Opuscula Math. 39, no. 3 (2019), 395–414
https://doi.org/10.7494/OpMath.2019.39.3.395 Opuscula Mathematica

REMARKS ON GLOBAL SOLUTIONS TO THE INITIAL-BOUNDARY VALUE PROBLEM FOR QUASILINEAR DEGENERATE PARABOLIC EQUATIONS WITH A NONLINEAR SOURCE TERM

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Communicated by Marius Ghergu

Abstract. We give an existence theorem of global solution to the initial-boundary value problem for $u_t - \mathrm{div}\{\sigma(|\nabla u|^2)\nabla u\} = f(u)$ under some smallness conditions on the initial data, where $\sigma(v^2)$ is a positive function of $v^2 \neq 0$ admitting the degeneracy property $\sigma(0) = 0$. We are interested in the case where $\sigma(v^2)$ has no exponent $m \geq 0$ such that $\sigma(v^2) \geq k_0|v|^m, k_0 > 0$. A typical example is $\sigma(v^2) = \log(1 + v^2)$. $f(u)$ is a function like $f = |u|^\alpha u$. A decay estimate for $\|\nabla u(t)\|_\infty$ is also given.

Keywords: degenerate quasilinear parabolic equation, nonlinear source term, Moser's method.

Mathematics Subject Classification: 35B40, 35D35, 58J35, 58K30.

1. INTRODUCTION

In this paper we consider the initial-boundary value problem for the quasilinear parabolic equation of the form:

$$u_t - \mathrm{div}\{\sigma(|\nabla u|^2)\nabla u\} = f(u) \text{ in } \Omega \times (0, \infty), \tag{1.1}$$

with the initial-boundary conditions

$$u(x, 0) = u_0(x) \text{ and } u(x, t)|_{\partial\Omega} = 0, \tag{1.2}$$

where Ω is a bounded domain in R^N with a smooth, say, C^3 class, boundary $\partial\Omega$.

Concerning $\sigma(v^2)$ we assume

Hypothesis A. $\sigma(\cdot)$ is a nonnegative function in $C^2((0,\infty)) \cap C([0,\infty))$, satisfying:
(1)

$$\sigma(v^2) + 2\sigma'(v^2)v^2 \geq k_0\sigma(v^2),$$

(2)

$$|\sigma'(v^2)|v^2 \leq k_1\sigma(v^2),$$

(3) $\sigma(v^2) \geq k_0$ if $|v| \geq 1$, and there exists $\nu \geq 0$ and $m \geq 0$ such that for any $K \geq 1$,

$$\sigma(v^2) \geq k_0 K^{-\nu}|v|^m \text{ if } |v| \leq K.$$

In the above k_0, k_1 are some positive constants.

We note that $\sigma(v^2) = |v|^m$ satisfies Hyp. A with $\nu = 0$ and $\sigma(v^2) = \log(1 + v^2)$ satisfies Hyp. A with $\nu = m = 2$.

Concerning the initial data and source term we assume

Hypothesis B.
(1) u_0 belongs to $W_0^{1,\infty}(\Omega)$,
(2) f is a Lipschitz continuous function on $[0,\infty)$ and satisfies

$$|f(u)| \leq k_1|u|^{\alpha+1} \text{ and } |f'(u)| \leq k_1|u|^{\alpha}, a.e.\ u.$$

When $\sigma(|\nabla u|^2) = |\nabla u|^m, m > 0$, related problems have been investigated by many authors from various points of view (cf. [2–7, 9, 16–21, 23] and the references cited therein). In particular the problem (1.1)–(1.2) which has a nonlinear source term has been considered by [20, 21] and [9] etc. Another typical example $\sigma = \log(1 + |\nabla u|^2)$, a logarithmic nonlinearity, has a similar property to $|\nabla u|^m$ in the sense that it is unboundedly growing as $|\nabla u|$ tends to ∞ and degenerate at $|\nabla u| = 0$. However, the nonlinearity $|\nabla u|^m$ is homogeneous and this property allows us to treat the problem very conveniently. Since $\sigma(|\nabla u|^2) = \log(1 + |\nabla u|^2)$ has not such a property, in particular, there is no exponent $m \geq 0$ such that $\sigma(|\nabla u|^2) \geq k_0|\nabla u|^m, k_0 > 0$, many useful techniques used for $|\nabla u|^m$ can not be applied directly to our problem here.

