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Dedicated to Prof. Dušan D. Repovš
on the occasion of his 65th birthday

ON SOLVABILITY OF ELLIPTIC BOUNDARY VALUE PROBLEMS VIA GLOBAL INVERTIBILITY

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Abstract. In this work we apply global invertibility result in order to examine the solvability of elliptic equations with both Neumann and Dirichlet boundary conditions.

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1. INTRODUCTION

In this note we study a unique solvability of the following boundary value problems

$$\begin{cases} -\Delta u = f(u) + h & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{D_h}$$

and

$$\begin{cases} -\Delta u = f(u) + h & \text{on } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{N_h}$$

where $f : \mathbb{R} \to \mathbb{R}$ is of class C^1, Ω is an open, bounded and connected subset of $\mathbb{R}^d$, $d \in \{1, 2, 3\}$, with a C^2–boundary and $h \in L^2(\Omega)$ is any fixed function. Solutions are understood in the H^2–sense, while the boundary value conditions are understood in the sense of a trace operator. The stability of problems (D_h) and (N_h) is understood as a C^1–differentiability of the mapping

$$L^2(\Omega) \ni h \longmapsto u_h \in H^2(\Omega),$$

where u_h is a solution to (D_h) or (N_h), respectively. To obtain such a result we show that associated solution operator, namely $u \longmapsto \Delta u + f \circ u$ is diffeomorphism. Such approach was considered in [1], where authors considered an operator defined between the spaces of Hölder continuous functions. While we are inspired by their results, we consider the solvability under somehow different assumptions since we do not restrict ourselves to relations between the first eigenvalue of the differential operator and the growth of the nonlinear term but we also investigate the interplay between the growth of the nonlinear term and other eigenvalues.

We have already considered global invertibility of mappings with applications to solvability of nonlinear boundary value problems in [2], [3] using the global inversion result due to [5]. The methodology used there was much more complicated and pertained to the application of tools from critical point theory, namely it required the Palais–Smale condition to be checked in addition to local invertibility and coercivity. Moreover, in the sources mentioned, as well as in [7], only relations between the growth of the nonlinear term and the first eigenvalue of the differential operator are used and only Dirichlet problems are considered.

2. PRELIMINARIES

2.1. SOBOLEV SPACES AND TRACE OPERATORS

Let $C_c^\infty(\Omega)$ stands for all smooth real functions with a compact support contained in Ω. We say that $u \in L^2(\Omega)$ belongs to $H^1(\Omega)$ if there exists $\nabla u = (\partial_{x_i} u)_{i=1}^d \in (L^2(\Omega))^d$ such that for every $i = 1, \ldots, d$ it holds

$$\int_\Omega u(x)\partial_{x_i}\varphi(x)dx = -\int_\Omega \partial_{x_i}u(x)\varphi(x)dx, \quad \varphi \in C_c^\infty(\Omega).$$

We say that $u \in H^2(\Omega)$ if $\partial_{x_i} u \in H^1(\Omega)$ for every $i = 1, \ldots, d$. Moreover, we put $H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}_{H^1}$ and $H_0^2(\Omega) = \overline{C_c^\infty(\Omega)}_{H^2}$

Let us recall that for every $u \in H^2(\Omega)$ there exists a unique representation $U \in C(\overline{\Omega})$ such that $u = U$ a.e. on $\overline{\Omega}$. Moreover, the embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$, given by $u \longmapsto U$ is compact. Since we identify elements of $L^2(\Omega)$ with they representations, the embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ is understood as an identity.

Since $\partial\Omega$ is of class C^2, then we can consider a surface measure s on $\partial\Omega$. Put

$$L^2(\partial\Omega) = \left\{ u : \partial\Omega \to \mathbb{R} : \int_{\partial\Omega} |u(x)|^2 ds(x) < \infty \right\}$$

and equip it with a natural norm

$$\|u\|_{L^2(\partial\Omega)} = \left(\int_{\partial\Omega} |u(x)|^2 ds(x) \right)^{\frac{1}{2}}.$$

