Opuscula Math. 39, no. 3 (2019), 425–431
https://doi.org/10.7494/OpMath.2019.39.3.425 Opuscula Mathematica

THE COMPLEXITY OF OPEN k-MONOPOLIES IN GRAPHS FOR NEGATIVE k

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Communicated by Mirko Horňák

Abstract. Let G be a graph with vertex set $V(G)$, $\delta(G)$ minimum degree of G and $k \in \left\{1 - \left\lceil \frac{\delta(G)}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta(G)}{2} \right\rfloor \right\}$. Given a nonempty set $M \subseteq V(G)$ a vertex v of G is said to be k-controlled by M if $\delta_M(v) \geq \frac{\delta_{V(G)}(v)}{2} + k$ where $\delta_M(v)$ represents the number of neighbors of v in M. The set M is called an open k-monopoly for G if it k-controls every vertex v of G. In this short note we prove that the problem of computing the minimum cardinality of an open k-monopoly in a graph for a negative integer k is NP-complete even restricted to chordal graphs.

Keywords: open k-monopolies, complexity, total domination.

Mathematics Subject Classification: 05C85, 05C69, 05C07.

1. INTRODUCTION, PRELIMINARIES AND THE RESULT

We consider in this work only finite simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The order of G is $n = |V(G)|$ and the size is $m = |E(G)|$. For a vertex $v \in V(G)$ is the set $N(v)$ called the *open neighborhood* of v and equals to $\{u \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of v is the set $N(v) \cup \{v\}$ denoted by $N[v]$. The *degree* of a vertex $v \in V(G)$ is $\delta_G(v) = |N(v)|$. The minimum degree of G is denoted by $\delta(G)$. The degree of v in $S \subseteq V(G)$ will be denoted by $\delta_S(v)$ and equals to the number of neighbors v has in S. A graph G is *chordal* if there exists no induced cycle C_t, $t > 3$, in G.

Let $M \subseteq V(G)$ and let $k \in \{1 - \lceil \frac{\delta(G)}{2} \rceil, \ldots, \lfloor \frac{\delta(G)}{2} \rfloor\}$. We say that $v \in V(G)$ is *k-controlled* by M if $\delta_M(v) \geq \frac{\delta_G(v)}{2} + k$. A set M is then called an *open k-monopoly* of G if every vertex of G is k-controlled by M. In addition, the cardinality of minimum open k-monopoly is called an *open k-monopoly number* of a graph G and is denoted by $\mathcal{M}_k(G)$. An open k-monopoly of G of cardinality $\mathcal{M}_k(G)$ is called an $\mathcal{M}_k(G)$-set.

Clearly $|N_G(v) \cap M| \geq \frac{\delta_G(v)}{2} + k$ is equivalent condition for M to be an open k-monopoly of G. If we replace the open neighborhood with the closed neighborhood and set $k = 0$, we obtain for every vertex v the condition $|N_G[v] \cap M| \geq \frac{\delta_G(v)}{2}$, which defines the *closed monopoly* of a graph G. A set $P \subseteq V(G)$ *total dominates* $V(G)$ if every vertex from $V(G)$ has a neighbor in P. The minimum cardinality of a total dominating set is called *total domination number* and is denoted by $\gamma_t(G)$.

The study of both, open and closed monopolies, has its motivation in [1], where an approach to several problems related to overcoming failures has a common notion of majorities. Their ideas are directed toward decreasing, as much as possible, the damage caused due to failed vertices; by maintaining copies of the most important data and performing a voting process among the participating processors in situations that failures occur; and by adopting as true those data stored at the majority of the processors that have not failed. This idea is also commonly used in some fault tolerant algorithms including agreement and consensus problems [2], diagnosis problems [17] or voting systems [5], among other applications and references.

The open k-monopolies were introduced recently in [11]. The equivalence between open k-monopoly number, $k \geq 1$, and signed total $(2k)$-domination was established (see [8, 14, 18] for the definition and complexity results). Similar the equivalence between open k-monopoly number, $k \geq 0$, and (global defensive $(2k)$-alliance and global offensive $(2k)$-alliance) was established (see [3, 7] for definitions and [4] for a recent survey). Several bounds and exact results followed for the direct product of graphs [10], for lexicographic product of graphs [12] and for strong product of graphs [13].

The study of closed monopoly started earlier, see the first [15] and the latest work [9], called monopolies there. Note that closed and open monopolies cannot be compared, since in a closed monopoly a vertex v also counts itself in controlling v, which is not the case in any open monopoly.

