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DESCRIPTION OF THE SCATTERING DATA FOR STURM–LIOUVILLE OPERATORS ON THE HALF-LINE

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Abstract. We describe the set of the scattering data for self-adjoint Sturm–Liouville operators on the half-line with potentials belonging to $L_1(\mathbb{R}_+, \rho(x)\,\mathrm{d}x)$, where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotonically nondecreasing function from some family $\mathscr{R}$. In particular, $\mathscr{R}$ includes the functions $\rho(x) = (1+x)^\alpha$ with $\alpha \geq 1$.

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1. INTRODUCTION

In the Hilbert space $L_2(\mathbb{R}_+)$, we consider the Schrödinger operator generated by the differential expression

$$\mathfrak{t}_q(f) := -f'' + qf$$

and the boundary condition

$$f(0) = 0$$

with the potential q belonging to the class

$$\mathcal{Q}_\rho := \{q \in L_1(\mathbb{R}_+, \rho(x)\,\mathrm{d}x) \mid \operatorname{Im} q = 0\}, \quad \rho \in \mathscr{R}_0.$$

Here $\mathscr{R}_0$ is the class of all monotonically nondecreasing weight functions $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $x \leq \rho(x)$ for all $x > 0$. In particular, the class $\mathscr{R}_0$ includes the weight function $\omega(x) := x$.

In the present paper, we study the problem of an efficient description of the scattering data for operators from the class $\mathcal{T}_\rho := \{T_q \mid q \in \mathcal{Q}_\rho\}$ (for more details on the operator T_q see Appendix A). For the class $\mathcal{T}_\omega$, such description was given by V.A. Marchenko [3]. As shown in [4], the scattering data for operators from the class

$\mathcal{T}_\omega$ can be efficiently described in terms of some functional Banach algebra introduced below. Our aim is to describe the class $\mathscr{R}$ of weight functions $\rho \in \mathscr{R}_0$ for which a result analogous to that can be obtained.

To formulate the main result of the paper, let us recall some definitions. The scattering function $S = S_q$ of the operator T_q is defined as

$$S(\lambda) := \frac{e(-\lambda)}{e(\lambda)}, \quad \lambda \in \mathbb{R},$$

where $e(\lambda) := e(\lambda, 0)$ and $e(\lambda, \cdot\,)$ is the Jost solution of the equation

$$-y'' + qy = \lambda^2 y, \quad \lambda \in \overline{\mathbb{C}_+} := \{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda \geq 0\}, \tag{1.1}$$

i.e., a solution of (1.1) satisfying the asymptotics

$$e(\lambda, x) = e^{\mathrm{i}\lambda x}(1 + o(1)), \quad x \to +\infty.$$

The spectrum of the operator T_q with $q \in \mathcal{Q}_\rho$ consists of the absolutely continuous part filling the whole positive half-axis and the point spectrum consisting of a finite number of negative simple eigenvalues (see, e.g., [3]). Let us enumerate these eigenvalues in the ascending order of their moduli and denote them by $-\kappa_s^2,\ s = 1, \ldots, n$, where $\kappa_s = \kappa_s(q) > 0$. To each eigenvalue $\lambda = -\kappa_s^2$, there correspond the eigenfunction $e(\mathrm{i}\kappa_s, \cdot)$ and the norming constant $m_s = m_s(q)$, which is defined as

$$m_s = \left(\int_0^\infty |e(\mathrm{i}\kappa_s, x)|^2 \, \mathrm{d}x \right)^{-\frac{1}{2}}.$$

The scattering data of the operator T_q are defined as the triple $\mathfrak{s}_q := (S_q, \vec{\kappa}_q, \vec{m}_q)$, where $\vec{\kappa}_q := (\kappa_s(q))_{s=1}^n,\ \vec{m}_q := (m_s(q))_{s=1}^n$. If $n = 0$, then $\mathfrak{s}_q := (S_q, 0, 0)$. Let us put

$$\Omega_n := \{(\kappa_1, \ldots, \kappa_n) \in \mathbb{R}_+^n \mid 0 < \kappa_1 < \cdots < \kappa_n\}, \quad n \in \mathbb{N}.$$

For an arbitrary open set $\mathcal{O} \subset \mathbb{R}$, we denote by $\mathrm{AC}(\mathcal{O})$ the set of all functions $f : \mathcal{O} \to \mathbb{C}$ that are absolutely continuous on each compact interval $\Delta \subset \mathcal{O}$. For an arbitrary $\rho \in \mathscr{R}_0$, let us denote by X_ρ the Banach space consisting of functions $u \in \mathrm{AC}(\mathbb{R} \setminus \{0\}) \cap L_1(\mathbb{R})$ with the norm

$$\|u\|_{X_\rho} := \int_{\mathbb{R}} \rho(|x|)|u'(x)| \, \mathrm{d}x < \infty.$$

Similarly, we denote by X_ρ^+ and X_ρ^- the Banach spaces consisting of $u_+ \in \mathrm{AC}(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$ and $u_- \in \mathrm{AC}(\mathbb{R}_-) \cap L_1(\mathbb{R}_-)$, respectively, with the norms

$$\|u_\pm\|_{X_\rho^\pm} := \int_{\mathbb{R}_\pm} \rho(|x|)|u_\pm'(x)| \, \mathrm{d}x < \infty.$$

Let us agree to identify the spaces $X_\rho^\pm$ with the subspaces $\{f \in X_\rho \mid f\,|_{\mathbb{R}_\mp} = 0\}$ in the space X_ρ. Then $X_\rho = X_\rho^+ \dotplus X_\rho^-$.

Recall that $\omega(x) = x$ and $\omega \le \rho$. Therefore, $X_\rho \subset X_\omega$ and $X_\rho^\pm \subset X_\omega^\pm$. As will be shown in Section 2 of this paper, the space X_ρ is continuously embedded in $L_1(\mathbb{R})$.

Consider the Banach space

$$\mathbf{B}_\rho := \{\alpha\mathbf{1} + \widehat{\varphi} \mid \alpha \in \mathbb{C},\ \varphi \in X_\rho\}$$

with the norm

$$\|\alpha\mathbf{1} + \widehat{\varphi}\|_{\mathbf{B}_\rho} := |\alpha| + \|\varphi\|_{X_\rho}. \tag{1.2}$$

Here $\mathbf{1}(x) \equiv 1$ and $\widehat{\varphi}$ is the Fourier transform of a function φ.

Definition 1.1. A weight function $\rho \in \mathscr{R}_0$ is called regular if

$$c(\rho) := \sup_{x>0} \rho(2x)/\rho(x) < \infty.$$

Denote by $\mathscr{R}$ the set of all regular functions $\rho \in \mathscr{R}_0$.

Theorem 1.2. *Let $\rho \in \mathscr{R}$. Then there is a norm on $\mathbf{B}_\rho$ (see the formula (3.1) below) equivalent to the norm (1.2) which turns $\mathbf{B}_\rho$ into a unital commutative Banach algebra in which the multiplication is the standard pointwise multiplication.*

The main result of this paper is:

Theorem 1.3. *Let $\rho \in \mathscr{R}$. Then the set $\{S_q \mid q \in \mathcal{Q}_\rho\}$ coincides with the set*

$$\mathcal{S}_\rho := \{S \in \mathbf{B}_\rho \mid S(\infty) = 1 \ \text{and}\ \forall \lambda \in \mathbb{R}\ \ S(\lambda)S(-\lambda) = |S(\lambda)| = 1\}.$$

The following result follows from Theorem 1.3.

