Opuscula Math. 39, no. 6 (2019), 839–862
https://doi.org/10.7494/OpMath.2019.39.6.839 Opuscula Mathematica

ON EXISTENCE AND GLOBAL ATTRACTIVITY OF PERIODIC SOLUTIONS OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS

Chuanxi Qian and Justin Smith

Communicated by Alexander Domoshnitsky

Abstract. Consider the delay differential equation with a forcing term

$$x'(t) = -f(t, x(t)) + g(t, x(t-\tau)) + r(t), \quad t \geq 0 \tag{$*$}$$

where $f(t,x) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, $g(t,x) : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions and ω-periodic in t, $r(t) : [0,\infty) \to \mathbb{R}$ is a continuous function and $\tau \in (0,\infty)$ is a positive constant. We first obtain a sufficient condition for the existence of a unique nonnegative periodic solution $\tilde{x}(t)$ of the associated unforced differential equation of Eq. $(*)$

$$x'(t) = -f(t, x(t)) + g(t, x(t-\tau)), \quad t \geq 0. \tag{$**$}$$

Then we obtain a sufficient condition so that every nonnegative solution of the forced equation $(*)$ converges to this nonnegative periodic solution $\tilde{x}(t)$ of the associated unforced equation $(**)$. Applications from mathematical biology and numerical examples are also given.

Keywords: delay differential equation, periodic solution, global attractivity.

Mathematics Subject Classification: 34K13, 34K20, 34K25.

1. INTRODUCTION

In applications, there are often external factors – either known or unknown – which affect the mathematical models. One such external factor that has been studied in related models is harvesting in a delayed recruitment model, see [3, 5, 7, 15], as well as a Holling's recruitment-delayed model with linear predator response

$$x'(t) = -\alpha(t)x(t) + g(t, x(t-\tau)) + \beta(t),$$

where $\beta(t) \not\equiv 0$, discussed in [18] and references therein.

Motivated by above observation and theoretical interest, our aim in this paper is to study the global attractivity of nonnegative solutions of the following more general nonlinear differential equation with a forcing term

$$x'(t) = -f(t, x(t)) + g(t, x(t-\tau)) + r(t), \quad t \geq 0, \tag{1.1}$$

where $f(t,x) : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and $g(t,x) : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions and ω-periodic in t, $r(t) : [0,\infty) \to \mathbb{R}$ is a continuous function and $\tau \in (0,\infty)$ is a positive constant.

As usual (see [10] for instance), with Eq. (1.1) an initial condition of the form

$$x(t) = \phi(t), \text{where } \phi \in C[[-\tau, 0], [0, \infty)] \tag{1.2}$$

is associated. By a solution $x(t)$ of Eq. (1.1), we mean x satisfies Eq. (1.1) with condition (1.2).

When $r(t) \equiv 0$, Eq. (1.1) becomes the following unforced equation

$$x'(t) = -f(t, x(t)) + g(t, x(t-\tau)), \quad t \geq 0. \tag{1.3}$$

In particular, when $f(t,x) = a(t)x$ where $a(t) : (-\infty, \infty) \to \mathbb{R}$ is a continuous and ω-periodic function, Eqs. (1.1) and (1.3) reduce to

$$x'(t) = -a(t)x(t) + g(t, x(t-\tau)) + r(t), \quad t \geq 0 \tag{1.4}$$

and

$$x'(t) = -a(t)x(t) + g(t, x(t-\tau)), \quad t \geq 0 \tag{1.5}$$

respectively.

The existence of periodic solutions of Eq. (1.5) and some related forms has been studied by numerous authors, see for example, [2–4, 8, 9, 14, 17] and the references cited therein. In these works, different techniques and approaches were used to establish the existence of periodic solutions. In particular, the positivity of Green's function of the periodic problem for linear first order functional differential equations (delay differential equations being a particular case) was studied first in [9], and then the ideals were developed in [2, 8]. In the next section of this paper, we will establish a sufficient condition for the existence of a periodic solution of the more general equation (1.3). By a periodic solution $\tilde{x}(t)$ of Eq. (1.3), we mean $\tilde{x}(t)$ is defined and periodic for $t \geq -\tau$, and satisfies Eq. (1.3) for $t \geq 0$.

While the global convergence of forced delay differential equations has been studied by some authors, such as [11] and the references cited therein, to the best of the authors' knowledge the results on the global attractivity of periodic solutions are relatively scarce. In a corresponding sense, attractivity can be interpreted as a sort of stability. Stability, which is of paramount importance in applications, has been studied and different methods have been developed by many authors. For instance, stability, based on nonoscillation of linear scalar functional differential equations of the first order, is one of the main topics in [8], the results of which were developed then in the book [1]. Recently, the global attractivity of periodic solutions of Eq. (1.5) was studied in [16] for $a(t), g(t,x) > 0$. In Section 3, we obtain a sufficient condition such that every nonnegative solution of the forced equation (1.1) converges to a nonnegative periodic solution of the associated unforced equation (1.3). When $r(t) \equiv 0$, Eq. (1.1) reduces to Eq. (1.3), this result extends the result found in [16] to include nonnegative $g(t,x)$ and $a(t)$ such that $a(t)$ may be zero or negative for some $t \in [0,\omega]$.

In Section 4 we apply these results to a differential equation model from mathematical biology, namely an extension of a blood cell production model discussed in [13], to consider external forcing factors such as the medical replacement of blood cells or administration of antibodies [12]. We also give examples to demonstrate the results from Section 2 and Section 3.

2. EXISTENCE OF A NONNEGATIVE PERIODIC SOLUTION

In this section we provide a sufficient condition for the existence of nonnegative periodic solutions of Eq. (1.3). The result is found using Schauder Fixed Point Theorem, which requires defining an operator T that satisfies certain conditions. The form of the operator T below is motivated by the integrating factor for linear equations. Observe that adding $a(t)x(t)$ to both sides of Eq. (1.3) gives us

$$x'(t) + a(t)x(t) = -f(t,x(t)) + a(t)x(t) + g(t,x(t-\tau)). \tag{2.1}$$

Multiplying both sides of (2.1) by $e^{\int_0^t a(u)du}$ and integrating both sides from t to $t+\omega$ yields

$$\begin{aligned} &e^{\int_0^{t+\omega} a(u)du} x(t+\omega) - e^{\int_0^t a(u)du} x(t) \\ &= \int_t^{t+\omega} e^{\int_0^s a(u)du} \left[-f(s,x(s)) + a(s)x(s) + g(s,x(s-\tau))\right] ds. \end{aligned} \tag{2.2}$$

As $a(t)$ is ω-periodic, (2.2) can be written as

$$\begin{aligned} &x(t+\omega) - e^{-\int_0^{\omega} a(u)du} x(t) \\ &= \int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du} \left[-f(s,x(s)) + a(s)x(s) + g(s,x(s-\tau))\right] ds. \end{aligned} \tag{2.3}$$

If $x(t)$ is a periodic solution with period ω, then $x(t+\omega) = x(t)$ and (2.3) becomes

$$\begin{aligned} &x(t) - e^{-\int_0^{\omega} a(u)du} x(t) \\ &= \int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du} \left[-f(s,x(s)) + a(s)x(s) + g(s,x(s-\tau))\right] ds, \end{aligned} \tag{2.4}$$

and thus,

$$x(t) = \frac{1}{1 - e^{-\int_0^\omega a(u)du}} \int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du} \left[-f(s, x(s)) + g(s, x(s-\tau))\right] ds.$$

With this motivation for an operator T, the following theorem provides a sufficient condition for the existence of nonnegative periodic solutions of Eq. (1.3).

Theorem 2.1. *Suppose there exists a continuous ω-periodic function $a(t)$ such that*

$$\int_0^\omega a(t)dt > 0 \tag{2.5}$$

and

$$f(t,x) \geq a(t)x, \quad x \geq 0 \tag{2.6}$$

and that $-f(t,x) + a(t)x$ is nonincreasing in x. Suppose also that $g(t,x)$ is nonincreasing in x and that there is a positive constant B such that

$$\int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du}[-f(s,B) + a(s)B + g(s,B)]ds \geq 0, \quad t \in [0,\omega] \tag{2.7}$$

and

$$\frac{1}{1 - e^{-\int_0^\omega a(s)ds}} \int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du} g(s,0)ds \leq B, \quad t \in [0,\omega]. \tag{2.8}$$

Then Eq. (1.3) has a nonnegative ω-periodic solution $\tilde{x}(t)$.

