LIGHTWEIGHT PATHS IN GRAPHS

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Abstract. Let $k$ be a positive integer, $G$ be a graph on $V(G)$ containing a path on $k$ vertices, and $w$ be a weight function assigning each vertex $v \in V(G)$ a real weight $w(v)$. Upper bounds on the weight $w(P) = \sum_{v \in V(P)} w(v)$ of $P$ are presented, where $P$ is chosen among all paths of $G$ on $k$ vertices with smallest weight.

Keywords: weighted graph, lightweight path.

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1. INTRODUCTION

We use standard terminology of graph theory and consider finite and simple graphs, where $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph $G$, respectively. It is well known that every planar graph $G$ contains a vertex $v$ such that the degree $d_G(v)$ of $v$ (in $G$) is at most 5. In 1955, Kotzig [7, 8] proved that every 3-connected planar graph $G$ contains an edge $uv$ such that $d_G(u) + d_G(v)$ is at most 13 in general and at most 11 in absence of 3-valent vertices. Moreover, these bounds are best possible. Given a positive integer $k$ and a graph $G$, a $k$-path of $G$ is a path of $G$ on $k$ vertices. Motivated by the previous results, for some positive integer $k$, upper bounds on a lightweight $k$-path of a planar graph were established, where the weight of a path $P$ of a graph $G$ is the sum of the degrees (in $G$) of the vertices of $P$. For example, Fabrici and Jendrol’ [4] proved that any 3-connected planar graph containing a $k$-path has a $k$-path of weight at most $5k^2$. This result has been strengthened by Fabrici, Harant, and Jendrol’ in [3] showing that the upper bound $5k^2$ can be replaced with $\frac{1}{2}k^2 + O(k)$ in general and with $k^2 + O(k)$ in the case of plane triangulations. Mohar [9] proved that any 4-connected planar graph of order at least $k$ contains a $k$-path of weight at most $6k - 1$, which is tight.

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Here the task is generalized by considering arbitrary graphs vertex-weighted by arbitrary real numbers. Let $w : V(G) \to \mathbb{R}$ be a fixed weight function assigning each vertex $v \in V(G)$ of a graph $G$ a real weight $w(v)$,

$$d_w = \frac{\sum_{v \in V(G)} w(v)}{|V(G)|}$$

be the average weight of $G$, and

$$w(P) = \sum_{v \in V(P)} w(v)$$

be the weight of a path $P$ of $G$.

In the sequel, we are interested (for some $k$) in a $k$-path $P$ of $G$ of smallest weight. Obviously, we may assume that $G$ is connected. If $G$ is a tree, then $P$ is a subpath of the (unique) path connecting two suitable leaves of $G$, thus, in this case it is easy to find $P$. Hence, throughout the paper, we assume that $G$ is a connected graph with size $m = |E(G)|$ at least $n = |V(G)|$. Let $\mathcal{H}(G)$ be the set of subgraphs $H$ of $G$ of positive size such that every component of $H$ is bridgeless. Since a cycle of $G$ is a bridgeless subgraph of $G$ of positive size, it follows that $\mathcal{H}(G)$ is not empty. By $\text{girth}(G)$ we denote the length of a shortest cycle of $G$.

The basic tool we use is the rotation of a $k$-path of $G$ around a cycle of $G$ on at least $k$ vertices. This idea was introduced by Mohar in [9]. Since a 4-connected planar graph $G$ contains a hamiltonian cycle $C$ ([10]), i.e. $C \in \mathcal{H}(G)$, the above mentioned result of Mohar follows from the forthcoming Theorem 2.1, which is our main result.

2. RESULTS AND PROOFS

**Theorem 2.1.** Let $t$ be a real number, $H \in \mathcal{H}(G)$, and $1 \leq k \leq \text{girth}(H)$. Then $H$ contains a $k$-path $P$ such that

$$w(P) \leq \frac{\sum_{v \in V(H)} d_H(v)w(v)}{2|E(H)|} k = \left( d_w + \frac{\sum_{v \in V(H)} d_H(v)(w(v) - d_w)}{2|E(H)|} \right) k.$$

Moreover, if $H$ is spanning, then

$$w(P) \leq \left( d_w + \frac{\sum_{v \in V(H)} (d_H(v) - t)(w(v) - d_w)}{2|E(H)|} \right) k.$$

**Proof.** Given a positive integer $s$, a cycle $s$-cover of a graph $G$ is a multiset of cycles of $G$ that each edge of $G$ is contained in exactly $s$ of these cycles.