Corollary 1.4. *Let $\rho \in \mathscr{R}$ and $n \in \mathbb{N}$ (resp. $n = 0$). A triple $(S, \vec{\kappa}, \vec{m})$ (resp. $(S, 0, 0)$), where $S : \mathbb{R} \to \mathbb{C}$, $\vec{\kappa} \in \Omega_n$, $\vec{m} \in \mathbb{R}_+^n$, is the scattering data of some $T \in \mathcal{T}_\rho$ if and only if $S \in \mathcal{S}_\rho$ and $[-\mathrm{ind} S/2] = n$, where $\mathrm{ind}\, S := ((\ln S)(\infty) - (\ln S)(-\infty))/2\pi\mathrm{i}$ and $[x]$ is the integer part of x.*

This paper is organized as follows. In Section 2, we study properties of the spaces X_ρ and their subspaces $X_\rho^\pm$. In Section 3, we consider properties of the algebra $\mathbf{B}_\rho$ and prove Theorem 1.2. In Section 4, we prove Theorem 1.3. Finally, in an Appendix, we give the explicit definition of the operator T_q.

2. PROPERTIES OF THE SPACES X_ρ

Denote by $\|\cdot\|_p$ the norm in the space $L_p(\mathbb{R})$, $p \in [1, \infty]$, and denote by $f * g$ the convolution of functions $f, g \in L_1(\mathbb{R})$, i.e.,

$$(f * g)(x) := \int_{\mathbb{R}} f(x - t) g(t)\, \mathrm{d}t, \quad x \in \mathbb{R}.$$

It is well known that the convolution is a commutative operation in $L_1(\mathbb{R})$ and that

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad f, g \in L_1(\mathbb{R}),$$

and

$$\widehat{f * g} = \widehat{f}\widehat{g},$$

where $\widehat{\varphi}$ is the Fourier transform of a function φ, i.e.,

$$\widehat{\varphi}(\lambda) := \int_{\mathbb{R}} e^{\mathrm{i}\lambda t} \varphi(t)\, \mathrm{d}t, \quad \lambda \in \mathbb{R}.$$

Let us denote by P_+ and P_- the projections in the space $L_1(\mathbb{R})$ acting by the formulas

$$(P_+ f)(x) := \chi_+(x) f(x), \quad (P_- f)(x) := \chi_-(x) f(x), \quad x \in \mathbb{R},$$

where χ_+ (resp. χ_-) is the indicator function of the half-line $\mathbb{R}_+$ (resp. of $\mathbb{R}_-$).

Remark 2.1. If $f, g \in L_1(\mathbb{R})$ and $P_- f = P_- g = 0$, then $P_-(f * g) = 0$ and

$$(f * g)(x) = \int_0^x f(x-t) g(t)\, \mathrm{d}t = \int_0^{x/2} f(x-t) g(t)\, \mathrm{d}t + \int_0^{x/2} g(x-t) f(t)\, \mathrm{d}t, \quad x > 0.$$

Clearly, P_+ and P_- are the projections in every space X_ρ ($\rho \in \mathscr{R}_0$). Moreover, $P_\pm X_\rho = X_\rho^\pm$ and

$$\|f\|_{X_\rho} = \|P_+ f\|_{X_\rho} + \|P_- f\|_{X_\rho}, \quad f \in X_\rho. \tag{2.1}$$

Note that the reflection operator Γ, given by the formula

$$(\Gamma f)(x) = f(-x), \quad x \in \mathbb{R},$$

is an isometry of X_ρ onto itself and maps the space X_ρ^+ (X_ρ^-) on X_ρ^- (X_ρ^+). Moreover,

$$(\Gamma f) * (\Gamma g) = \Gamma(f * g), \quad f, g \in L_1(\mathbb{R}). \tag{2.2}$$

Next, denote by Λ_ρ the operator acting on the space $L_{1,\mathrm{loc}}(\mathbb{R})$ by the formula

$$(\Lambda_\rho f)(x) := \rho(|x|) f(x), \quad x \in \mathbb{R}.$$

Lemma 2.2. *Let $\rho \in \mathscr{R}_0$. Then*

(i) *the space X_ρ is continuously embedded in $L_1(\mathbb{R})$ and*

$$\|u\|_1 \leq \|u\|_{X_\rho}, \quad u \in X_\rho; \tag{2.3}$$

(ii) *the operator Λ_ρ maps continuously the space X_ρ into $L_\infty(\mathbb{R})$ and*

$$\|\Lambda_\rho u\|_\infty \leq \|u\|_{X_\rho}, \quad u \in X_\rho. \tag{2.4}$$

Proof. Clearly, it suffices to prove the estimates (2.3), (2.4), and only for $u \in X_\rho^+$. Fix an arbitrary $u \in X_\rho^+$. Since $u(x)$ vanishes at $+\infty$ and thus

$$|u(x)| \le \int_x^\infty |u'(t)|\,\mathrm{d}t, \quad x \in \mathbb{R}_+,$$

we have

$$\rho(x)|u(x)| \le \rho(x)\int_x^\infty |u'(t)|\,\mathrm{d}t \le \int_x^\infty \rho(t)|u'(t)|\,\mathrm{d}t, \quad x \in \mathbb{R}_+, \tag{2.5}$$

and

$$\int_0^\infty |u(x)|\,\mathrm{d}x \le \int_0^\infty\int_x^\infty |u'(t)|\,\mathrm{d}t\,\mathrm{d}x = \int_0^\infty t|u'(t)|\,\mathrm{d}t \le \int_0^\infty \rho(t)|u'(t)|\,\mathrm{d}t.$$

Using these estimates, we obtain (2.3) and (2.4). □

Consider the spaces

$$Y^\pm := \{f \in X_\rho^\pm \mid f \text{ has compact support and } f \in C^1(\mathbb{R}_\pm \cup \{0\})\}.$$

Lemma 2.3. *Let $\rho \in \mathscr{R}_0$. Then the set Y^+ (resp. Y^-) is everywhere dense in the space X_ρ^+ (resp. in X_ρ^-).*

Proof. Obviously, it suffices to prove the statement for the set Y^+ only. Take $f \in X_\rho^+$ and consider the sequence $f_n := \theta_n f$ ($n \in \mathbb{N}$), where the functions $\theta_n : \mathbb{R} \to [0,1]$ are defined as

$$\theta_n(x) := \begin{cases} 1, & \text{if } 0 \le x \le n, \\ 2 - x/n, & \text{if } n < x \le 2n, \\ 0, & \text{if } x < 0 \text{ or } x > 2n. \end{cases}$$

It is easily seen that each function f_n belongs to X_ρ^+, has compact support and

$$\|f - f_n\|_{X_\rho} = \int_0^\infty \rho(t)|f'(t) - f_n'(t)|\,\mathrm{d}t \le \int_n^\infty \rho(t)|f'(t)|\,\mathrm{d}t + \frac{1}{n}\int_n^{2n} \rho(t)|f(t)|\,\mathrm{d}t.$$

It follows from (2.5) that

$$\rho(x)|f(x)| \le \int_n^\infty \rho(t)|f'(t)|\,\mathrm{d}t, \quad x \ge n.$$

Thus

$$\|f - f_n\|_{X_\rho} \le 2\int_n^\infty \rho(t)|f'(t)|\,\mathrm{d}t$$

and hence $f_n \overset{X_\rho}{\to} f$ as $n \to \infty$.

