &:= \left\{\mathbf{x} \in \mathbb{R}^n \,\middle|\, \exists\, \mathbf{y} \in E : \mathbf{x} = \mathbf{y} \pm h\mathbf{e}^k\right\}, \\ E_h &:= \left\{\mathbf{x} \in E \,\middle|\, \operatorname{dist}(\mathbf{x}, \partial E) > h\right\}. \end{aligned}$$

Let $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a measurable tensor field (or a vector field or a real-valued function) and $h > 0$. Then we define the *difference quotients of* $\mathbf{G}$ as follows:

$$d^{\pm}_{h,k}\mathbf{G}(\mathbf{x}) := \frac{\mathbf{G}(\mathbf{x} \pm h\mathbf{e}^k) - \mathbf{G}(\mathbf{x})}{h}, \qquad \mathbf{x} \in \mathbb{R}^n.$$

We will also use the notation $\Delta^{\pm}_{h,k}\mathbf{G}(\mathbf{x}) := h\, d^{\pm}_{h,k}\mathbf{G}(\mathbf{x})$. It is well-known (cf. [11, Sec. 5.8]) that for $\mathbf{G} \in W^{1,1}(\mathbb{R}^n)$ one has

$$\lim_{h\to 0+} d^{\pm}_{h,k}\mathbf{G}(\mathbf{x}) = \partial_k\mathbf{G}(\mathbf{x}) \qquad \text{for a.e. } \mathbf{x} \in \mathbb{R}^n,$$

and

$$\nabla d^{\pm}_{h,k}\mathbf{G}(\mathbf{x}) = d^{\pm}_{h,k}\nabla\mathbf{G}(\mathbf{x}) \qquad \text{for a.e. } \mathbf{x} \in \mathbb{R}^n.$$

Moreover, if $d^{\pm}_{h,k}\mathbf{G} \in L^p(E_{h_0})$ for all $h_0 > 0$ and for all $0 < h < h_0$ it holds

$$\int_{E_{h_0}} |d^{\pm}_{h,k}\mathbf{G}(\mathbf{x})|^p \, d\mathbf{x} \le c_1, \tag{2.4}$$

then $\partial_k\mathbf{G}$ exists in the sense of distributions and satisfies

$$\int_{E} |\partial_k\mathbf{G}(\mathbf{x})|^p \, d\mathbf{x} \le c_1. \tag{2.5}$$

Let us introduce now the tangential derivatives near the boundary. To simplify the notation we fix $P \in \partial\Omega$, $h \in (0, \frac{R_P}{16})$, and simply write $\xi := \xi_P$, $a := a_P$. We use the standard notation $\mathbf{x} = (x', x_3)^\top$ and denote by $\mathbf{e}^i, i = 1, 2, 3$ the canonical orthonormal basis in $\mathbb{R}^3$. In the following lower-case Greek letters take values 1, 2. For a function g with $\operatorname{spt} g \subset \operatorname{spt} \xi$ we define for $\alpha = 1, 2$

$$g_\tau(x', x_3) = g_{\tau_\alpha}(x', x_3) := g\big(x' + h\,\mathbf{e}^\alpha, x_3 + a(x' + h\,\mathbf{e}^\alpha) - a(x')\big)\,,$$

and if $\Delta^+ g := g_\tau - g$, we define tangential divided differences by $d^+ g := h^{-1}\Delta^+ g$. It holds that, if $g \in W^{1,1}(\Omega)$, then we have for $\alpha = 1, 2$

$$d^+ g \to \partial_\tau g = \partial_{\tau_\alpha} g := \partial_\alpha g + \partial_\alpha a\, \partial_3 g \qquad \text{as } h \to 0, \tag{2.6}$$

almost everywhere in $\operatorname{spt}\xi$, (cf. [15, Sec. 3]). In addition, uniform L^q-bounds for $d^+ g$ imply that $\partial_\tau g$ belongs to $L^q(\operatorname{spt}\xi)$. More precisely, if we define, for $0 < h < R_P$

$$\Omega_{P,h} := \left\{\mathbf{x} \in \Omega_P : x \in B^2_{R_P - h}(0)\right\},$$

and if $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^3)$, then

$$\int\limits_{\Omega_{P,h}} |d^+ f|^q\, d\mathbf{x} \le c \int\limits_{\Omega_P} |\partial_\tau f|^q\, d\mathbf{x}.$$

Moreover, if $d^+ f \in L^q(\Omega_{P,h_0})$, for all $0 < h_0 < R_P$ and if it holds

$$\exists\, c_1 > 0 : \quad \int\limits_{\Omega_{P,h_0}} |d^+ f|^q\, d\mathbf{x} \le c_1 \qquad \forall\, h_0 \in (0, R_P) \text{ and } \forall\, h \in (0, h_0), \tag{2.7}$$

then $\partial_\tau f \in L^q(\Omega_P)$ and

$$\int\limits_{\Omega_{P,}} |\partial_\tau f|^q\, d\mathbf{x} \le c_1. \tag{2.8}$$

We recall some auxiliary lemmas related to these objects, see [5]. For simplicity we denote $\nabla a := (\partial_1 a, \partial_2 a, 0)^\top$ and use the operations $(\cdot)_\tau$, $(\cdot)_{-\tau}$, $\Delta^+(\cdot)$, $\Delta^+(\cdot)$, $d^+(\cdot)$ and $d^-(\cdot)$ also for vector-valued and tensor-valued functions, intended as acting component-wise.

Lemma 2.4. *Let* $\mathbf{v} \in W^{1,1}(\Omega)$ *such that* $\operatorname{spt}\mathbf{v} \subset \operatorname{spt}\xi$. *Then, for small enough* $h > 0$

$$\begin{aligned}
\nabla d^\pm \mathbf{v} &= d^\pm \nabla \mathbf{v} + (\partial_3 \mathbf{v})_\tau \otimes d^\pm \nabla a,\\
\mathbf{D} d^\pm \mathbf{v} &= d^\pm \mathbf{D}\mathbf{v} + (\partial_3 \mathbf{v})_\tau \overset{s}{\otimes} d^\pm \nabla a,\\
\operatorname{div} d^\pm \mathbf{v} &= d^\pm \operatorname{div}\mathbf{v} + (\partial_3 \mathbf{v})_{\pm\tau} d^\pm \nabla a,\\
\nabla \mathbf{v}_{\pm\tau} &= (\nabla \mathbf{v})_{\pm\tau} + (\partial_3 \mathbf{v})_{\pm\tau} d^\pm \nabla a,
\end{aligned}$$

where $\overset{s}{\otimes}$ *is the symmetric tensor product.*

The following variant of integration per parts will be often used.

