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ON THE IMAGINARY PART OF COUPLING RESONANCE POINTS

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Abstract. We prove for rank one perturbations that the imaginary part of a coupling resonance point is inversely proportional by a factor of -2 to the rate of change of the scattering phase, as a function of the coupling variable, evaluated at the real part of the resonance point. This equality is analogous to the Breit–Wigner formula from quantum scattering theory. For more general relatively trace class perturbations, we also give a formula for the spectral shift function in terms of coupling resonance points, non-real and real.

Keywords: scattering matrix, scattering phase, resonance point, Breit–Wigner formula.

Mathematics Subject Classification: 47A40, 47A55, 47A70.

1. INTRODUCTION

By a *coupling resonance point* we mean a pole of the scattering matrix $S(z; H_r, H_0)$, where $H_r = H_0 + rV$, as a function of the *coupling variable* r for a fixed (complex) value of *energy* z. More precisely, given a self-adjoint operator H_0 and a relatively compact self-adjoint operator V, a coupling resonance point r_z of the triple $(z; H_0, V)$ is defined as a pole of the meromorphic operator valued function

$$r \mapsto R_z(H_r) = R_z(H_0)(1 + rVR_z(H_0))^{-1},$$

where $H_r = H_0 + rV$ and $R_z(H) = (H - z)^{-1}$. This function is meromorphic by the analytic Fredholm theorem (see e.g. [22, Theorem 1.8.2]) and r_z is resonant iff $-r_z^{-1}$ is an eigenvalue of the compact operator $VR_z(H_0)$. Coupling resonance points, when considered as functions of z, are branches of multivalued analytic functions (of Herglotz type).

Before continuing, a few words of context are worthwhile. The above definition is different from the usual definition of a resonance point (as e.g. in [12, 13, 16, 17, 23]) which comes from the physics literature, as a pole of the analytic continuation –

through the essential spectrum – of the scattering matrix, or, which is the same, of the sandwiched resolvent, considered as a function of energy. For clarity, we will call these energy poles of the scattering matrix *energy resonance points*.

Although energy resonances are not within the scope of this paper, we note that there is a simple connection between coupling and energy resonances, namely, the zeros of the analytic continuation of coupling resonance points r_z (considered as functions of z) are energy resonance points.

In order for analytic continuation through the essential spectrum to be possible the Hamiltonian H_0 is to obey certain strict conditions, usual for the smooth approach to scattering theory, such as Kato smoothness (for a definition see e.g. [22]), or is to belong to a class of differential operators with smooth coefficients. On the other hand, the definition of coupling resonances does not require any smoothness assumptions. This allows us to consider an arbitrary initial self-adjoint operator H_0 and conditions under which the energy resonances are unlikely to exist.

The reason for our interest in coupling resonance points is that those which, as $\operatorname{Im} z \to 0$, have limit values lying on the real axis, are closely connected to the intriguing phenomenon of the flow of singular spectrum inside the essential spectrum.

Briefly, the notion of the flow of discrete spectrum, often called spectral flow, which has been studied by many (e.g. [2, 6, 14, 19]) with a focus on its topological invariance, is usually not defined in a way that allows its extension to the essential spectrum. On the other hand, the spectral shift function (SSF) is defined on the entire real axis (see e.g. [10, 18]). Outside of the essential spectrum, the SSF coincides with spectral flow and is naturally integer-valued. Whereas within the essential spectrum, its values take into account the movement, or rather phase shift, of absolutely continuous spectrum and can be any real number. More detail is provided by the absolutely continuous $\xi^{(a)}(\lambda; H_1, H_0)$ and singular $\xi^{(s)}(\lambda; H_1, H_0)$ parts of the SSF (see e.g. [3, 7]), which can be defined (assuming the perturbation is of relatively trace class type) by

$$\xi^{\#}(\varphi; H_1, H_0) = \int_0^1 \operatorname{Tr}\left(E_r^{\#}(\operatorname{supp}\varphi) V \varphi(H_r)\right)\, dr, \quad \varphi \in C_c(\mathbb{R}), \tag{1.1}$$

where the placeholder # should be replaced by (a) or (s) respectively, in which case $E_r^{(a)}$ and $E_r^{(s)}$ denote the absolutely continuous and singular parts of the spectral measure E_r corresponding to the self-adjoint operator H_r. More precisely, (1.1) defines an absolutely continuous measure whose density function is, by definition, $\xi^{\#}(\lambda; H_1, H_0)$. The SSF $\xi(\lambda; H_1, H_0)$ itself can be defined in the same way by the Birman–Solomyak formula ([9]), which is obtained by removing # altogether from (1.1).