It remains to prove that every function $u \in X_\rho^+$ of compact support can be approximated by elements from Y^+ in the norm of X_ρ. Let $u \in X_\rho^+$ be a function of compact support. Fix an arbitrary non-negative function $\phi \in C^\infty(\mathbb{R})$ for which

$$\operatorname{supp}\phi \subset [0,1], \qquad \int_{\mathbb{R}} \phi(t)\,\mathrm{d}t = 1.$$

Obviously, for an arbitrary $\varepsilon > 0$, the function

$$u_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon}\int_{\mathbb{R}} u(t)\phi\left(\frac{t-x}{\varepsilon}\right)\,\mathrm{d}t, & \text{if } \ x \geq 0,\\ 0, & \text{if } \ x < 0, \end{cases}$$

belongs to Y^+. Note that for $x > 0$,

$$u(x) - u_\varepsilon(x) = \int_0^1 \left(u(x) - u(x+\varepsilon y)\right)\,\phi(y)\,\mathrm{d}y,$$

and

$$\rho(x)\,\frac{\mathrm{d}}{\mathrm{d}x}(u(x) - u(x+\varepsilon y)) = v(x) - v(x+\varepsilon y) + v(x+\varepsilon y)\,m_\varepsilon(x,y),$$

where $v(x) := \rho(x)u'(x)$ and $m_\varepsilon(x,y) := 1 - \frac{\rho(x)}{\rho(x+\varepsilon y)}$. Thus

$$\|u - u_\varepsilon\|_{X_\rho} \leq \int_0^\infty\int_0^1 |v(x) - v(x+\varepsilon y)|\phi(y)\,\mathrm{d}y\,\mathrm{d}x + \int_0^\infty\int_0^1 |v(x+\varepsilon y)|\,m_\varepsilon(x,y)\phi(y)\,\mathrm{d}y\,\mathrm{d}x.$$

Since $v \in L_1(\mathbb{R})$, $0 \leq m_\varepsilon \leq 1$, and $m_\varepsilon(x,y) \to 0$ as $\varepsilon \to 0$ almost everywhere on $\mathbb{R}_+ \times [0,1]$, we conclude that $u_\varepsilon \overset{X_\rho}{\to} u$ as $\varepsilon \to +0$. □

Proposition 2.4. *Let $\rho \in \mathscr{R}$ and $c = c(\rho)$. Then for an arbitrary $f, g \in X_\rho$, the convolution $f * g$ belongs to X_ρ and*

$$\|f * g\|_{X_\rho} \leq 4c\|f\|_{X_\rho}\|g\|_{X_\rho}. \tag{2.6}$$

Proof. Note that in view of Definition 1.1,

$$\rho(2x) \leq c\rho(x), \quad x > 0. \tag{2.7}$$

1) Let $f, g \in Y^+$. Then (see Remark 2.1) $(f*g)(x) = 0$ for $x < 0$ and

$$(f*g)'(x) = f(x/2)g(x/2) + \int_0^{x/2} f'(x-t)g(t)\,\mathrm{d}t + \int_0^{x/2} g'(x-t)f(t)\,\mathrm{d}t, \quad x > 0.$$

Using this fact and the estimate (2.7), we obtain that for $x > 0$

$$\rho(x)|(f*g)'(x)| \le c\rho(x/2)|f(x/2)|\,|g(x/2)| + c\int_0^{x/2} \rho(x-t)|f'(x-t)|\,|g(t)|\,\mathrm{d}t + c\int_0^{x/2} \rho(x-t)|g'(x-t)|\,|f(t)|\,\mathrm{d}t.$$

Therefore, taking into account (2.3) and (2.4), we get that for all $f, g \in Y^+$,

$$\|f*g\|_{X_\rho} \le 2c\|\Lambda_\rho f\|_\infty \|g\|_1 + c\|f\|_{X_\rho}\|g\|_1 + c\|g\|_{X_\rho}\|f\|_1 \le 4c\|f\|_{X_\rho}\|g\|_{X_\rho}. \quad (2.8)$$

2) Since the reflection operator Γ maps Y^+ onto Y^- and is an isometry of the spaces X_ρ, taking into account (2.2) and (2.8), we obtain that

$$\|f*g\|_{X_\rho} \le 4c\|f\|_{X_\rho}\|g\|_{X_\rho}, \quad f, g \in Y^-. \quad (2.9)$$

3) Let $f \in Y^+$ and $g \in Y^-$. Then

$$\rho(x)|(f*g)'(x)| \le \rho(x)\int_{-\infty}^{0} |f'(x-t)|\,|g(t)|\,\mathrm{d}t \le \int_{-\infty}^{0} \rho(x-t)|f'(x-t)|\,|g(t)|\,\mathrm{d}t$$

for $x > 0$ and

$$\rho(|x|)|(f*g)'(x)| \le \rho(|x|)\int_{0}^{\infty} |g'(x-t)|\,|f(t)|\,\mathrm{d}t \le \int_{0}^{\infty} \rho(|x-t|)|g'(x-t)|\,|f(t)|\,\mathrm{d}t$$

for $x < 0$. Since $c \ge 1$, using the estimate (2.3), we get

$$\|f*g\|_{X_\rho} \le \|f\|_{X_\rho}\|g\|_1 + \|g\|_{X_\rho}\|f\|_1 \le 2c\|f\|_{X_\rho}\|g\|_{X_\rho}, \quad f \in Y^+,\ g \in Y^-. \quad (2.10)$$

4) Let $f, g \in Y^+ \oplus Y^-$ and $f_\pm := P_\pm f$, $g_\pm := P_\pm g$. Then

$$f*g = f_+ * g_+ + f_- * g_- + f_+ * g_- + f_- * g_+.$$

Taking into account (2.9), (2.10) and (2.1), we obtain

$$\|f*g\|_{X_\rho} \le 4c\|f\|_{X_\rho}\|g\|_{X_\rho}, \quad f, g \in Y^+ \oplus Y^-. \quad (2.11)$$

Let $f, g \in X_\rho$ and $u = f * g$. In view of Lemma 2.3, there exist sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $Y^+ \oplus Y^-$ converging in X_ρ to f and g, respectively. It follows from (2.11) that the sequence $(f_n * g_n)_{n\in\mathbb{N}}$ is Cauchy in X_ρ and

$$\|f_n * g_n\|_{X_\rho} \le 4c\|f_n\|_{X_\rho}\|g_n\|_{X_\rho}, \quad n \in \mathbb{N}.$$

Since the space X_ρ is complete and continuously embedded in $L_1(\mathbb{R})$, we conclude that the sequence $(f_n * g_n)_{n\in\mathbb{N}}$ converges in X_ρ to some $u \in X_\rho$. Thus, letting $n \to \infty$, we get that $\|f*g\|_{X_\rho} \le 4c\|f\|_{X_\rho}\|g\|_{X_\rho}$, and the proof is complete. □

3. PROPERTIES OF THE SPACES $\mathbf{B}_\rho$

Let us consider the classical Wiener algebra (see, e.g., [7, 8]), i.e., the commutative Banach algebra

$$\mathbf{A} := \{\alpha\mathbf{1} + \widehat{\varphi} \mid \alpha \in \mathbb{C}, \quad \varphi \in L_1(\mathbb{R})\}$$

with the norm

$$\|\alpha\mathbf{1} + \widehat{\varphi}\|_{\mathbf{A}} := |\alpha| + \|\varphi\|_1.$$

The multiplication in $\mathbf{A}$ is the usual pointwise multiplication and

$$\|fg\|_{\mathbf{A}} \le \|f\|_{\mathbf{A}}\|g\|_{\mathbf{A}}, \quad f, g \in \mathbf{A}.$$

It is known that every function $f \in \mathbf{A}$ is continuous on $\mathbb{R} \cup \{\infty\}$.

In the algebra $\mathbf{A}$, we consider the closed subalgebras

$$\mathbf{A}^+ := \{f = \alpha\mathbf{1} + \widehat{h} \mid \alpha \in \mathbb{C},\ h \in L_1(\mathbb{R}),\ h\big|_{\mathbb{R}_-} = 0\},$$
$$\mathbf{A}_0 := \{f = \widehat{h} \mid h \in L_1(\mathbb{R})\}, \quad \mathbf{A}_0^+ := \mathbf{A}_0 \cap \mathbf{A}^+.$$

Remark 3.1. Each function $\varphi \in \mathbf{A}^+$ is the restriction onto $\mathbb{R}$ of a function Φ which is analytic in the upper half-plane $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+} \cup \{\infty\}$. We will identify the functions φ and Φ.

The following statement follows from the well known results of Wiener (see, e.g., [2], Chapter VIII, 6) and is an analogue of classical Wiener's lemma.