Henning shown in [8] that signed total 1-domination problem is NP-complete even restricted to bipartite or chordal graphs. This work was continued by Liang in [14], where the NP-completeness of signed total k-domination problem was shown for $k \geq 2$. Consequently, by results from [11] (see Theorem 1), the open k-monopoly problem is also NP-complete for every $k \geq 1$. In addition in [11] it was shown that open 0-monopoly is NP-complete as well. Hence, it remains to investigate the complexity of open k-monopolies for $1 - \lceil \frac{\delta(G)}{2} \rceil \leq k \leq -1$, which is as follows.

Problem: **OPEN k-MONOPOLY**
INSTANCE: A graph G and positive integer $t \leq |V(G)|$.
QUESTION: Is $\mathcal{M}_k(G) \leq t$?

We follow a similar approach as in [6] for signed total 1-domination and later in [11] for open 0-monopoly problem. In the next section we give a proof of the following theorem.

Theorem 1.1. *Problem OPEN k-MONOPOLY, $k \in \{1 - \lceil \frac{\delta(G)}{2} \rceil, \ldots, -1\}$, is NP-complete, even when restricted to chordal graphs.*

We will show the polynomial time reduction from the total domination set problem which is as follows.

Problem: **TOTAL DOMINATION SET (TDS)**
INSTANCE: A graph G and a positive integer $j \leq |V(G)|$.
QUESTION: Is $\gamma_t(G) \leq j$?

Recall that the total domination set problem is NP-complete even when restricted to chordal graphs (see [16]).

2. PROOF OF THEOREM 1.1

It is obvious that open k-monopoly, $k \in \{1 - \lceil \frac{\delta(G)}{2} \rceil, \ldots, -1\}$, is a member of NP since for a given set M with $|M| \leq t$ we can check in polynomial time for each vertex x of a graph G if x is k-controlled by M.

Let G be a graph of order n and size m. We separate for k the odd and the even case. They are similar, but they do not follow the same pattern in general and therefore, cannot be combined into one. We present both of them in all details.

Case 1. $k = -2\ell$, for a positive integer ℓ. We construct a graph H from G by adding isomorphic graphs F_x^i, $x \in V(G)$ and $1 \leq i \leq \delta_G(x) + 2k - 2$, to G. For every $x \in V(G)$ and $1 \leq i \leq \delta_G(x) + 2k - 2$ graph F_x^i consists of a vertex w_x^i, of two complete graphs $A_x^{i_1}$ and $A_x^{i_2}$ isomorphic to $K_{4\ell+1}$ and of two complete graphs $B_x^{i_1}$ and $B_x^{i_2}$ isomorphic to $K_{2\ell+1}$. In addition, vertex w_x^i is adjacent to every vertex of $A_x^{i_1}$ and of $A_x^{i_2}$, every vertex of $B_x^{i_1}$ is adjacent to every vertex of $A_x^{i_1}$ and every vertex of $B_x^{i_2}$ is adjacent to every vertex of $A_x^{i_2}$. (One possible interpretation of every F_x^i is that in a path P_5 both leaves are blown in to complete subgraphs $K_{2\ell+1}$ and their supports into complete subgraphs $K_{4\ell+1}$.) Now we add an edge between x and vertex w_x^i for each $1 \leq i \leq \delta_G(x) + 2k - 2$.

Hence, to obtain H from G we added

$$\sum_{x \in V(G)} (\delta_G(x) + 2k - 2)(-6k + 5) = 2m(5 - 6k) + 2n(-6k^2 + 11k - 5)$$

vertices. Clearly this can be done in polynomial time. Also, if G is chordal graph, so is H. Next we claim that