Lemma 2.5. *Let* $\operatorname{spt} g \cup \operatorname{spt} f \subset \operatorname{spt} \xi$ *and* h *small enough. Then*

$$\int_\Omega f g_{-\tau}\, d\mathbf{x} = \int_\Omega f_\tau g\, d\mathbf{x}.$$

Consequently, $\int_\Omega f d^+ g\, d\mathbf{x} = \int_\Omega (d^- f) g\, d\mathbf{x}$. *Moreover, if in addition* f *and* g *are smooth enough and at least one vanishes on* $\partial\Omega$, *then*

$$\int_\Omega f \partial_\tau g\, d\mathbf{x} = -\int_\Omega (\partial_\tau f) g\, d\mathbf{x}.$$

2.4. EXISTENCE OF WEAK SOLUTIONS

The existence of weak solutions to the boundary value problem (1.1) is easily proved by simplifying (thanks the lack of the convective term) the approach in [10]. The following theorem holds true.

Theorem 2.6. *Let* $p \geq 6/5$ *and let be given* $\mathbf{u}_0 \in L^2_\sigma(\Omega)$ *and* $\mathbf{f} = \operatorname{div} \boldsymbol{\mathcal{F}}$ *with* $\boldsymbol{\mathcal{F}} \in L^{p'}(I \times \Omega)$. *Then, there exists a unique weak solution*

$$\mathbf{u} \in L^\infty(I; L^2_\sigma(\Omega)) \cap L^p(I; W^{1,p}_{0,\sigma}(\Omega)),$$

such that

$$\frac{d}{dt}\int_\Omega \mathbf{u} \cdot \boldsymbol{\varphi}\, d\mathbf{x} + \int_\Omega \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}\boldsymbol{\varphi}\, d\mathbf{x} = -\int_\Omega \boldsymbol{\mathcal{F}} \cdot \nabla \boldsymbol{\varphi}\, d\mathbf{x} \qquad \forall\, \boldsymbol{\varphi} \in W^{1,p}_{0,\sigma}(\Omega). \tag{2.9}$$

Proof. The result can be obtained by a Galerkin approximation and an appropriate limit on the approximate solutions. Since the argument is rather standard we just write the a priori estimates. Consider the Galerkin approximation $\mathbf{u}^n : [0,T] \to V_n$ (with $V_n \subseteq L^p_\sigma(\Omega)$, such that dim $V_n = n$) which satisfies the system of ordinary differential equations

$$\frac{d}{dt}\int_\Omega \mathbf{u}^n \cdot \boldsymbol{\varphi}^n\, d\mathbf{x} + \int_\Omega \mathbf{S}(\mathbf{D}\mathbf{u}^n) \cdot \mathbf{D}\boldsymbol{\varphi}^n\, d\mathbf{x} = -\int_\Omega \boldsymbol{\mathcal{F}} \cdot \nabla \boldsymbol{\varphi}^n\, d\mathbf{x} \qquad \forall\, \boldsymbol{\varphi}^n \in V_n. \tag{2.10}$$

Testing with $\mathbf{u}^n \in V_n$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^n\|_2^2 + \int_\Omega \varphi(|\mathbf{D}\mathbf{u}^n|)\, d\mathbf{x} \leq C \|\boldsymbol{\mathcal{F}}\|_{p'}^{p'}.$$

This implies that, if $\mathbf{u}_0 \in L^2(\Omega)$, then

$$\mathbf{u}^n \in L^\infty(I; L^2(\Omega)) \quad \text{and} \quad \varphi(|\mathbf{D}\mathbf{u}^n|) \in L^1(I \times \Omega).$$

Then, Korn's inequality and the definition of $\mathbf{F}$ imply also that

$$\nabla \mathbf{u}^n \in L^{p'}(I \times \Omega) \quad \text{and} \quad \mathbf{F}(\mathbf{D}\mathbf{u}^n) \in L^2(I \times \Omega),$$

and all estimates are with bounds independent of $n \in \mathbb{N}$. The estimates are then inherited by the limit. By comparison, we also get the following information on the time derivative of the weak solution

$$\frac{\partial \mathbf{u}}{\partial t} \in L^{p'}(I; (W^{1,p}_{0,\sigma}(\Omega))^*).$$

In particular this implies that one can take the difference of two solutions starting from the same data, and use the difference as test function to show that they coincide, due to the assumption (2.1a) on the stress tensor and using Gronwall's lemma. □

The above result concerns only the velocity $\mathbf{u}$, but it is possible to reconstruct a pressure. The pressure can be introduced exactly as in Wolf [17, Thm 2.6] to show the following result.

Theorem 2.7. *Let* $\mathbf{u}$ *be a weak solution to* (1.1). *Then, since* $\mathbf{S}(\mathbf{D}\mathbf{u}) \in L^{p'}(I; L^{p'}(\Omega))$ *and the solution is at least such that* $\mathbf{u} \in C_w(I; L^2(\Omega))$, *there exist unique* (*if the zero mean value is imposed*) *scalar functions* $p_0, \widetilde{p}_h$ *with*

$$\begin{aligned} p_0 &\in L^{p'}(I; A^{p'}) \subset L^{p'}(I; L^{p'}(\Omega)), \\ \widetilde{p}_h &\in C_w(I; B^{p'}) \subset C_w(I; L^{p'}(\Omega)), \end{aligned}$$

where $A^{p'}$ *is the closure in* $L^{p'}(\Omega)$ *of* $\Delta\varphi$ *for* $\varphi \in C_0^\infty(\Omega)$, *while* $B^{p'}$ *is the subspace of* $L^{p'}(\Omega)$ *made of harmonic functions, such that for all* $\boldsymbol{\varphi} \in C_0^\infty([0,T) \times \Omega)$ *holds*

$$\begin{aligned} -\int_0^T \int_\Omega \mathbf{u} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}\boldsymbol{\varphi}\, d\mathbf{x}dt &= \int_0^T \int_\Omega p_0 \operatorname{div} \boldsymbol{\varphi} - \widetilde{p}_h \frac{\partial \operatorname{div} \boldsymbol{\varphi}}{\partial t}\, d\mathbf{x}dt \\ &\quad + \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0)\, d\mathbf{x} - \int_0^T \int_\Omega \boldsymbol{\mathcal{F}} \cdot \nabla \boldsymbol{\psi}\, d\mathbf{x}dt. \end{aligned}$$

With the result from Theorem 2.7 we have already identified a pressure field, as sum of a "regular" part and one which is represented by a time derivative. We will show later that the pressure is indeed more regular, especially the part with $\widetilde{p}_h$. This can be obtained, with a similar argument, once we have a better knowledge of the time derivative (cf. (2.12)). The relevant fact is that the regularity of the pressure (at least that π is in $L^{p'}(I \times \Omega)$) is needed to infer the regularity of the second-order space-derivatives.