Since $C^\infty(\overline{\Omega})$ is dense in $H^2(\Omega)$, then we define *trace operators*

$$\gamma_0, \gamma_1 : H^2(\Omega) \supset C^\infty(\overline{\Omega}) \to L^2(\partial\Omega)$$

given by the formulas

$$\begin{aligned} \gamma_0 u(x) &= u(x) \text{ for } x \in \partial\Omega, \\ \gamma_1 u(x) &= \partial_\nu u(x) \text{ for } x \in \partial\Omega, \end{aligned}$$

where ∂_ν stands for outward normal derivative. Operators γ_0 and γ_1 are continuous with respect to H^2–topology, see [4]. Therefore, each of them has a unique extension on whole $H^2(\Omega)$, denoted again by γ_0 and γ_1. For every $u \in H_0^1(\Omega)$ conditions $u \in H_0^1(\Omega)$ and $\gamma_0 u = 0$ coincide. Moreover condition $u = 0$ *on* $\partial\Omega$, from now on, is understood as $\gamma_0 u = 0$. Analogously, condition $\partial_\nu u = 0$ *on* $\partial\Omega$ is equivalent with $\gamma_1 u = 0$.

Theorem 2.1 ([4])**.** *Assume that there exists $u \in H_0^1(\Omega)$ and $v \in L^2(\Omega)$ such that*

$$\int_\Omega (\nabla u(x)|\nabla\varphi(x))\, dx + \int_\Omega u(x)\varphi(x) = \int_\Omega v(x)\varphi(x)dx, \quad \varphi \in H_0^1(\Omega).$$

Then $u \in H^2(\Omega)$ and $\gamma_0 u = 0$. In particular, $-\Delta u + u = v$.

Theorem 2.2 ([4])**.** *Assume that there exists $u \in H^1(\Omega)$ and $v \in L^2(\Omega)$ such that*

$$\int_\Omega (\nabla u(x)|\nabla\varphi(x))dx + \int_\Omega u(x)\varphi(x) = \int_\Omega v(x)\varphi(x)dx, \quad \varphi \in H^1(\Omega).$$

Then $u \in H^2(\Omega)$ and $\gamma_1 u = 0$. In particular, $-\Delta u + u = v$.

2.2. VARIATIONAL CALCULUS

Consider a continuous function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and define $G : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$G(x, s) = \int_0^s g(x, \tau)d\tau, \quad x \in \Omega, \ s \in \mathbb{R}.$$

Fix $h \in L^2(\Omega)$ and define a functionals $\mathcal{E}_D : H_0^1(\Omega) \to \mathbb{R}$ and $\mathcal{E}_N : H^1(\Omega) \to \mathbb{R}$ by

$$\mathcal{E}_D(u) = \mathcal{E}_N(u) = \tfrac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - \tfrac{1}{2} \int_\Omega G(x, u(x))dx - \int_\Omega h(x)u(x)dx$$

Then $\mathcal{E}_D$ and $\mathcal{E}_N$ are of class C^1 with derivatives

$$\begin{aligned} \langle \mathcal{E}_D'(u), v\rangle &= \langle \mathcal{E}_N'(u), v\rangle \\ &= \int_\Omega (\nabla u(x)|\nabla v(x))dx - \int_\Omega g(x, u(x))v(x)dx - \int_\Omega h(x)v(x)dx. \end{aligned}$$

for all v from $H_0^1(\Omega)$ and $H^1(\Omega)$, respectively. We consider two boundary value problems associated with $\mathcal{E}_D$ and $\mathcal{E}_N$, respectively,

$$\begin{cases} -\Delta u = g(x,u) + h & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

and

$$\begin{cases} -\Delta u = g(x,u) + h & \text{on } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Following the Fermat Rule and Theorems 2.1 and 2.2 we obtain

Proposition 2.3. *Every critical point of $\mathcal{E}_D$ is a solution to* (2.1). *Analogously, every critical point of $\mathcal{E}_N$ is a solution to* (2.2).

From the Browder–Minty Theorem we get the following result.

Proposition 2.4. *Assume that $\mathcal{E}$ is of class C^1 and there exists $c > 0$ such that*

$$\langle \mathcal{E}''(u)v|v\rangle \geq c\|v\|^2, \quad u, v \in X. \tag{2.3}$$

Then $\mathcal{E}$ has a unique critical point.