It turns out that the singular SSF is a.e. integer-valued and can be relatively simply described in terms of coupling resonance points by the *total resonance index* (see [7, Theorem 1.3], [4, Theorem 6.3.2])

$$\xi^{(s)}(\lambda; H_1, H_0) = \sum_{r_\lambda \in [0,1]} \operatorname{ind}_{res}(\lambda; H_{r_\lambda}, V), \tag{1.2}$$

where the resonance index $\mathrm{ind}_{res}(\lambda; H_{r_\lambda}, V)$ is defined as the difference $N_+ - N_-$, with $N_\pm$ being the number of coupling resonance points corresponding to $(\lambda + iy, H_0, V)$ which converge to $r_\lambda \in \mathbb{R}$ from the half-plane $\mathbb{C}_\pm$ as $y \to 0^+$. Only a finite number of resonance indices for $r_\lambda \in [0,1]$ can be non-zero and the total resonance index is their sum. More information about the resonance index and its interpretation can be found e.g. in [4, 5].

Various similar notions to coupling resonances can be found in the literature (e.g. Regge poles, dilation analytic resonances), but the similarities are for our purposes largely superficial. To a limited extent coupling resonances do however appear in the literature, in connection with rank-one perturbations and Weyl m-functions (see e.g. [20]), and in connection with random Schrödinger operators (see e.g. [1, Theorem 5.3]) which do not necessarily obey the smoothness conditions to allow energy resonances. We note that there is a sense in which embedded singular spectrum belongs to things such as random Schrödinger operators rather than to Schrödinger operators with decreasing potentials.

Another point of difference with the existing literature [10, 12] is that we work with derivatives of the scattering matrix with respect to the coupling variable, not with respect to energy. This can be seen as another aspect of the same point as above: we focus on the coupling variable instead of energy. Since the scattering matrix $S(\lambda; H_r, H_0)$ is normally defined for a fixed pair of operators H_0, H_r and a.e. real value $z = \lambda + i0$ of energy, a rigorous justification of this change of viewpoint requires careful treatment of the null set of exceptional real energy values λ and their dependence on r. For this reason we rely on the constructive approach to scattering theory found in [3, 4, 7] which was developed for this purpose.

Under certain conditions on the pair (H_0, V), which ensure the existence of scattering theory (namely the limiting absorption principle, see e.g. [22, Chapter 6]), coupling resonance points r_z have limit values $r_{\lambda+i0}$ for a.e. $\lambda \in \mathbb{R}$. When $r_{\lambda+i0}$ is real, it has several interpretations, as discussed in detail in the introduction of [4]. One such interpretation is that one of the scattering phases $\theta_j(\lambda; r)$ of the scattering matrix $S(\lambda; H_r, H_0)$ (by which we mean that $e^{i\theta_j(\lambda;r)}$ is an eigenvalue of $S(\lambda; H_r, H_0)$), when considered as an analytic function of the coupling variable r, suffers a sudden jump by an integer multiple of 2π when r crosses $r_{\lambda+i0}$ (in fact such a jump is only revealed when λ is perturbed slightly to $\lambda + i\epsilon$).

After having concentrated on the case $r_{\lambda+i0} \in \mathbb{R}$, this paper originated from a curiosity about what happens if the imaginary part is non-zero. It turns out that it has a meaning in the spirit of the Breit-Wigner formula from quantum scattering theory, which concerns the situation of an energy resonance near the real axis (see e.g. [11, Chapter XVIII], [21, Chapter 13]). In this paper for rank-one perturbations, in which case there is only one non-zero scattering phase $\theta_1(\lambda; r)$, we prove the formula:

$$\frac{\partial \theta_1(\lambda; r)}{\partial r}\Big|_{r=\mathrm{Re}\, r_{\lambda+i0}} = -\frac{2}{\mathrm{Im}\, r_{\lambda+i0}}, \quad \text{a.e. } \lambda \in \mathbb{R}. \tag{1.3}$$