2.5. EXISTENCE OF TIME REGULAR SOLUTIONS

We first prove an existence result for time regular solutions. For such solutions the time derivative belongs to some Lebesgue space and it is not just a distribution, and thus the solution satisfies the equations in the sense explained in (2.11).

Theorem 2.8. *Let be given* $\mathbf{u}_0 \in W^{1,2}_{0,\sigma}(\Omega)$ *such that* $\operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{u}_0) \in L^2(\Omega)$. *Let* $p \geq \frac{6}{5}$ *and assume* $\delta \in [0,\delta_0]$ (*in this theorem also the degenerate case can be considered*). *Let also* $\mathbf{f} \in W^{1,2}(I;L^2(\Omega))$. *Then, there exists a unique solution* $\mathbf{u}$ *of* (1.1) *such that*

$$\begin{aligned}&\mathbf{u} \in W^{1,\infty}(I;L^2(\Omega)),\\ &\mathbf{F}(\mathbf{D}\mathbf{u}) \in W^{1,2}(I;L^2(\Omega)),\end{aligned}$$

and for all $\boldsymbol{\varphi} \in L^p(I;W^{1,p}_{0,\sigma}(\Omega))$ *it holds*

$$\int_0^T\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot\boldsymbol{\varphi}\,d\mathbf{x}dt + \int_0^T\int_\Omega \mathbf{S}(\mathbf{D}\mathbf{u})\cdot\mathbf{D}\boldsymbol{\varphi}\,d\mathbf{x}dt = \int_0^T\int_\Omega \mathbf{f}\cdot\boldsymbol{\varphi}\,d\mathbf{x}dt. \tag{2.11}$$

By interpolation $\mathbf{u}$, $\mathbf{F}(\mathbf{D}\mathbf{u}) \in C(I;L^2(\Omega))$, *hence initial datum is attained strongly in* $L^2(\Omega)$

Proof. Following the same argument used to prove existence of weak solutions, we reason on the Galerkin approximations. We differentiate the approximate system (2.10) with respect to time and multiply by $\frac{\partial \mathbf{u}^n}{\partial t}$ to get, thanks to Assumption (2.1a),

$$\frac{d}{dt}\Big\|\frac{\partial \mathbf{u}^n}{\partial t}\Big\|_2^2 + \Big\|\frac{\partial \mathbf{F}(\mathbf{D}\mathbf{u}^n)}{\partial t}\Big\|_2^2 \leq c\left(\Big\|\frac{\partial \mathbf{f}}{\partial t}\Big\|_2^2 + \Big\|\frac{\partial \mathbf{u}^n}{\partial t}\Big\|_2^2\right),$$

hence it follows that if $\operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{u}_0) \in L^2(\Omega)$ then, uniformly w.r.t. $n \in \mathbb{N}$,

$$\frac{\partial \mathbf{u}^n}{\partial t} \in L^\infty(I;L^2(\Omega)) \quad \text{and} \quad \frac{\partial \mathbf{F}(\mathbf{D}\mathbf{u}^n)}{\partial t} \in L^2(I;L^2(\Omega)). \tag{2.12}$$

Passing to the limit as $n \to +\infty$, it follows that there exists a unique solution $\mathbf{u}$, which inherits the regularity of the approximations and thus is a time regular solution (clearly it is also a weak solution). □

Remark 2.9. In the case in which the stress tensor is derived from a potential Φ one can also test with $\frac{\partial \mathbf{u}^n}{\partial t}$ to get (as intermediate step)

$$\Big\|\frac{\partial \mathbf{u}^n}{\partial t}\Big\|_2^2 + \frac{d}{dt}\Phi(\mathbf{u}^n) \leq c\|\mathbf{f}\|_2^2,$$

hence it follows that if $\nabla\mathbf{u}_0 \in L^p(\Omega)$ then, uniformly w.r.t. $n \in \mathbb{N}$,

$$\frac{\partial \mathbf{u}^n}{\partial t} \in L^2(I;L^2(\Omega)), \quad \nabla\mathbf{u}^n \in L^\infty(I;L^p(\Omega)), \quad \mathbf{F} \in L^\infty(I;L^2(\Omega)).$$

Next, by using the improved regularity of the time regular solution, we can deduce some further regularity of the pressure. To this end we use the regularity of the steady Stokes system.

Theorem 2.10. *Under the assumptions of Theorem 2.8, and if in addition* $\mathbf{f} \in L^{p'}(I\times\Omega)$*, there exists a unique (if zero mean value is assumed) pressure* $\pi \in L^{p'}(I\times\Omega)$*, such that for all* $\boldsymbol{\varphi} \in L^p(I; W_0^{1,p}(\Omega))$

$$\int_0^T\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot\boldsymbol{\varphi} + \mathbf{S}(\mathbf{D}\mathbf{u})\cdot\mathbf{D}\boldsymbol{\varphi}\,d\mathbf{x}dt = \int_0^T\int_\Omega \pi\,\mathrm{div}\,\boldsymbol{\varphi} + \mathbf{f}\cdot\boldsymbol{\varphi}\,d\mathbf{x}dt. \tag{2.13}$$

Proof. As in [17] one can show that there exists $\widetilde{p}$ such that

$$\widetilde{p} \in C_w(I; L^{p'}(\Omega)),$$

and satisfying, for all $\boldsymbol{\psi} \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}\int_\Omega (\mathbf{u}(t)-\mathbf{u}_0)\cdot\boldsymbol{\psi} + \Big(\int_0^t \mathbf{S}(\mathbf{D}\mathbf{u}(\tau))\,d\tau\Big)\cdot\mathbf{D}\boldsymbol{\psi}\,d\mathbf{x} \\ = \int_0^t\int_\Omega \widetilde{p}\,\mathrm{div}\,\boldsymbol{\psi}\,d\mathbf{x}d\tau - \int_0^t\int_\Omega \boldsymbol{\mathcal{F}} : \nabla\boldsymbol{\psi}\,d\mathbf{x}d\tau.\end{aligned} \tag{2.14}$$

Now we proceed as in [10, Thm. 2.2] and consider the steady Stokes problem

$$\begin{aligned} -\Delta\mathbf{V}_1(t) + \nabla\pi_1(t) &= \int_0^t \mathbf{f}(\tau)\,d\tau - \mathrm{div}\int_0^t \mathbf{S}(\mathbf{D}\mathbf{u}(\tau))\,d\tau && \text{in } \Omega,\\ \mathrm{div}\,\mathbf{V}_1(t) &= 0 && \text{in } \Omega,\\ \mathbf{V}_1(t) &= \mathbf{0} && \text{on } \partial\Omega,\end{aligned}$$

where $t \in I$ is treated as a parameter.