Lemma 3.2 (Wiener)**.** *An element $f \in \mathbf{A}$ ($f \in \mathbf{A}^+$) is invertible in the algebra $\mathbf{A}$ (resp., in $\mathbf{A}^+$) if and only if f does not vanish on $\mathbb{R} \cup \{\infty\}$ (resp., in $\overline{\mathbb{C}_+} \cup \{\infty\}$).*

Remark 3.3. Since $\widehat{X_\rho}$ and X_ρ are isometric, then $\widehat{X_\rho}$ and $\mathbf{B}_\rho$ are Banach spaces. It follows from (2.3) that the space $\widehat{X_\rho}$ is continuously embedded in $\mathbf{A}_0$. Thus the algebra $\mathbf{B}_\rho$ is continuously embedded in $\mathbf{A}$.

Proof of Theorem 1.2. Let $\rho \in \mathscr{R}$ and $f, g \in X_\rho$. In view of Proposition 2.4, the convolution $f * g$ belongs to X_ρ. Since $\widehat{f * g} = \widehat{f}\,\widehat{g}$, the product $\widehat{f}\widehat{g}$ belongs to $\widehat{X_\rho}$. Thus $\widehat{X_\rho}$ is a complex algebra. By the definition of $\mathbf{B}_\rho$,

$$\mathbf{B}_\rho = \widehat{X}_\rho \dotplus \{\alpha\mathbf{1} \mid \alpha \in \mathbb{C}\}.$$

Hence $\mathbf{B}_\rho$ is a complex algebra with unit $\mathbf{1}$.

Let c be the constant from Definition 1.1. Obviously, the formula

$$\|\alpha\mathbf{1} + \widehat{\varphi}\|_{\rho,c} := |\alpha| + 4c\|\varphi\|_{X_\rho}, \quad \alpha \in \mathbb{C}, \varphi \in X_\rho, \tag{3.1}$$

defines a norm on $\mathbf{B}_\rho$ which is equivalent to the norm (1.2). We now show that $\mathbf{B}_\rho$ with the norm $\|\cdot\|_{\rho,c}$ is a Banach algebra with unit. Clearly, it suffices to prove that the norm $\|\cdot\|_{\rho,c}$ satisfies the multiplicative inequality. Let $f = \alpha\mathbf{1} + \widehat{\varphi}$ and $g = \beta\mathbf{1} + \widehat{\psi}$, where $\alpha, \beta \in \mathbb{C}$ and $\varphi, \psi \in X_\rho$. Then

$$\|fg\|_{\rho,c} \le |\alpha||\beta| + |\beta|\|\widehat{\varphi}\|_{\rho,c} + |\alpha|\|\widehat{\psi}\|_{\rho,c} + \|\widehat{\varphi}\widehat{\psi}\|_{\rho,c}.$$

It follows from the inequality (2.6) that

$$\|\widehat{\varphi}\widehat{\psi}\|_{\rho,c} = 4c\|\varphi * \psi\|_{X_\rho} \leq 16c^2\|\varphi\|_{X_\rho}\|\psi\|_{X_\rho} \leq \|\widehat{\varphi}\|_{\rho,c}\|\widehat{\psi}\|_{\rho,c}.$$

Thus

$$\|fg\|_{\rho,c} \leq (|\alpha| + \|\widehat{\varphi}\|_{\rho,c})(|\beta| + \|\widehat{\psi}\|_{\rho,c}) = \|f\|_{\rho,c}\|g\|_{\rho,c}$$

as claimed. □

In the algebra $\mathbf{B}_\rho$, we consider the closed subalgebras $\mathbf{B}_\rho^+ := \mathbf{B}_\rho \cap \mathbf{A}^+$.

Lemma 3.4.

(i) *Let $\rho \in \mathscr{R}$ and b be a rational function that has only simple zeros and does not vanish on $\mathbb{R} \cup \{\infty\}$. Then $1/b \in \mathbf{B}_\rho$.*

(ii) *Let $\rho \in \mathscr{R}$ and $u \in Y^+$ and, moreover, assume that the function $g = \mathbf{1} + \widehat{u}$ does not vanish in $\overline{\mathbb{C}_+} \cup \{\infty\}$. Then $1/g \in \mathbf{B}_\rho^+$.*

Proof. Let the conditions of (i) be satisfied. Then

$$\frac{1}{b(\lambda)} = c_0 + \sum_{j=1}^{n} \frac{c_j}{\lambda + \alpha_j}, \quad \lambda \in \mathbb{R},$$

where $\{c_j\}_{j=0}^n \subset \mathbb{C}$ and $\{\alpha_j\}_{j=1}^n \subset \mathbb{C} \setminus \mathbb{R}$. Thus, it suffices to show that the functions $f_\alpha(\lambda) = (\lambda + \alpha)^{-1}$ with $\alpha \in \mathbb{C}_+$ belong to $\mathbf{B}_\rho^+$. Note that f_α is the Fourier transform of the function $u_\alpha(x) := -ie^{i\alpha x}\chi_+(x)$. Since $\lim\limits_{x\to+\infty} \rho(x)e^{-\gamma x} = 0$ for $\gamma > 0$, then $f_\alpha \in \widehat{X_\rho^+}$.

Let the conditions of (ii) be satisfied. We consider the function $v(x) := \mathrm{i}u(x) + \mathrm{i}u'(x)$ $(x \neq 0)$. This function belongs to $L_2(\mathbb{R})$, has compact support and

$$\widehat{v}(\lambda) = \mathrm{i}\widehat{u}(\lambda) + \mathrm{i}\int_{\mathbb{R}} e^{\mathrm{i}\lambda x}u'(x)\,\mathrm{d}x = (\lambda + \mathrm{i})\widehat{u}(\lambda) - \mathrm{i}(u(+0) - u(-0)).$$

Thus

$$\widehat{u}(\lambda) = \frac{\mathrm{i}(u(+0) - u(-0))}{\lambda + \mathrm{i}} + \frac{\widehat{v}(\lambda)}{\lambda + \mathrm{i}}, \quad \lambda \in \mathbb{C}.$$

Using this fact, we conclude that

$$\widehat{u}(\lambda) = o(\lambda^{-1}), \quad \lambda \to \infty,$$

uniformly in each strip $\{z \in \mathbb{C} \mid |\mathrm{Im} z| < \gamma\}$ $(\gamma > 0)$. Thus

$$\frac{1}{g(\lambda)} = 1 - \widehat{u}(\lambda) + \frac{\widehat{u}(\lambda)^2}{1 + \widehat{u}(\lambda)} = 1 - \widehat{u}(\lambda) + h(\lambda),$$

where the function h is analytic in some half-plane $\{z \in \mathbb{C} \mid \mathrm{Im} z > -\delta\}$ $(\delta > 0)$ and

$$\sup_{|y|<\delta} \int_{\mathbb{R}} |(x + \mathrm{i}y)h(x + \mathrm{i}y)|^2\,\mathrm{d}x < \infty. \tag{3.2}$$

Therefore, it suffices to show that $h \in \widehat{X_\rho^+}$. It follows from (3.2) that $h = \widehat{w}$, where w belongs to the Sobolev space $W_2^1(\mathbb{R})$. From known properties of the Fourier transform (see, e.g., [6, Chapter 5]), we obtain that

$$2\pi \int_{\mathbb{R}} e^{-2y\xi} |w'(\xi)|^2 \, \mathrm{d}\xi = \int_{\mathbb{R}} |(x + \mathrm{i}y) h(x + \mathrm{i}y)|^2 \, \mathrm{d}x, \quad y \in (-\delta, \delta).$$

Using this fact and (3.2), we get that

$$J(y) := \int_{\mathbb{R}} e^{2y|\xi|} |w'(\xi)|^2 \, \mathrm{d}\xi < \infty, \quad y \in (0, \delta).$$