Proof. Let $x(t)$ be a continuous function defined for $t \geq -\tau$ and let

$$X = \{x : x \text{ satisfies } x(t+\omega) = x(t)\}.$$

Then X is a normed vector space with the usual linear operations and norm $\|x\| = \sup_{0 \leq t \leq \omega} |x(t)|$. Let Λ be a subset of X defined by

$$\Lambda = \{x : x \in X \text{ with } 0 \leq x(t) \leq B\},$$

for some positive constant $B > 0$. It is easy to see that Λ is a compact and convex subset of X. Now, define a mapping T on Λ as the following: for each $x \in \Lambda$,

$$Tx(t) = \begin{cases} \frac{1}{1-\hat{a}} \int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du}[-f(s,x(s)) + a(s)x(s) + g(s,x(s-\tau))]ds, \ t \geq 0, \\ Tx(t+n_0\omega), \ -\tau \leq t < 0, \end{cases} \tag{2.9}$$

where $\hat{a} = e^{-\int_0^\omega a(s)ds}$ and n_0 is a positive integer such that $n_0\omega \geq \tau$. Clearly, T is continuous since f, g and a are continuous. We now show that $T : \Lambda \to \Lambda$. By noting

$-f(t,x)+a(t)x$ and $g(t,x)$ are nonincreasing in x, and (2.7) and (2.8) hold, it is easy to see that

$$Tx(t) \geq \frac{1}{1-\hat{a}} \int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du}[-f(s,B)+a(s)B+g(s,B)]ds \geq 0,$$

and

$$Tx(t) \leq \frac{1}{1-\hat{a}} \int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du} g(s,0)ds \leq B.$$

Next, we note that

$$\begin{aligned}
Tx(t+\omega) &= \frac{1}{1-\hat{a}} \int_{t+\omega}^{t+2\omega} e^{-\int_s^{t+2\omega} a(u)du}[-f(s,x(s))+a(s)x(s)+g(s,x(s-\tau))]ds \\
&= \frac{1}{1-\hat{a}} \int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u+\omega)du}\big[-f(s+\omega,x(s+\omega)) \\
&\qquad\qquad + a(s+\omega)x(s+\omega) \\
&\qquad\qquad + g(s+\omega,x(s+\omega-\tau))\big]ds \\
&= \frac{1}{1-\hat{a}} \int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du}[-f(s,x(s)+a(s)x(s)+g(s,x(s-\tau))]ds = Tx(t).
\end{aligned}$$

Hence $Tx(t)$ is ω-periodic and so $Tx \in \Lambda$. By the Schauder Fixed Point Theorem, T has a fixed point $\tilde{x} \in \Lambda$. We claim that $\tilde{x}$ is a solution of Eq. (1.3). By noting f, g and $\tilde{x}$ are ω-periodic in t, and $T\tilde{x}(t) = \tilde{x}(t)$, we see that

$$\begin{aligned}
\frac{d}{dt}T\tilde{x}(t) &= \frac{1}{1-\hat{a}}\Big[-f(t+\omega,\tilde{x}(t+\omega)) + a(t+\omega)\tilde{x}(t+\omega) + g(t+\omega,\tilde{x}(t+\omega-\tau)) \\
&\quad - e^{-\int_t^{t+\omega} a(u)du}[-f(t,\tilde{x}(t)) + a(t)x(t) + g(t,\tilde{x}(t-\tau))] \\
&\quad - a(t+\omega)\int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du}[-f(s,\tilde{x}(s)) + a(s)\tilde{x}(s) + g(s,\tilde{x}(s-\tau))]ds\Big] \\
&= \frac{1}{1-\hat{a}}\Big[(-f(t,x(t)) + a(t)\tilde{x}(t) + g(t,\tilde{x}(t-\tau)))\big(1-e^{-\int_t^{t+\omega} a(u)du}\big)\Big] \\
&\quad - a(t)\tilde{x}(t) \\
&= -f(t,\tilde{x}(t)) + a(t)\tilde{x}(t) + g(t,\tilde{x}(t-\tau)) - a(t)\tilde{x}(t) \\
&= -f(t,T\tilde{x}(t)) + g(t,T\tilde{x}(t-\tau)).
\end{aligned}$$

Hence, $T\tilde{x}(t)$ satisfies Eq. (1.3). Thus, $T\tilde{x}(t) = \tilde{x}(t)$ is a $\omega-$*periodic* solution of Eq. (1.3). The proof is complete. □

3. GLOBAL ATTRACTIVITY OF PERIODIC SOLUTIONS

In this section we establish a sufficient condition for which every nonnegative solution of Eq. (1.1) converges to a nonnegative periodic solution of the associated unforced Eq. (1.3).

First we introduce a lemma that will be needed later in the proof of the main result.

Lemma 3.1. *Suppose $a(t)$ is a continuous, ω-periodic function such that* (2.1) *holds. Then for any fixed constant $\alpha \geq 0$, there exists a constant $M > 0$ such that*

$$\int_{\alpha}^{t} e^{-\int_s^t a(u)du} ds \leq M, \quad t \geq \alpha.$$

Proof. Let $t = n\omega + t^*$, where $n \in \mathbb{N}$ and t^* is such that $0 \leq t^* < \omega$. Then it follows that

$$\int_{\alpha}^{t} e^{-\int_s^t a(u)du} ds = \int_{\alpha}^{n\omega+t^*} e^{-\int_s^{n\omega+t^*} a(u)du} ds$$

$$= \int_{\alpha}^{\alpha+\omega} e^{-\int_s^{n\omega+t^*} a(u)du} ds + \int_{\alpha+\omega}^{\alpha+2\omega} e^{-\int_s^{n\omega+t^*} a(u)du} ds + \dots$$
$$+ \int_{\alpha+n^*\omega}^{n\omega+t^*} e^{-\int_s^{n\omega+t^*} a(u)du} ds,$$

where $n^* \in \mathbb{N}$, $n^* \leq n$ is the largest positive integer such that

$$\alpha + n^*\omega \leq n\omega + t^*$$

and

$$0 \leq n\omega + t^* - (\alpha + n^*\omega) \leq \omega.$$

From the above equality we have

$$\int_{\alpha}^{t} e^{-\int_s^t a(u)du} ds = \int_0^{\omega} e^{-\int_{s+\alpha}^{n\omega+t^*} a(u)du} ds + \ldots + \int_0^{\omega} e^{-\int_{s+\alpha+(n^*-1)\omega}^{n\omega+t^*} a(u)du} ds + \int_0^{n\omega+t^*-(\alpha+n^*\omega)} e^{-\int_{s+\alpha+n^*\omega}^{n\omega+t^*} a(u)du} ds$$

$$\leq \int_0^{\omega} \left(\sum_{j=0}^{n^*} e^{-\int_{s+\alpha+(n^*-j)\omega}^{n\omega+t^*} a(u)du} \right) ds$$

$$\leq \left(\frac{1 - e^{-(n^*+1)\int_0^{\omega} a(u)du}}{1 - e^{-\int_0^{\omega} a(u)du}} \right) \int_0^{\omega} e^{-\int_{s+\alpha+n^*\omega}^{n\omega+t^*} a(u)du} ds$$

$$\leq \frac{1}{1 - e^{-\int_0^{\omega} a(u)du}} \int_0^{\omega} e^{-\int_{s+\alpha+n^*\omega}^{n\omega+t^*} a(u)du} ds$$

$$= \frac{1}{1 - e^{-\int_0^{\omega} a(u)du}} \int_0^{\omega} e^{-\int_{s}^{n\omega+t^*-(\alpha+n^*\omega)} a(u+\alpha)du} ds$$

For the sake of notation, set $\gamma = n\omega + t^* - (\alpha + n^*\omega)$. As $0 \leq \gamma \leq \omega$ and $a(u+\alpha) = a(u+\beta)$, where $0 \leq \beta \leq \omega$,

$$\int_{\alpha}^{t} e^{-\int_s^t a(u)du} ds \leq \left(\frac{1}{1 - e^{-\int_0^{\omega} a(u)du}} \right) \max_{\substack{0\leq\gamma\leq\omega \\ 0\leq\beta\leq\omega}} \left\{ \int_0^{\omega} e^{-\int_s^{\gamma} a(u+\beta)du} ds \right\} = M.$$

