Opuscula Math. 39, no. 3 (2019), 415–423
https://doi.org/10.7494/OpMath.2019.39.3.415 Opuscula Mathematica

METRIC DIMENSION OF ANDRÁSFAI GRAPHS

S. Batool Pejman, Shiroyeh Payrovi, and Ali Behtoei

Communicated by Andrzej Żak

Abstract. A set $W \subseteq V(G)$ is called a resolving set, if for each pair of distinct vertices $u, v \in V(G)$ there exists $t \in W$ such that $d(u,t) \neq d(v,t)$, where $d(x,y)$ is the distance between vertices x and y. The cardinality of a minimum resolving set for G is called the metric dimension of G and is denoted by $\dim_M(G)$. This parameter has many applications in different areas. The problem of finding metric dimension is NP-complete for general graphs but it is determined for trees and some other important families of graphs. In this paper, we determine the exact value of the metric dimension of Andrásfai graphs, their complements and $And(k)\Box P_n$. Also, we provide upper and lower bounds for $dim_M(And(k)\Box C_n)$.

Keywords: resolving set, metric dimension, Andrásfai graph, Cayley graph, Cartesian product.

Mathematics Subject Classification: 05C12, 05C25.

1. INTRODUCTION

Throughout this paper all graphs are finite, simple and undirected. Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. The distance between two vertices $x, y \in V$ is the length of a shortest path between them and is denoted by $d_G(x, y)$, or $d(x, y)$ for convenience. The neighborhood of x is $N(x) = \{y \in V : d(x, y) = 1\}$ and the diameter of G is $diam(G) = \max\{d(x, y) : x, y \in V\}$. It is well known that almost all graphs have diameter 2. The notations $\overline{G}$ and $Line(G)$ stand for the complement graph and the line graph of G, respectively. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex $v \in V$, the k-vector $r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the *metric representation* of v with respect to W (the *code* of v, for convenience). The set W is called a *resolving set* for G if distinct vertices of G have distinct metric representations with respect to W. The cardinality of a minimum resolving set is the *metric dimension* of G and is denoted by $\dim_M(G)$. A graph with metric dimension k is called *k-dimensional.* These concepts were introduced by Slater in 1975 when he was working with U.S. Sonar and Coast Guard Loran stations and he described the usefulness of these concepts, [22]. Independently, Harary and Melter [12] discovered these concepts. They have applications in many areas including network discovery and verification [2], robot navigation [15], problems of pattern recognition and image processing [16], coin weighing problems [21], strategies for the Mastermind game [9], combinatorial search and optimization [21]. Determining the metric dimension of different families of graphs, operations and products, or characterizing n-vertex graphs with a specified metric dimension are fascinating problems and atracts the attention of many researchers. The problem of finding metric dimension is NP-Complete for general graphs but the metric dimension of trees can be obtained using a polynomial time algorithm [15]. It is not hard to see that for each n-vertex graph G we have $1 \leq \dim_M(G) \leq n-1$. Khuller *et al.* [15] and Chartrand *et al.* [7] proved that $\dim_M(G) = 1$ if and only if G is a path P_n. Chartrand *et al.* [7] proved that for $n \geq 2$, $\dim_M(G) = n-1$ if and only if G is the complete graph K_n. The metric dimension of each complete t-partite graph with n vertices is $n-t$. They also provided a characterization of graphs of order n with metric dimension $n-2$, see [7]. Graphs of order n with metric dimension $n-3$ are characterized in [14]. Béla Bollobás studied the metric dimension of random graphs [5]. C*á*ceres *et al.* [6], and independently Peters–Fransen and Oellermann [18], have studied this parameter for the Cartesian product of graphs. They show that

$$\dim_M(G) \leq \dim_M(G\Box P_n) \leq \dim_M(G)+1$$

and

$$\dim_M(G) \leq \dim_M(G\Box C_n) \leq \begin{cases} \dim_M(G)+1 & n \text{ odd}, \\ \dim_M(G)+2 & n \text{ even}. \end{cases}$$

Bailey and Cameron [1] have computed the exact value of the metric dimension for diameter 2 Kneser and Johnson graphs. Fijav*ž* and Mohar studied this parameter for Paley graphs [11]. Chau *et al.* in [8] determined $\dim_M$ for some circulant graphs and their Cartesian products. Salman *et al.* studied this parameter for the Cayley graphs on cyclic groups [20]. In [17] and [10] the metric dimension of Cayley digraphs for the groups which are direct product of some cyclic groups is investigated. Imran studied the metric dimension of barycentric subdivision of Cayley graphs in [13]. Each cycle graph C_n is a 2-dimensional graph. In [23] some properties of 2-dimensional graphs are obtained. All of 2-trees with metric dimension two are characterized in [3], 2-dimensional Cayley graphs on Abelian groups are characterized in [24] and 2-dimensional Cayley graphs on dihedral groups are characterized in [4]. For more results in this subject or related subjects see [1].