Since $r_{\lambda+i0}$ is a pole of the scattering matrix, this formula is in agreement with the Breit–Wigner formula, with the difference that the phase is considered as a function

of the coupling variable instead of energy. The proof turns out to be surprisingly simple and is based on an elementary fact:

$$e^{i\theta(r)} = \frac{1 - r/(\alpha - i\beta)}{1 - r/(\alpha + i\beta)} \quad \Longrightarrow \quad \frac{d\theta(r)}{dr}\Big|_{r=\alpha} = -\frac{2}{\beta}.$$

Also established here is a more general formula (see Theorem 2.2), which applies in the case that V is a certain kind of relatively trace class perturbation. This allows us to present a new formula for the SSF purely in terms of coupling resonance points:

$$\xi(\lambda; H_1, H_0) = \frac{1}{2\pi} \int_0^1 \sum_{j=1}^{\infty} \frac{2\beta_\lambda^j}{|r - \alpha_\lambda^j|^2 + |\beta_\lambda^j|^2}\, dr + \sum_{r_\lambda \in [0,1]} \mathrm{ind}_{res}(\lambda; H_{r_\lambda}, V), \qquad (1.4)$$

where $r_{\lambda+i0}^j = \alpha_\lambda^j + i\beta_\lambda^j$ are the limits of coupling resonance points with non-zero β_λ^j.

For comparison, we note that a similar formula in terms of the energy variable can be found in [12, (1.3)], which is anologous to the derivative of (1.4) with respect to the coupling variable. Some other analogous considerations in terms of energy appear in [10, §9.3] along with further references.

Throughout the rest of the paper, since we will only be concerned with *coupling* resonance points, we refer to them simply as by *resonance points*, leaving "coupling" implicit.

2. RESULTS

Theorem 2.1. *If H_0 is a self-adjoint operator and V is a rank one self-adjoint operator, then for a.e. $\lambda \in \mathbb{R}$ the formula* (1.3) *holds. In this formula, for z outside the essential spectrum of H_0, the number r_z is the unique pole of the meromorphic function $\mathbb{C} \ni s \mapsto VR_z(H_s)$ with $r_{\lambda+i0} = \lim_{y\to 0^+} r_{\lambda+iy}$, and $e^{i\theta_1(\lambda;r)}$ is the unique non-trivial eigenvalue of the scattering matrix $S(\lambda; H_r, H_0)$.*

Before proceeding to the proof we discuss its context and sketch an important lemma whose details lie outside the scope of this paper. Since V has rank one, the default premise of this theorem – the limiting absorption principle – holds; this is necessary for the existence of the scattering matrix and therefore also the scattering phase. Because the proof involves considering these objects as functions of the coupling variable, it requires the constructive approach to stationary scattering theory given in [3, 7] and outlined in the introduction to [4]. In this approach, objects such as the wave matrices $w_\pm(\lambda; H_r, H_0)$ and scattering matrix $S(\lambda; H_r, H_0)$ are defined by explicit formulas for all r except a discrete set, as long as λ belongs to a pre-defined set of full Lebesgue measure in $\mathbb{R}$, which comes from the limiting absorption principle. Moreover, the scattering matrix can be differentiated with respect to r, as discussed further below.

The setting for the proof of Theorem 2.1 and its generalisation Theorem 2.2 is outlined by the following assumptions which are common within scattering theory.

(1) H_0 is a self-adjoint operator on a (separable complex) Hilbert space $\mathcal{H}$.
(2) V is a symmetric form on $\mathcal{H}$, which admits the decomposition $V = F^*JF$ on the form domain of H_0, i.e.

$$V\colon (f,g) \mapsto \langle Ff, JFg\rangle\,, \quad f,g \in \operatorname{dom}|H_0|^{1/2},$$

where $F\colon \mathcal{H} \to \mathcal{K}$ is a closed operator and J is a self-adjoint bounded operator on $\mathcal{K}$.
(3) The sandwiched resolvent $T_z(H_0) = FR_z(H_0)F^*$, where $R_z(H) = (H-z)^{-1}$ is the resolvent of H, is (or more precisely extends to) a compact operator for some (and thus for any) $z \notin \sigma(H_0)$, the spectrum of H_0.
(4) F is bounded or H_0 is semi-bounded.

Without loss of generality we assume that F has trivial kernel (by extending to the kernel as a compact operator) and that $F\operatorname{dom}|H_0|^{1/2}$ is dense in $\mathcal{K}$. Given conditions (1), (2), and (4), condition (3) is equivalent to the relative compactness of F with respect to $|H_0|^{1/2}$.