For instance, for any 2-connected planar graph, the faces provide a cycle 2-cover of the graph: each edge belongs to exactly two faces.

It is an unsolved problem (posed by G. Szekeres and P.D. Seymour and known as the Cycle Double Cover Conjecture), whether every bridgeless graph has a cycle
2-cover, however, Bermond, Jackson, and Jaeger [1] proved that every bridgeless graph has a cycle 4-cover.

For the proof of Theorem 2.1, we first construct a non-empty multiset $\Pi$ of $k$-paths of $H$ and show that the arithmetical mean of all values $w(P)$ taken over all $P \in \Pi$ equals $\frac{\sum_{v \in V(H)} d_H(v)w(v)}{2|E(H)|}k$.

Consider an arbitrary component $F$ of $H$. If $F$ consists of a single vertex only, then $F$ does not contribute to the expression $\frac{\sum_{v \in V(H)} d_H(v)w(v)}{2|E(H)|}$. Since $H \in \mathcal{H}(G)$, we may assume that $|V(F)| \geq 3$ and that $F$ is bridgeless.

For a cycle $C$ of a fixed cycle 4-cover of $F$, let $R_C$ be the set of $k$-paths rotating around $C$ (note that $|V(C)| \geq \text{girth}(H)$). If $C$ is an $i$-cycle, then $|R_C| = i$. For the multiset $\Pi_F = \bigcup_C R_C$ of $k$-paths we have $|\Pi_F| = \sum_C |R_C| = 4|E(F)|$. Let $\Pi = \bigcup_{F \in \mathcal{V}(F) \geq 3} \Pi_F$ and it follows $|\Pi| = 4|E(H)|$.

Every vertex $v \in V(F)$ belongs to exactly $\frac{4d_H(v)}{2} = 2d_H(v)$ cycles of the cycle 4-cover of $F$, thus, $v \in V(F)$ belongs to exactly $2 \cdot d_H(v)k$ paths of $\Pi$, hence,

$$\sum_{P \in \Pi} w(P) = \left(2 \sum_{v \in V(H)} d_H(v)w(v)\right)k.$$ 

Eventually, the equality

$$\frac{\sum_{v \in V(H)} d_H(v)w(v)}{2|E(H)|} = d_w + \frac{\sum_{v \in V(H)} d_H(v)(w(v) - d_w)}{2|E(H)|}$$

is clear and, if $H$ is spanning, then

$$d_w + \frac{\sum_{v \in V(G)} d_H(v)(w(v) - d_w)}{2|E(H)|} = d_w + \frac{\sum_{v \in V(G)} (d_H(v) - t)(w(v) - d_w)}{2|E(H)|}$$

because

$$\sum_{v \in V(G)} (w(v) - d_w) = 0.$$

We remark, that in the second part of Theorem 2.1 the assumption that $H$ is spanning is not really a restriction. To see this, let $v$ be a vertex of $G$ not belonging to $H$. Then, as already mentioned, adding $v$ to $H$ as an additional component of $H$ consisting of $v$ only preserves all assumptions on $H$ and does not change the value of $\frac{\sum_{v \in V(H)} d_H(v)w(v)}{2|E(H)|}$.

If $G$ has a hamiltonian cycle $H$, then it follows by Theorem 2.1 that, for all $1 \leq k \leq n$, $G$ contains a $k$-path $P$ such that $w(P) \leq d_w \cdot k$. Clearly, if $w(v) = d_w$ for all $v \in V(G)$ (in this case $G$ is called $w$-regular) or $k = n$ (i.e. $P$ is a hamiltonian path of $G$), then the last inequality is tight.

At first, we prove Corollary 2.2 and Corollary 2.3 and show how Theorem 2.1 can be used to present inequalities $w(P) \leq c \cdot k$ for a $k$-path $P$ of $G$, where $c$ is a constant (depending on $G$ and on $w$ only) less than $d_w$. We have seen that this is possible only if $G$ is not $w$-regular and $k < n$. 

Lightweight paths in graphs
An edge $e = uv$ of $G$ is $w$-good if $f(e) = 2d_w - w(u) - w(v) > 0$. Note that $w$-regular graphs do not contain $w$-good edges. On the other hand, it is easy to choose $G$ and $w$ such that all edges of $G$ are $w$-good: let $G$ be a star and $w(v) = w(u) + 1$ if $v \in V(G) \setminus \{u\}$, where $u$ is the central vertex of $G$.