2.3. ON THE LAPLACE OPERATOR

Following [6], we consider

$$\mathcal{D}(\Delta_D) = \left\{u \in H^2(\Omega) : \gamma_0(u) = 0\right\},$$
$$\mathcal{D}(\Delta_N) = \left\{u \in H^2(\Omega) : \gamma_1(u) = 0\right\}.$$

Observe that $\mathcal{D}(\Delta_D) = H^2(\Omega) \cap H_0^1(\Omega)$ and $\mathcal{D}(\Delta_N) = H_0^2(\Omega) \oplus \{C\}_{C\in\mathbb{R}}$. Define operators $\Delta_D : L^2(\Omega) \supset \mathcal{D}(\Delta_D) \to L^2(\Omega)$ and $\Delta_N : L^2(\Omega) \supset \mathcal{D}(\Delta_N) \to L^2(\Omega)$ by

$$\Delta_D u = \Delta u \text{ for } u \in \mathcal{D}(-\Delta_D) \quad \text{and} \quad \Delta_N u = \Delta u \text{ for } u \in \mathcal{D}(\Delta_N).$$

Let $L \in \{-\Delta_D, -\Delta_N\}$. We recall that set $\rho(L) \subset \mathbb{R}$ stands for the resolvent of L while $\sigma(L) = \mathbb{R} \setminus \rho(L)$ for spectrum of L, see [4].

Following [4] and [6] sets $\sigma(-\Delta_D)$ and $\sigma(-\Delta_N)$ are discrete, that is they can be write on the following form: $\sigma(-\Delta_D) = \{\delta_n\}_{n\in\mathbb{N}}$ and $\sigma(-\Delta_N) = \{\eta_n\}_{n\in\mathbb{N}} \cup \{0\}$, where $\delta_n \nearrow \infty$ and $\eta_n \nearrow \infty$. Moreover $\sigma(-\Delta_D) \subset (0,\infty)$ and $\sigma(-\Delta_N) \subset [0,\infty)$. Therefore we can put $\sigma(L) = \{\lambda_n\}_{n\in\mathcal{N}}$. Here $\mathcal{N} = \mathbb{N}$ when $L = -\Delta_D$ and $\mathcal{N} = \mathbb{N} \cup \{0\}$ with $\lambda_0 = 0$ when $L = -\Delta_N$.

Moreover, for every $n \in \mathcal{N}$ there exists $e_n \in L^2(\Omega)$ such that $Le_n = \lambda_n e_n$ and $(e_i)_{i\in\mathcal{N}}$ is an orthonormal basis of $L^2(\Omega)$. As a consequence, for every $u = \sum_{i\in\mathcal{N}} a_i e_i \in L^2(\Omega)$ we have

$$Lu = L\sum_{i\in\mathcal{N}} a_i e_i = \sum_{i=1}^{\infty} \lambda_i a_i e_i.$$

Note that λ_1 is the smallest positive element of $\sigma(L)$. Moreover we put $\lambda_{\min} = \min \sigma(L)$. The *Poincaré inequality* reads

$$\lambda_1 \int_\Omega |u(x)|^2 dx \leq \int_\Omega |\nabla u(x)|^2 dx, \quad u \in \mathcal{D}(\Delta_D).$$

Note that similar result is not true on $\mathcal{D}(\Delta_N)$.

2.4. DIFFERENTIABILITY AND GLOBAL INVERTIBILITY

Let us consider an open sets $U \subset \mathbb{E}$ and $V \subset \mathbb{F}$, where $\mathbb{E}$ and $\mathbb{F}$ are real Banach spaces. We say that a C^1–mapping $\boldsymbol{p}$ is a C^1*–diffeomorphism* of U onto V if $\boldsymbol{p}|_U$ is a homeomorphism of U onto V and $(\boldsymbol{p}|_U)^{-1}$ is of class C^1 on V. A mapping is *diffeomorphism* if it is a diffeomorphism of $\mathbb{E}$ onto $\mathbb{F}$. Moreover, we say that $\boldsymbol{p}$ is a *local diffeomorphism* if for every point $u \in \mathbb{E}$ there exists its neighbourhood U such that $\boldsymbol{p}$ is a diffeomorphism of U onto $\boldsymbol{p}(U)$.