In the present paper we are interested in the global existence to the problem (1.1)–(1.2) for a class of nonlinear functions $\sigma(v^2)$ in Hyp. A. Our class of $\sigma(v^2)$ includes

$$\sigma(|\nabla u|^2) = \{\log(1 + |\nabla u|^{m_1})\}^{m_2} \quad \text{with} \quad m_1, m_2 > 0\ (\nu = m = m_1 m_2),$$

$$\sigma(|\nabla u|^2) = |\nabla u|^{m_1} \log(1 + |\nabla u|^{m_2}) \quad \text{with} \quad m_1, m_2 > 0,\ (\nu = m_2, m = m_1 + m_2)$$

and

$$\sigma(|\nabla u|^2) = |\nabla u|^{m_1}/\sqrt{1 + |\nabla u|^{2m_2}} \quad \text{with} \quad m_1 \geq m_2 > 0\ (\nu = m_2, m = m_1)$$

etc. These examples have no exponent m such that $\sigma(|\nabla u|^2) \geq k_0|\nabla u|^m, k_0 > 0$. We note that in Hyp. A, no explicite growth order of $\sigma(v^2)$ is assumed as $|v| \to \infty$.

Recently, in the second part of [13], we have considered the problem (1.1)–(1.2) with $\sigma(|\nabla u|^2) = \log(1+|\nabla u|^2)$ and $f(u) \equiv 0$ and shown global existence of a solution $u(t) \in L^\infty([0,\infty); W_0^{1,p_0}) \cap W^{1,2}([0,\infty); L^2)$ for $u_0 \in W_0^{1,p_0}, p_0 > 2$, and further a stronger solution $u(t) \in L^\infty([0,\infty); W_0^{1,\infty}) \cap W^{1,2}([0,\infty); L^2)$ for $u_0 \in W_0^{1,\infty}$. Subsequently in [15] we have considered the problem (1.1)–(1.2) with $f(u) \equiv 0$ for a general nonlinearity as in Hyp. A and for the initial data u_0 in a weaker space W_0^{1,p_0}, $p_0 > m+2$, or more weakly in L^r with some $r \geq 1$. In [15] it is proved for the case $u_0 \in L^r$ that the problem (1.1)–(1.2) admits a unique solution in the class $L^\infty_{loc}((0,\infty); W_0^{1,\infty}) \cap W^{1,2}_{loc}((0,\infty); L^2) \cap C([0,\infty); L^r)$, and also the estimates for $\|\nabla u(t)\|_\infty$ near $t=0$ and $t=\infty$ showing a smoothing effect and a decay property, respectively, are given. In [15] a growth condition on $\sigma(v^2)$ for large $|v|$ is also assumed.

To control the nonlinear source term $f(u)$ it is desirable to get an estimate for $\|\nabla u(t)\|_\infty$. Indeed, if we could establish an estimate like $\|\nabla u(t)\|_\infty \leq K, K > 0$, we know by Hyp. A (3) that

$$\int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2 dx \geq CK^{-\nu}\|\nabla u(t)\|_{m+2}^{m+2}$$

and by this we can expect to control the nonlinear source term $f(u)$. Our purpose of this paper to show that such a conjecture is true. The frame work of the argument to derive various a priori estimates required for the global existence, in particular, the estimate for $|\nabla u(t)\|_\infty$, is similar to the ones in [13,15]. However, to derive the estimate $\|\nabla u(t)\|_\infty$ we need to control the term $f(u)$ by $\|\nabla u(t)\|_\infty$ itself, and the argument is delicate and never trivial. For our purpose we assume $u_0 \in W_0^{1,\infty}$ and we seek for a solution in the class $L^\infty([0,\infty); W_0^{1,\infty}) \cap W^{1,2}([0,\infty); L^2)$.