**Corollary 2.2.** Let $C$ be a hamiltonian cycle of $G$, $M$ be a non-empty set of $w$-good chords of $C$, $H$ be the subgraph of $G$ with $V(H) = V(C)$ and $E(H) = E(C) \cup M$. If $1 \leq k \leq \text{girth}(H)$, then there is a $k$-path $P$ of $G$ such that

$$w(P) \leq \left( d_w - \frac{\sum_{e \in M} f(e)}{2(n + |M|)} \right) k < d_w \cdot k.$$

**Proof.** By Theorem 2.1 with $t = 2$, it follows

$$w(P) \leq \left( d_w + \frac{\sum_{v \in V(G)} (d_H(v) - 2)(w(v) - d_w)}{2|E(H)|} \right) k,$$

for all $1 \leq k \leq \text{girth}(G)$. Note that $d_H(v) - 2$ is the number of edges in $M$ incident with $v \in V(H)$.

If each vertex $v \in V(H)$ sends the value $w(v) - d_w$ to each edge of $M$ incident with $v$, then

$$\sum_{v \in V(H)} (d_H(v) - 2)(w(v) - d_w) = \sum_{w \in M} (w(u) + w(v) - 2d_w)$$

and, therefore,

$$w(P) \leq \left( d_w + \frac{\sum_{w \in M} (w(u) + w(v) - 2d_w)}{2|E(H)|} \right) k = \left( d_w - \frac{\sum_{e \in M} f(e)}{2(n + |M|)} \right) k < d_w \cdot k.$$

Throughout the paper, let $C_w$ be a cycle of $G$ such that

$$\frac{\sum_{v \in V(C_w)} (w(v) - d_w)}{|V(C_w)|} \leq \frac{\sum_{v \in V(C)} (w(v) - d_w)}{|V(C)|}$$

for all cycles $C$ of $G$. It is easy to see that $C_w$ even can be a hamiltonian cycle of $G$: let $G$ be obtained from a cycle $C$ and an additional chord of $C$ and $w(v) = d_G(v)$ for $v \in V(G)$.

**Corollary 2.3.** If $G$ contains at least $n$ $w$-good edges and $1 \leq k \leq |V(C_w)|$, then there is a $k$-path $P$ of $G$ such that

$$w(P) \leq \left( d_w + \frac{\sum_{v \in V(C_w)} (w(v) - d_w)}{|V(C_w)|} \right) k < d_w \cdot k.$$
Proof. Obviously, \( G \) contains a cycle \( C \) containing \( w \)-good edges only, thus,
\[
\frac{\sum_{v \in V(C_w)} (w(v) - d_w)}{|V(C_w)|} \leq \frac{\sum_{v \in V(C)} (w(v) - d_w)}{|V(C)|} = \frac{\sum_{e \in E(C)} (-f(e))}{2|V(C)|} < 0.
\]
We are done by Theorem 2.1 with \( H = C_w \).

Next, we ask which subgraph \( H_w \in \mathcal{H}(G) \) in Theorem 2.1 is the best one, i.e.
\[
\frac{\sum_{v \in V(H_w)} d_{H_w}(v)w(v)}{2|E(H_w)|} \leq \frac{\sum_{v \in V(H)} d_H(v)w(v)}{2|E(H)|}
\]
for all subgraphs \( H \in \mathcal{H}(G) \).

**Theorem 2.4.** \( H_w = C_w \) and if \( H \in \mathcal{H}(G) \), then \( H \) contains a cycle \( C \) such that
\[
\frac{\sum_{v \in V(C)} w(v)}{|V(C)|} \leq \frac{\sum_{v \in V(H)} d_H(v)w(v)}{2|E(H)|}.
\]