Let us denote by $\mathrm{Isom}(\mathbb{E}, \mathbb{F})$ the space of all linear and continuous bijections of $\mathbb{E}$. Recall that a C^1–mapping $\boldsymbol{p} : \mathbb{E} \to \mathbb{F}$ is a local diffeomorphism if and only if $\boldsymbol{p}'(u) \in \mathrm{Isom}(\mathbb{E}, \mathbb{F})$ for every $u \in \mathbb{E}$, see [1].

The mapping $\boldsymbol{p} : \mathbb{E} \to \mathbb{F}$ is called *proper* if $\boldsymbol{p}^{-1}(K)$ is compact for every compact $K \subset \mathbb{F}$. Every homeomorphims is a proper map.

Proposition 2.5 ([8])**.** *Take $\boldsymbol{p}, \boldsymbol{c} : \mathbb{E} \to \mathbb{F}$. If $\boldsymbol{p}$ is proper, $\boldsymbol{c}$ is strongly continuous, that is $\boldsymbol{c}(u_n) \to \boldsymbol{c}(u_0)$ whenever $u_n \rightharpoonup u_0$, and $\boldsymbol{p} - \boldsymbol{c}$ is cercive, that is*

$$\lim_{\|u\| \to \infty} \|\boldsymbol{p}(u) - \boldsymbol{c}(u)\| = \infty,$$

then operator $\boldsymbol{p} - \boldsymbol{c}$ is proper.

Theorem 2.6 ([1])**.** *Assume that $\boldsymbol{p} : \mathbb{E} \to \mathbb{F}$ is continuous and locally invertible, that is, every point $u \in \mathbb{E}$ possesses an open neighbourhood U such that $\boldsymbol{p}|_U$ is a homeomorphism of U onto $\boldsymbol{p}(U)$, then the following conditions are equivalent:*

(i) *$\boldsymbol{p}$ is proper,*
(ii) *$\boldsymbol{p}$ is a homeomorphism of $\mathbb{E}$ onto $\mathbb{F}$.*

Proposition 2.7. *Assume that $\boldsymbol{p}$ is proper and of class C^1. If $\boldsymbol{p}'(u) \in \mathrm{Isom}(\mathbb{E}, \mathbb{F})$ for every $u \in \mathbb{E}$, then $\boldsymbol{p}$ is a C^1–diffeomorphism.*

3. MAIN RESULT

We investigate which perturbations $\boldsymbol{p} : \mathcal{D}(L) \to L^2(\Omega)$ make $L - \boldsymbol{p}$ into a C^1–diffeomorphism between $\mathcal{D}(L)$ and $L^2(\Omega)$. Conditions will be given in terms of a derivative of a perturbation and of a spectrum of L. Note that $-\Delta_D$ is already a smooth diffeomorphism as a linear, bounded and bijective mapping. However, note that this is not the case for $-\Delta_N$ since $\ker \Delta_N$ consists of all constant functions. We start with linear perturbations then turning to the nonlinear ones.

3.1. LINEAR PERTURBATION

Let $\psi : \overline{\Omega} \to \mathbb{R}$ be a continuous function. We define $\boldsymbol{\psi} \in \mathcal{B}(L^2(\Omega))$ by

$$(\boldsymbol{\psi} u)(x) = \psi(x)u(x), \quad u \in L^2(\Omega),\ x \in \Omega \text{ (a.e)}.$$

Theorem 3.1. *If $\psi\left(\overline{\Omega}\right) \subset \rho(L)$, then $L - \boldsymbol{\psi}$ is an isomorphism of $\mathcal{D}(L)$ onto $L^2(\Omega)$.*

Proof. Since $L - \boldsymbol{\psi}$ is bounded and linear, it is enough to show that it is bijective. Denote

$$\alpha := \min_{x \in \overline{\Omega}} \{\psi(x)\} \quad \text{and} \quad \beta := \max_{x \in \overline{\Omega}} \{\psi(x)\}.$$

We divide this proof into two disjoint cases.