It is an interesting problem to show the global existence of a solution to the problem (1.1)–(1.2) in the space $L^\infty_{loc}((0,\infty); W_0^{1,\infty}) \cap W^{1,2}_{loc}((0,\infty); L^2) \cap C([0,\infty); L^r)$ under the weaker condition on the initial data $u_0 \in L^r$ with some r, $r \geq 1$. But, this problem seems to be very difficult because in this case, $\|\nabla u(t)\|_\infty$ and $\|f(u(t))\|_\infty$ should be unbounded near $t = 0$ and we encounter several complicate problems in controlling the term $f(u)$ by $-\text{div}\{\sigma(|\nabla u|^2)\nabla u\}$. This problem is left as an open problem for a future research.

We employ some ideas used in [13] (see also [15]) to show the boundedness and decay of $\|\nabla u(t)\|_\infty$ based on Moser's iteration method and a "loan" method (a continuity principle). However, as is stated above, it is a very delicate task to derive a priori estimate of $\|\nabla u(t)\|_\infty$ since the nonlinear term $f(u)$ gives a significant effect. The source term $f(u)$ is sometimes called a blowing-up term because we can not generally expect global existence for parabolic equations with such a nonlinear term if the initial data are large (cf. [6]). For our purpose we require a smallness condition on $\|u_0\|_r$ with some $r \geq 2$, but no direct smallness condition is imposed on $\|\nabla u_0\|_\infty$.

2. STATEMENT OF THE RESULT

We use only familiar function spaces and omit their definitions. But, we note that $u \in W_0^{1,\infty}$ iff $u \in W_0^{1,p}$ for any $p \geq 1$ and $|\nabla u| \in L^\infty$. We denote by $\|\cdot\|_p$ the L^p norm on Ω. We use $\|\cdot\|$ for $\|\cdot\|_p$ if $p = 2$ and the inner product in L^2 is denoted by $(\cdot,\cdot)$.

We set

$$\Gamma(t) = \int_\Omega \left(\int_0^{|\nabla u(t)|^2} \sigma(\eta) d\eta \right) dx \quad \text{and} \quad \tilde{\Gamma}(t) = \int_\Omega \sigma(|\nabla u(t)|^2)|\nabla u(t)|^2 dx.$$

By Hyp. B (1)–(2), we see that

$$\tilde{k}_0 \tilde{\Gamma}(t) \le \Gamma(t) \le \tilde{k}_1 \tilde{\Gamma}(t) \tag{2.1}$$

with some $\tilde{k}_0, \tilde{k}_1 > 0$. For a proof of (2.1), see [15].

Theorem 2.1. *We make Hyp. A and Hyp. B. Assume that $\alpha > m, r \ge 2$ and further, one of the following conditions is satisfied:*
(1)

$$2(m+2)N(\alpha-m)/(2(m+1)(m+2)-mN)^+ \le r \le 2(\alpha+1) \le (m+2)N/(N-m-2)^+, \tag{2.2}$$

(2)

$$2(\alpha+1) \le r \le (m+2)N/(N-m-2)^+ \tag{2.3}$$

and
(3)

$$r \ge \max\{2(\alpha+1), (m+2)N/(N-m-2)^+, 2(m+2)N(\alpha-m)/(2(m+1)(m+2)-mN)^+\}. \tag{2.4}$$

Additionally, assume that $r > N\alpha/2$ if $N \ge 3$ and $r > \alpha$ if $N = 1, 2$. Then there exists a constant

$$K = K(\|u_0\|_r, \Gamma(0), \|\nabla u_0\|_\infty) > \max\{\|\nabla u_0\|_\infty, 1\}$$

continuously depending on $\|u_0\|_r, \Gamma(0)$ and $\|\nabla u_0\|_\infty$ such that under the smallness conditions

$$CK^\nu \|u_0\|_r^{\alpha-m} < 1$$

and

$$C(1+\Gamma(0))^{m/2(m+2)} K^{(m+1)\nu/(m+2)} \|u_0\|_r^{\alpha-m} < 1$$

with some $C > 0$ independent of u_0 and K, the problem (1.1)–(1.2) admits a unique solution $u(t) \in L^\infty([0,\infty); W_0^{1,\infty}) \cap W^{1,2}([0,\infty); L^2)$, satisfying the estimates