Recall that the *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1\Box G_2$, is a graph with vertex set $V(G_1)\times V(G_2) := \{(u,v) : u \in V(G_1), v \in V(G_2)\}$, in which (u,v) is adjacent to (u',v') whenever $u=u'$ and $vv' \in E(G_2)$, or $v=v'$ and $uu' \in E(G_1)$. Note that the vertex set of $G_1\Box G_2$ can be arranged in $|V(G_2)|$ rows and $|V(G_1)|$ columns. Also if G_1 and G_2 are connected, then $G_1\Box G_2$ is connected. Let H be a group and let S be a subset of H that is closed under taking inverse and does not contain the identity element. Recall that the *Cayley graph* $Cay(H,S)$ is a simple graph whose vertex set is H and two vertices u and v are adjacent in it when

$uv^{-1} \in S$, see [19]. For any integer $k \geq 1$, the *Andrásfai* graph $And(k)$ is the Cayley graph $Cay(\mathbb{Z}_{3k-1}, S)$ where $\mathbb{Z}_{3k-1} = \{1, 2, \ldots, 3k-1 = 0\}$ is the additive group of integers modulo $3k-1$ and $S = \{1, 4, 7, \ldots, 3k-2\}$ is the subset of $\mathbb{Z}_{3k-1}$ consisting of the elements congruent to 1 modulo 3. Note that $And(1)$ is a path with two vertices, $And(2)$ is isomorphic to the 5-cycle and $And(3)$ is a *Möbius ladder*. It is well known that $And(k)$ is a reduced (twin free), circulant, vertex transitive, triangle-free and k-regular graph whose diameter is two for $k \geq 2$.

In this paper, we determine the exact value of the metric dimension of Andrásfai graphs, their complements and $And(k)\Box P_n$. Also, we prove that $k \leq dim_M(And(k)\Box C_n) \leq k+1$.

2. MAIN RESULTS

Note that if W is a resolving set for G, then for each $v \in V \setminus W$ the set $W \cup \{v\}$ is a larger resolving set for G. Also, when G is a graph with diameter 2, then $W \subseteq V$ is a resolving set for G if and only if for each pair of distinct vertices $u, v \in V \setminus W$ there exist $w \in W$ such that $\{d(w,u), d(w,v)\} = \{1, 2\}$.

Theorem 2.1. *Let $k \geq 1$ be an integer. Then, the metric dimension of the Andrásfai graph $And(k)$ is k.*

Proof. By investigation, it is easy to see that $\dim_M(And(k)) = k$ for $k \in \{1, 2, 3\}$. Hence, we assume that $k \geq 4$. Note that $And(k) = Cay(\mathbb{Z}_{3k-1}, S)$ where $S = \{1, 4, 7, \ldots, 3k-2\}$. Since $1 \in S$, $And(k)$ contains the Hamiltonian cycle $1, 2, 3, \ldots, 3k-1$ and we can consider a drawing of it in such a way that vertices are consecutively ordered clockwise around a cycle. Hereafter, all of vertex numbers will be considered in modulo $3k-1$. It is straightforward to check that the vertex $0 = 3k-1$ is adjacent to every vertex in S and each vertex $x \neq 0$ has at least one non-adjacent vertex in S. Consider the subset $\{t, t+3\}$ of S with $1 \leq t \leq 3k-5$. We have $d(t+1, t) = 1 = d(t+2, t+3)$ and $d(t+1, t+3) = 2 = d(t+2, t)$. Also, for each vertex $y \notin \{t, t+1, t+2, t+3\}$ we have $d(y, t) = 1$ if and only if $d(y, t+3) = 1$ (because in modulo $3k-1$, we have $t - y \in S$ if and only if $t + 3 - y \in S$). Therefore, $r(t+1|S) \neq r(t+2|S)$, $r(y|S) \neq r(t+1|S)$ and $r(y|S) \neq r(t+2|S)$. This means that two vertices $t+1$ and $t+2$ have unique codes among the vertices of $And(k)$. Since for each vertex $0 \neq x \notin S$ there exists $1 \leq t \leq 3k-5$ such that $x = t+1$ or $x = t+2$, the code of x is unique. Hence, S is a resolving set for $And(k)$ and this implies that $\dim_M(And(k)) \leq |S| = k$.