If $\beta < \lambda_{\min}$ then functional $\mathcal{E} : X \to \mathbb{R}$, where $X \in \{H^1(\Omega), H_0^1(\Omega)\}$, given by

$$\mathcal{E}(u) = \tfrac{1}{2} \int\limits_{\Omega} |\nabla u(x)|^2 dx - \tfrac{1}{2} \int\limits_{\Omega} \psi(x)|u(x)|^2 dx - \int\limits_{\Omega} h(x)u(x)dx, \quad u \in X.$$

satisfies condition (2.3) for some $c > 0$. Hence, assertion follows by Proposition 2.4.

Let $\lambda_{n-1} < \alpha \leq \beta < \lambda_n$ for some $n \in \mathcal{N}$. We put $\lambda = \frac{\lambda_n + \lambda_{n-1}}{2}$. Define operators $\Lambda : \mathcal{D}(L) \to L^2(\Omega)$ and $A : L^2(\Omega) \to L^2(\Omega)$ by

$$\begin{aligned} \Lambda u &= Lu - \lambda u, \quad u \in \mathcal{D}(L), \\ Au &= \boldsymbol{\psi} u - \lambda u, \quad u \in \mathcal{D}(L). \end{aligned} \tag{3.1}$$

Then for every $u = \sum_{i \in \mathcal{N}} \alpha_i e_i$ we have

$$\|\Lambda u\|^2 = \left\| \sum_{i \in \mathcal{N}} (\lambda_i - \lambda)\alpha_i e_i \right\|^2 = \sum_{i \in \mathcal{N}} |\lambda_i - \lambda|^2 \alpha_i^2 \geq \tfrac{|\lambda_n - \lambda_{n-1}|^2}{4} \|u\|^2.$$

Therefore Λ^{-1} is $\frac{2}{\lambda_n - \lambda_{n-1}}$–Lipschitz. Since

$$\|Au\| = \left(\int\limits_{\Omega} |(\psi(x) - \lambda)u(x)|^2 dx \right)^{\frac{1}{2}} \leq \max_{x \in \overline{\Omega}} |\psi(x) - \lambda| \|u\|.$$

Then A is $\left(\frac{\lambda_n - \lambda_{n-1}}{2} - \epsilon\right)$–Lipschitz, where $\epsilon = \operatorname{dist}\left(\sigma(L), \psi\left(\overline{\Omega}\right)\right) > 0$. Equip $\mathcal{D}(L)$ with the equivalent norm $\|\cdot\|_\Lambda := \|\Lambda \cdot\|$. Then, for every $h \in L^2(\Omega)$, the mapping $\Phi : \mathcal{D}(L) \to \mathcal{D}(L)$ given by the formula

$$\Phi(u) = \Lambda^{-1}(Gu + h)$$

is a contraction as a composition of $\left(\frac{\lambda_n - \lambda_{n-1}}{2} - \epsilon\right)$–Lipschitz mapping, isomtery and $\frac{2}{\lambda_n - \lambda_{n-1}}$–Lipschitz function. The assertion now follows by the Banach Contraction Principle. $\square$

3.2. AUTONOMOUS PERTURBATIONS

Consider a C^1–function $f : \mathbb{R} \to \mathbb{R}$ and the associated operator $\boldsymbol{f} : \mathcal{D}(L) \to L^2(\Omega)$:

$$\boldsymbol{f}(u)(x) := f(u(x)), \quad u \in \mathcal{D}(L),\ x \in \Omega \text{ (a.e.)}.$$

Note that operator $\boldsymbol{f}$ is compact due to the compact embedding $\mathcal{D}(L) \hookrightarrow C(\overline{\Omega})$. Moreover, $\boldsymbol{f}$ is of class C^1 with a derivative given by the formula

$$\left(\boldsymbol{f}'(u)v\right)(x) = f'(u(x))v(x), \quad u, v \in \mathcal{D}(L),\ x \in \Omega \text{ (a.e.)}.$$

Proposition 3.2. *Assume that* $f'(\mathbb{R}) \subset \rho(L)$. *Then* $L - \boldsymbol{f}'(u)$ *is invertible for every* $u \in \mathcal{D}(L)$.