$$\|u(t)\|_r \le (\|u_0\|_r^{-m} + CmK^{-\nu} t)^{-1/m} \le \|u_0\|_r,$$

$$\Gamma(t) \le \left(\Gamma(0)^{-m/(m+2)} + mC^{-1}(1+\Gamma(0))^{-m/(m+2)} t\right)^{-(m+2)/m}, \quad 0 \le t < \infty,$$

$$\int_0^\infty \|u_t(s)\|^2 ds \le C\Gamma(0)$$

and

$$\|\nabla u(t)\|_\infty \le C(\|u_0\|_r, \Gamma(0), \|\nabla u_0\|_\infty) < K, \quad 0 \le t < \infty.$$

Further we have the decay estimate

$$\|\nabla u(t)\|_\infty \le \tilde{C}_1 (1+t)^{-1/m}$$

with some constant $\tilde{C}_1 = C(\|u_0\|_r, \|\nabla u_0\|_\infty)$. (*When* $m = 0$ *the estimates for* $\Gamma(t)$ *and* $\|\nabla u(t)\|_\infty$ *should be replaced by* $\Gamma(t) \le \Gamma(0)e^{-\lambda t}$ *and* $\|\nabla u(t)\|_\infty \le \tilde{C}_1 e^{-\lambda t}$, *respectively, with some* $\lambda > 0$.)

For the definition of our weak solution see (8.2) and (8.3) below. Needless to say, if $f(u) \equiv 0$ the conditions $\alpha > m$, (2.2), (2.3), (2.4) and any smallness condition on u_0 are unnecessary.

3. A PRIORI ESTIMATE FOR $\|u(t)\|_r$ UNDER THE ASSUMPTION $\|\nabla u(t)\|_\infty \le K$

Let $u(t)$ be an assumed appropriate smooth solution of the problem (1.1)–(1.2) which will assure the calculations below. We assume that $\|\nabla u_0\|_\infty < K$ for some $K > 1$. Then we may assume that

$$\|\nabla u(t)\|_\infty \le K, \quad 0 \le t \le T_0, \tag{3.1}$$

for some $T_0 > 0$. Our final aim is to derive the estimate $\|\nabla u(t)\|_\infty < K, 0 \le t \le T_0$. If this is true we know by a continuity principle that $\|\nabla u(t)\|_\infty < K$ on $[0, \infty)$. (For convenience we call such an argument as "loan" method. See [10–12, 14] for other types of such argument.) We begin with deriving a boundedness estimate for $\|u(t)\|_r, 0 \le t \le T_0$, under (3.1) and a smallness condition on $\|u_0\|_r$. Our result and the proof will be applied in fact to approximate classical solutions u_ϵ for some approximate problems (see Section 9). But, for simplicity of notation and also to make the essential points clear we carry out the estimations for an assumed smooth solution of the original problem (1.1)–(1.2). We denote by C a general positive constant independent of u_0 and K which may be different from line to line.

Multiplying the equation by $|u|^{r-2}u, r \ge 2$, and integrating it we have

$$\frac{1}{r}\frac{d}{dt}\|u\|_r^r + (r-1)\int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2)|\nabla u|^2|u|^{r-2}dx \le k_1\|u\|_{\alpha+r}^{\alpha+r}$$

and by the assumption (3.1) and Hyp. A (3),

$$\frac{1}{r}\frac{d}{dt}\|u\|_r^r + \frac{1}{CK^\nu}\|\nabla|u|^{(r+m)/(m+2)}\|_{m+2}^{m+2} \le k_1\|u\|_{\alpha+r}^{\alpha+r} \tag{3.2}$$

with some $C = C(r) > 0$. Here, we know by the Gagliardo–Nirenberg inequality,

$$\|u\|_{\alpha+r} \le C^{1/r}\|u\|_r^{1-\theta_0}\|\nabla|u|^{(r+m)/(m+2)}\|_{m+2}^{(m+2)\theta_0/(r+m)}$$