In order to complete the proof, it is sufficient to show that $|W| \geq k$ for each resolving set W of $And(k)$. Suppose on the contrary that there exists a resolving set W of $And(k)$ with $|W| < k$. By including some additional vertices to W (if it is necessary) we can assume that $|W| = k-1$. If there exists a subset of four (clockwise) consecutive vertices $T = \{i, i+1, i+2, i+3\}$ such that $T \cap W = \emptyset$, then for each vertex $j \notin T$ we have $d(j, i) = 1$ if and only if $d(j, i+3) = 1$ (because, $i - j \in S$ if and only if $i + 3 - j \in S$). This implies that two vertices i and $i+3$ have the same metric representations with respect to W, which contradicts the resolvability of W. Now, assume that $W = \{i_1, i_2, \ldots, i_{k-1}\}$, where

$$1 \le i_1 < i_2 < \ldots < i_{k-1} \le 3k - 1.$$

For each $i_j \in W$ (and with the assumption that $i_k = i_1$) let

$$B_{i_j} = \{i_j, i_j + 1, i_j + 2, \ldots, i_{j+1}\} \setminus \{i_j, i_{j+1}\}.$$

Note that $B_{i_j} = \emptyset$ just when $i_j + 1 = i_{j+1}$ and that $B_{i_j} \cap W = \emptyset$ for each $i_j \in W$. Also, using previous facts we have

$$\bigcup_{j=1}^{k-1} B_{i_j} = \mathbb{Z}_{3k-1} \setminus W, \quad \{|B_{i_j}| : \ i_j \in W\} \subseteq \{0, 1, 2, 3\}.$$

For each $s \in \{0, 1, 2, 3\}$ let β_s be the number of blocks B_{i_j} with $|B_{i_j}| = s$. Thus, using the fact $|W| = k - 1$ we see that

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 = k - 1$$

and

$$0\beta_0 + 1\beta_1 + 2\beta_2 + 3\beta_3 = \left| \bigcup_{j=1}^{k-1} B_{i_j} \right| = (3k - 1) - (k - 1) = 2k.$$

Therefore,

$$\begin{aligned} -2\beta_0 - \beta_1 + \beta_3 &= (0\beta_0 + 1\beta_1 + 2\beta_2 + 3\beta_3) - 2(\beta_0 + \beta_1 + \beta_2 + \beta_3) \\ &= (2k) - 2(k - 1) = 2. \end{aligned}$$

This implies that $\beta_3 = 2 + 2\beta_0 + \beta_1 \ge 2$. Specially, $\beta_3 > \beta_0 + \beta_1$. Now the Pigeonhole Principle implies that there exist two blocks $B_{i_j}, B_{i_{j'}}$ of size 3 such that between them in at least one direction (clockwise or counterclockwise) only blocks of size 2 (if any exists) are located. Since $And(k)$ is vertex transitive, without loss of generality and for convenience, we can assume that $i_j = 1$ (i.e. $B_{i_j} = B_1$) and $B_{i_{j'}}$ is located (in clockwise direction) after $\ell \ge 0$ blocks of size two (when $\ell \ge 1$ they are $B_5, B_8, \ldots, B_{3\ell+2}$), i.e. $B_{i_{j'}} = B_{3\ell+5}$. Note that for the case $\ell = 0$ two blocks $B_{i_j} = B_1$ and $B_{i_{j'}} = B_5$ are consecutive. Therefore,

$$W \cap \{1, 2, 3, \ldots, 3\ell + 9\} = \{1, 5, \ldots, 3\ell + 5, 3\ell + 9\}.$$

Now consider two vertices $x = 2 \in B_1$ and $y = 3\ell + 8 \in B_{3\ell+5}$. Since $y = x + 3(\ell + 2)$, for each $z \notin \{1, 2, 3, \ldots, 3\ell + 9\}$ we have $d(z, x) = 1$ if and only if $d(z, y) = 1$. Also, it is straightforward to check that

$$N(x) \cap \{1, 5, \ldots, 3\ell + 5, 3\ell + 9\} = \{1, 3\ell + 9\} = N(y) \cap \{1, 5, \ldots, 3\ell + 5, 3\ell + 9\}.$$

This means that $r(x|W) = r(y|W)$, which is a contradiction. Therefore, $|W| \ge k$ and this completes the proof. $\square$

Note that $And(1)$ is a 2-vertex path and hence, its complement $\overline{And(1)}$ is disconnected. $\overline{And(2)}$ is a 5-cycle and its metric dimension is 2. Also, for each $k \geq 3$ the complement of $And(k)$ is a connected $(2k-2)$-regular graph and its diameter is two.