Using the Cauchy–Schwarz inequality, we derive that

$$\left(\int_{\mathbb{R}} e^{y|\xi|} |w'(\xi)| \, \mathrm{d}\xi \right)^2 \le J(u) \int_{\mathbb{R}} e^{2(y-u)|\xi|} \, \mathrm{d}\xi < \infty, \quad 0 < y < u < \delta.$$

Since $\lim\limits_{x \to +\infty} \rho(x) e^{-yx} = 0$ for $y > 0$, we conclude that $w \in X_\rho^+$, and hence $h \in \widehat{X_\rho^+}$. The proof is complete. □

Lemma 3.5. *Let $\rho \in \mathscr{R}$, $c = c(\rho)$, $u \in Y^+$ and $\|u\|_1 \le 1/4c$. Then the function $g = \mathbf{1} + \widehat{u}$ is invertible in the algebra $\mathbf{B}_\rho^+$ and, moreover,* (*see* (3.1))

$$\|1/g\|_{\rho,c} \le 4 \|g\|_{\rho,c}.$$

Proof. Since $c \ge 1$, we conclude that the element $g = \mathbf{1} + \widehat{u}$ is invertible in the algebra $\mathbf{A}^+$ and, moreover, $1/g = \mathbf{1} + \widehat{v}$, where $v \in L_1(\mathbb{R})$ and

$$\|v\|_1 = \|1/g - \mathbf{1}\|_{\mathbf{A}} \le \sum_{n=1}^{\infty} \|\widehat{u}\|_{\mathbf{A}}^n = \frac{\|\widehat{u}\|_{\mathbf{A}}}{1 - \|\widehat{u}\|_{\mathbf{A}}} = \frac{\|u\|_1}{1 - \|u\|_1} \le \frac{1}{2c}. \tag{3.3}$$

In view of the Wiener Lemma and Lemma 3.4, we obtain that $v \in X_\rho^+$. Since $(\mathbf{1} + \widehat{u})(\mathbf{1} + \widehat{v}) = \mathbf{1}$, we have that $u + v + u * v = 0$. Taking into account that $u \in Y^+$ and $v \in X_\rho^+$, we get the equality

$$u(x) + v(x) + \int_0^x u(x - t) v(t) \, \mathrm{d}t = 0, \quad x > 0,$$

from which we can easily see that $v \in C^1[0, \infty)$. We represent the convolution $u * v$ in the form $u * v = w_1 + w_2$, where (see Remark 2.1)

$$w_1(x) := \int_0^{x/2} u(x - t) v(t) \, \mathrm{d}t, \quad w_2(x) := \int_0^{x/2} v(x - t) u(t) \, \mathrm{d}t, \quad x \ge 0,$$

and $w_1(x) = w_2(x) = 0$ for $x < 0$. It is clear that $w_1, w_2 \in C^1[0,\infty)$ and

$$w_1'(x) = \frac{1}{2}u(x/2)v(x/2) + \int_0^{x/2} u'(x-t)v(t)\,\mathrm{d}t, \quad x > 0,$$

$$w_2'(x) = \frac{1}{2}u(x/2)v(x/2) + \int_0^{x/2} v'(x-t)u(t)\,\mathrm{d}t, \quad x > 0.$$

Let us estimate the norm $\|w_1\|_{X_\rho}$. Taking into account the inequality (2.7), we have that for an arbitrary $x > 0$,

$$\rho(x)|w_1'(x)| \le \frac{c}{2}|\rho(x/2)u(x/2)||v(x/2)| + c\int_0^{x/2} \rho(x-t)|u'(x-t)||v(t)|\,\mathrm{d}t.$$

Thus, using (2.4) and (3.3), we get

$$\|w_1\|_{X_\rho} \le c\|u\|_{X_\rho}\|v\|_1 + c\|u\|_{X_\rho}\|v\|_1 \le 2c\|u\|_{X_\rho}\|v\|_1 \le \|u\|_{X_\rho}. \tag{3.4}$$

Similarly, we obtain that

$$\|w_2\|_{X_\rho} \le 2c\|v\|_{X_\rho}\|u\|_1 \le \frac{1}{2}\|v\|_{X_\rho}. \tag{3.5}$$

It is easily seen that $\|v\|_{X_\rho} \le \|u\|_{X_\rho} + \|w_1\|_{X_\rho} + \|w_2\|_{X_\rho}$. Taking into account (3.4) and (3.5), we obtain that $\|v\|_{X_\rho} \le 4\|u\|_{X_\rho}$, so that

$$\|1/g\|_{\rho,c} = 1 + 4c\|v\|_{X_\rho} \le 4(1 + 4c\|u\|_{X_\rho}) = 4\|g\|_{\rho,c}$$

as claimed. □

The main result of this section is following analogue of the Wiener Lemma.

Theorem 3.6. *Let $\rho \in \mathscr{R}$. Then $g \in \mathbf{B}_\rho^+$ is invertible in the Banach algebra $\mathbf{B}_\rho^+$ if and only if g does not vanish in $\overline{\mathbb{C}_+} \cup \{\infty\}$.*

Proof. Let g be invertible in the algebra $\mathbf{B}_\rho^+$. Since $\mathbf{B}_\rho^+ \subset \mathbf{A}^+$, the element g is invertible in the algebra $\mathbf{A}^+$. Thus, in view of Wiener Lemma, g does not vanish in $\overline{\mathbb{C}_+} \cup \{\infty\}$.

Conversely, let $g \in \mathbf{B}_\rho^+$ not vanish in $\overline{\mathbb{C}_+} \cup \{\infty\}$. From Wiener Lemma, we can conclude that $1/g \in \mathbf{A}^+$. Let us show that $1/g \in \mathbf{B}_\rho^+$. Without loss of generality, we can assume that $g = \mathbf{1} + \widehat{u}$, where $u \in X_\rho^+$.

First, we consider the case $\|u\|_1 \le 1/4c$. By Lemma 2.3, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in Y^+ converging to u in X_ρ^+. Since the space X_ρ is continuously embedded in $L_1(\mathbb{R})$, we can assume that $\|u_n\|_1 \le 1/4c$ for all $n \in \mathbb{N}$. Let $g_n := \mathbf{1} + \widehat{u}_n$, $n \in \mathbb{N}$. Then the sequence $(g_n)_{n\in\mathbb{N}}$ converges to g in $\mathbf{B}_\rho^+$ and, in view of Lemma 3.5,

$$1/g_n \in \mathbf{B}_\rho^+, \quad \|1/g_n\|_{\rho,c} \le 4\|g_n\|_{\rho,c}, \quad n \in \mathbb{N}.$$

Since the sequence $(1/g_n)_{n\in\mathbb{N}}$ is bounded in $\mathbf{B}_\rho^+$, we conclude (see, e.g., [5, Chapter 10]) that $1/g \in \mathbf{B}_\rho^+$.

Now we consider the general case when $g = \mathbf{1} + \widehat{u}$, $u \in X_\rho^+$ and g does not vanish in $\overline{\mathbb{C}_+} \cup \{\infty\}$. By Lemma 2.3, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in Y^+ converging to u in X_ρ^+. Since X_ρ is continuously embedded in $L_1(\mathbb{R})$, we can assume that all functions $g_n := \mathbf{1} + \widehat{u}_n$ $(n \in \mathbb{N})$ do not vanish in $\overline{\mathbb{C}_+} \cup \{\infty\}$, so that (see Lemma 3.4) $1/g_n \in \mathbf{B}_\rho^+$ for all n. Hence (see Theorem 1.2) the sequence $f_n := g/g_n$ $(n \in \mathbb{N})$ belongs to the space $\mathbf{B}_\rho^+$ and, obviously, converges to $\mathbf{1}$ in the space $\mathbf{A}^+$. Using this fact, we conclude that $f_n = \mathbf{1} + \widehat{v}_n$, where the sequence $(v_n)_{n\in\mathbb{N}}$ belongs to X_ρ^+ and converges to zero in $L_1(\mathbb{R})$. Thus (see Lemma 3.5) $1/f_n \in \mathbf{B}_\rho^+$ for sufficiently large n. Let $1/f_m \in \mathbf{B}_\rho^+$ for some $m \in \mathbb{N}$. Since $1/g = 1/g_m \cdot 1/f_m$, in view of Theorem 1.2, we arrive at the conclusion that $1/g \in \mathbf{B}_\rho^+$ and the proof is complete. □

4. PROOF OF THEOREM 1.3.

First, we prove two auxiliary Lemmas that are generalizations of the similar Lemmas in [3, Chapter 3].