with

$$\theta_0 = \frac{m+r}{m+2}\left(\frac{1}{r} - \frac{1}{\alpha+r}\right)\left(\frac{1}{N} - \frac{1}{m+2} + \frac{m+r}{m+2}\cdot\frac{1}{r}\right)^{-1}$$
$$= \alpha(r+m)N/(\alpha+r)(mN+(m+2)r)$$

provided that $\theta_0 \le 1$. We make the assumption $r \ge (\alpha - m)N/(m+2)$. Then we see that $\theta_0 < 1$ and $(m+2)(\alpha+r)\theta_0/(r+m) \le m+2$, and hence, using the inequality

$$\|u\|_r \le C^{-1}\|\nabla|u|^{(r+m)/(m+2)}\|^{(m+2)/(r+m)}$$

if necessary, we have

$$\|u\|_{\alpha+r}^{\alpha+r} \le C\|u\|_r^{\alpha-m}\|\nabla|u|^{(r+m)/(m+2)}\|^{m+2}.$$

It follows from this and (3.2) that

$$\frac{1}{r}\frac{d}{dt}\|u\|_r^r + \left(\frac{1}{CK^\nu} - C\|u(t)\|_r^{\alpha-m}\right)\|\nabla|u|^{(r+m)/(m+2)}\|_{m+2}^{m+2} \le 0$$

with some $C > 0$. Therefore, assuming $\alpha > m$ and $C^2K^\nu\|u_0\|_r^{\alpha-m} < 1/2$ we see

$$\frac{1}{r}\frac{d}{dt}\|u\|_r^r + \frac{1}{2CK^\nu}\|\nabla|u|^{(r+m)/(m+2)}\|_{m+2}^{m+2} \le 0.$$

Since $\|\nabla|u|^{(r+m)/(m+2)}\|_{m+2}^{m+2} \ge C^{-1}\|u\|_r^{r+m}$, we obtain the estimate

$$\|u(t)\|_r \le (\|u_0\|_r^{-m} + C^{-1}mK^{-\nu}t)^{-1/m} \le \|u_0\|_r, \quad 0 \le t \le T_0. \tag{3.3}$$

(We have changed C^2 by C.)

We conclude the following assertion.

Proposition 3.1. *Let $\alpha > m$ and $r \ge (\alpha - m)N/(m+2)$. In addition to* (3.1) *we assume*

$$CK^\nu\|u_0\|_r^{\alpha-m} < 1 \tag{3.4}$$

with some constant $C > 0$. Then, for the assumed smooth solution $u(t)$ with $u(0) = u_0$ we have the estimate (3.3)

4. A PRIORI ESTIMATE FOR $\Gamma(t)$

We proceed to the estimation for

$$\Gamma(t) = \int_\Omega \left(\int_0^{|\nabla u(t)|^2} \sigma(\tau)d\tau \right) dx$$

for assumed smooth solution $u(t)$. We assume that $\alpha > m$, $r \geq 2$ and one of the conditions (1), (2) and (3) in Theorem 2.1 hold. We note that the condition $r \geq (\alpha - m)/(m+2)$ in Proposition 3.1 is automatically satisfied under these assumptions and we can use the estimate (3.3) if (3.4) holds.

Multiplying the equation by $u_t(t)$ and integrating it we have

$$\frac{1}{2}\frac{d}{dt}\Gamma(t) + \|u_t(t)\|^2 = (f(u), u_t) \leq \|f(u)\|\|u_t\|$$

and

$$\frac{d}{dt}\Gamma(t) + \|u_t(t)\|^2 \leq \|f(u)\|^2. \tag{4.1}$$

Next, multiplying the equation by $u(t)$ and integrating it we see that

$$\tilde{\Gamma}(t) = -(u_t, u) + (f(u), u) \leq (\|u_t\| + \|f(u)\|)\|u\|. \tag{4.2}$$