Standard L^q-results of regularity for the steady linear Stokes problem (see for instance Galdi [12]) imply that there exists a unique strong solution $(\mathbf{V}_1(t), \pi_1(t))_{t\in I}$, with $\pi_1(t)$ such that $\int_\Omega \pi_1(t)\,d\mathbf{x} = 0$ for all $t \in I$ and moreover

$$\pi_1 \in W^{1,p'}(I; L^{p'}(\Omega)).$$

Next, let $\mathbf{V}_2(t) \in W^{2,2}(\Omega)\cap W^{1,2}_{0,\sigma}(\Omega)$ and $\pi_2(t) \in W^{1,2}(\Omega)\cap L^2_0(\Omega)$ be the unique strong solution of

$$\begin{aligned} -\Delta\mathbf{V}_2(t) + \nabla\pi_2(t) &= -\mathbf{u}(t) + \mathbf{u}_0 && \text{in } \Omega,\\ \mathrm{div}\,\mathbf{V}_2(t) &= 0 && \text{in } \Omega,\\ \mathbf{V}_2(t) &= \mathbf{0} && \text{on } \partial\Omega,\end{aligned}$$

where $t \in I$ is treated again as a parameter.

Using the standard L^2-regularity results for the Stokes problem it follows that

$$\|\pi_2(t)\|_{1,2} \le \|\mathbf{u}(t)-\mathbf{u}_0\|_{L^2} \qquad t \in I.$$

Then, by considering time-increments $t+h$, with $h>0$ small enough such that $t+h<T$, we can consider that same system with t replaced by $t+h$. Taking the difference between the two systems we get

$$\begin{aligned} -\Delta(\mathbf{V}_2(t+h)-\mathbf{V}_2(t))+\nabla(\pi_2(t+h)-\pi_2(t)) &= -\mathbf{u}(t+h)+\mathbf{u}(t) && \text{in } \Omega,\\ \operatorname{div}(\mathbf{V}_2(t+h)-\mathbf{V}_2(t)) &= 0 && \text{in } \Omega,\\ \mathbf{V}_2(t+h)-\mathbf{V}_2(t) &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned}$$

It follows that, after division by $h>0$,

$$\Big\|\frac{\pi_2(t+h)-\pi_2(t)}{h}\Big\|_{1,2} \le \Big\|\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h}\Big\|_2 \le \Big\|\frac{\partial \mathbf{u}(t)}{\partial t}\Big\|_2 \qquad \forall\, t\in I,$$

hence, by using the argument as in (2.4)–(2.5) but applied to finite differences with respect to the time variable, we have that $\nabla\pi_2\in W^{1,\infty}(I;L^2)$. Observe that this is implied by the fact that $\mathbf{u}$ is a solution such that the time derivative $\frac{\partial \mathbf{u}}{\partial t}$ belongs to $L^\infty(I;L^2(\Omega))$. The latter estimates on the pressure π_2 implies, by the usual Sobolev embedding valid in three-space dimensions $W^{1,2}(\Omega)\subset L^6(\Omega)$, that

$$\Big\|\frac{\pi_2(t+h)-\pi_2(t)}{h}\Big\|_6 \le \Big\|\frac{\partial \mathbf{u}(t)}{\partial t}\Big\|_2 \le \Big\|\frac{\partial \mathbf{u}}{\partial t}\Big\|_{L^\infty(I;L^2)} \qquad \forall\, t\in I,$$

hence, by using (2.4)-(2.5), that

$$\pi_2\in W^{1,\infty}(I;L^6(\Omega))\subset W^{1,p'}(I;L^{p'}(\Omega)), \qquad \text{since } p\ge\frac{6}{5}.$$

Finally, observe that

$$\begin{aligned} -\Delta(\mathbf{V}_1(t)+\mathbf{V}_2(t))+\nabla(\pi_1(t)+\pi_2(t)-\widetilde{p}) &= 0 && \text{in } \Omega,\\ \operatorname{div}(\mathbf{V}_1(t)+\mathbf{V}_2(t)) &= 0 && \text{in } \Omega,\\ \mathbf{V}_1(t)+\mathbf{V}_2(t) &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned}$$

which implies that $\widetilde{p}=\pi_1(t)+\pi_2(t)$, hence plugging into the system (2.14) $\boldsymbol{\psi}(t)=\partial_t\boldsymbol{\varphi}(t)$ and integrating by parts in time as in [10], we find that $\mathbf{u}$ and

$$\pi := \frac{\partial \pi_1}{\partial t}+\frac{\partial \pi_2}{\partial t}\in L^{p'}(I\times\Omega)$$

satisfy (2.13). □

3. PROOF OF THE MAIN RESULT

The proof of Theorem 1.1 essentially consists of a proper localization and the usage of difference quotients, which yields the stated regularity, if the constants of the various inequalities are uniform in the increment. This applies to tangential derivatives, by using (2.7)–(2.8), and all derivatives in the interior situation. Next, the usage of the equations point-wise allows us to prove also the regularity in the normal direction.

Our approach, which is an adaption to the unsteady problem of that one in [5], requires a proper characterization of the boundary. Results are a technical improvement of those in [6] for the parabolic system (1.2), due to divergence-free constraint.

We first prove a result concerning the regularity of tangential spatial derivatives near the boundary and the interior regularity.

Proposition 3.1. *Let the tensor field* $\mathbf{S}$ *in* (1.1) *have* (p,δ)*-structure for some* $p \in (1,2]$*, and* $\delta \in (0,\infty)$*, and let* $\mathbf{F}$ *be the associated tensor field to* $\mathbf{S}$*. Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with* $C^{2,1}$ *boundary and let* $\mathbf{u}_0 \in W^{1,2}_{0,\sigma}(\Omega)$ *and* $\mathbf{f} \in L^{p'}(I \times \Omega)$*. Then, the unique time-regular solution* $\mathbf{u}$ *of the problem* (1.1) *satisfies for all* $t \in I$