Lemma 4.1. *Let $\rho \in \mathscr{R}_0$ and $\varphi \in L_r(\mathbb{R}_+)$ $(r \in [1,\infty])$. If a function $\psi \in X_\rho^+$ is such that the function g is given by*

$$g(x) := \varphi(x) + \int_0^\infty \varphi(t)\psi(x+t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+, \tag{4.1}$$

belongs to the space X_ρ^+, then $\varphi \in X_\rho^+$.

Proof. Let $g, \psi \in X_\rho^+$. Since $X_\rho^+ \subset L_1(\mathbb{R}_+)$, then (see [4], Lemma 3.1) $\varphi \in L_1(\mathbb{R}_+)$. Taking into account the equalities

$$g(x) = -\int_x^\infty g'(\xi)\,\mathrm{d}\xi, \quad \psi(x) = -\int_x^\infty \psi'(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}_+,$$

(4.1) can be represented as

$$\varphi(x) = -\int_x^\infty g'(\xi)\,\mathrm{d}\xi + \int_0^\infty \varphi(t)\int_x^\infty \psi'(\xi+t)\,\mathrm{d}\xi\,\mathrm{d}t. \tag{4.2}$$

Since

$$\int_0^\infty\int_0^\infty |\psi'(\xi+t)|\,\mathrm{d}\xi\,\mathrm{d}t = \int_0^\infty t|\psi'(t)|\;\mathrm{d}t \le \|\psi\|_{X_\rho^+},$$

applying Fubini's theorem to the iterated integral in (4.2), we get

$$\varphi(x) = -\int_x^\infty \left(g'(\xi) - \int_0^\infty \varphi(t)\psi'(\xi+t)\,\mathrm{d}t \right) \mathrm{d}\xi, \quad x \in \mathbb{R}_+.$$

Consequently, the function φ belongs to $\mathrm{AC}(\mathbb{R}_+)$ and

$$\varphi'(x) = g'(x) - \int_0^\infty \varphi(t)\psi'(x+t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+.$$

Thus

$$\int_0^\infty \rho(x)|\varphi'(x)|\,\mathrm{d}x \le \int_0^\infty \rho(x)|g'(x)|\,\mathrm{d}x + \int_0^\infty\int_0^\infty |\varphi(t)|\,|\rho(x+t)\psi'(x+t)|\,\mathrm{d}t\,\mathrm{d}x,$$

and, therefore, $\|\varphi\|_{X_\rho^+} \le \|g\|_{X_\rho^+} + \|\varphi\|_1\|\psi\|_{X_\rho^+} < \infty$. □

Lemma 4.2. *Let $\rho \in \mathscr{R}_0$ and $\varphi \in L_1(\mathbb{R}_+)$ and $\psi \in X_\rho^+$ be related via*

$$\varphi(x) + \psi(x) + \int_0^\infty \varphi(t)\psi(x+t)\,\mathrm{d}t = 0, \quad x \in \mathbb{R}_+. \tag{4.3}$$

If the function f is given by the formula

$$f(\lambda) = 1 + \int_0^\infty \varphi(t)e^{\mathrm{i}\lambda t}\,\mathrm{d}t, \quad \lambda \in \mathbb{R},$$

and $f(0) = 0$, then there exists $g \in \mathbf{B}_\rho^+$ such that $f(\lambda) = \frac{\lambda}{\lambda+\mathrm{i}}\,g(\lambda)$.

Proof. Let the conditions of the lemma be satisfied. From Lemma 4.1, it follows that $\varphi \in X_\rho^+$ and thus $f \in \mathbf{B}_\rho^+$. Let us show that the function

$$h(x) := \int_x^\infty \varphi(t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+,$$

belongs to X_ρ^+. Note that it follows from the condition $f(0) = 0$ that $h(0) = -1$. Consider the auxiliary function

$$\Phi(x) := \int_0^\infty h'(t)\int_{x+t}^\infty \psi(\xi)\,\mathrm{d}\xi\,\mathrm{d}t, \quad x \ge 0. \tag{4.4}$$

Integrating by parts, we obtain that

$$\Phi(x) = \int_x^\infty \psi(\xi)\,\mathrm{d}\xi + \int_0^\infty h(t)\psi(x+t)\,\mathrm{d}t. \tag{4.5}$$

On the other hand, it follows from (4.4) that

$$\Phi(x) = -\int_0^\infty \varphi(t) \int_x^\infty \psi(y+t)\,\mathrm{d}y\,\mathrm{d}t = -\int_x^\infty \int_0^\infty \varphi(t)\psi(y+t)\,\mathrm{d}t\,\mathrm{d}y. \tag{4.6}$$

Taking into account (4.3), (4.5) and (4.6), we get

$$\int_x^\infty \psi(\xi)\,\mathrm{d}\xi + \int_0^\infty h(t)\psi(x+t)\,\mathrm{d}t = \int_x^\infty (\varphi(y)+\psi(y))\,\mathrm{d}y$$

and, therefore,

$$h(x) + \int_0^\infty h(t)(-\psi(x+t))\,\mathrm{d}t = 0, \quad x \in \mathbb{R}_+.$$

Since $h \in L_\infty(\mathbb{R}_+)$ and $-\psi \in X_\rho^+$, in view of Lemma 4.1, we conclude that $h \in X_\rho^+$. Consequently, the function

$$g_1(\lambda) := \mathrm{i}\int_0^\infty h(t)e^{\mathrm{i}\lambda t}\,\mathrm{d}t, \quad \lambda \in \mathbb{R},$$

belongs to $\mathbf{B}_\rho^+$. Integrating by parts, we get

$$\lambda g_1(\lambda) = \int_0^\infty h(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{\mathrm{i}\lambda t}\right) = -h(0) + \int_0^\infty \varphi(t)e^{\mathrm{i}\lambda t}\,\mathrm{d}t = f(\lambda).$$

Let $g(\lambda) := f(\lambda) + \mathrm{i}g_1(\lambda)$. Since $g_1, f \in \mathbf{B}_\rho^+$, we deduce that $g \in \mathbf{B}_\rho^+$. Moreover, $\lambda(\lambda+\mathrm{i})^{-1}g(\lambda) = \lambda g_1(\lambda) = f(\lambda)$. □

Below, we list some facts from [3, Chapter 3]. Let $q \in \mathcal{Q}_\omega$ and

$$\sigma(x) := \int_x^\infty |q(\xi)|\,\mathrm{d}\xi, \quad \sigma_1(x) := \int_x^\infty \xi|q(\xi)|\,\mathrm{d}\xi.$$

1°. The solution of the Jost equation (1.1) can be represented in the form

$$e(\lambda, x) = e^{\mathrm{i}\lambda x} + \int_x^\infty K(x,t)e^{\mathrm{i}\lambda t}\,\mathrm{d}t, \quad \lambda \in \overline{\mathbb{C}_+}, \quad x \in \mathbb{R}_+,$$

where the kernel K is continuous on the set $\Omega := \{(x,t) \in \mathbb{R}_+^2 \mid x \le t\}$ and

$$|K(x,t)| \le \sigma\left(\frac{x+t}{2}\right)\exp\{\sigma_1(x)\}, \quad (x,t) \in \Omega.$$

2°. For $\lambda \in \mathbb{R} \setminus \{0\}$, the estimate for the derivative of the Jost solution

$$|e'(\lambda, x) - \mathrm{i}\lambda e^{\mathrm{i}\lambda x}| \leq \sigma(x) \exp\{\sigma_1(x)\}, \quad x \in \mathbb{R}_+, \tag{4.7}$$

holds, and the formula

$$\omega(\lambda, x) := \frac{e(-\lambda, 0)e(\lambda, x) - e(\lambda, 0)e(-\lambda, x)}{2\mathrm{i}\lambda}, \quad x \in \mathbb{R}_+, \tag{4.8}$$

defines a solution of the equation (1.1) satisfying

$$\omega(\lambda, x) = x(1 + o(1)), \quad \omega'(\lambda, x) = 1 + o(1), \quad x \to +0. \tag{4.9}$$

3°. The function $\overline{\mathbb{C}_+} \setminus \{0\} \ni \lambda \mapsto e(\lambda) := e(\lambda, 0)$ has a finite number of zeros which are simple and lie on the imaginary line.