The proof of the lemma is complete. □

Theorem 3.2. *Assume that there exists a continuous ω-periodic function $a(t)$ such that (2.1) and (2.2) hold and $-f(t,x) + a(t)x$ is nonincreasing in x. Suppose that $g(t,x)$ is nonincreasing in x and L-Lipschitz for each $t \in [0,\omega]$, i.e. there exists a continuous function $L(t)$ such that for any $x_1, x_2 \geq 0$,*

$$|g(t,x_1) - g(t,x_2)| \leq L(t)|x_1 - x_2|, \quad t \in [0,\omega]. \tag{3.1}$$

Suppose also that either

$$a(t) > 0 \text{ and } \int_t^{t+\tau} L(s) e^{-\int_s^{t+\tau} a(u)du} ds < 1, \quad t \in [0,\omega] \tag{3.2}$$

or

$$\int_t^{t+\tau+\omega} L(s)e^{-\int_s^{t+\tau+\omega} a(u)du} ds < 1, \quad t \in [0, \omega] \tag{3.3}$$

and that

$$\lim_{t\to\infty} r(t) = 0. \tag{3.4}$$

Then if Eq. (1.3) *has a nonnegative ω-periodic solution $\tilde{x}(t)$, then $\tilde{x}(t)$ is the global attractor of every nonnegative solution $x(t)$ of Eq.* (1.1)*, that is,*

$$\lim_{t\to\infty} (x(t) - \tilde{x}(t)) = 0. \tag{3.5}$$

Proof. Let $x(t)$ be a nonnegative solution of Eq. (1.1) and let $y(t) = x(t) - \tilde{x}(t)$. Then $y(t)$ satisfies the equation

$$y'(t) = -f(t, \tilde{x}(t)+y(t))+g(t, \tilde{x}(t-\tau)+y(t-\tau))+f(t, \tilde{x}(t))-g(t, \tilde{x}(t-\tau))+r(t). \tag{3.6}$$

First, assume that $y(t)$ does not oscillate about zero. Then, $y(t)$ is either eventually positive or eventually negative. Assume $y(t)$ is eventually positive. The proof for the case that $y(t)$ is eventually negative is similar and will be omitted. Hence there is a number $t_0 \geq 0$ such that $y(t) > 0$ for $t \geq t_0$. As $-f(t, x) + a(t)x$ is nonincreasing in x, we have

$$-f(t, \tilde{x}(t) + y(t)) + a(t)(\tilde{x}(t) + y(t)) \leq -f(t, \tilde{x}(t)) + a(t)\tilde{x}(t), \quad t \geq t_0,$$

thus

$$-f(t, \tilde{x}(t) + y(t)) + f(t, \tilde{x}(t)) \leq -a(t)y(t), \quad t \geq t_0.$$

Then by noting that g is nonincreasing in x, from (3.6) and the above result we see that $y'(t) \leq -a(t)y(t) + r(t),\ t \geq t_0 + \tau$ and so it follows that

$$y(t) \leq e^{-\int_{t_0+\tau}^t a(u)du} y(t_0 + \tau) + \int_{t_0+\tau}^t e^{-\int_s^t a(u)du} |r(s)|ds, \quad t \geq t_0 + \tau.$$

By noting $\int_{t_0+\tau}^t a(u)du \to \infty$ as $t \to \infty$, we see that

$$e^{-\int_{t_0+\tau}^t a(u)du} y(t_0 + \tau) \to 0 \text{ as } t \to \infty.$$

We now need to show

$$\lim_{t\to\infty} \int_{t_0+\tau}^t e^{-\int_s^t a(u)du} |r(s)|ds = 0. \tag{3.7}$$

As (3.4) holds, there exists fixed constant $M > 0$ and $T_1 > 0$ such that for each $\varepsilon > 0$, $t \geq T_1$ implies that

$$|r(t)| < \frac{\varepsilon}{2M}.$$

We see that

$$\int_{t_0+\tau}^{t} e^{-\int_s^t a(u)du}|r(s)|ds = \int_{t_0+\tau}^{T_1} e^{-\int_s^t a(u)du}|r(s)|ds + \int_{T_1}^{t} e^{-\int_s^t a(u)du}|r(s)|ds$$

$$\leq e^{-\int_0^t a(u)du} \int_{t_0+\tau}^{T_1} e^{\int_0^s a(u)du}|r(s)|ds + \frac{\varepsilon}{2M}\int_{T_1}^{t} e^{-\int_s^t a(u)du}ds.$$

$\int_{t_0+\tau}^{T_1} e^{\int_0^s a(u)du}|r(s)|ds < \infty$ as $a(t)$ and $r(t)$ are continuous, and $e^{-\int_0^t a(u)du} \to 0$ as $t \to \infty$ when (3.2) or (3.3) hold. Thus, $e^{-\int_0^t a(u)du} \int_{t_0+\tau}^{T_1} e^{\int_0^s a(u)du}|r(s)|ds \to 0$ as $t \to \infty$. That is, there exists $T_2 > 0$ such that for each $\varepsilon > 0$, $t \geq T_2$ implies that

$$\int_{t_0+\tau}^{T_1} e^{-\int_s^t a(u)du}|r(s)|ds < \frac{\varepsilon}{2}.$$

From Lemma 1, with $\alpha = T_1$, we see that $\int_{T_1}^{t} e^{-\int_s^t a(u)du}ds \leq M$ as (2.1) holds. Thus for $t \geq T_1$, $\int_{T_1}^{t} e^{-\int_s^t a(u)du}|r(s)|ds < \frac{\varepsilon}{2}$. Let $T = \max\{T_1, T_2\}$. Then for $t \geq T$, $\int_{t_0+\tau}^{t} e^{-\int_s^t a(u)du}|r(s)|ds$ can be written instead as

$$\begin{aligned}&\int_{t_0+\tau}^{t} e^{-\int_s^t a(u)du}|r(s)|ds\\ &= \int_{t_0+\tau}^{T} e^{-\int_s^t a(u)du}|r(s)|ds + \int_{T}^{t} e^{-\int_s^t a(u)du}|r(s)|ds < \varepsilon\end{aligned}$$

and (3.7) holds. Thus $y(t) \to 0$ as $t \to \infty$, and (3.5) holds.

Next, consider the case where $y(t)$ oscillates about 0. Let $\{t_n\}$ and $\{s_n\}$ be sequences of t such that

$$\begin{cases} y(t_1) = 0, & \\ y(t) \geq 0 \text{ for } t_{2k-1} \leq t \leq t_{2k}, & \\ y(t) \leq 0 \text{ for } t_{2k} \leq t \leq t_{2k+1}, & k = 1, 2, \ldots \\ t_k \leq s_k \leq t_{k+1}, & \\ y(t) \text{ has relative extrema at } s_k, & \end{cases}$$

To show (3.5) holds, it suffices to show that $|y(s_n)| \to 0$ as $n \to \infty$.

We claim that for $t_1 \le t \le t_2$,

$$\begin{aligned} y(t) \le &\int_{t_1}^{t} e^{-\int_s^t a(u)du} |g(s, \tilde{x}(s-\tau) + y(s-\tau)) - g(s, \tilde{x}(s-\tau))| ds \\ &+ \int_{t_1}^{t} e^{-\int_s^t a(u)du} |r(s)| ds. \end{aligned} \tag{3.8}$$

As $y(t) \ge 0$ for $t_1 \le t \le t_2$,

$$-f(t, \tilde{x}(t) + y(t)) + a(t)(\tilde{x}(t) + y(t)) \le -f(t, \tilde{x}(t)) + a(t)\tilde{x}(t)$$

and so

$$f(t, \tilde{x}(t) + y(t)) + f(t, \tilde{x}(t)) \le -a(t)y(t).$$

Combining this result with (3.6) gives

$$y'(t) \le -a(t)y(t) + g(t, \tilde{x}(t-\tau) + y(t-\tau)) - g(t, \tilde{x}(t-\tau)) + r(t).$$

Multiplying the inequality by $e^{\int_0^t a(s)ds}$ gives

$$\begin{aligned} &\left(e^{\int_0^t a(s)ds} y(t)\right)' \\ &\le e^{\int_0^t a(s)ds} \left[g(t, \tilde{x}(t-\tau) + y(t-\tau)) - g(t, \tilde{x}(t-\tau))\right] + e^{\int_0^t a(s)ds} r(t). \end{aligned}$$

Integrating from t_1 to t yields

$$\begin{aligned} |y(t)| = y(t) \le &\int_{t_1}^{t} e^{-\int_s^t a(u)du} |g(s, \tilde{x}(s-\tau) + y(s-\tau)) - g(s, \tilde{x}(s-\tau))| ds \\ &+ \int_{t_1}^{t} e^{-\int_s^t a(u)du} |r(s)| ds \end{aligned}$$

and (3.8) holds.