$$
\begin{aligned}
&\|\xi_0^2 \nabla \mathbf{u}(t)\|_2^2 + \int_0^t \int_\Omega \xi_0^2 |\nabla \mathbf{F}(\mathbf{Du})|^2 + \xi_0^2 |\nabla \pi|^2 \, d\mathbf{x}ds \\
&\le c(\|\mathbf{u}_0\|_{1,2}, \|\operatorname{div} \mathbf{S}(\mathbf{Du}_0)\|_2, \|\mathbf{f}\|_{L^{p'}(I\times\Omega)}, \|\xi_0\|_{2,\infty}, \delta^{-1}) \,, \\
&\|\xi_P^2 \partial_\tau \mathbf{u}(t)\|_2^2 + \int_0^t \int_\Omega \xi_P^2 |\partial_\tau \mathbf{F}(\mathbf{Du})|^2 \, d\mathbf{x}ds \\
&\le c(\|\mathbf{u}_0\|_{1,2}, \|\operatorname{div} \mathbf{S}(\mathbf{Du}_0)\|_2, \|\mathbf{f}\|_{L^{p'}(I\times\Omega)}, \|\xi_P\|_{2,\infty}, \|a_P\|_{C^{2,1}}, \delta) \,.
\end{aligned}
\tag{3.1}
$$

provided that in the local description of the boundary there holds $r_P < C_1$ *in* $(b3)$*, where* $\xi_P(\mathbf{x})$ *is a cut-off function with support in* Ω_P *and for arbitrary* $P \in \partial\Omega$ *the tangential derivative is defined locally in* Ω_P *by* (2.6)*. In addition* $\xi_0(\mathbf{x})$ *is a cut-off function with support in the interior of* Ω*.*

Moreover, the pressure satisfies

$$
\begin{aligned}
&\int_0^T \int_\Omega \xi_{00}^2 |\nabla \pi|^2 \, d\mathbf{x}ds \\
&\le c(\delta^{p-2}, \|\xi_{00}\|_{2,\infty}) \Big(\|\mathbf{u}_0\|_{1,2}^2 + \|\operatorname{div} \mathbf{S}(\mathbf{Du}_0)\|_2^2 + \int_0^T \|\mathbf{f}\|_{p'}^{p'} ds \Big) \,, \\
&\int_0^T \int_\Omega \xi_P^2 |\partial_\tau \pi|^2 \, d\mathbf{x}ds \\
&\le c(\delta^{p-2}, \|\xi_P\|_{2,\infty}, \|a_P\|_{C^{2,1}}) \Big(\|\operatorname{div} \mathbf{S}(\mathbf{Du}_0)\|_2^2 + \|\mathbf{u}_0\|_{1,2}^2 + \int_0^T \|\mathbf{f}(s)\|_{p'}^{p'} \, ds \Big),
\end{aligned}
\tag{3.2}
$$

Remark 3.2. We warn the reader that $c(\delta)$ only indicates that the constant c depends on δ and it is such that $c(\delta) \le c(\delta_0)$ for all $\delta \le \delta_0$.

Remark 3.3. In this section special care has to be taken to distinguish the partial time-derivative $\frac{\partial \mathbf{u}}{\partial t}$, from the tangential space-derivative $\partial_\tau \mathbf{u}$.

Proof of Proposition 3.1. As usual in the study of boundary regularity we need to localize and to use appropriate test functions. Consequently, let us fix $P \in \partial\Omega$ and in Ω_P use $\xi := \xi_P$, $a := a_P$, while $h \in (0, \frac{R_P}{16})$, as in Section 2.3. Since we deal with time regular solutions, we can multiply (1.1) by $\mathbf{v}$

$$\mathbf{v} = d^-(\xi\, \boldsymbol{\psi}),$$

(more precisely ξ is extended by zero for $\mathbf{x} \in \Omega \backslash \Omega_P$, in order to have a global function over Ω) with $\boldsymbol{\psi} \in W_0^{1,2}(\Omega)$ and integrate by parts over Ω. We get, with the help of Lemma 2.4 and Lemma 2.5, the following equality for a.e $t \in I$

$$\begin{aligned}
&\int_\Omega d^+ \frac{\partial \mathbf{u}}{\partial t} \cdot (\xi\, \boldsymbol{\psi}) + d^+ \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}(\xi\, \boldsymbol{\psi}) + \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \big((\partial_3(\xi\, \boldsymbol{\psi}))_{-\tau} \overset{s}{\otimes} d^- \nabla a \big) \\
&- \int_\Omega \pi \operatorname{div} d^-(\xi\, \boldsymbol{\psi})\, d\mathbf{x} \\
&= \int_\Omega \mathbf{f} \cdot d^-(\xi\, \boldsymbol{\psi})\, d\mathbf{x}\,.
\end{aligned} \tag{3.3}$$

Due to the fact that $\mathbf{u} \in W^{1,2}_{0,\operatorname{div}}(\Omega)$ we can set

$$\boldsymbol{\psi} = \xi\, d^+(\mathbf{u}_{|\widetilde{\Omega}_P})$$

in Ω_P (and zero outside), hence as a test function we use the following vector field

$$\mathbf{v} = d^-(\xi^2 d^+(\mathbf{u}_{|\widetilde{\Omega}_P})),$$

where

$$\widetilde{\Omega}_P := \frac{1}{2}\Omega_P,$$

for the definition recall (2.3). Since $\boldsymbol{\psi}$ has zero trace on $\partial\Omega_P$, we get that (for small enough $h > 0$) the vector $\mathbf{v}(t)$ belongs to $W_0^{1,p}(\Omega_P)$, for a.e. $t \in I$. Using Lemma 2.4 and Lemma 2.5 we thus get the following identity

$$\begin{aligned}
&\int_\Omega \xi^2 d^+ \frac{\partial \mathbf{u}}{\partial t} \cdot d^+\mathbf{u} + \xi^2 d^+ \mathbf{S}(\mathbf{Du}) \cdot d^+ \mathbf{Du}\, d\mathbf{x} \\
&= -\int_\Omega \mathbf{S}(\mathbf{Du}) \cdot \big(\xi^2 d^+ \partial_3 \mathbf{u} - (\xi_{-\tau} d^- \xi + \xi d^- \xi)\partial_3 \mathbf{u}\big) \overset{s}{\otimes} d^- \nabla a \, d\mathbf{x} \\
&\quad - \int_\Omega \mathbf{S}(\mathbf{Du}) \cdot \xi^2 (\partial_3 \mathbf{u})_\tau \overset{s}{\otimes} d^- d^+ \nabla a - \mathbf{S}(\mathbf{Du}) \cdot d^- \big(2\xi \nabla \xi \overset{s}{\otimes} d^+ \mathbf{u}\big)\, d\mathbf{x} \\
&\quad + \int_\Omega \mathbf{S}((\mathbf{Du})_\tau) \cdot \big(2\xi \partial_3 \xi d^+ \mathbf{u} + \xi^2 d^+ \partial_3 \mathbf{u}\big) \overset{s}{\otimes} d^+ \nabla a \, d\mathbf{x} \\
&\quad - \int_\Omega \pi \big(\xi^2 d^- d^+ \nabla a - (\xi_{-\tau} d^- \xi + \xi d^- \xi) d^- \nabla a\big) \cdot \partial_3 \mathbf{u}\, d\mathbf{x} \\
&\quad - \int_\Omega \pi \big(d^-(2\xi \nabla \xi \cdot d^+ \mathbf{u}) - \xi^2 d^+ \partial_3 \mathbf{u} \cdot d^+ \nabla a\big)\, d\mathbf{x} \\
&\quad + \int_\Omega \pi_\tau \big(2\xi \partial_3 \xi d^+ \mathbf{u} + \xi^2 d^+ \partial_3 \mathbf{u}\big) \cdot d^+ \nabla a \, d\mathbf{x} \\
&\quad + \int_\Omega \mathbf{f} \cdot d^-(\xi^2 d^+ \mathbf{u})\, d\mathbf{x} =: \sum_{j=1}^{15} I_j \,.
\end{aligned} \tag{3.4}$$