$$\mathcal{M}_k(H) = 2m(-k + 3) - 2n(k^2 - 4k + 3) + \gamma_t(G).$$

To prove this, let M be a minimum open k-monopoly of H. Choose an arbitrary subgraph F_x^i added to G. There are vertices of three different types in F_x^i: w_x^i, a vertex v from $A_x^{i_j} \cong K_{4\ell+1}$, $j \in \{1, 2\}$, and a vertex u from $B_x^{i_j} \cong K_{2\ell+1}$, $j \in \{1, 2\}$. It is easy to see that $\delta_H(u) = 6\ell + 1$, $\delta_H(v) = 6\ell + 2$ and $\delta_H(w_x^i) = 8\ell + 3$. Hence, to k-control u one needs to have at least $\frac{6\ell+1}{2} - 2\ell = \ell + \frac{1}{2} \leq \ell + 1$ vertices of M in the neighborhood of u. Similar, one needs at least $\frac{6\ell+2}{2}-2\ell{=}\ell+1$ vertices of M in the neighborhood of v and $\frac{8\ell+3}{2}-2\ell{=}2\ell+\frac{3}{2}\leq 2\ell+2$ vertices of M in the neighborhood of w_x^i. Suppose that $w_x^i\notin M$. Then every vertex from $(A_x^{i_j}\cup B_x^{i_j})\cap M$, $j\in\{1,2\}$, needs $\ell+1$ neighbors in M, which gives altogether at least $2\ell+4$ vertices in $V(F)\cap M$, since this holds also for vertices in M. This contradicts the minimality of M since $2\ell+3=-k+3$ vertices in $V(F)\cap M$ are enough: w_x^i and any $\ell+1$ subset of $A_x^{i_j}$ for every $j\in\{1,2\}$. By this, every $x\in V(G)$ has $\delta_G(x)+2k-2$ neighbors in M outside of G. Since $\delta_H(x)=2\delta_G(x)+2k-2$, x needs an additional neighbor in $M\cap V(G)=P$ to be k-controlled by M. Hence, P forms a total dominating set of G and so $\gamma_t(G)\leq|P|$. Altogether

$$\begin{aligned}\mathcal{M}_k(H)=|M|=|P|+\sum_{x\in V(G)}(\delta_G(x)+2k-2)(-k+3)\\ \geq\gamma_t(G)+2m(-k+3)-2n(k^2-4k+3).\end{aligned}$$

On the other hand, suppose S is a $\gamma_t(G)$-set of G. Let $C_x^{i_1}$ and $C_x^{i_2}$ be a fixed subset of $A_x^{i_1}$ and $A_x^{i_2}$, respectively, of cardinality $\ell+1$ in every subgraph F of $H-V(G)$. Furthermore, let

$$C=\bigcup_{x\in V(G),1\leq i\leq\delta_G(x)+2k-2}(C_x^{i_1}\cup C_x^{i_2}\cup\{w_x^i\}).$$

We will show that $M=S\cup C$ is an open k-monopoly for H. Let x be a vertex from $V(G)$ in H with $\delta_H(x)=2\delta_G(v)+2k-2$. Since S is a $\gamma_t(G)$ set, x has at least one neighbor in S and additional $\delta_G(x)+2k-2$ vertices in M in $V(H)-V(G)$. Altogether x has at least $\delta_G(x)+2k-1$ neighbors in M. Since $\delta_G(x)+2k-1\geq\frac{2\delta_G(x)+2k-2}{2}+k$, x is k-controlled by M. Let F be any subgraph of $H-V(G)$. A vertex u from $B_x^{i_j}$, $j\in\{1,2\}$ has $\ell+1$ neighbors in $C_x^{i_j}$ and with this in M. Therefore u is k-controlled by M. Let v be from $A_x^{i_j}$, $j\in\{1,2\}$. If $v\in C_x^{i_j}$, then v has ℓ neighbors in $C_x^{i_j}$ and additional neighbor w_x^i, which is enough to be k-controlled by M. If $v\notin C_x^{i_j}$, then v has even one neighbor more in M and is also k-controlled by M. Finally, w_x^i is adjacent to all vertices of $C_x^{i_1}\cup C_x^{i_2}$, which gives at least $2\ell+2$ neighbors in M and also w_x^i is k-controlled by M. The next calculation ends the proof of the claim:

$$\begin{aligned}\mathcal{M}_k(H)&\leq|M|=|S|+|C|\\ &=\gamma_t(G)+\sum_{v\in V(G)}(\delta_G(v)+2k-2)(-k+3)\\ &=\gamma_t(G)+2m(-k+3)-2n(k^2-4k+3).\end{aligned}$$

Therefore, we have that if $i=2m(-k+3)+n(-2k^2+8k-6)$, then $\gamma_t(G)\leq j$ if and only if $\mathcal{M}_k(H)\leq j+i$ and the proof is completed for even k.

Case 2. $k=-2\ell+1$, for a positive integer ℓ. We have a similar construction as before. For every vertex $x\in V(G)$ and integer i, $1\leq i\leq\delta_G(x)+2k-2$, let F_x^i be of the following structure. Graph F_x^i consists of a vertex w_x^i, of two complete graphs $A_x^{i_1}$ and

$A_x^{i_2}$ isomorphic to $K_{4\ell}$ and of two complete graphs $B_x^{i_1}$ and $B_x^{i_2}$ isomorphic to $K_{2\ell}$. In addition, vertex w_x^i is adjacent to every vertex of $A_x^{i_1}$ and of $A_x^{i_2}$, every vertex of $B_x^{i_1}$ is adjacent to every vertex of $A_x^{i_1}$ and every vertex of $B_x^{i_2}$ is adjacent to every vertex of $A_x^{i_2}$. (Again a possible interpretation of every F_x^i is that in a path P_5 both leaves are blown in to complete subgraphs $K_{2\ell}$ and their supports into complete subgraphs $K_{4\ell}$.) Now we add an edge between x and vertex w_x^i for each $1 \leq i \leq \delta_G(x) + 2k - 2$.