As $g(t, x)$ is ω-periodic in t and (3.10) holds, we see that for $x_1, x_2 \ge 0$,

$$|g(t, x_1) - g(t, x_2)| \le L(t)|x_1 - x_2|, \tag{3.9}$$

where $L(t) = L(t + \omega)$. It then follows from (3.8) that when $t_1 \le t \le t_2$,

$$
\begin{aligned}
y(t) &\le \int_{t_1}^{t} e^{-\int_s^t a(u)du} |g(s,\tilde{x}(s-\tau)+y(s-\tau)) - g(s,\tilde{x}(s-\tau))| ds \\
&\quad + \int_{t_1}^{t} e^{-\int_s^t a(u)du} |r(s)| ds \\
&\le \int_{t_1}^{t} e^{-\int_s^t a(u)ds} L(s) |y(s-\tau)| ds + \int_{t_1}^{t} e^{-\int_s^t a(u)du} |r(s)| ds.
\end{aligned} \tag{3.10}
$$

Suppose (3.2) holds. By the periodic properties of $a(t)$ and $L(t)$, from (3.2) there exists a positive constant $c < 1$ such that

$$
\int_{t}^{t+\tau} L(s) e^{-\int_s^{t+\tau} a(u)du} ds \le c. \tag{3.11}
$$

We now claim for all $t_1 \le t \le t_2$,

$$
y(t) \le c \cdot \max_{t_1-\tau \le s \le t_1} |y(s)| + \int_{t_1}^{t} e^{-\int_s^t a(u)du} |r(s)| ds. \tag{3.12}
$$

In particular, as $t_1 \le s_1 \le t_2$,

$$
y(s_1) \le c \cdot \max_{t_1-\tau \le s \le t_1} |y(s)| + \int_{t_1}^{s_1} e^{-\int_s^{s_1} a(u)du} |r(s)| ds. \tag{3.13}
$$

To this end we consider two cases: $t_2 \le t_1 + \tau$ and $t_2 > t_1 + \tau$. For $t_2 \le t_1 + \tau$, $t - \tau \le t_1$ for all $t_1 \le t \le t_2$. From (3.10) we have

$$
\begin{aligned}
y(t) &\le \int_{t_1}^{t} L(s) e^{-\int_s^t a(u)du} |y(s-\tau)| ds + \int_{t_1}^{t} e^{-\int_s^t a(u)du} |r(s)| ds \\
&\le \int_{t-\tau}^{t} L(s) e^{-\int_s^t a(u)du} ds \cdot \max_{t_1-\tau \le s \le t_1} |y(s)| + \int_{t_1}^{t} e^{-\int_s^t a(u)du} |r(s)| ds.
\end{aligned}
$$

From (3.11) we see that (3.12) and (3.13) hold. Next consider the case $t_2 > t_1 + \tau$. In this case we must consider two different possibilities: $t_1 \le t \le t_1 + \tau$ and $t_1 + \tau < t \le t_2$. If $t_1 \le t \le t_1 + \tau$, as we have shown in the previous case, (3.12) holds. So we only need to consider t such that $t_1 + \tau < t \le t_2$. As $y(t) > 0$ when $t_1 + \tau < t < t_2$, then as before we see that

$$
-f(t,\tilde{x}(t)+y(t)) + f(t,\tilde{x}(t)) \le -a(t)y(t).
$$

As $y(t-\tau) \geq 0$ when $t_1 + \tau < t \leq t_2$, and $g(\cdot, x)$ is nonincreasing, we see that

$$g(t, \tilde{x}(t-\tau) + y(t-\tau)) - g(t, \tilde{x}(t-\tau)) \leq 0.$$

By noting the above results, (3.6) yields

$$\begin{aligned} y'(t) &= -f(t, \tilde{x}(t) + y(t)) + g(t, \tilde{x}(t-\tau) + y(t-\tau)) + f(t, \tilde{x}(t)) - g(t, \tilde{x}(t-\tau)) + r(t) \\ &\leq -a(t)y(t) + r(t). \end{aligned}$$

For $t_1 + \tau < t \leq t_2$, as $y(t) \geq 0$, we have

$$y(t) \leq e^{-\int_{t_1+\tau}^{t} a(u)du} y(t_1+\tau) + \int_{t_1+\tau}^{t} e^{-\int_s^t a(u)du} |r(s)| ds, \quad t_1 + \tau < t \leq t_2.$$

As (3.12) holds for $t = t_1 + \tau$ we have

$$y(t_1 + \tau) \leq c \cdot \max_{t_1 - \tau \leq s \leq t_1} |y(s)| + \int_{t_1}^{t_1+\tau} e^{-\int_s^{t_1+\tau} a(u)du} |r(s)| ds$$

Combining the two inequalities above yields

$$\begin{aligned} y(t) &\leq e^{-\int_{t_1+\tau}^{t} a(u)du} \left(c \cdot \max_{t_1-\tau \leq s \leq t_1} |y(s)| + \int_{t_1}^{t_1+\tau} e^{-\int_s^{t_1+\tau} a(u)du} |r(s)| ds \right) \\ &\quad + \int_{t_1+\tau}^{t} e^{-\int_s^t a(u)du} |r(s)| ds \\ &\leq e^{-\int_{t_1+\tau}^{t} a(u)du} \cdot c \cdot \max_{t_1-\tau \leq s \leq t_1} |y(s)| + \int_{t_1}^{t_1+\tau} e^{-\int_s^t a(u)du} |r(s)| ds \\ &\quad + \int_{t_1+\tau}^{t} e^{-\int_s^t a(u)du} |r(s)| ds \\ &\leq c \cdot \max_{t_1-\tau \leq s \leq t_1} |y(s)| + \int_{t_1}^{t} e^{-\int_s^t a(u)du} |r(s)| ds, \quad t_1 + \tau < t \leq t_2. \end{aligned}$$

So (3.12) holds for $t_1 \leq t \leq t_2$, and thus (3.13) holds. Now consider $t_2 \leq t \leq t_3$. By a similar argument that is omitted, we may show that for $t_2 \leq t \leq t_3$,

$$y(t) \geq -c \cdot \max_{t_2-\tau \leq s \leq t_2} |y(s)| - \int_{t_2}^{t} e^{-\int_s^t a(u)du} |r(s)| ds, \quad t_2 \leq t \leq t_3.$$

In particular,

$$y(s_2) \geq -c \cdot \max_{t_2-\tau \leq s \leq t_2} |y(s)| - \int_{t_2}^{s_2} e^{-\int_s^{s_2} a(u)du} |r(s)| ds. \tag{3.14}$$

Combining (3.13) and (3.14) gives

$$|y(s_n)| \leq c \cdot \max_{t_n-\tau \leq s \leq t_n} |y(s)| + \int_{t_n}^{s_n} e^{-\int_s^{s_n} a(u)du} |r(s)| ds, \quad n = 1, 2.$$

Then by the Method of Steps we have

$$|y(s_n)| \leq c \cdot \max_{t_n-\tau \leq s \leq t_n} |y(s)| + \int_{t_n}^{s_n} e^{-\int_s^{s_n} a(u)du} |r(s)| ds, \quad n = 1, 2, \dots \tag{3.15}$$

Define subsequences $\{t_{n_m}\}$ of $\{t_n\}$ and $\{s_{n_m}\}$ of $\{s_n\}$ such that

$$\begin{cases} \text{for all } n \geq n_1,\ t_n - \tau \geq 0, \\ \text{for all } n \geq n_{m+1},\ t_n - \tau \geq t_{n_m}, \\ t_{n_m} \leq s_{n_m} \leq t_{n_{m+1}}, \text{ and} \\ |y(s_{n_m})| \geq |y(s)|, \quad t_{n_m} \leq s \leq t_{n_{m+1}}, \end{cases} \qquad m = 1, 2, \dots \tag{3.16}$$