The integrals I_j with $j = 1, \ldots, 7$ can be estimated exactly as in [5, (4.8)–(4.13)] by using the growth properties (2.1b) of the stress tensor $\mathbf{S}$ and it follows that

$$\begin{aligned}
\sum_{j=1}^{7} |I_j| \le\,& c\,(\varepsilon^{-1}, \|a\|_{C^{2,1}}, \|\xi\|_{2,\infty}) \int_{\Omega \cap \operatorname{spt} \xi} \varphi(|\mathbf{Du}|) + \varphi(|\nabla \mathbf{u}|)\, d\mathbf{x} \\
&+ 4\varepsilon \, \|\xi\|_{1,\infty} \int_\Omega \varphi(\xi |d^+ \nabla \mathbf{u}|) + \varphi(\xi |\nabla d^+ \mathbf{u}|)\, d\mathbf{x}.
\end{aligned}$$

The integrals involving the pressure I_j, with $j = 8, \ldots, 14$ can be estimated as in [5, (4.14)–(4.19)] and it follows that

$$\begin{aligned}
\sum_{j=8}^{14} |I_j| \le\,& c\,(\varepsilon^{-1}, \|a\|_{C^{2,1}}, \|\xi\|_{2,\infty}) \int_{\Omega \cap \operatorname{spt} \xi} \varphi^*(|\pi|) + \varphi(|\nabla \mathbf{u}|)\, d\mathbf{x} \\
&+ 4\varepsilon (1 + \|\xi\|_{1,\infty}) \int_\Omega \varphi(\xi |d^+ \nabla \mathbf{u}|) + \varphi(\xi |\nabla d^+ \mathbf{u}|)\, d\mathbf{x},
\end{aligned}$$

while

$$\big|I_{15}\big| \le c(\varepsilon^{-1}, \|\xi\|_{1,\infty}) \int\limits_{\Omega\cap\operatorname{spt}\xi} \varphi^*(|\mathbf{f}|) + \varphi(|\nabla\mathbf{u}|)\, d\mathbf{x} + \varepsilon \int\limits_{\Omega\cap\operatorname{spt}\xi} \varphi(\xi|\nabla d^+\mathbf{u}|)\, d\mathbf{x}\,.$$

Then, we use the inequality

$$\int\limits_{\Omega} \varphi\big(\xi|\nabla d^+\mathbf{u}|\big) + \varphi\big(\xi|d^+\nabla\mathbf{u}|\big)\, d\mathbf{x} \le c \int\limits_{\Omega} \xi^2 \big|d^+\mathbf{F}(\mathbf{Du})\big|^2\, d\mathbf{x}$$
$$+ c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int\limits_{\Omega\cap\operatorname{spt}\xi} \varphi\big(|\nabla\mathbf{u}|\big)\, d\mathbf{x}\,,$$

proved in [5, Lemma 3.11]. It follows from (3.4), by collecting the estimates for $\mathcal{I}_j$, $j = 1, \dots, 15$ and finally by choosing $\varepsilon > 0$ small enough (in order to absorb terms in the left-hand side) that

$$\frac{d}{dt}\,\frac{1}{2} \int\limits_{\Omega} \xi^2 |d^+\mathbf{u}(t)|^2 + \int\limits_{\Omega} \xi^2 \big|d^+\mathbf{F}(\mathbf{Du}(t))\big|^2 + \varphi(\xi|d^+\nabla\mathbf{u}(t)|) + \varphi(\xi|\nabla d^+\mathbf{u}(t)|)\, d\mathbf{x}$$
$$\le c\,(\varepsilon^{-1}, \|a\|_{C^{2,1}}, \|\xi\|_{2,\infty}) \int\limits_{\Omega\cap\operatorname{spt}\xi} \varphi^*(|\mathbf{f}|) + \varphi(|\nabla\mathbf{u}|) + \varphi^*(|\pi|)\, d\mathbf{x}\,,$$

for a.e. $t \in I$, where we also used also that $d^+\frac{\partial\mathbf{u}}{\partial t} = \frac{\partial d^+\mathbf{u}}{\partial t}$.

Integration over $[0, t]$, the a priori estimates from Theorem 2.8, and the result on the summability of π proved in Theorem 2.10 finally show that for $t \in I$

$$\frac{1}{2} \int\limits_{\Omega} \xi^2 |d^+\mathbf{u}(t)|^2\, d\mathbf{x} + \int\limits_0^t \int\limits_{\Omega} \xi^2 \big|d^+\mathbf{F}(\mathbf{Du}(s))\big|^2 + \varphi(\xi|d^+\nabla\mathbf{u}(s)|) + \varphi(\xi|\nabla d^+\mathbf{u}(s)|)\, d\mathbf{x}ds$$
$$\le \frac{1}{2}\|\mathbf{u}_0\|_{1,2}^2 + c(\|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}, \delta)\Big(\|\operatorname{div}\mathbf{S}(\mathbf{Du}_0)\|_2^2 + \int\limits_0^T \|\mathbf{f}(s)\|_{p'}^{p'}\, ds\Big),$$

from which $(3.1)_2$ follows by standard arguments, since the estimate is independent of $h > 0$.