Proof. Let $u \in \mathcal{D}(L)$. Then

$$(L - \boldsymbol{f}'(u))v(x) = -\Delta v(x) - f'(u(x))v(x), \quad v \in \mathcal{D}(L),\ x \in \Omega.$$

Since $u \in C(\overline{\Omega})$, then assumptions of Theorem 3.1 are satisfied and the assertion follows. □

Although local invertibility is a necessary condition for global invertibility, it is not sufficient. In other words, there exists a locally invertible C^1–mappings, which are not diffeomorphisms. Moreover, we can find examples of such mappings of the form $L - \boldsymbol{f}$ for some C^1–function f.

Example 3.3. Take $\Omega = (0, \pi)$ and let $L = -\Delta_D$. Consider $f(u) = u + e^{-u}$. Then $f'(\mathbb{R}) = (-\infty, 1)$ and hence $f'(\mathbb{R}) \subset \rho(L) = (-\infty, 1)$ since $\sigma(L) = \{n^2\}_{n\in\mathbb{N}}$. Then $L - \boldsymbol{f}'(u)$ is invertible by Proposition 3.2. Nevertheless, for the sequence $(u_n)_n$, where $u_n(x) = n\sin(x)$ for $x \in (0, \pi)$ which has the property $\|u_n\| \to \infty$ we have

$$\|Lu_n - \boldsymbol{f}(u_n)\|^2 = \int_0^\pi \left|-\ddot{u}_n(x) - u(x) - e^{-u(x)}\right|^2 dx = \int_0^\pi e^{-2n\sin(x)}dx \to 0,$$

when $n \to \infty$. Therefore $L - \boldsymbol{f}$ cannot be even a homeomorphism.

Example 3.4. For $\Omega = (0, \pi)$ and $L = -\Delta_N$ we take $f(u) = e^{-u}$. Then arguing as in Example 3.3 we conclude that $L - \boldsymbol{f}$ is locally invertible. Nevertheless, taking sequence $u_n(x) = n$ for $x \in (0, \pi)$, we immediately obtain $\|Lu_n - \boldsymbol{f}(u_n)\| \to 0$ when $n \to \infty$. Therefore $L - \boldsymbol{f}$ is not a homeomorphism as well.

Lemma 3.5. *Assume that* $\sup f'(\mathbb{R}) < 0$. *Then operator* $-\Delta_N - \boldsymbol{f}$ *is coercive.*

Proof. Denote $\beta := \sup f'(\mathbb{R})$. Then, one has

$$|f(\xi)| \geq -\beta|\xi| - |f(0)|, \quad \xi \in \mathbb{R}$$

and hence, for every $u \in \mathcal{D}(\Delta_D)$,

$$\|\boldsymbol{f}(u)\| = \left(\int_\Omega |f(u(x))|^2 dx\right)^{\frac{1}{2}} \geq |\beta|\|u\| - |f(0)|\sqrt{|\Omega|}.$$

and, since $\gamma_1 u = 0$

$$\begin{aligned}\langle Lu|\boldsymbol{f}(u)\rangle &= \int\limits_\Omega (-\Delta u(x))\, f(u(x))dx \\ &= \int\limits_\Omega f'(u(x))|\nabla u(x)|^2 dx + \int\limits_{\partial\Omega} (\gamma_1 u(x))\, f(u(x))ds(x) < 0.\end{aligned}$$

Therefore, for every $u \in \mathcal{D}(\Delta_N)$, we obtain

$$\begin{aligned}\|Lu - \boldsymbol{f}(u)\|^2 &= \|Lu\|^2 - 2\langle Lu|\boldsymbol{f}(u)\rangle + \|\boldsymbol{f}(u)\|^2 \\ &\geq \|Lu\|^2 + |\beta|\|u\|^2 - C\end{aligned}$$

for some $C > 0$. □

Lemma 3.6. *Assume that* $\sup f'(\mathbb{R}) < \lambda_1$. *Then operator* $-\Delta_D - \boldsymbol{f}$ *is coercive.*