As (3.4) holds, $\sup_{s \geq t_n}\{|r(s)|\} \to 0$ as $n \to \infty$. From Lemma 3.1, with $\alpha = t_n$ and $s_n \geq t_n$,

$$\int_{t_n}^{s_n} e^{-\int_s^{s_n} a(u)du} ds \leq M, \quad n = 1, 2, \dots$$

So

$$\lim_{n\to\infty} \int_{t_n}^{s_n} e^{-\int_s^{s_n} a(u)du} |r(s)| ds \leq \lim_{n\to\infty} \left(\sup_{s \geq t_n} \{|r(s)|\} \int_{t_n}^{s_n} e^{-\int_s^{s_n} a(u)du} ds \right) = 0.$$

Then there exists positive constant δ such that $c + \delta < 1$ and

$$\int_{t_n}^{s_n} e^{-\int_s^{s_n} a(u)du} |r(s)| ds < \delta^m, \quad n \geq n_m.$$

From the above result, together with (3.15) we have

$$|y(s_n)| \leq c \cdot \max_{t_n-\tau \leq s \leq t_n} |y(s)| + \delta^m, \quad n \geq n_m. \tag{3.17}$$

Let $B = \max_{0 \le s \le t_{n_1}} |y(s)|$. We claim that

$$|y(s_n)| \le (c+\delta)^m (B+1), \quad n \ge n_m. \tag{3.18}$$

First we show that the result is true for $m = 1$. When $n = n_1$, $t_n - \tau \ge 0$ and

$$\begin{aligned} |y(s_{n_1})| &\le c \cdot \max_{t_{n_1} - \tau \le s \le t_{n_1}} |y(s)| + \delta \\ &\le c \cdot \max_{0 \le s \le t_{n_1}} |y(s)| + \delta \\ &\le cB + \delta \\ &\le (c+\delta)(B+1). \end{aligned}$$

Next we assume

$$|y(s_n)| \le (c+\delta)(B+1), \quad n_1 \le n \le k$$

and show this result holds for $k \le n \le k+1$. As $t_k \le s_k \le t_{k+1}$, from the above assumption we see that $|y(s)| \le B+1$ for $t_{n_1} \le s \le t_{k+1}$. Since $B = \max_{0 \le s \le t_{n_1}} |y(s)|$ we have $|y(s)| \le B+1$ for $0 \le s \le t_{k+1}$. From (3.17), for $k \le n \le k+1$,

$$\begin{aligned} |y(s_n)| &\le c \cdot \max_{t_n - \tau \le s \le t_n} |y(s)| + \delta \\ &\le c \cdot \max_{0 \le s \le t_{k+1}} |y(s)| + \delta \\ &\le c\,(B+1) + \delta \\ &\le (c+\delta)(B+1). \end{aligned}$$

Therefore, by induction, (3.18) holds when $m = 1$.

Next, assume that

$$|y(s_n)| \le (c+\delta)^k (B+1), \quad n \ge n_k.$$

We are going to show that

$$|y(s_n)| \le (c+\delta)^{k+1} (B+1), \quad n \ge n_{k+1}.$$

From (3.16), (3.17), and the above assumption we see that

$$\begin{aligned} |y(s_n)| &\le c \cdot \max_{t_n - \tau \le s \le t_n} |y(s)| + \delta^{k+1} \\ &\le c \cdot \max_{t_{n_k} \le s \le t_n} |y(s)| + \delta^{k+1} \\ &\le c \cdot (c+\delta)^k (B+1) + \delta^{k+1}, \quad n \ge n_{k+1}. \end{aligned}$$

As

$$c \cdot (c+\delta)^k (B+1) + \delta^{k+1} \le (c+\delta)^{k+1} (B+1),$$

the above inequalities yield

$$|y(s_n)| \le (c+\delta)^{k+1} (B+1), \quad n \ge n_{k+1}.$$

Thus, by induction, we see that (3.18) holds. This implies that $y(s_n) \to 0$ as $n \to \infty$, and thus (3.5) holds.

Next suppose first that (3.3) holds. By the periodic property of $a(t)$ and $L(t)$, there exists a positive constant $d < 1$ such that

$$\int_{t}^{t+\tau+\omega} L(s)e^{-\int_s^{t+\tau+\omega} a(u)du} ds \le d. \tag{3.19}$$

We claim that

$$|y(t)| \le d \cdot \max_{t_1-\tau\le s\le t_1+\omega} |y(s)| + \int_{t_1}^{t} e^{-\int_s^t a(u)du}|r(s)|ds, \quad t_1 \le t \le t_2. \tag{3.20}$$

In particular, as $t_1 \le s_1 \le t_2$ we will show that

$$|y(s_1)| \le d \cdot \max_{t_1-\tau\le s\le t_1+\omega} |y(s)| + \int_{t_1}^{s_1} e^{-\int_s^{s_1} a(u)du}|r(s)|ds. \tag{3.21}$$

To this end, we consider two cases: $t_2 \le t_1 + \tau + \omega$ and $t_2 > t_1 + \tau + \omega$. When $t_2 \le t_1 + \tau + \omega$, then for any $t_1 \le t \le t_2$, we have $t - \tau - \omega \le t_1$, and so (3.8) yields

$$\begin{aligned}
|y(t)| = y(t) &\le \int_{t_1}^{t} e^{-\int_s^t a(u)du}|g(s, \tilde{x}(s-\tau) + y(s-\tau)) - g(s, \tilde{x}(s-\tau))|ds \\
&\quad + \int_{t_1}^{t} e^{-\int_s^t a(u)du}|r(s)|ds \\
&\le \int_{t_1}^{t} L(s)e^{-\int_s^t a(u)du}|y(s-\tau)|ds + \int_{t_1}^{t} e^{-\int_s^t a(u)du}|r(s)|ds \\
&\le \int_{t-\tau-\omega}^{t} L(s)e^{-\int_s^t a(u)du}ds \cdot \max_{t_1-\tau\le s\le t_1+\omega} |y(s)| + \int_{t_1}^{t} e^{-\int_s^t a(u)du}|r(s)|ds.
\end{aligned}$$

By noting (3.19) we see that (3.20), and thus (3.21), hold.

Next, consider $t_2 > t_1 + \tau + \omega$. When $t_1 < t \le t_1 + \tau + \omega$, as we have shown in the previous case, (3.20) holds. So we need only to consider t such that $t_1 + \tau + \omega < t \le t_2$. By a similar argument as used when (3.2) holds,

$$y'(t) \le -a(t)y(t) + r(t), \quad t_1 + \tau < t \le t_2. \tag{3.22}$$

For $t_1 + \tau < t \le t_2$, as $y(t) \ge 0$, (3.22) gives

$$|y(t)| = y(t) \le e^{-\int_{t_1+\tau}^t a(s)ds} y(t_1+\tau) + \int_{t_1+\tau}^{t} e^{-\int_s^t a(u)du}|r(s)|ds, \quad t_1 + \tau < t \le t_2.$$

For $t = t_1 + \tau + \omega$, the inequality above yields

$$y(t_1+\tau+\omega) \le e^{-\int_{t_1+\tau}^{t_1+\tau+\omega} a(s)ds} y(t_1+\tau) + \int\limits_{t_1+\tau}^{t_1+\tau+\omega} e^{-\int_s^{t_1+\tau+\omega} a(u)du} |r(s)| ds.$$

From (3.20) we see that

$$y(t_1+\tau) \le d \cdot \max_{t_1-\tau \le s \le t_1+\omega} |y(s)| + \int\limits_{t_1}^{t_1+\tau} e^{-\int_s^{t_1+\tau} a(u)du} |r(s)| ds,$$

and

$$\begin{aligned} e^{-\int_{t_1+\tau}^{t_1+\tau+\omega} a(s)ds} y(t_1+\tau) &\le e^{-\int_{t_1+\tau}^{t_1+\tau+\omega} a(s)ds} \cdot d \cdot \max_{t_1-\tau \le s \le t_1+\omega} |y(s)| \\ &\quad + \int\limits_{t_1}^{t_1+\tau} e^{-\int_s^{t_1+\tau+\omega} a(u)du} |r(s)| ds \\ &\le d \cdot \max_{t_1-\tau \le s \le t_1+\omega} |y(s)| \\ &\quad + \int\limits_{t_1}^{t_1+\tau} e^{-\int_s^{t_1+\tau+\omega} a(u)du} |r(s)| ds. \end{aligned}$$

Thus

$$y(t_1+\tau+\omega) \le d \cdot \max_{t_1-\tau \le s \le t_1+\omega} |y(s)| + \int\limits_{t_1}^{t_1+\tau+\omega} e^{-\int_s^{t_1+\tau+\omega} a(u)du} |r(s)| ds.$$