Theorem 2.2. *For each $k \geq 2$ we have $\dim_M(\overline{And(k)}) = k$.*

Proof. Let W be a non-empty ordered subset of $\mathbb{Z}_{3k-1}$ and let $v \in \mathbb{Z}_{3k-1}$ be an arbitrary vertex. Assume that the metric representation of vertex v with respect to W in $And(k)$ is $r(v|W) = (w_1, w_2, \ldots, w_k)$ and the metric representation of v with respect to W in $\overline{And(k)}$ is $\bar{r}(v|W) = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k)$. Since both graphs $And(k)$ and $\overline{And(k)}$ have diameter two, for each $i \in \{1, 2, \ldots, |W|\}$ we have

$$\bar{w}_i = \begin{cases} 0 & w_i = 0, \\ 1 & w_i = 2, \\ 2 & w_i = 1. \end{cases}$$

This means that for each vertex u we have $r(v|W) = r(u|W)$ if and only if $\bar{r}(v|W) = \bar{r}(u|W)$. Thus, there is a one-to-one correspondence between the vectors $\{r(v|W) \mid v \in V(And(k)\}$ and the vectors $\{\bar{r}(v|W) \mid v \in V(\overline{And(k)}\}$ by a switching on non-zero components. In the proof of Theorem 2.1 we see that S is a minimum resolving set for $And(k)$ and hence, $|\{\overline{r(v|S)} \mid v \in V(And(k)\}| = |V(And(k)|$. Therefore, S is a minimum resolving set for $\overline{And(k)}$ and the result follows. $\square$

In the following theorem we determine $\dim_M(And(k)\square P_n)$ and $\dim_M(\overline{And(k)}\square P_n)$.

Theorem 2.3. *For each $k \geq 1$ and $n \geq 2$ we have*

$$\dim_M(And(k)\square P_n) = \dim_M(\overline{And(k)}\square P_n) = k.$$

Specifically, the metric dimension of the prism generated by $And(k)$ or its complement is k.

Proof. Assume that $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. Hence,

$$V(And(k)\square P_n) = \bigcup_{t=1}^{n} \{(1, v_t), (2, v_t), \ldots, (3k-1, v_t)\}$$

and the induced subgraph of $And(k)\square P_n$ on the set $\{(1, v_t), (2, v_t), \ldots, (3k-1, v_t)\}$ is isomorphic to $And(k)$ for each $t \in \{1, 2, \ldots, n\}$. Using Corollary 3.2 in [6] and Theorem 2.1 we see that

$$k = \max\{\dim_M(And(k)), \dim_M(P_n)\} \leq \dim_M(And(k)\square P_n).$$

Let

$$W = \{(1, v_1), (4, v_1), (7, v_1), \ldots, (3k-2, v_1)\} = S \times \{v_1\}.$$

We want to show that W is a resolving set for $And(k)\square P_n$. For each $i, j \in \mathbb{Z}_{3k-1}$ and for each $t, t' \in \{1, 2, \ldots, n\}$ it is easy to see that

$$d_{And(k)\square P_n}((i, v_t), (j, v_{t'})) = d_{And(k)}(i, j) + |t - t'|,$$

which implies that

$$r((i, v_t)|W) = r(i|S) + (t - 1, t - 1, \ldots, t - 1).$$

Note that except the vertex $(0, v_1)$ whose metric representation with respect to W is the all 1 vector $(1, 1, \ldots, 1)$, the metric representation of each vertex (i, v_1) has at least one component equal to 2 and we have $r((i, v_1)|W) \in \{0, 1, 2\}^k$. Similarly, for each $t \in \{2, 3, \ldots, n\}$, except the vertex $(0, v_t)$ whose metric representation is $(t, t, \ldots t)$, the metric representation of each vertex (i, v_t) has at least one component equal to $t+1$ and $r((i, v_t)|W) \in \{t-1, t, t+1\}^k$. By the proof of Theorem 2.1, S is a minimum resolving set for $And(k)$. Note that by Lemma 3.1 in [6] the projection of W onto each copy of $And(k)$ in $And(k)\square P_n$ (i.e. the induced subgraph on each row) resolves the vertices of that copy (row). Therefore, each pair of distinct vertices (i, v_t) and $(j, v_{t'})$ (with $t = t'$ or $t \neq t'$) have distinct metric representations with respect to W. Hence, W is a minimum resolving set for $And(k)\square P_n$ and $\dim_M(And(k)\square P_n) = k$. Using an argument similar to the proof of Theorem 2.2, we can show that $\dim_M(\overline{And(k)\square P_n}) = k$. □