Now, setting $\Omega_1 = \{x \in \Omega | |\nabla u(x,t)| \leq 1\}$ and $\Omega_2 = \Omega \setminus \Omega_1$, we see (cf. [15])

$$\begin{aligned} \|\nabla u(t)\|^2 &\leq C\left(\int\limits_{\Omega_1} \sigma(|\nabla u|^2)|\nabla u|^2 dx\right)^{2/(m+2)} + C\int\limits_{\Omega_2} \sigma(|\nabla u|^2)\nabla u|^2 dx \\ &\leq C\left(\tilde{\Gamma}^{2/(m+2)}(t) + \tilde{\Gamma}(t)\right), \end{aligned} \tag{4.3}$$

where we have used Hyp. A (3), with $K = 1$.

Using also the Poincaré inequality $\|u\| \leq C\|\nabla u\|$ it follows from (4.2) and (4.3) and (4.1) that

$$\begin{aligned} \tilde{\Gamma}^{(m+1)/(m+2)}(t) &\leq C(\|u_t(t)\| + \|f(u)\|)(1 + \tilde{\Gamma}(t)^{m/2(m+2)}) \\ &\leq C\left((\|f(u)\|^2 - \frac{d}{dt}\Gamma(t))^{1/2} + \|f(u)\|\right)(1 + \tilde{\Gamma}(t)^{m/2(m+2)}). \end{aligned} \tag{4.4}$$

Since $\Gamma(t)$ and $\tilde{\Gamma}(t)$ are mutually equivalent we obtain from (4.4)

$$\frac{d}{dt}\Gamma(t) + \frac{\Gamma^{2(m+1)/(m+2)}(t)}{C(1+\Gamma(t))^{m/(m+2)}} \leq 2\|f(u)\|^2. \tag{4.5}$$

We shall estimate the term $\|f(u)\|^2$. For this we use the inequality

$$\|\nabla u\|_{m+2}^{m+2} \leq CK^\nu \int\limits_\Omega \sigma(|\nabla u|^2)|\nabla u|^2 dx \leq CK^\nu\Gamma(t), \quad 0 \leq t \leq T_0. \tag{4.6}$$

For possible cases to be able to control $\|f(u)\|$ by $\|u\|_r$ and $\|\nabla u(t)\|_{m+2}$ we encounter three cases:

(1) $r \leq 2(\alpha+1) \leq (m+2)N/(N-m-2)^+$,
(2) $2(\alpha+1) \leq r \leq (m+2)N/(N-m-2)^+$,
(3) $r > \max\{2(\alpha+1), (m+2)N/(N-m-2)^+\}$.

Case (1). We have

$$\|f(u)\|^2 \le C\|u\|_{2(\alpha+1)}^{2(\alpha+1)} \le C\|u\|_r^{2(\alpha+1)(1-\theta_1)}\|\nabla u\|_{m+2}^{2(\alpha+1)\theta_1}$$

with

$$\theta_1 = \Big(\frac{1}{r} - \frac{1}{2(\alpha+1)}\Big)\Big(\frac{1}{N} + \frac{1}{r} - \frac{1}{m+2}\Big)^{-1}.$$

We require the condition $2(\alpha+1)\theta_1 \le 2(m+1)$, that is,

$$N < 2(m+1)(m+2)/m \text{ and } r \ge 2(m+2)N(\alpha-m)/(2(m+1)(m+2)-mN) \quad (4.7)$$

Then, using the inequalities $\|u\|_r \le C\|\nabla u\|_{m+2}$ and (4.6) we have from (2.3),

$$\begin{aligned}\|f(u)\|^2 &\le C\|u\|_r^{2(\alpha-m)}\|\nabla u\|_{m+2}^{2(m+1)} \\ &\le CK^{2\nu(m+1)/(m+2)}\|u_0\|_r^{2(\alpha-m)}\Gamma(t)^{2(m+1)/(m+2)}.\end{aligned} \quad (4.8)$$

Case (2). In this case we have easily,

$$\begin{aligned}\|f(u)\|^2 &\le C\|u\|_r^{2(\alpha+1)} \le C\|u\|_r^{2(\alpha-m)}\|\nabla u\|_{m+2}^{2(m+1)} \\ &\le CK^{2\nu(m+1)/(m+2)}\|u_0\|_r^{2(\alpha-m)}\Gamma(t)^{2(m+1)/(m+2)},\end{aligned}$$

where we have used (4.6) again.