Similarly, for any positive k such that $k < \omega$ and $t_1 + \tau + \omega + k \le t_2$ we have

$$\begin{aligned} y(t_1+\tau+\omega+k) &\le e^{-\int_{t_1+\tau+k}^{t_1+\tau+\omega+k} a(s)ds} y(t_1+\tau+k) + \\ &\quad \int\limits_{t_1+\tau+k}^{t_1+\tau+\omega+k} e^{-\int_s^{t_1+\tau+\omega+k} a(u)du} |r(s)| ds \\ &\le d \cdot \max_{t_1-\tau \le s \le t_1+\omega} |y(s)| + \int\limits_{t_1}^{t_1+\tau+\omega+k} e^{-\int_s^{t_1+\tau+\omega+k} a(u)du} |r(s)| ds. \end{aligned}$$

If $k \ge \omega$, we write $k = k^* + n\omega$, where $k^* < \omega$, $n = 1, 2, \ldots$. Then for $k \ge \omega$ such that $t_1 + \tau + \omega + k \le t_2$, $y(t_1+\tau+\omega+k) = y(t_1+\tau+(n+1)\omega+k^*)$. From the above inequalities we have

$$
\begin{aligned}
y(t_1+\tau+\omega+k) &\le e^{-\int_{t_1+\tau+n\omega+k^*}^{t_1+\tau+(n+1)\omega+k^*} a(s)ds} y(t_1+\tau+n\omega+k^*) \\
&\quad + \int\limits_{t_1+\tau+n\omega+k^*}^{t_1+\tau+(n+1)\omega+k^*} e^{-\int_s^{t_1+\tau+(n+1)\omega+k^*} a(u)du} |r(s)|ds \\
&\le e^{-\int_{t_1+\tau+(n-1)\omega+k^*}^{t_1+\tau+(n+1)\omega+k^*} a(s)ds} y(t_1+\tau+(n-1)\omega+k^*) \\
&\quad + \int\limits_{t_1+\tau+(n-1)\omega+k^*}^{t_1+\tau+(n+1)\omega+k^*} e^{-\int_s^{t_1+\tau+(n+1)\omega+k^*} a(u)du} |r(s)|ds \\
&\vdots \\
&\le e^{-\int_{t_1+\tau+k^*}^{t_1+\tau+(n+1)\omega+k^*} a(s)ds} y(t_1+\tau+k^*) \\
&\quad + \int\limits_{t_1+\tau+k^*}^{t_1+\tau+(n+1)\omega+k^*} e^{-\int_s^{t_1+\tau+(n+1)\omega+k^*} a(u)du} |r(s)|ds.
\end{aligned}
$$

and thus

$$
\begin{aligned}
y(t_1+\tau+\omega+k) &= y(t_1+\tau+(n+1)\omega+k^*) \\
&\le d\cdot \max_{t_1-\tau\le s\le t_1+\omega} |y(s)| \\
&\quad + \int\limits_{t_1}^{t_1+\tau+(n+1)\omega+k^*} e^{-\int_s^{t_1+\tau+(n+1)\omega+k^*} a(u)du} |r(s)|ds.
\end{aligned}
$$

So then for all $t_1+\tau+\omega < t \le t_2$, as t is of the form $t_1+\tau+(n+1)\omega+k^*$, we see from the above inequalities that

$$
y(t) \le d\cdot \max_{t_1-\tau\le s\le t_1+\omega} |y(s)| + \int\limits_{t_1}^{t} e^{-\int_s^t a(u)du} |r(s)|ds.
$$

and (3.20) holds for $t_1 \le t \le t_2$. As $t_1 \le s_1 \le t_2$, (3.21) holds as well. By an argument similar to that found in the proof of the case where (3.2) holds, we can show that

$$
|y(s_n)| \le d\cdot \max_{t_n-\tau\le s\le t_n+\omega} |y(s)| + \int\limits_{t_n}^{s_n} e^{-\int_s^{s_n} a(u)du} |r(s)|ds, \quad n=1,2,\dots \tag{3.23}
$$

We choose a positive constant η such that $d+\eta<1$. From Lemma 3.1, in light of (3.16), (3.23) becomes

$$
|y(s_n)| \le d\cdot \max_{t_n-\tau\le s\le t_n+\omega} |y(s)| + \eta^m, \quad n \ge n_m.
$$

Again by an argument similar to when (3.2) holds, we show that $y(s_n) \to 0$ as $n \to \infty$. Thus (3.5) holds and the proof is complete. $\square$

The following result is a direct consequence of combining Theorem 2.1 and Theorem 3.2.

Theorem 3.3. *Suppose there exists a continuous ω-periodic function $a(t)$ such that* (2.1) *and* (2.2) *hold and that $-f(t,x)+a(t)x$ is nonincreasing in x. Suppose also that there is a positive constant $B>0$ such that* (2.7) *and* (2.8) *hold. Suppose also that $g(t,x)$ is nonincreasing in x and that there exists continuous function $L(t)$ such that* (3.1) *and either* (3.2) *or* (3.3) *holds. Then Eq.* (1.3) *has a unique nonnegative ω-periodic solution $\tilde{x}(t)$, where $\tilde{x}(t)$ is a global attractor of all nonnegative solutions of Eq.* (1.3)*. Suppose also that $r(t)$ satisfies* (3.4)*. Then $\tilde{x}(t)$ is a global attractor of all nonnegative solutions $x(t)$ of Eq.* (1.1)*.*

When $g(t,x)=b(t)h(x)$, where $b(t):[0,\infty)\to[0,\infty)$ is a continuous ω-periodic function and $h(x):[0,\infty)\to[0,\infty)$ is a continuous function, Eq. (1.1) and Eq. (1.3) reduce to

$$x'(t) = -f(t,x(t)) + b(t)h(x(t-\tau)) + r(t), \quad t \ge 0 \tag{3.24}$$

and

$$x'(t) = -f(t,x(t)) + b(t)h(x(t-\tau)), \quad t \ge 0 \tag{3.25}$$

respectively. Clearly, $g(t,x)$ is L-Lipschitz with $L(t)=b(t)$. Hence, the following conclusion is a direct consequence of Theorem 3.3.

Corollary 3.4. *Suppose there exists a continuous ω-periodic function $a(t)$ such that* (2.1) *and* (2.2) *hold and that $-f(t,x)+a(t)x$ is nonincreasing in x. Suppose also that there is a positive constant $B>0$ such that*

$$\int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)du}\left[-f(s,B)+a(s)B+b(s)h(B)\right]ds \ge 0, \quad t\in[0,\omega]$$

and

$$\frac{1}{1-e^{-\int_0^{\omega} a(s)ds}} \int_t^{t+\omega} e^{-\int_s^{t+\omega} a(u)} b(s)h(0)ds \le B, \quad t\in[0,\omega].$$

Suppose also that $h(x)$ is nonincreasing in x and that either

$$a(t)>0 \text{ and } \int_t^{t+\tau} b(s)e^{-\int_s^{t+\tau} a(u)du}ds < 1$$

or

$$\int_t^{t+\tau+\omega} b(s)e^{-\int_s^{t+\tau+\omega} a(u)du}ds < 1$$

and that

$$\lim_{t\to\infty} r(t) = 0.$$

Then Eq. (3.25) *has a unique nonnegative ω-periodic solution $\tilde{x}(t)$, and $\tilde{x}(t)$ is a global attractor of all nonnegative solutions of Eq.* (3.24) *and Eq.* (3.25)*.*

When $f(t,x) = a(t)x$, Eq. (1.1) and Eq. (1.3) reduce to

$$x'(t) = -a(t)x(t) + g(t, x(t-\tau)) + r(t) \tag{3.26}$$

and

$$x'(t) = -a(t)x(t) + g(t, x(t-\tau)), \tag{3.27}$$

respectively. As (2.2) cleary holds and $-f(t,x) + a(t)x = 0$ is obviously nonincreasing, (2.7) holds for any $B > 0$ and (2.8) holds for B large enough, the following is a direct consequence of Theorem 3.3.