Hence, to obtain H from G we added

$$\sum_{x \in V(G)} (\delta_G(x) + 2k - 2)(-6k + 7) = 2m(7 - 6k) + 2n(-6k^2 + 13k - 7)$$

vertices. Clearly this can be done in polynomial time. Also, if G is chordal graph, so is H. Next we claim that

$$\mathcal{M}_k(H) = 2m(-k + 4) - 2n(k^2 - 5k + 4) + \gamma_t(G).$$

To prove this, let M be an open k-monopoly set of H. Let x be an arbitrary vertex of G. In F_x^i added to G there are vertices of three different types: w_x^i, a vertex v from $A_x^{i_j} \cong K_{4\ell}$, $j \in \{1, 2\}$, and a vertex u from $B_x^{i_j} \cong K_{2\ell}$, $j \in \{1, 2\}$. It is easy to see that $\delta_H(u) = 6\ell - 1$, $\delta_H(v) = 6\ell$ and $\delta_H(w_x^i) = 8\ell + 1$. To k-control u one needs to have at least $\frac{6\ell-1}{2} - 2\ell + 1 = \ell + \frac{1}{2} \leq \ell + 1$ vertices of M in the neighborhood of u. Similar, one needs at least $\frac{6\ell}{2} - 2\ell + 1 = \ell + 1$ vertices of M in the neighborhood of v and $\frac{8\ell+1}{2} - 2\ell + 1 = 2\ell + \frac{3}{2} \leq 2\ell + 2$ vertices of M in the neighborhood of w_x^i. Suppose that $w_x^i \notin M$. Then every vertex from $(A_x^{i_j} \cup B_x^{i_j}) \cap M$, $j \in \{1, 2\}$, needs $\ell + 1$ neighbors in M. Thus, also a vertex from M needs $\ell + 1$ neighbors in M, which gives altogether at least $2\ell + 4$ vertices in $V(F) \cap M$. This contradicts M being an open k-monopoly set, since $2\ell + 3 = -k + 4$ vertices in $V(F) \cap M$ are enough: w_x^i and any $\ell + 1$ subset of $A_x^{i_j}$ for every $j \in \{1, 2\}$. By this, every x has $\delta_G(x) + 2k - 2$ neighbors in M outside of G. Since $\delta_H(x) = 2\delta_G(x) + 2k - 2$, x needs an additional neighbor in $M \cap V(G) = P$ to be k-controlled by M. Hence, P forms a total dominating set of G and so $\gamma_t(G) \leq |P|$.

Altogether

$$\begin{aligned}
\mathcal{M}_k(H) = |M| = |P| + \sum_{x \in V(G)} (\delta_G(x) + 2k - 2)(-k + 4) \\
\geq \gamma_t(G) + 2m(-k + 4) - 2n(k^2 - 5k + 4).
\end{aligned}$$

Suppose now that S is a $\gamma_t(G)$-set of G. Sets $C_x^{i_1}, C_x^{i_2}$ and C are as in the previous case. We will show that $M = S \cup C$ is an open k-monopoly for H. The proof that M k-controls every vertex of H is exactly the same as in Case 1. Hence the next calculation ends the proof of the claim:

$$\begin{aligned}
\mathcal{M}_k(H) &\leq |M| = |S| + |C| \\
&= \gamma_t(G) + \sum_{x \in V(G)} (\delta_G(x) + 2k - 2)(-k + 4) \\
&= \gamma_t(G) + 2m(-k + 4) - 2n(k^2 - 5k + 4).
\end{aligned}$$

Therefore, we have that if $i = 2m(-k+4) - 2n(k^2 - 5k + 4)$, then $\gamma_t(G) \le j$ if and only if $\mathcal{M}_k(H) \le j + i$ and the proof is completed also for odd k.

Acknowledgements
This work was partially supported by Slovenian research agency under the grants P1-0297 and J1-9109.

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Received: January 26, 2018.
Revised: September 18, 2018.
Accepted: September 20, 2018.