Proof. Denote $\beta := \sup(f'(\mathbb{R}) \cup \{0\})$. Note that for every $u \in \mathcal{D}(\Delta_D)$ one has

$$\begin{aligned}\langle Lu|\boldsymbol{f}(u)\rangle &= \int\limits_\Omega -\Delta u(x) f(u(x))dx \\ &= \int\limits_\Omega f'(u(x))|\nabla u(x)|^2 dx + \int\limits_{\partial\Omega} \gamma_1 u(x) f(u(x))ds(x) \\ &\leq \beta \int\limits_\Omega |\nabla u(x)|^2 dx + \int\limits_{\partial\Omega} \gamma_1 u(x) f(u(x))ds(x).\end{aligned}$$

Using the Poincaré inequality and the Stokes Theorem we obtain

$$\begin{aligned}\lambda_1 \left(\int\limits_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}} \left(\int\limits_\Omega |\nabla u(x)|^2 dx\right)^{\frac{1}{2}} &\leq \int\limits_\Omega |\nabla u(x)|^2 dx = \int\limits_\Omega -\Delta u(x) u(x) dx \\ &\leq \left(\int\limits_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}} \left(\int\limits_\Omega |\Delta u(x)|^2 dx\right)^{\frac{1}{2}}\end{aligned}$$

for every $u \in \mathcal{D}(\Delta_D)$. Moreover,

$$\begin{aligned}\int\limits_{\partial\Omega} \gamma_1 u(x) f(u(x))ds(x) &\leq \left(\int\limits_{\partial\Omega} |\gamma_1 u(x)|^2 ds(x)\right)^{\frac{1}{2}} \left(\int\limits_{\partial\Omega} |f(0)|^2 ds(x)\right)^{\frac{1}{2}} \\ &\leq \|\gamma_1\|_{\mathcal{D}(\Delta_D)\to L^2(\Omega)} |f(0)| \sqrt{s(\partial\Omega)} \left(\int\limits_\Omega |\Delta u(x)|^2 dx\right)^{\frac{1}{2}}\end{aligned}$$

Here

$$\|\gamma_1\|_{\mathcal{D}(\Delta_D)\to L^2(\Omega)} = \sup\left\{\left(\int\limits_{\partial\Omega} |\gamma_1 u(x)|^2\right)^{\frac{1}{2}} : \left(\int\limits_{\Omega} |\Delta u(x)|^2\right)^{\frac{1}{2}} = 1\right\} < \infty.$$

Taking the above into account, we obtain that there exists a constant $C > 0$ such that

$$\langle Lu|\boldsymbol{f}(u)\rangle \leq \tfrac{\beta}{\lambda_1}\|Lu\|^2 + C\|Lu\|.$$

Define

$$\Xi := \left\{u \in H^2(\Omega) \cap H_0^1(\Omega) : \|\boldsymbol{f}(u)\| \leq \sqrt{\tfrac{\beta}{\lambda_1}}\|Lu\|\right\}.$$

Taking $u \in \Xi$ we instantly obtain that

$$\|Lu - \boldsymbol{f}(u)\| \geq \|Lu\| - \|\boldsymbol{f}(u)\| \geq \left(1 - \sqrt{\tfrac{\beta}{\lambda_1}}\right)\|Lu\|.$$

On the other hand, if $u \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \setminus \Xi$, then

$$\|Lu - \boldsymbol{f}(u)\|^2 = \|Lu\|^2 - 2\langle Lu|\boldsymbol{f}(u)\rangle + \|\boldsymbol{f}(u)\|^2 \geq \tfrac{\lambda_1 - \beta}{\lambda_1}\|Lu\|^2 - 2C\|Lu\|$$

for every $u \in \mathcal{D}(\Delta_D)$. □

Lemma 3.7. *Assume that* $\overline{f'(\mathbb{R})} = [\alpha, \beta] \subset (\rho(L) \cap (\lambda_{\min}, \infty))$. *Then* $L - \boldsymbol{f}$ *is coercive.*

Proof. Note that $[\alpha, \beta] \subset (\lambda_{n-1}, \lambda_n)$ for some $n \in \mathbb{N}$. Take $\lambda = \frac{\lambda_{n-1}+\lambda_n}{2}$ and Λ given by (3.1). We obtain that for every $u \in \mathcal{D}(L)$ one has

$$\|Lu - \boldsymbol{f}(u)\| \geq \|Lu - \lambda u\| - \|\boldsymbol{f}(u) - \lambda u\| = \|\Lambda u\| - \|\boldsymbol{f}(u) - \lambda u\|.$$