4°. The kernel K is a solution of the Marchenko equation

$$F(x + t) + K(x, t) + \int_x^\infty K(x, \xi) F(\xi + t) \, \mathrm{d}\xi = 0, \quad (x, t) \in \Omega, \tag{4.10}$$

with F given by

$$F(x) := \sum_{s=1}^n m_s e^{-\kappa_s x} + F_S(x), \quad x \geq 0, \tag{4.11}$$

where

$$F_S(x) := \frac{1}{2\pi} \int_{\mathbb{R}} (1 - S(\lambda)) e^{\mathrm{i}\lambda x} \, \mathrm{d}\lambda, \quad x \in \mathbb{R}. \tag{4.12}$$

5°. The function F belongs to the class $\mathrm{AC}(\mathbb{R}_+)$ and there exists a constant $C_1 > 0$ such that

$$|F'(2x) - q(x)/4| \leq C_1 \sigma^2(x), \quad x > 0. \tag{4.13}$$

Lemma 4.3. *Let $q \in \mathcal{Q}_\omega$ and the function F be given by formula* (4.11)*. Then for each $\rho \in \mathscr{R}$ the function q belongs to the class $\mathcal{Q}_\rho$ if and only if $F \in X_\rho^+$.*

Proof. 1) Let $\rho \in \mathscr{R}$ and $q \in \mathcal{Q}_\rho$. Then for an arbitrary $\gamma \geq 0$,

$$\rho(x)\sigma(x) \leq \int_x^\infty \rho(t)|q(t)| \, \mathrm{d}t \leq \int_\gamma^\infty \rho(t)|q(t)| \, \mathrm{d}t, \quad x \geq \gamma,$$

and

$$\int_\gamma^\infty \sigma(x) \, \mathrm{d}x = \int_\gamma^\infty \int_x^\infty |q(t)| \, \mathrm{d}t \, \mathrm{d}x \leq \int_\gamma^\infty t|q(t)| \, \mathrm{d}t < \infty.$$

Thus

$$\int_\gamma^\infty \rho(x)\sigma^2(x)\,\mathrm{d}x \le \left(\int_\gamma^\infty \rho(t)|q(t)|\,\mathrm{d}t\right)\left(\int_\gamma^\infty \sigma(x)\,\mathrm{d}x\right)$$
$$\le \left(\int_\gamma^\infty \rho(t)|q(t)|\,\mathrm{d}t\right)\left(\int_\gamma^\infty t|q(t)|\,\mathrm{d}t\right) < \infty. \tag{4.14}$$

It follows from (4.13) that

$$|F'(2x)| \le |q(x)| + C_1\sigma^2(x), \quad x > 0.$$

Using this estimate and (2.7), we get

$$\int_0^\infty \rho(2x)|F'(2x)|\,\mathrm{d}x \le c\int_0^\infty \rho(x)|F'(2x)|\,\mathrm{d}x$$
$$\le c\int_0^\infty \rho(x)|q(x)|\,\mathrm{d}x + c\,C_1\int_0^\infty \rho(x)\sigma^2(x)\,\mathrm{d}x < \infty,$$

and hence $F \in X_\rho^+$ as claimed.

2) Let $q \in \mathcal{Q}_\omega$ and $F \in X_\rho^+$. It follows from (4.13) that

$$|q(x)| \le 4|F'(2x)| + 4C_1\sigma^2(x), \quad x > 0. \tag{4.15}$$

Let us fix $\gamma > 0$ for which

$$\int_\gamma^\infty t|q(t)|\,\mathrm{d}t \le \frac{1}{8C_1}, \tag{4.16}$$

and put

$$\rho_n(x) := \min\{\rho(x), n + x\}, \quad x \ge 0, \quad n \in \mathbb{N}.$$

Obviously, that $\rho_n \in \mathscr{R}$. Using the estimate (4.15), we obtain that for an arbitrary $n \in \mathbb{N}$,

$$\int_\gamma^\infty \rho_n(x)|q(x)|\,\mathrm{d}x \le 4\int_\gamma^\infty \rho_n(2x)|F'(2x)|\,\mathrm{d}x + 4C_1\int_\gamma^\infty \rho_n(x)\,\sigma^2(x)\,\mathrm{d}x. \tag{4.17}$$

From (4.14) and (4.16), we deduce that

$$4C_1\int_\gamma^\infty \rho_n(x)\,\sigma^2(x)\,\mathrm{d}x \le 4C_1\int_\gamma^\infty \xi|q(\xi)|\,\mathrm{d}\xi\int_\gamma^\infty \rho_n(t)|q(t)|\,\mathrm{d}t \le \frac{1}{2}\int_\gamma^\infty \rho_n(t)|q(t)|\,\mathrm{d}t.$$

Thus, in view of (4.17), we get

$$\int_{\gamma}^{\infty} \rho_n(x)|q(x)|\,\mathrm{d}x \le 8\int_{\gamma}^{\infty} \rho_n(2x)|F'(2x)|\,\mathrm{d}x \le 4\int_{0}^{\infty} \rho(x)|F'(x)|\,\mathrm{d}x.$$

Using the monotone convergence theorem, we have

$$\int_{\gamma}^{\infty} \rho(x)|q(x)|\,\mathrm{d}x \le 4\int_{0}^{\infty} \rho(x)|F'(x)|\,\mathrm{d}x < \infty,$$

and hence $q \in \mathcal{Q}_\rho$. □

Proof of Theorem 1.3. First, we prove sufficiency. Let $\rho \in \mathscr{R}$, $S \in \mathcal{S}_\rho$ and $n := [-\operatorname{ind} S/2]$. Since $\mathcal{S}_\rho \subset \mathcal{S}_\omega$, in view of the results of [4], we conclude that S is the scattering function for some operator T_q with $q \in \mathcal{Q}_\omega$. Since $S \in \mathcal{S}_\rho$, the function F_S (see (4.12)) belongs to the space X_ρ. Therefore, the function F, given by the formula (4.11), belongs to the space X_ρ^+. In view of Lemma 4.3, we have that $q \in \mathcal{Q}_\rho$ so that every function $S \in \mathcal{S}_\rho$ is the scattering function of some operator T_q with $q \in \mathcal{Q}_\rho$ as claimed.

Let us prove necessity. Let $q \in \mathcal{Q}_\rho$. We need to prove that $S_q \in \mathcal{S}_\rho$. Since $q \in \mathcal{Q}_\rho$, in view of Lemma 4.3, we conclude that $F \in X_\rho^+$. It follows from the Marchenko equation (4.10) that

$$F(t) + K(0,t) + \int_{0}^{\infty} K(0,\xi)F(\xi+t)\,\mathrm{d}\xi = 0, \quad t > 0.$$

Thus in view of Lemma 4.1 the function $\mathbb{R}_+ \ni t \mapsto K(0,t)$ belongs to the space X_ρ^+ and, therefore, the Jost function

$$e(\lambda) = 1 + \int_{0}^{\infty} K(0,t)e^{\mathrm{i}\lambda t}\,\mathrm{d}t, \quad \lambda \in \overline{\mathbb{C}_+},$$

belongs to the space $\mathbf{B}_\rho^+$.