These conditions (1)–(4) imply that the perturbed operator $H_r := H_0 + rV$, $r \in \mathbb{R}$, is well-defined, as an operator-sum if F is bounded or a form-sum if H_0 is semi-bounded. In fact we will need the following strengthened version of condition (3).

(3′) The sandwiched resolvent $T_z(H_0) = FR_z(H_0)F^*$ is (extends to) a trace class operator for some (and thus any) $z \notin \sigma(H_0)$.

Given (3′), it follows from the second resolvent identity that $T_z(H_r)$ also belongs to the trace class for any $r \in \mathbb{R}$ and $z \in \mathbb{C}\setminus\mathbb{R}$. This condition implies the limiting absorption principle in the following sense: the set of points $\lambda \in \mathbb{R}$ for which the uniform limit $T_{\lambda+i0}(H_r) := \lim_{y\to 0^+} T_{\lambda+iy}(H_r)$ exists, has full Lebesgue measure in $\mathbb{R}$. In addition, the limit of the imaginary part $\operatorname{Im} T_{\lambda+i0}(H_r) = \lim_{y\to 0^+} \operatorname{Im} T_z(H_r)$ exists in the trace class norm for a.e. $\lambda \in \mathbb{R}$. The full set of points λ for which both of these limits exist will be denoted by $\Lambda(H_r, F)$. Note that the limit of the real part of $T_{\lambda+i0}(H_r)$ may not exist in the trace-class, even if F is Hilbert–Schmidt.

If V is an operator which is relatively compact with respect to H_0, then $F = \sqrt{|V|}$ and $J = \operatorname{sgn} V$ satisfy the conditions (1)–(3). In this case for $z \in \mathbb{C}\setminus\mathbb{R}$, the compact operators $VR_z(H_0)$ and $JT_z(H_0)$ share the same non-zero eigenvalues. It follows that (coupling) resonance points r_z corresponding to z can be defined as poles of the meromorphic function

$$r \mapsto T_z(H_r) = T_z(H_0)(1 + rJT_z(H_0))^{-1},$$

extending the definition given above. This definition also makes sense for $z = \lambda + i0$ provided λ belongs to $\Lambda(H_0, F)$. What's more, for $\lambda \in \Lambda(H_0, F)$, it happens that $\lambda \in \Lambda(H_r, F)$ if and only if $r \in \mathbb{R}$ is non-resonant at λ.

Note that if V is finite-rank, then so can be the choice of F (before extending to the kernel as a Hilbert-Schmidt operator), hence in this case conditions $(3')$ and (4) are satisfied. More generally, condition $(3')$ will be satisfied if the (necessarily bounded) operator $F(|H_0|^{1/2}+1)^{-1}$ belongs to the Hilbert–Schmidt class.

We now briefly review how the scattering matrix can be realised as a function of the coupling variable by taking a constructive approach to stationary scattering theory (for more information see e.g. [7] or the introduction to [4]). A fibre Hilbert space $\mathfrak{h}_\lambda(H_0)$ is defined for any $\lambda \in \Lambda(H_0, F)$ as the closed range

$$\mathfrak{h}_\lambda(H_0) = \operatorname{cl}\left(\operatorname{ran}\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}\right) \subset \mathcal{K}.$$

These fibre Hilbert spaces give rise to a direct integral

$$\begin{aligned}\mathcal{H}(H_0) &:= \int\limits^{\oplus}_{\Lambda(H_0,F)} \mathfrak{h}_\lambda(H_0)\, d\lambda \\ &= \big\{ f \in L_2(\Lambda(H_0,F), \mathcal{K}) : f(\lambda) \in \mathfrak{h}_\lambda(H_0) \text{ for a.e. } \lambda \in \Lambda(H_0,F) \big\}.\end{aligned}$$