To prove estimate (3.2) for $\partial_\tau\pi$, we adapt the approach in [5, Sec. 3] and start with the following inequality obtained adding and subtracting the mean value

$$\int\limits_{\Omega} \xi^2 |d^+\pi|^2\, d\mathbf{x} \le 2 \int\limits_{\Omega} |\xi\, d^+\pi - \langle \xi d^+\pi\rangle_\Omega|^2\, d\mathbf{x} + \frac{2}{|\Omega|} \Big|\int\limits_{\Omega} \xi\, d^+\pi\, d\mathbf{x}\Big|^2, \tag{3.5}$$

in order to take advantage of the Poincaré inequality. The second term on the right-hand side is treated as follows

$$\frac{2}{|\Omega|} \Big|\int\limits_{\Omega} \xi\, d^+\pi\, d\mathbf{x}\Big|^2 = \frac{2}{|\Omega|} \Big| \int\limits_{\Omega\cap\operatorname{spt}\xi} \pi\, d^-\xi\, d\mathbf{x}\Big|^2 \le 2\, \|\xi\|_{1,\infty}^2 \int\limits_{\Omega\cap\operatorname{spt}\xi} |\pi|^2\, d\mathbf{x}\,,$$

where we used Lemma 2.5. The first term on the right-hand side of (3.5) is treated with the help of (cf. [3, Lemma 4.3])

$$\|q\|_{L^2_0(G)} \le c \sup_{\|\mathbf{v}\|_{W^{1,2}_0(G)}\le 1} \langle q, \operatorname{div}\mathbf{v}\rangle.$$

We re-write (3.3), using Lemma 2.4 and Lemma 2.5 to get for all $\boldsymbol{\psi} \in W^{1,2}_0(\Omega)$

$$\begin{aligned}
&\int_\Omega \xi\, d^+\pi \operatorname{div}\boldsymbol{\psi}\, d\mathbf{x}\\
&= \int_\Omega \xi\, d^+\mathbf{S}(\mathbf{Du})\cdot \mathbf{D}\boldsymbol{\psi} + \mathbf{S}(\mathbf{Du})\cdot d^-(\nabla\xi \overset{s}{\otimes} \boldsymbol{\psi}) - \mathbf{S}((\mathbf{Du})_\tau)\cdot \big(\partial_3(\xi\boldsymbol{\psi}) \overset{s}{\otimes} d^+\nabla a\big)\, d\mathbf{x}\\
&\quad + \int_\Omega \pi_\tau\, \partial_3(\xi\boldsymbol{\psi})\cdot d^+\nabla a - \pi\, d^-(\nabla\xi\cdot\boldsymbol{\psi}) - \mathbf{f}\cdot d^-(\xi\boldsymbol{\psi}) + \frac{\partial\mathbf{u}}{\partial t}\cdot d^-(\xi\boldsymbol{\psi})\, d\mathbf{x}\\
&=: \sum_{k=1}^{7} J_k\,.
\end{aligned}$$

We follow exactly the same approach as in [5, p. 857-858] to control J_k with $k = 1, \dots, 6$, while J_7 is simply estimated by Schwarz inequality. This proves that

$$\int_\Omega \xi^2 \big|d^+\pi\big|^2\, d\mathbf{x} \le c(\delta^{p-2}, \|\xi\|_{2,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega\cap \operatorname{spt}\xi} |\mathbf{f}|^2 + |\pi|^2 + \varphi(|\nabla\mathbf{u}|) + \Big|\frac{\partial\mathbf{u}}{\partial t}\Big|^2\, d\mathbf{x}.$$

By using the Young inequality and being $p < 2$, this shows that

$$\int_\Omega \xi^2 \big|d^+\pi\big|^2\, d\mathbf{x} \le c(\delta^{p-2}, \|\xi\|_{2,\infty}, \|a\|_{C^{1,1}}) \left[(1+\delta^p)|\Omega| + \int_{\Omega\cap\operatorname{spt}\xi} |\mathbf{f}|^{p'} + |\pi|^{p'} + \Big|\frac{\partial\mathbf{u}}{\partial t}\Big|^2 d\mathbf{x}\right],$$

for a.e. $t \in I$. Hence, integration over I, using the a-priori estimates for $\mathbf{u}$ and $\frac{\partial\mathbf{u}}{\partial t}$, and π already proved, yields

$$\int_0^T\!\!\int_\Omega \xi^2 \big|d^+\pi\big|^2\, d\mathbf{x}ds \le c(\delta^{p-2}, \|\xi\|_{2,\infty}, \|a\|_{C^{1,1}}, T, |\Omega|) \left[\|\mathbf{u}_0\|^2_{1,2} + \int_0^T \|\mathbf{f}(s)\|^{p'}_{p'} ds\right],$$

which is the second estimate in (3.2). The same procedure, with many simplifications, can be used in the interior of Ω for difference quotients in all directions $\mathbf{e}^i$, $i = 1, 2, 3$.

In fact, by choosing $h \in \Big(0, \frac{1}{2}\operatorname{dist}(\operatorname{spt}\xi_{00}, \partial\Omega)\Big)$ and mainly with the same steps as before, this leads to

$$\int_0^T \int_\Omega \xi_{00}^2 \big|\nabla \mathbf{F}(\mathbf{Du})\big|^2 + \varphi(\xi_{00}|\nabla^2 \mathbf{u}|)\, d\mathbf{x} ds$$
$$\leq c(\|\xi_{00}\|_{2,\infty})\Big(\|\mathbf{u}_0\|_{1,2}^2 + \|\operatorname{div}\mathbf{S}(\mathbf{Du}_0)\|_2^2 + \int_0^T \|\mathbf{f}\|_{p'}^{p'}\, ds\Big)\,,$$

and

$$\int_0^T \int_\Omega \xi_{00}^2 \big|\nabla \pi\big|^2\, d\mathbf{x} ds$$
$$\leq c(\delta^{p-2}, \|\xi_{00}\|_{2,\infty})\Big(\|\mathbf{u}_0\|_{1,2}^2 + \|\operatorname{div}\mathbf{S}(\mathbf{Du}_0)\|_2^2 + \int_0^T \|\mathbf{f}\|_{p'}^{p'} ds\Big)\,,$$

where ξ_{00} is any cut-off function with compact support contained in Ω. This proves the first estimates in (3.1) and (3.2). $\square$

Moreover, from (3.1) and (3.2) we can infer the following result which will be used to study the regularity of the derivatives in the x_3 (locally the normal direction) variable.

Corollary 3.4. *Under the assumptions of Proposition 3.1 we obtain that* $\mathbf{F}(\mathbf{D}) \in L^2(I; W^{1,2}_{loc}(\Omega))$, $\mathbf{u} \in L^p(I; W^{2,p}_{loc}(\Omega))$, *and* $\pi \in L^2(I; W^{1,2}_{loc}(\Omega))$. *This implies, in particular, that the system* (1.1) *holds almost everywhere in* $I \times \Omega$.