Corollary 3.5. *Assume $\int_0^\omega a(t)dt > 0$ and suppose that $g(t,x)$ is nonincreasing in x and that there exists a continuous function $L(t)$ such that* (3.1) *and either* (3.2) *or* (3.3) *holds. Suppose also that* (3.4) *holds. Then Eq.* (3.27) *has a unique nonnegative ω-periodic solution $\tilde{x}(t)$, and $\tilde{x}(t)$ is a global attractor of all nonnegative solutions of Eq.* (3.26) *and Eq.* (3.27)*.*

When $f(t,x) = a(t)x$ and $g(t,x) = b(t)h(x)$ as above, Eq. (1.1) and Eq. (1.3) reduce to

$$x'(t) = -a(t)x(t) + b(t)h(x(t-\tau)) + r(t) \tag{3.28}$$

and

$$x'(t) = -a(t)x(t) + b(t)h(x(t-\tau)), \tag{3.29}$$

respectively. The following conclusion is a direct result of combining Corollary 3.4 and Corollary 3.5, which will be useful in the next section.

Corollary 3.6. *Assume $\int_0^\omega a(t)dt > 0$ and suppose that $h(x)$ is nonincreasing in x and that*

$$a(t) > 0 \text{ and } \int_t^{t+\tau} b(s)e^{-\int_s^{t+\tau} a(u)du} ds < 1$$

or

$$\int_t^{t+\tau+\omega} b(s)e^{-\int_s^{t+\tau+\omega} a(u)du} ds < 1$$

and that (3.4) *holds. Then Eq.* (3.29) *has a unique nonnegative ω-periodic solution $\tilde{x}(t)$, and $\tilde{x}(t)$ is a global attractor of all nonnegative solutions of Eq.* (3.28) *and Eq.* (3.29)*.*

4. APPLICATIONS AND EXAMPLES

In this section, we apply results found in the last section to equations derived from mathematical biology and give examples to demonstrate results found in the previous section.

Consider the differential equation

$$x'(t) = -a(t)x(t) + \frac{b(t)}{1 + x^\gamma(t-\tau)} + r(t), \quad t \geq 0, \tag{4.1}$$

where $a(t)$ is a continuous ω-periodic function, $b(t)$ is a nonnegative continuous ω-periodic function, $r(t)$ is a continuous function and τ is a positive constant. When $r(t) \equiv 0$ Eq. (4.1) reduces to

$$x'(t) = -a(t)x(t) + \frac{b(t)}{1 + x^\gamma(t-\tau)}, \quad t \geq 0. \tag{4.2}$$

When $a(t) \equiv a$, $b(t) \equiv b$ are positive constants, (4.1) and (4.2) become

$$x'(t) = -ax(t) + \frac{b}{1 + x^\gamma(t-\tau)} + r(t), \quad t \geq 0 \tag{4.3}$$

and

$$x'(t) = -ax(t) + \frac{b}{1 + x^\gamma(t-\tau)}, \quad t \geq 0, \tag{4.4}$$

respectively. Eq. (4.4) is a model used to study blood cell production. When $r(t) \not\equiv 0$, the function $r(t)$ may represent the medical replacement rate of blood cells or destruction rate of blood cells due to administration of antibodies as a function of t, see [6, 12] and references cited therein. While the global attractivity of positive solutions of Eq. (4.4) has been studied quite extensively, results are relatively scarce when considering the model with periodic coefficients or the model with a nonzero forcing term as previously described.

We see that Eq. (4.1) is in the form of Eq. (3.29) with $h(x) = \frac{1}{1+x^\gamma}$. By combining results from [16] for Eq. (4.2) with Corollary 3.6 in the previous section we have the following result.

Corollary 4.1. *Suppose that*

$$a(t) > 0, \ \gamma = 1, \ \textit{and} \ \int_t^{t+\tau} b(s)e^{-\int_s^{t+\tau} a(u)du} ds < 1, \quad t \in [0, \omega],$$

or

$$a(t) > 0, \ \gamma > 1, \ \textit{and} \ \frac{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}}{4\gamma} \int_t^{t+\tau} b(s)e^{-\int_s^{t+\tau} a(u)du} ds < 1, \quad t \in [0, \omega],$$

or

$$\int_0^{\omega} a(t)dt > 0, \quad \gamma = 1, \; and \quad \int_t^{t+\tau+\omega} b(s)e^{-\int_s^{t+\tau+\omega} a(u)du} ds < 1, \quad t \in [0,\omega],$$

or

$$\int_0^{\omega} a(t)dt > 0, \quad \gamma > 1, \; and$$

$$\frac{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}}{4\gamma} \int_t^{t+\tau+\omega} b(s)e^{-\int_s^{t+\tau+\omega} a(u)du} ds < 1, \quad t \in [0,\omega],$$

and that $r(t)$ *satisfies* (3.4)*. Then Eq.* (4.2) *has a unique nonnegative* ω*-periodic solution* $\tilde{x}(t)$ *and every nonnegative solution* $x_1(t)$ *of Eq.* (4.1) *and* $x_2(t)$ *of Eq.* (4.2) *converges to* $\tilde{x}(t)$*.*

The following are examples to further demonstrate the results of Theorem 3.3 and its corollaries.

Example 4.2. Consider the differential equations

$$x'(t) = -(0.1\sin 12t + 0.1)\left(\frac{10x^2(t)+9x(t)+1}{9x(t)+7}\right) + \frac{0.1\cos 4t}{1+x^2(t-1)} + te^{-2t} \tag{4.5}$$

and

$$x'(t) = -(0.1\sin 12t + 0.1)\left(\frac{10x^2(t)+9x(t)+1}{9x(t)+7}\right) + \frac{0.1\cos 4t}{1+x^2(t-1)}. \tag{4.6}$$

Eq. (4.5) takes the form of Eq. (3.24) with $f(t,x) = (0.1\sin 12t + 0.1)\left(\frac{10x^2(t)+9x(t)+1}{9x(t)+7}\right)$, $b(t) = 0.1\cos 4t + 0.1$, $h(x) = \frac{1}{1+x^2}$, $r(t) = te^{-2t}$, $\tau = 1$ and $\omega = \frac{\pi}{2}$. When $r(t) \equiv 0$, Eq. (4.5) reduces to Eq. (4.6) which takes the form of Eq. (3.25). For $a(t) = 0.1\sin 12t + 0.1$, we see that $\int_0^{\omega} a(t)dt > 0$, while $r(t)$ satisfies $\lim\limits_{t\to\infty} r(t) = 0$. As $h'(x) = \frac{-2x}{(1+x^2)^2} < 0$, $h(x)$ is nonincreasing in x for $x \geq 0$. We numerically verify that (2.7) and (2.8) hold for $B = 1.2$ and that

$$\int_t^{t+1+\frac{\pi}{2}} (0.1\cos 4s + 0.1)e^{-\int_s^{t+1+\frac{\pi}{2}} (0.1\sin 12u+0.1)du} ds < 1, \quad t \in \left[0, \frac{\pi}{2}\right].$$

From Corollary 3.4, we see that Eq. (4.6) has a unique nonnegative $\frac{\pi}{2}$-periodic solution $\tilde{x}(t)$, and every nonnegative solution of Eq. (4.5) and Eq. (4.6) converges to $\tilde{x}(t)$. The solution for a particular initial condition is shown in Figure 1.

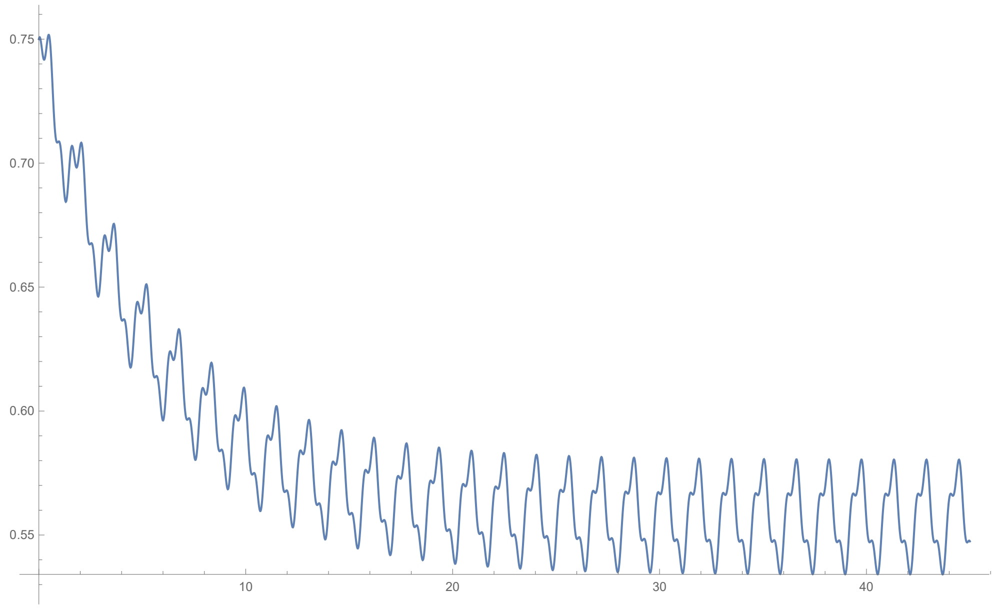

Fig. 1. Graph of solution $x(t)$ of Eq. (4.5) with initial function $\phi(t) = 0.1t + 0.75$