Proof. Let \( \mathcal{C} = \{C_1, \ldots, C_t\} \) be a cycle 4-cover of \( H \). In the proof of Theorem 2.1, we have seen that \( |V(C_1)| + \ldots + |V(C_t)| = 4|E(H)| \) and that a vertex \( v \in V(G) \) belongs to exactly \( 2d_H(v) \) cycles of \( \mathcal{C} \), thus,
\[
2 \sum_{v \in V(H)} d_H(v)w(v) = \left( \sum_{v \in V(C_1)} w(v) \right) + \ldots + \left( \sum_{v \in V(C_t)} w(v) \right)
\]
and
\[
\frac{\sum_{v \in V(H)} d_H(v)w(v)}{2|E(H)|} = \frac{(\sum_{v \in V(C_1)} w(v)) + \ldots + (\sum_{v \in V(C_t)} w(v))}{|V(C_1)| + \ldots + |V(C_t)|}.
\]
Let \( \mathcal{C} = \{C_1, \ldots, C_t\} \) be ordered such that
\[
\frac{\sum_{v \in V(C_1)} w(v)}{|V(C_1)|} \leq \ldots \leq \frac{\sum_{v \in V(C_t)} w(v)}{|V(C_t)|}.
\]
It follows
\[
\frac{\sum_{v \in V(C_1)} w(v)}{|V(C_1)|} \leq \frac{(\sum_{v \in V(C_1)} w(v)) + \ldots + (\sum_{v \in V(C_t)} w(v))}{|V(C_1)| + \ldots + |V(C_t)|}
\]
(can be seen easily by induction on \( t \)) and \( H_w = C_w \).

By Theorem 2.4, the best upper bound on the weight of a lightweight \( k \)-path presented by Theorem 2.1 is obtained if \( H \in \mathcal{H}(G) \) is a cycle \( C_w,k \) from the set \( \mathcal{C}(G,k) \) of cycles of \( G \) on at least \( k \) vertices such that
\[
\frac{\sum_{v \in V(C_w,k)} w(v)}{|V(C_w,k)|} \leq \frac{\sum_{v \in V(C)} w(v)}{|V(C)|}
\]
for \( C \in \mathcal{C}(G,k) \).
It is clear that $C_w$ is such a cycle $C_{w,k}$ if $k \leq |V(C_w)|$.

It is known that, if $0 < c \leq 1$ is a fixed absolute constant, then the problem to
decide whether a graph $G$ contains a cycle on at least $c \cdot n$ vertices is NP-complete.
Thus, the problem to find a cycle $C_{w,k}$ is hard if $k$ is large because the problem whether
$G$ contains a cycle on at least $k$ vertices is a subproblem.

Using the observation

$$\frac{\sum_{v \in V(C_w)} w(v)}{|V(C_w)|} = \frac{\sum_{uv \in E(C_w)} (w(u) + w(v))}{|E(C_w)|}$$

and the polynomality of the forthcoming undirected minimum mean cycle problem,
it follows that $C_w$ can be found in polynomial time.

**Undirected minimum mean cycle problem:** Given an undirected graph $G$,
$\sigma : E(G) \to R$, find a cycle $C$ in $G$ whose mean weight $\frac{\sum_{e \in E(C)} \sigma(e)}{|E(C)|}$ is minimum.

There is an $O(n^5)$-algorithm solving the undirected minimum mean cycle
problem ([6]), moreover, the time complexity can be improved to $O(n^2m + n^3 \log n)$
(see also [5]).

We remark that this problem becomes already hard if $C$ has to contain a specified
vertex $v$ of $G$. To see this, let $\sigma(e) = 1$ if $e$ is incident with $v$, $\sigma(e) = 0$ otherwise, and
$C$ contain $v$. Then

$$\frac{\sum_{e \in E(C)} \sigma(e)}{|E(C)|} = \frac{2}{|E(C)|},$$

thus, $C$ is a hamiltonian cycle of $G$ if and only if $G$ is hamiltonian. It is known, that
the decision problem, whether a graph is hamiltonian, is NP-complete.

Corollary 2.5 presents easily calculable upper bounds on $\sum_{v \in V(C_w)} \frac{w(v) - d_w}{|V(C_w)|}$ (see
Corollary 2.3) and on $\sum_{e \in V(C_w)} \frac{w(e)}{|V(C_w)|}$ (see Theorem 2.4) if the girth of $G$ is known.

**Corollary 2.5.** If the edges $e_1, \ldots, e_m$ of $G$ are ordered such that $f(e_1) \geq \ldots \geq f(e_m)$,
then

$$\frac{\sum_{v \in V(C_w)} w(v)}{|V(C_w)|} = d_w + \frac{\sum_{v \in V(C_w)} (w(v) - d_w)}{|V(C_w)|} \leq d_w - \frac{f(e_{n - \text{girth}(G)} + \ldots + f(e_n))}{2 \text{girth}(G)}.$$