The *evaluation operator* $\mathcal{E}_\lambda(H_0)$ is defined on the range of F^* by

$$\mathcal{E}_\lambda(H_0) = \sqrt{\pi^{-1}\operatorname{Im} T_{\lambda+i0}(H_0)}(F^*)^{-1}$$

and the collection $\mathcal{E}(H_0) = \{\mathcal{E}_\lambda(H_0) : \lambda \in \Lambda(H_0, F)\}$, considered as an operator from $\mathcal{H}$ to $\mathcal{H}(H_0)$ defined on $\operatorname{ran} F^*$, extends to a partial isometry which diagonalises the absolutely continuous part of H_0 ([7, Theorem 5.1], [3, Theorem 3.4.2]):

$$\mathcal{E}(H_0)\colon \mathcal{H} \to \mathcal{H}(H_0), \qquad H_0^{(a)} = \mathcal{E}^*(H_0) M_\lambda \mathcal{E}(H_0),$$

where M_λ denotes the operator of multiplication by λ. For any non-resonant r, the wave matrices $w_\pm(\lambda; H_r, H_0)$ may then be defined as unitary transforms from $\mathfrak{h}_\lambda(H_0)$ to $\mathfrak{h}_\lambda(H_r)$, which are uniquely determined for $f, g \in \operatorname{ran} F^*$ by

$$\langle \mathcal{E}_\lambda(H_r) f, w_\pm(\lambda; H_r, H_0)\mathcal{E}_\lambda(H_0) g\rangle = \lim_{y\to 0^+} \frac{y}{\pi} \langle R_{\lambda+iy}(H_r) f, R_{\lambda+iy}(H_0) g\rangle\,.$$

The scattering matrix $S(\lambda; H_r, H_0) = w_+^*(\lambda; H_r, H_0) w_-(\lambda; H_r, H_0)$ can be shown to satisfy the stationary formula

$$S(\lambda; H_r, H_0) = 1 - 2ir\sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)} J (1 + rT_{\lambda+i0}(H_0)J)^{-1} \sqrt{\operatorname{Im} T_{\lambda+i0}(H_0)}, \quad (2.1)$$

for any $\lambda \in \Lambda(H_0, F)$ and non-resonant $r \in \mathbb{R}$. This allows the scattering matrix, for fixed such λ, to be considered as a function of r. Although its factor $(1+rT_{\lambda+i0}(H_0)J)^{-1}$ is meromorphic with poles at resonance points, since the scattering matrix is unitary and hence bounded for non-resonant $r \in \mathbb{R}$, it admits analytic continuation to a neighbourhood of the real axis.

A significant part of the proof of Theorem 2.1 (see [7, Theorem 5.7], [3, Theorem 7.3.3]) is the following fact, obtained from (2.1). The derivative of the scattering matrix at any non-resonant $r \in \mathbb{R}$ is given by

$$\begin{aligned}&\frac{dS(\lambda; H_r, H_0)}{dr}\\ &= -2i\, w_+(\lambda; H_0, H_r)\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}J\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\\ &\quad \times w_+(\lambda; H_r, H_0)S(\lambda; H_r, H_0).\end{aligned} \tag{2.2}$$

Assuming condition $(3')$, this derivative can be taken in the trace class norm.

Proof of Theorem 2.1. Let λ be a real number from the full set $\Lambda(H_0, F)$. Since λ is fixed, we write $S(r)$ for $S(\lambda; H_r, H_0)$ and $\theta_j(r)$ for $\theta_j(\lambda; r)$. Since the scattering matrix is unitary, it follows from (2.2) that for any non-resonant $r \in \mathbb{R}$,

$$S'(r)S^{-1}(r) = -2i\, w_+(\lambda; H_0, H_r)\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}J\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\, w_+(\lambda; H_r, H_0). \tag{2.3}$$

Further, since $w_+(\lambda; H_r, H_0)w_+(\lambda; H_0, H_r) = 1_{\mathfrak{h}_\lambda(H_r)}$ (see [7, Theorem 5.3], [3, Corollary 5.3.8]), taking traces of both sides of the equality (2.3) gives

$$\operatorname{Tr}\left(S'(r)S^{-1}(r)\right) = -2i \operatorname{Tr}\left(\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}J\sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)}\right). \tag{2.4}$$

The trace on the right can be interpreted to be associated to the trace class operators on the whole space $\mathcal{K}$, rather than the fibre Hilbert space $\mathfrak{h}_\lambda(H_r)$, since $\ker \sqrt{\operatorname{Im} T_{\lambda+i0}(H_r)} = \mathfrak{h}_\lambda(H_r)^\perp$. It follows using condition $(3')$ that the equality (2.4) can be rewritten as

$$\begin{aligned}\operatorname{Tr}\left(S'(r)S^{-1}(r)\right) &= -2i \operatorname{Tr}\left(J \operatorname{Im} T_{\lambda+i0}(H_r)\right)\\ &= -\lim_{y\to 0^+} 2i \operatorname{Tr}\left(J \operatorname{Im} T_{\lambda+iy}(H_r)\right)\\ &= -\lim_{y\to 0^+} \operatorname{Tr}\left(JT_{\lambda+iy}(H_r)\right) - \operatorname{Tr}\left(JT_{\lambda-iy}(H_r)\right).\end{aligned} \tag{2.5}$$