Example 4.3. Consider the differential equations

$$x'(t) = -\left(0.15 \sin\left(\frac{t}{2}\right) + 0.10\right) x(t) + \frac{0.1\cos(\frac{t}{2}) + 0.1}{1 + x^{1.5}(t-2)} + te^{-t}, \quad t \geq 0 \tag{4.7}$$

and

$$x'(t) = -\left(0.15 \sin\left(\frac{t}{2}\right) + 0.10\right) x(t) + \frac{0.1\cos(\frac{t}{2}) + 0.1}{1 + x^{1.5}(t-2)}, \quad t \geq 0. \tag{4.8}$$

Eq. (4.7) takes the form of Eq. (4.1) with $a(t) = 0.15\sin\left(\frac{t}{2}\right) + 0.10$, $b(t) = 0.1\cos\left(\frac{t}{2}\right) + 0.1$, $r(t) = te^{-t}$, $\gamma = 1.5$, $\tau = 2$, and $\omega = 4\pi$. When $r(t) \equiv 0$, Eq. (4.7) reduces to Eq. (4.8), which takes the form of Eq. (4.2). Clearly, $\int_0^\omega a(t)dt > 0$ and $\lim_{t\to\infty} r(t) = 0$. As $\gamma > 1$ we numerically verify that

$$\begin{aligned}
&\frac{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}}{4\gamma} \int_t^{t+\tau+\omega} b(s)e^{-\int_s^{t+\tau+\omega} a(u)du} ds \\
&= \frac{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}}{4\gamma} \int_t^{t+2+4\pi} \left(0.1\cos\left(\frac{s}{2}\right) + 0.1\right) e^{-\int_s^{t+2+4\pi}\left(0.15\sin\left(\frac{u}{2}\right)+0.10\right)du} ds \\
&< 1, \quad t \in [0, 4\pi].
\end{aligned}$$

From Corollary 4.1, we see that Eq. (4.8) has a unique nonnegative 4π-periodic solution $\tilde{x}(t)$ and every nonnegative solution of Eq. (4.7) and Eq. (4.8) converges to $\tilde{x}(t)$. The solution for a particular initial condition is shown in Figure 2.

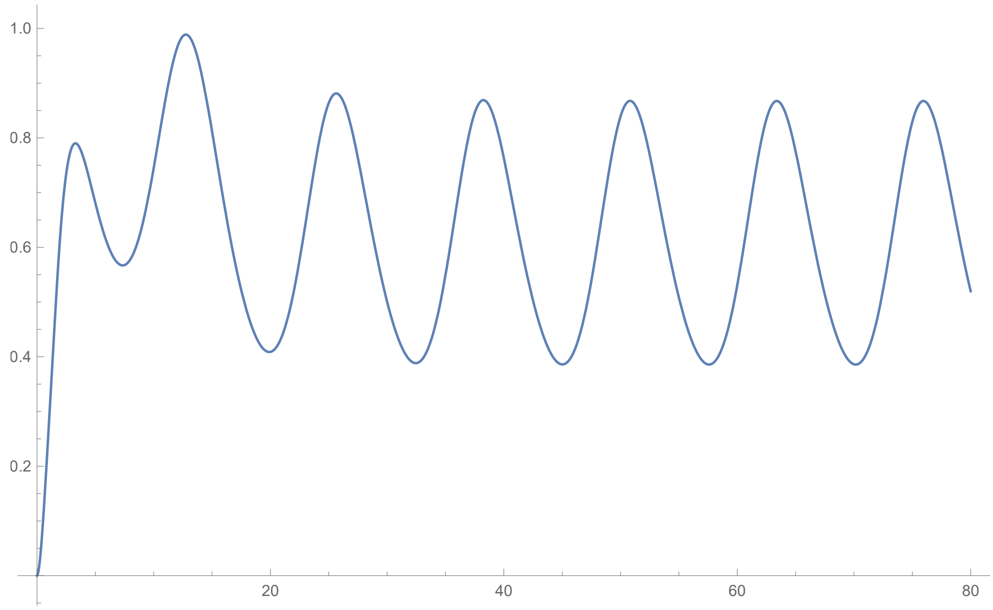

Fig. 2. Graph of solution $x(t)$ of Eq. (4.7) with initial function $\phi(t) = 2t^2$

Acknowledgements

The authors are indebted to the referee for carefully reading the manuscript and providing helpful suggestions.

REFERENCES

[1] R.P. Agarwal, L. Berezansky, E. Braverman, A. Domoshnitsky, *Nonoscillation Theory of Functional Differential Equations with Applications*, Springer, New York, 2012.

[2] L. Berezansky, E. Braverman, A. Domoshnitsky, *First order functional differential equations: nonoscillation and positivity of Green's functions*, Funct. Diff. Equ. **15** (2008), 57–94.

[3] L. Berezansky, E. Braverman, L. Idels, *Nicholson's blowflies differential equations revisited: Main results and open problems*, Appl. Math. Model. **34** (2010), 1405–1417.

[4] L. Berezansky, E. Braverman, L. Idels, *Mackey–Glass model of hematopoiesis with monotone feedback revisited*, Appl. Math. Comput. **219** (2013), 4892–4907.

[5] F. Brauer, C. Castillo-Chávez, *Mathematical models in population biology and epidemiology*, Texts in Applied Mathematics, vol. 40, Springer-Verlag, New York, 2001.

[6] E. Braverman, S.H. Saker, *On a difference equation with exponentially decreasing nonlinearity*, Discrete Dyn. Nat. Soc. (2011), Art. ID 147926, 17 pp.

[7] H.-S. Ding, M.-X. Ji, *Pseudo-almost periodic solutions for a discrete Nicholson's blowflies model with harvesting term*, Adv. Difference Equ. (2016), Paper no. 289, 11 pp.

[8] A. Domoshnitsky, *Maximum principles and nonoscillation intervals for first order Volterra functional differential equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **15** (2008), 769–814.

[9] S.A. Gusarenko, A.I. Domoshnitskii, *Asymptotic and oscillation properties of first order linear scalar functional-differential equations*, Differentsial'nye Uravneniya **25** (1989), 2090–2103.

[10] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations*, Oxford University Press, New York, 1991.

[11] D.D. Hai, C. Qian, *On global convergence of forced nonlinear delay differential equations and applications*, Differ. Equ. Appl. **9** (2017), 13–28.

[12] G. Kiss, G. Röst, *Controlling Mackey–Glass chaos*, Chaos **27** (2017), 114321, 7 pp.

[13] M. Mackey, L. Glass, *Oscillation and chaos in physiological control systems*, Science **197** (1977), 287–289.

[14] R. Olach, *Positive periodic solutions of delay differential equations*, Appl. Math. Lett. **26** (2013), 1141–1145.

[15] S. Padhi, S. Pati, R. Kumar, *Positive periodic solutions of Nicholson's blowflies model with harvesting*, PanAmerican Math. Journal **24** (2014), 15–26.

[16] C. Qian, *Global attractivity of periodic solutions in a delay differential equation*, Comm. in Appl. Anal. **18** (2014), 253–260.

[17] A. Wan, D. Jiang, X. Xu, *A new existence theory for positive periodic solutions to functional differential equations*, Comput. Math. Appl. **47** (2004), 1257–1262.

[18] W. Wang, *Positive periodic solutions of delayed Nicholson's blowflies models with a nonlinear density-dependent mortality term*, Appl. Math. Model. **36** (2012), 4708–4713.

Chuanxi Qian
qian@math.msstate.edu

Mississippi State University
Department of Mathematics and Statistics
Mississippi State, MS 39762, U.S.A.

Justin Smith

Mississippi State University
Department of Mathematics and Statistics
Mississippi State, MS 39762, U.S.A.

Received: February 26, 2019.
Revised: August 8, 2019.
Accepted: August 12, 2019.