3.1. REGULARITY IN THE NORMAL DIRECTION

By following the same approach as in [5, Sec. 3.2], since we already proved that the equations can be rigorously treated in the point-wise sense, we consider the first two equations of the system (1.1) written as follows in $I \times \Omega$

$$\begin{aligned} &- \partial_{\gamma 3}\mathbf{S}_{\alpha 3}\partial_3\mathbf{D}_{\gamma 3} - \partial_{3\gamma}\mathbf{S}_{\alpha 3}\partial_3\mathbf{D}_{3\gamma} \\ &= f_\alpha - \frac{\partial u_\alpha}{\partial t} + \partial_\alpha \pi + \partial_{33}\mathbf{S}_{\alpha 3}\partial_3\mathbf{D}_{33} + \partial_{\gamma\sigma}\mathbf{S}_{\alpha 3}\partial_3\mathbf{D}_{\gamma\sigma} + \partial_{kl}\mathbf{S}_{\alpha\beta}\partial_\beta\mathbf{D}_{kl} =: \mathfrak{f}_\alpha\,. \end{aligned} \tag{3.6}$$

We multiply (3.6) point-wise by $-\mathfrak{b}_\alpha := \partial_3\mathbf{D}_{\alpha 3}$ to obtain

$$2\kappa_0(p)\,\varphi''(|\mathbf{Du}|)|\mathfrak{b}|^2 \leq 2A_{\alpha\gamma}\mathfrak{b}_\gamma\mathfrak{b}_\alpha \leq |\mathfrak{f}|\,|\mathfrak{b}| \quad \text{a.e. in } I \times \Omega.$$

By using the same argument as in the cited reference (mainly the growth properties of $\mathbf{S}$ from (2.1b)) the right-hand side $\mathfrak{f}$ of (3.6) can be bounded as follows in $I \times \Omega_P$:

$$|\mathfrak{f}| \leq c\left(|\mathbf{f}| + \left|\frac{\partial \mathbf{u}}{\partial t}\right| + |\partial_\tau \pi| + \|\nabla a\|_\infty |\partial_3 \pi| + \varphi''(|\mathbf{Du}|)\left(|\partial_\tau \nabla \mathbf{u}| + \|\nabla a\|_\infty |\nabla^2 \mathbf{u}|\right)\right),$$

where the constant c depends only on the characteristics of $\mathbf{S}$. To estimate the partial derivative $\partial_3\pi$ we use again the equations, to write point-wise in $I\times\Omega$ that $\partial_3\pi=-f_3+\frac{\partial u_3}{\partial t}-\partial_j\mathbf{S}_{3j}$ and to obtain

$$|\partial_3\pi|\le|\mathbf{f}|+\left|\frac{\partial\mathbf{u}}{\partial t}\right|+c\,\varphi''(|\mathbf{Du}|)|\nabla^2\mathbf{u}|\qquad\text{a.e. in } I\times\Omega\,.$$

Hence, there exists a constant C_1, depending only on the characteristics of $\mathbf{S}$, such that a.e. in $I\times\Omega_P$ it holds

$$\begin{aligned}&\varphi''(|\mathbf{Du}|)|\nabla^2\mathbf{u}|\\&\le C_1\Big((1+\|\nabla a\|_\infty)\Big(|\mathbf{f}|+\Big|\frac{\partial\mathbf{u}}{\partial t}\Big|\Big)+|\partial_\tau\pi|+\varphi''(|\mathbf{Du}|)\big(|\partial_\tau\nabla\mathbf{u}|+\|\nabla a\|_\infty|\nabla^2\mathbf{u}|\big)\Big)\,.\end{aligned}\tag{3.7}$$

Next, we choose the open sets Ω_P small enough (that is we choose the radii R_P small enough) in such a way that

$$\|\nabla a_P(x)\|_{L^\infty(\Omega_P)}\le r_P\le\frac{1}{2C_1}=:C_2\,.$$

Thus, we can absorb the last term from the right-hand-side of (3.7) in the left-hand side, which yields a.e. in $I\times\Omega_P$

$$\varphi''(|\mathbf{Du}|)|\nabla^2\mathbf{u}|\le c\left(|\mathbf{f}|+\left|\frac{\partial\mathbf{u}}{\partial t}\right|+|\partial_\tau\pi|+\varphi''(|\mathbf{Du}|)|\partial_\tau\nabla\mathbf{u}|\right).\tag{3.8}$$

We next recall that for smooth enough $\mathbf{u}$

$$\sqrt{\varphi''(|\mathbf{Du}|)}|\nabla^2\mathbf{u}|\sim|\nabla\mathbf{F}(\mathbf{Du})|\,.$$

Thus, after multiplying both sides of (3.8) by $\xi\sqrt{\varphi''(|\mathbf{Du}|)}$ and raising both sides to the p-th power, we get a.e. in $I\times\Omega_P$

$$\xi^p|\nabla\mathbf{F}(\mathbf{Du})|^p\le c\,\xi^p\varphi''(|\mathbf{Du}|)^{-\frac{p}{2}}\Big(|\mathbf{f}|^p+\left|\frac{\partial\mathbf{u}}{\partial t}\right|^p+|\partial_\tau\pi|^p\Big)+c\,\xi^p\varphi''(|\mathbf{Du}|)^{\frac{p}{2}}|\partial_\tau\nabla\mathbf{u}|^p\,.$$

Furthermore, the integral

$$\int_0^T\int_\Omega\xi^p\varphi''(|\mathbf{Du}|)^{-\frac{p}{2}}\Big(|\mathbf{f}|^p+\left|\frac{\partial\mathbf{u}}{\partial t}\right|^p+|\partial_\tau\pi|^p\Big)\,d\mathbf{x}ds\,,$$

is finite, due to the assumptions on $\mathbf{f}$ and on the square integrability in $I\times\Omega_P$ of $\frac{\partial\mathbf{u}}{\partial t}$ and of $\xi_P\partial_\tau\pi$, already proved in Theorem 2.8 and Proposition 3.1. Next, by observing that

$$\int_0^T\int_\Omega\xi^p\varphi''(|\mathbf{Du}|)^{\frac{p}{2}}|\partial_\tau\nabla\mathbf{u}|^p\;d\mathbf{x}ds\le c\,\delta^{\frac{(p-2)p}{2}}\int_0^T\int_\Omega\Big[\varphi(|\xi\,\partial_\tau\nabla\mathbf{u}|)+\delta^p\Big]\,d\mathbf{x}ds,$$

with Proposition 3.1 imply that also the second term in the last inequality is integrable.

Hence, we proved that

$$\int_0^T \int_\Omega \xi^p |\nabla \mathbf{F}(\mathbf{Du})|^p \, d\mathbf{x}ds \le c,$$

that is $\nabla \mathbf{F}(\mathbf{Du}) \in L^p(I \times \Omega_P)$, which proves the missing local estimate.

Finally, the properties of the finite covering and the results of the previous section imply that

$$\nabla \mathbf{F}(\mathbf{Du}) \in L^p(I \times \Omega) \cap L^2_{loc}(I \times \Omega).$$

Thus, all assertions of Theorem 1.1 are proved.

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