Let $n \geq 3$ be an integer. Using Theorem 8.6 and Theorem 8.4 in [6] we see that

$$\dim_M(And(1)\square C_n) = \dim_M(K_2\square C_n) = \begin{cases} 2 & n \text{ is odd,} \\ 3 & n \text{ is even} \end{cases}$$

and

$$\dim_M(And(2)\square C_n) = \dim_M(C_5\square C_n) = 3.$$

Proposition 2.4. *If $k \geq 3$ and $n \geq 3$, then $k \leq \dim_M(And(k)\square C_n) \leq k + 1$.*

Proof. Corollary 3.2 in [6] using Theorem 2.1 implies that

$$k = \max\{\dim_M(C_n), \dim_M(And(k))\} \leq dim_M(And(k)\square C_n).$$

For the upper bound, assume that

$$V(C_n) = \{v_1, v_2, \ldots, v_n\}, \quad E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$$

and let

$$W' = \{(1, v_1), (4, v_1), (7, v_1), \ldots, (3k - 2, v_1)\}, \quad W = W' \cup \{(1, v_2)\}.$$

Using the structure of the Cartesian product of two graphs, for each $i, j \in V(And(k))$ and for each $t, t' \in \{1, 2, \ldots, n\}$ we have

$$d_{And(k)\square C_n}((i, v_t), (j, v_{t'})) = d_{And(k)}(i, j) + \min\{|t - t'|, n - |t - t'|\}.$$

Note that (using the proof of Theorem 2.1 and Lemma 3.1 in [6]) the projection of W' onto each copy of $And(k)$ in $And(k)\square C_n$ (i.e. the induced subgraph on each row) resolves the vertices of that copy and the projection of W' onto each copy of C_n in $And(k)\square C_n$ (each column) resolves its vertices. Also, for each i and for each $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$ we have

$$r\left((i, v_{1+t})|W'\right) = r\left((i, v_1)|W'\right) + (t, t, \ldots, t) = r\left((i, v_{n-t+1})|W'\right).$$

Thus, distinct vertices in $\{(i, v_t) \mid i \in V(And(k)),\ 1 \leq t \leq \lfloor \frac{n}{2} \rfloor + 1\}$ have distinct metric representations with respect to W' (and hence, with respect to W) and distinct vertices in

$$\left\{(i, v_t)|\ i \in V(And(k)),\ t \in \{1, n, n-1, n-2, \ldots, \left\lceil \frac{n}{2} \right\rceil + 1\}\right\}$$

have distinct metric representations with respect to W' (and hence, with respect to W). These facts imply that for each i and for each $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$ the metric representation of the vertex (i, v_{t+1}) with respect to W' in $And(k)\square C_n$ is just equal to the code of (i, v_{n-t+1}) and no other vertex (note that when n is even and $t = \frac{n}{2}$ the vertex (i, v_{t+1}) coincides with (i, v_{n-t+1})). Since

$$d_{And(k)\square C_n}((i, v_{t+1}), (1, v_2)) = d_{And(k)}(i, 1) + t - 1$$

and

$$d_{And(k)\square C_n}((i, v_{n-t+1}), (1, v_2)) = d_{And(k)}(i, 1) + \min\{n - t - 1, t + 1\}$$

for distinct vertices (i, v_{t+1}) and (i, v_{n-t+1}) we have

$$r((i, v_{t+1})|W) \neq r((i, v_{n-t+1})|W).$$

Hence, W is a resolving set for $And(k)\square C_n$ and $\dim_M(And(k)\square C_n) \leq k + 1$. $\square$

Acknowledgements

We would like to express our deepest gratitude to the referee and the editor for their invaluable comments and suggestions which improved the quality of this paper.

REFERENCES

[1] R.F. Bailey, P.J. Cameron, *Base size, metric dimension and other invariants of groups and graphs*, Bull. London Math. Soc. **43** (2011), 209–242.