Case (3). We see, in both cases $2(\alpha+1) \ge (m+2)N/(N-m-2)^+$ and $2(\alpha+1) \le (m+2)N/(N-m-2)^+$,

$$\|f(u)\|^2 \le C\|u\|_{(m+2)N/(N-m-2)^+}^{2(\alpha+1)\theta_3}\|u\|_r^{2(\alpha+1)(1-\theta_3)}$$

with

$$\theta_3 = \Big(\frac{1}{2(\alpha+1)} - \frac{1}{r}\Big)\Big(\frac{N-m-2}{(m+2)N} - \frac{1}{r}\Big)^{-1}.$$

Under the condition (4.7) we see $2(\alpha+1)\theta_3 \ge 2(m+1)$ and by the condition $r \ge (m+2)N/(N-m-2)$ we get also in this case the inequality (4.8), where we use $\|u\|_{(m+2)N/(N-m-2)^+} \le C\|u\|_r$ if necessary.

It follows from (4.5) and the above estimate (4.8) that

$$\frac{d}{dt}\Gamma(t) + \frac{\Gamma^{2(m+1)/(m+2)}(t)}{C(1+\Gamma(t))^{m/(m+2)}} \le CK^{2\nu(m+1)/(m+2)}\|u_0\|_r^{2(\alpha-m)}\Gamma(t)^{2(m+1)/(m+2)}, \quad (4.9)$$

for $0 \le t \le T_0$. We may assume for any $\delta > 0$ that

$$\Gamma(t) \le \Gamma(0) + \delta, \quad 0 \le t \le T_\delta,$$

for some $T_\delta, 0 < T_\delta \le T_0$. Then we see from (4.9) that

$$\begin{aligned}&\frac{d}{dt}\Gamma(t) + \Big(\frac{1}{C(1+\Gamma(0)+\delta)^{m/(m+2)}} - CK^{2(m+1)\nu/(m+2)}\|u_0\|_r^{2(\alpha-m)}\Big) \\ &\times \Gamma^{2(m+1)/(m+2)}(t) \le 0,\end{aligned} \quad (4.10)$$

for $0 \le t \le T_\delta$. Thus, under the condition

$$C(1+\Gamma(0)+\delta)^{m/2(m+2)}K^{(m+1)\nu/(m+2)}\|u_0\|_r^{\alpha-m} < 1, \tag{4.11}$$

$\Gamma(t)$ is monotone decreasing and we have

$$\Gamma(t) \le \Gamma(0) < \Gamma(0)+\delta, \quad 0 \le t \le T_\delta, \tag{4.12}$$

which means that we can take $T_\delta = T_0$ and (4.12) is in fact valid on $[0, T_0]$. Since $\delta > 0$ is arbitrary we conclude that under the assumption (4.11) with $\delta = 0$ the estimate $\Gamma(t) \le \Gamma(0), 0 \le t \le T_0$, holds. Further, we make a little stronger assumption (4.11) with $\delta = 0$ and with C replaced by $2C^2$. Then, instead of (4.10), we obtain

$$\frac{d}{dt}\Gamma(t) + \frac{1}{2C(1+\Gamma(0))^{m/(m+2)}}\Gamma^{2(m+1)/(m+2)}(t) \le 0, \quad 0 \le t \le T_0,$$

which implies

$$\Gamma(t) \le \left(\Gamma^{-m/(m+2)}(0) + \frac{mt}{C(1+\Gamma(0))^{m/(m+2)}}\right)^{-(m+2)/m}. \tag{4.13}$$

(We have replaced $2(m+2)C$ by C.) When $m = 0$ we have $\Gamma(t) \le \Gamma(0)e^{-\lambda t}$ with some $\lambda > 0$ for (4.13).