1) Suppose that $e(0) \ne 0$. Then, in view of 3°, the function e has a finite number of zeros in $\overline{\mathbb{C}_+} \cup \{\infty\}$. All these zeros are simple and can be represented as $z = \mathrm{i}\kappa_j$, where $\{\kappa_j\}_{j=1}^n \subset \mathbb{R}_+$. Let us consider the Blaschke product

$$b(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \mathrm{i}\kappa_j}{\lambda + \mathrm{i}\kappa_j} \tag{4.18}$$

and the functions

$$f(\lambda) := \frac{e(-\lambda)}{b(\lambda)}, \quad g(\lambda) := \frac{e(\lambda)}{b(\lambda)}, \quad \lambda \in \mathbb{R}.$$

It follows from Lemma 3.4 and Theorem 1.2 that $f, g \in \mathbf{B}_\rho$. Obviously, $g \in \mathbf{A}^+$, and thus $g \in \mathbf{B}_\rho^+$. Moreover, the function g does not vanish in $\overline{\mathbb{C}_+} \cup \{\infty\}$. Therefore, in view of Theorem 3.6, we obtain that $1/g \in \mathbf{B}_\rho^+$. Since $S = f/g$ and $\mathbf{B}_\rho$ is an algebra, we deduce that $S \in \mathbf{B}_\rho$.

2) Suppose that $e(0) = 0$. Taking into account (4.10) and Lemma 4.2, we get that $e(\lambda) = \frac{\lambda}{\lambda+\mathrm{i}} h(\lambda)$, where $h \in \mathbf{B}_\rho^+$. Let us show that $h(0) \neq 0$. It follows from (4.7) that there exists $C > 0$ such that $|e'(\lambda, x)| \le C$ for $x \in \mathbb{R}_+$ and $\lambda \in [-1, 1] \setminus \{0\}$. Thus (see (4.8))

$$|\omega'(\lambda, x)| \le C(|h(-\lambda)| + |h(\lambda)|), \quad x \in \mathbb{R}_+, \quad \lambda \in [-1, 1] \setminus \{0\}.$$

Therefore, taking into account (4.9), we have

$$1 = \lim_{x \to +0} |\omega'(\lambda, x)| \le C(|h(-\lambda)| + |h(\lambda)|), \quad \lambda \in [-1, 1] \setminus \{0\}.$$

Since the function h is continuous, we obtain that $h(0) \neq 0$. In view of 3°, the function h has a finite number of zeros in $\overline{\mathbb{C}_+} \cup \{\infty\}$. All these zeros are simple and can be represented as $z = \mathrm{i}\kappa_j$, where $\{\kappa_j\}_{j=1}^n \subset \mathbb{R}_+$. Let us consider the functions

$$f(\lambda) := \frac{\lambda + \mathrm{i}}{\lambda - \mathrm{i}} \frac{h(-\lambda)}{b(\lambda)}, \quad g(\lambda) := \frac{h(\lambda)}{b(\lambda)}, \quad \lambda \in \mathbb{R},$$

where b is the Blaschke product given by the formula (4.18). It follows from Lemma 3.4 and Theorem 1.2 that $f, g \in \mathbf{B}_\rho$. Obviously, $g \in \mathbf{B}_\rho^+$ and the function g does not vanish in $\overline{\mathbb{C}_+} \cup \{\infty\}$. It follows from Theorem 3.6 that $1/g \in \mathbf{B}_\rho$. Since $S = f/g$ and $\mathbf{B}_\rho$ is an algebra, we arrive at the conclusion that $S \in \mathbf{B}_\rho$. Therefore, the proof is complete. □

APPENDIX

A. OPERATOR T_q

In this appendix, we will give the explicit definition of the operator T_q.

We denote by C_0^∞ the linear space of all functions on the half-line with compact support that are infinitely often differentiable. Also we denote by W_2^1 the Sobolev space of functions $f \in \mathrm{AC}[0, \infty)$ for which

$$\|f\|_{W_2^1}^2 := \int_0^\infty (|f(x)|^2 + |f'(x)|^2) \, \mathrm{d}x < \infty.$$

Let q be a locally integrable real-valued function on $\mathbb{R}_+$ and

$$\int_0^\infty x|q(x)| \, \mathrm{d}x < \infty. \tag{A.1}$$

We consider the symmetric sesqulinear forms $\mathfrak{t}_0$ and $\mathfrak{q}$ that are defined on the common domain $W^1_{2,0} := \{f \in W^1_2 \mid f(0) = 0\}$ by the formulas

$$\mathfrak{t}_0[f,g] := \int_0^\infty f'(x)\,\overline{g'(x)}\,\mathrm{d}x, \quad \mathfrak{q}[f,g] := \int_0^\infty q(x) f(x)\,\overline{g(x)}\,\mathrm{d}x.$$

Note that the form $\mathfrak{t}_0$ is nonnegative and closed (see [1], Ch.VI-§1.3). We will show that the form $\mathfrak{q}$ is $\mathfrak{t}_0$-bounded (see [1], Ch.VI-§1.6). We represent the function q (see (A.1)) as the sum $q_1 + q_2$, where $q_1 \in C_0^\infty$ and q_2 satisfies the following condition:

$$\int_0^\infty x|q_2(x)|\,\mathrm{d}x \le b < 1.$$

Using the Cauchy–Schwarz inequality, we get that for $f \in W^1_{2,0}$

$$|f(x)|^2 = \left|\int_0^x f'(t)\,\mathrm{d}t\right|^2 \le x \int_0^x |f'(t)|^2\,\mathrm{d}t \le x\,\mathfrak{t}_0[f], \quad x \in \mathbb{R}_+,$$

where $\mathfrak{t}_0[f] := \mathfrak{t}_0[f,f]$. Thus for all $f \in W^1_{2,0}$

$$|\mathfrak{q}[f]| \le \int_0^\infty |q_1(x)||f(x)|^2\,\mathrm{d}x + \int_0^\infty |q_2(x)||f(x)|^2\,\mathrm{d}x \le a\|f\|^2 + b\,\mathfrak{t}_0[f],$$

where $a := \max|q_1(x)|$. Consequently, the form $\mathfrak{q}$ is $\mathfrak{t}_0$-bounded with $b < 1$. Therefore (see [1, Chapter VI, §1.6]), the symmetric form $\mathfrak{t} = \mathfrak{t}_0 + \mathfrak{s}$ is bounded from below and closed. By the first representation theorem (see [1, Chapter VI, §2.1]), there exists the unique self-adjoint operator T_q that is associated with $\mathfrak{t}$. Its domain consists of functions $f \in W^1_{2,0}$ for which there exists $h \in L_2(\mathbb{R}_+)$ such that

$$\mathfrak{t}[f,g] = (h \mid g), \quad g \in W^1_{2,0}. \tag{A.2}$$

If (A.2) holds, then $T_q f = h$. Let $f \in \operatorname{dom} T_q$. Then for some $h \in L_2(\mathbb{R}_+)$

$$(f' \mid g') = (h - qf \mid g), \quad g \in C_0^\infty.$$

Thus we have that $-f'' = h - qf$ in the sense of distribution theory. It means that $f' \in \mathrm{AC}(0,\infty)$ and $(-f'' + qf) = h \in L_2(0,\infty)$. Therefore,

$$\operatorname{dom} T_q := \{f \in W^1_{2,0} \mid f' \in \mathrm{AC}(0,\infty),\ (-f'' + qf) \in L_2(\mathbb{R}_+)\}$$

and

$$T_q f := -f'' + qf, \quad f \in \operatorname{dom} T_q.$$

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