[2] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalák, L.S. Ram, *Network discovery and verification*, IEEE J. Sel. Areas Commun. **24** (2006), 2168–2181.

[3] A. Behtoei, A. Davoodi, M. Jannesari, B. Omoomi, *A characterization of some graphs with metric dimension two*, Discrete Math. Algorithm. Appl. **09** (2017), 1–15.

[4] A. Behtoei, Y. Golkhandy Pour, *On two-dimensional Cayley graphs*, Alg. Struc. Appl. **4** (2017) 1, 43–50.

[5] B. Bollobás, *Metric dimension for random graphs*, Electron. J. Combin. **20** (2013), 1–19.

[6] J. Caceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara, D.R. Wood, *On the metric dimension of Cartesian products of graphs*, SIAM J. Discrete Math. **21** (2007), 423–441.

[7] G. Chartrand, L. Eroh, M.A. Johnson, O.R. Ollermann, *Resolvability in graphs and the metric dimension of a graph*, Discrete Appl. Math. **105** (2000), 99–113.

[8] K. Chau, S. Gosselin, *The metric dimension of circulant graphs and their cartesian products*, Opuscula Math. **37** (2017), 509–534.

[9] V. Chvátal, *Mastermind*, Combinatorica **3** (1983), 325–329.

[10] M. Fehr, S. Gosselin, O.R. Oellermann, *The metric dimension of Cayley digraphs*, Discrete Math. **306** (2006), 31–41.

[11] G. Fijavž, B. Mohar, *Rigidity and separation indices of Paley graphs*, Discrete Math. **289** (2004), 157–161.

[12] F. Harary, R.A. Melter, *On the metric dimension of a graph*, Ars Combin. **2** (1976), 191–195.

[13] M. Imran, *On the metric dimension of barycentric subdivision of Cayley graphs*, Acta Math. Appl. Sin. Engl. Ser. **32** (2016), 1067–1072.

[14] M. Janessari, B. Omoomi, *Characterization of n-vertex graphs with metric dimension* $n-3$, Math. Bohem. **139** (2014), 1–23.

[15] S. Khuller, B. Raghavachari, A. Rosenfeld, *Landmarks in graphs*, Discrete Appl. Math. **70** (1996), 217–229.

[16] R.A. Melter, I. Tomescu, *Metric bases in digital geometry*, Comput. Gr. Image Process. **25** (1984), 113–121.

[17] O.R. Oellermann, C.D. Pawluck, A. Stokke, *The metric dimension of Cayley digraphs of abelian groups*, Ars Comb. **81** (2006), 97–111.

[18] J. Peters–Franser, O.R. Oellermann, *The metric dimension of Cartesian products of graphs*, Util. Math. **69** (2006), 33–41.

[19] R. Rosiek, M. Woźniak, *A note introducing Cayley graphs and group-coset graphs generated by graph packings*, Opuscula Math. **24** (2004), 203–221.

[20] M. Salman, I. Javaid, M.A. Chaudhry, *Resolvability in circulant graphs*, Acta Math. Sin. (Engl. Ser.) **29** (2012), 1851–1864.

[21] A. Sebo, E. Tannier, *On metric generators of graphs*, Math. Oper. Res. **29** (2004), 383–393.

[22] P.J. Slater, *Leaves of trees*, Congr. Numer. **14** (1975), 549–559.

[23] G. Sudhakara, A.R. Hemanth Kumar, *Graphs with metric dimension two – A characterization*, World Academy of Science, Engineering and Technology **36** (2009), 621–626.

[24] E. Vatandoost, A. Behtoei, Y. Golkhandy Pour, *Cayley graphs with metric dimension two – A characterization*, https://arxiv.org/abs/1609.06565.

S. Batool Pejman
b.pejman@edu.ikiu.ac.ir

Imam Khomeini International University
Department of Mathematics, Faculty of Science
P.O. Box: 34148-96818, Qazvin, Iran

Shiroyeh Payrovi
shpayrovi@ikiu.ac.ir

Imam Khomeini International University
Department of Mathematics, Faculty of Science
P.O. Box: 34148-96818, Qazvin, Iran

Ali Behtoei
a.behtoei@sci.ikiu.ac.ir

Imam Khomeini International University
Department of Mathematics, Faculty of Science
P.O. Box: 34148-96818, Qazvin, Iran

Received: September 25, 2017.
Revised: August 19, 2018.
Accepted: August 21, 2018.