Since, for every $\xi \in \mathbb{R}$,

$$|f(\xi) - \lambda\xi| = \left|\int\limits_0^{\xi} (f'(\tau) - \lambda)\, d\tau + f(0)\right| \leq \left(\tfrac{\lambda_n - \lambda_{n-1}}{2} - \epsilon\right)|\xi| + |f(0)|,$$

where $\epsilon = \min\{\alpha - \lambda_{n-1}, \lambda_n - \beta\}$, it follows

$$\|\boldsymbol{f}(u) - \lambda u\| \leq \left(\tfrac{\lambda_n - \lambda_{n-1}}{2} - \epsilon\right)\|u\| + \sqrt{|\Omega|}|f(0)|.$$

Hence, $\|\Lambda u\| \geq \frac{\lambda_n - \lambda_{n-1}}{2}\|u\|$ and we have

$$\|Lu - \boldsymbol{f}(u)\| \geq \epsilon\|\Lambda u\|.$$

It is enough to observe that Λ is coercive since $\Lambda \in \operatorname{Isom}\big(\mathcal{D}(L), L^2(\Omega)\big)$. □

Theorem 3.8. *If $\overline{f'(\mathbb{R})} \subset \rho(L)$, then operator $L - \boldsymbol{f}$ is a C^1–diffeomorphism of $\mathcal{D}(L)$ onto $L^2(\Omega)$.*

Proof. Due to the Proposition 3.2, the operator $L - \boldsymbol{f}$ is a local diffeomorphism. Therefore, by Proposition 2.5, it is enough to observe that $\|L - \boldsymbol{f}\|$ is coercive which follows from Lemmas 3.5, 3.6 and 3.7. □

Theorem 3.9. *Let $h \in L^2(\Omega)$.*

(i) *If $\overline{f'(\mathbb{R})} \subset \rho(\Delta_D)$, then problem (D_h) has exactly one solution.*
(ii) *If $\overline{f'(\mathbb{R})} \subset \rho(\Delta_N)$, then problem (N_h) has exactly one solution.*

4. FINAL COMMENTS AND EXAMPLES

Let us recall

Theorem 4.1 ([1])**.** *Assume that a function f satisfies the following assumptions:*

(i) *$f(s) \geq 0$ for $s \in \mathbb{R}$,*
(ii) *there exists $\alpha < \lambda_1$ and $\omega > 0$ such that $f(s) \leq \alpha s + \omega$ for $s \geq 0$,*
(iii) *$f'(s) < \lambda_1$ for $s \in \mathbb{R}$.*

Then, for every $a \in (0,1)$ and $h \in C^{0,a}(\overline{\Omega})$ there exists a unique $C^{2,a}(\overline{\Omega})$–solution to the problem (D_h).

As it was mentioned, the result obtained in this paper is an extension of Theorem 4.1 under an assumption $d \leq 3$, since it allows us to omit a restrictive condition: $f(s) \geq 0$. Moreover, the classical results can not be used to directly obtain Theorem 3.8.

The Banach Fixed Point Theorem requires a Lipschitz continuity of f which is not assumed. To apply direct method of calculus of variations or Browder–Minty Theorem one needs to define a functional or operator on whole $H^1(\Omega)$. To show that it can not be possible we consider the problem

$$\begin{cases} \Delta u = e^{2e^u} + \alpha u & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega = \overline{B}\left(0, \frac{1}{2}\right) \subset \mathbb{R}^2$ and $\alpha < \lambda_1$. Then f satisfies assumptions of Theorem 3.8. On the other hand, taking $u(x) = \ln|\ln|x||$ we obtain $u \in H_0^1(\Omega)$, see [6], and

$$\int\limits_{\Omega} e^{2e^{u(x)}} dx = \int\limits_{\Omega} \frac{dx}{|x|^2} = \infty.$$

Therefore any proper operator acting between $H_0^1(\Omega)$ and $\left(H_0^1(\Omega)\right)^*$ can not be defined on whole $H_0^1(\Omega)$.

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