We recall that a resonance point r_z corresponding to the triple $(z; H_0, V)$ is a complex number such that $-r_z^{-1}$ is an eigenvalue of the compact operator $JT_z(H_0)$, or in this case equivalently of $VR_z(H_0)$. Since by the premise V has rank 1, the operator $VR_z(H_r)$ has only one eigenvalue. Therefore, with $z = \lambda + iy$ and using the fact that $\bar{r}_z = r_{\bar{z}}$, we find that

$$\begin{aligned}\operatorname{Tr}\left(S'(r)S^{-1}(r)\right) &= -\lim_{y\to 0^+} \left((r - r_z)^{-1} - (r - \bar{r}_z)^{-1}\right)\\ &= -\lim_{y\to 0^+} \frac{-\bar{r}_z + r_z}{(r - r_z)(r - \bar{r}_z)}\\ &= -\lim_{y\to 0^+} \frac{2i \operatorname{Im} r_z}{(r - r_z)(r - \bar{r}_z)}.\end{aligned}$$

Taking the limit and replacing r by $\operatorname{Re} r_{\lambda+i0}$ (assuming it is not resonant, i.e. $\operatorname{Im} r_{\lambda+i0} \neq 0$),

$$\operatorname{Tr}\left(S'(r)S^{-1}(r)\right)\big|_{r=\operatorname{Re} r_{\lambda+i0}} = -\frac{2i}{\operatorname{Im} r_{\lambda+i0}}.$$

On the other hand, since $\operatorname{rank}(V) = 1$, the scattering matrix $S(r)$ is one-dimensional. Hence, it is the operator of multiplication by its eigenvalue: $S(r) = e^{i\theta_1(r)} \cdot 1_{\mathfrak{h}_\lambda(H_0)}$. It follows that

$$\begin{aligned}\operatorname{Tr}\left(S'(r)S^{-1}(r)\right)\big|_{r=\operatorname{Re} r_{\lambda+i0}} &= \frac{de^{i\theta_1(r)}}{dr}e^{-i\theta_1(r)}\big|_{r=\operatorname{Re} r_{\lambda+i0}} \\ &= i\theta_1'(\operatorname{Re} r_{\lambda+i0}).\end{aligned}$$

Comparing the last two formulas completes the proof. □

Scattering phases are well-known to be closely related to the SSF $\xi(\lambda; H_1, H_0)$ by the Birman–Krein formula (see e.g. [8, 10])

$$\det S(\lambda; H_1, H_0) = \exp(-2\pi i \xi(\lambda; H_1, H_0)).$$

It follows that the SSF is, up to an integer term, equal to the sum of scattering phases scaled by $-(2\pi)^{-1}$. The undetermined integer term is the singular SSF $\xi^{(s)}(\lambda; H_1, H_0)$ ([3, 7]). While the absolutely continuous SSF $\xi^{(a)} = \xi - \xi^{(s)}$ has the representation ([7, §5.3], [3, Theorem 9.2.2])

$$\xi^{(a)}(\lambda; H_1, H_0) = -\frac{1}{2\pi}\sum_{j=1}^{\infty}\theta_j(\lambda; 1), \tag{2.6}$$

where $e^{i\theta_j(\lambda;r)}$, $r \in [0, 1]$, are the eigenvalues of the scattering matrix $S(\lambda; H_r, H_0)$, which are continuously enumerated with phases chosen so that $\theta_j(\lambda; 0) = 0$.

In the proof of Theorem 2.1 the following equality is derived:

$$\theta_1'(\lambda; r) = -\frac{2\beta_\lambda}{|r - \alpha_\lambda|^2 + |\beta_\lambda|^2},$$

where $\alpha_\lambda := \operatorname{Re} r_{\lambda+i0}$, and $\beta_\lambda := \operatorname{Im} r_{\lambda+i0}$. If the phase $\theta_1(\lambda; r)$ is chosen so that $\theta_1(\lambda; 0) = 0$, then integrating gives the formula

$$\theta_1(\lambda; 1) = -\int_0^1 \frac{2\beta_\lambda}{|r - \alpha_\lambda|^2 + |\beta_\lambda|^2}\, dr.$$

As mentioned in the introduction, the singular SSF $\xi^{(s)}(\lambda; H_1, H_0)$ is known to be given in terms of resonance points by (1.2). The following generalisation of Theorem 2.1 shows that the absolutely continuous SSF can also be expressed in terms of resonance points.