**Proof.** Recall that $m \geq n$. Obviously, the subgraph $F$ of $G$ with $V(F) = V(G)$ and
$E(F) = \{e_1, \ldots, e_n\}$ contains a cycle $C$. It follows

$$\frac{\sum_{v \in V(C_w)} w(v)}{|V(C_w)|} \leq \frac{\sum_{v \in V(C)} w(v)}{|V(C)|} = \frac{\sum_{e \in E(C)} (2d_w - f(e))}{2|E(C)|}.$$

Note that $|E(C)| \geq \text{girth}(G)$ and that $2d_w - f(e_1) \leq \ldots \leq 2d_w - f(e_n)$. 
Thus, 

\[
\sum_{e \in E(C)} (2d_w - f(e)) \leq \frac{(2d_w - f(e_{n-|E(C)|+1})) + \ldots + (2d_w - f(e_n))}{2|E(C)|}
\leq \frac{(2d_w - f(e_{n-girth(G)+1})) + \ldots + (2d_w - f(e_n))}{2girth(G)}
= d_w - \frac{f(e_{n-girth(G)+1}) + \ldots + f(e_n)}{2girth(G)}.
\]

If \( G \) itself is bridgeless, then \( G \in \mathcal{H}(G) \) and, by Theorem 2.1, it follows that \( G \) contains a \( k \)-path \( P \) such that

\[ w_G(P) \leq \frac{\sum_{v \in V(G)} d_G(v)w(v)}{2m} k \]

for \( 1 \leq k \leq \text{girth}(G) \). Figure 1 presents a graph \( G_0 \) showing that this is not true, if \( G \) contains bridges (let \( w(v) = d_G(v) \) for \( v \in V(G) \)).

![Fig. 1. The graph \( G_0 \)](example_image)

Obviously, \( w_G(P) \leq \Delta_w k \) for each \( k \)-path \( P \) of \( G \), if \( \Delta_w = \max_{v \in V(G)} w(v) \). Theorem 2.6 shows, how this trivial bound can be improved if \( G \) is bridgeless, \( 1 \leq k \leq \text{girth}(G) \), and \( G \) is not \( w \)-regular. Therefore, let \( \delta \) be the minimum degree of \( G \) and \( \Sigma_w = \sum_{v \in V(G)} w(v) \).

**Theorem 2.6.** If \( G \) is a bridgeless graph of positive size \( m \) and \( 1 \leq k \leq \text{girth}(G) \), then \( G \) contains a \( k \)-path \( P \) such that

\[ w(P) \leq \left( \Delta_w - \frac{\delta}{2m} (\Delta_w n - \Sigma_w) \right) k. \]
Proof. By Theorem 2.1, it follows with $H = G$ that

\[
  w(P) \leq \sum_{v \in V(G)} d_G(v) w(v) k
  = \frac{1}{2m} \left( \Delta w \sum_{v \in V(G)} d_G(v) - \sum_{v \in V(G)} \left( \Delta w - w(v) d_G(v) \right) \right) k
  \leq \frac{1}{2m} \left( \Delta w \sum_{v \in V(G)} d_G(v) - \delta \sum_{v \in V(G)} \left( \Delta w - w(v) \right) \right) k
  = \left( \Delta w - \frac{\delta}{2m} (\Delta w n - \sum w) \right) k.
\]

\[\square\]

Corollary 2.7 is a consequence of Theorem 2.6 if $w(v) = d_G(v)$ for $v \in V(G)$ or $w(v) = -d_G(v)$ for $v \in V(G)$.

**Corollary 2.7.** If $G$ is a bridgeless graph of positive size, $1 \leq k \leq \text{girth}(G)$, $\Delta$ and $d$ are the maximum degree and the average degree of $G$, respectively, then $G$ contains a $k$-path $P$ and a $k$-path $Q$ such that

\[
  \sum_{v \in V(P)} d_G(v) \leq \left( \Delta - \delta \left( \frac{\Delta}{d} - 1 \right) \right) k \quad \text{and} \quad \sum_{v \in V(Q)} d_G(v) \geq \delta \left( 2 - \frac{2}{d} \right) k.
\]

Obviously, $\Delta - \delta \left( \frac{\Delta}{d} - 1 \right) \leq \Delta$ and $\delta \left( 2 - \frac{2}{d} \right) \geq \delta$ with equality if and only if $G$ is regular. The same holds for the inequalities $d \leq \Delta - \delta \left( \frac{\Delta}{d} - 1 \right)$ and $d \geq \delta \left( 2 - \frac{2}{d} \right)$ because they are equivalent to $(\Delta - d)(d - \delta) \geq 0$ and $(d - \delta)^2 \geq 0$, respectively.

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