Theorem 2.2. *Let H_0 and $V = F^*JF$ satisfy conditions* (1), (2), (3′), *and* (4), *and let $\theta_j(\lambda; 1)$ be as in* (2.6). *Then for any λ from the set $\Lambda(H_0, F)$ of full measure in $\mathbb{R}$,*

$$\sum_{j=1}^{\infty}\theta_j(\lambda; 1) = -\int_0^1 \sum_{j=1}^{\infty}\frac{2\beta_\lambda^j}{|r - \alpha_\lambda^j|^2 + |\beta_\lambda^j|^2}\, dr, \tag{2.7}$$

where $r^j_{\lambda+i0} = \alpha^j_\lambda + i\beta^j_\lambda$ is the jth resonance point with non-zero β^j_λ, that is, $(r - r^j_{\lambda+i0})^{-1}$ is the jth eigenvalue of $JT_{\lambda+i0}(H_r)$.

Combining (2.6) and (2.7) gives

$$\xi^{(a)}(\lambda; H_1, H_0) = \frac{1}{2\pi} \int_0^1 \sum_{j=1}^{\infty} \frac{2\beta_\lambda^j}{|r - \alpha_\lambda^j|^2 + |\beta_\lambda^j|^2}\, dr.$$

By adding (1.2) we obtain the formula (1.4) for the SSF, which shows that resonance points $r_{\lambda+i0}^j$ with zero imaginary part β_λ^j still contribute to the SSF in the form of resonance indices.

Proof of Theorem 2.2. The equality (2.5) holds by the same argument as before. Since $JT_z(H_r)$ is trace class for $z = \lambda + iy$, $y > 0$, its eigenvalues $(r - r_z^j)^{-1}$ are summable and from (2.5) we have

$$\begin{aligned} -2i \operatorname{Tr}(J \operatorname{Im} T_{\lambda+i0}(H_r)) &= -\lim_{y\to 0^+} \left(\sum_{j=1}^{\infty} (r - r_z^j)^{-1} - \sum_{j=1}^{\infty} (r - \bar{r}_z^j)^{-1} \right) \\ &= -\lim_{y\to 0^+} \left(\sum_{j=1}^{\infty} \frac{2i \operatorname{Im} r_z^j}{(r - r_z^j)(r - \bar{r}_z^j)} \right) \\ &= -\sum_{j=1}^{\infty} \frac{2i\beta_\lambda^j}{|r - \alpha_\lambda^j|^2 + |\beta_\lambda^j|^2}. \end{aligned}$$

The interchange of limit and sum in the last equality holds since each sum is equal to the trace of $2iJ \operatorname{Im} T_z(H_r)$, which converges to $2iJ \operatorname{Im} T_{\lambda+i0}(H_r)$ in the trace class. Dividing by $-2\pi i$ and integrating, the proof is completed by combining (2.6) with the equality

$$\xi^{(a)}(\lambda; H_1, H_0) = \frac{1}{\pi} \int_0^1 \operatorname{Tr}(J \operatorname{Im} T_{\lambda+i0}(H_r))\, dr,$$

for whose proof we refer to [7, Theorem 3.2] (also see [3, §8]). □

We conclude with a small related note. Following from condition (3′) and the proof of Theorem 2.2 is the equality

$$\operatorname{Tr}(\operatorname{Im} A_{\lambda+i0}(H_r)) = \sum_{j=1}^{\infty} \operatorname{Im} \sigma_\lambda^j(r), \tag{2.8}$$

where $\sigma_\lambda^j(r)$ are the eigenvalues of the operator $A_{\lambda+i0}(H_r) := JT_{\lambda+i0}(H_r)$. The root vectors of $A_{\lambda+i0}(H_r)$ do not depend on the choice of non-resonant r (see [4, Proposition 3.1.2]) and we note that if $J = 1$, which may be assumed in the case that $V \geq 0$, the equality (2.8) implies that the system of root vectors is complete; in fact these are equivalent conditions by [15, Theorem V-2.1].

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