Finally, from (4.1), (4.8), (4.13) and the assumption (4.11) with $\delta = 0$ we obtain

$$\int_0^{T_0} \|u_t(t)\|^2 dt \le \Gamma(0) + CK^{2(m+1)\nu/(m+2)}\Gamma(0)\|u_0\|_r^{2(\alpha-m)} \le C\Gamma(0). \tag{4.14}$$

From the above argument we conclude the following.

Proposition 4.1. *Let $\alpha > m$, $r \ge 2$ and make one of the assumptions (1), (2) and (3) in Theorem 2.1. Then, under the assumptions:*

$$\|\nabla u(t)\|_\infty \le K, \quad 0 \le t \le T_0,$$

and

$$CK^\nu\|u_0\|_r^{\alpha-m} < 1 \text{ and } CK^{(m+1)\nu/(m+2)}(1+\Gamma(0))^{m/2(m+2)}\|u_0\|_r^{\alpha-m} < 1 \tag{4.15}$$

with some $C > 0$, the assumed smooth solution $u(t)$ of the problem (1.1)–(1.2) satisfies the estimate (4.13) for $0 \le t \le T_0$. The estimate (4.14) also holds.

5. BOUNDEDNESS AND DECAY ESTIMATES FOR $\|u(t)\|_\infty$

Based on the assumptions in Proposition 4.1 we derive here an estimate for $\|u(t)\|_\infty$, $0 \le t \le T_0$.

Multiplying the equation by $|u|^{p-2}u, p \ge 2$, and integrating it we have

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + (p-1)\int_\Omega \sigma([\nabla u|^2)|\nabla u|^2|u|^{p-2}dx \le C\|u\|_{p+\alpha}^{p+\alpha}. \tag{5.1}$$

We see by Hyp. A (3) (cf. [15]),

$$\begin{aligned}\frac{4}{p^2}\|\nabla(|u|^{(p-2)/2}u)\|^2 &= \int_\Omega |\nabla u|^2|u|^{p-2}dx \\ &\le C\left(\int_{\Omega_1} \sigma(|\nabla u|^2)|\nabla u|^2|u|^{p-2}dx + \int_{\Omega_1} |u|^{p-2}dx\right) \\ &\quad + C\int_{\Omega_2} \sigma(|\nabla u|^2)|\nabla u|^2|u|^{p-2}dx\end{aligned} \tag{5.2}$$

and from (5.1) we have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + \frac{\epsilon_0(p-1)}{p^2}\|\nabla|u|^{p/2}\|^2 \\ &\le C(\|u\|_{p+\alpha}^{p+\alpha} + \|u\|_{p-2}^{p-2}) \le C(\|u\|_{p+\alpha}^{p+\alpha} + 1)\end{aligned} \tag{5.3}$$

with some $\epsilon_0 > 0$. Further, under the conditions $r > \frac{N\alpha}{2}$ if $N \ge 3$ and $r > \alpha$ if $N = 1, 2$, we can take p, q with $q > 1$ and $p \ge r$ such that

$$\max\{r, p+\alpha, pr/(r-\alpha)\} < q \le pN/(N-2)^+, \quad (1 \ll q < \infty \text{ if } N = 2).$$

Thus, we can define $\theta_i, 0 < \theta_i < 1, i = 1, 2, 3$, by

$$\theta_1 = \frac{\alpha}{p+\alpha}, \theta_2 = \frac{p}{p+\alpha}\left(1 - \frac{q\alpha}{r(q-p)}\right) \text{ and } \theta_3 = \frac{p}{p+\alpha}\cdot\frac{q\alpha}{r(q-p)}.$$

Then we have, by the Hölder and Sobolev inequalities,

$$\begin{aligned}\|u\|_{p+\alpha}^{p+\alpha} &\le \|u\|_r^{(p+\alpha)\theta_1}\|u\|_p^{(p+\alpha)\theta_2}\|u\|_q^{(p+\alpha)\theta_3} \\ &\le C\|u_0\|_r^{(p+\alpha)\theta_1}\|u\|_p^{(p+\alpha)\theta_2}\|\nabla|u|^{p/2}\|^{2(p+\alpha)\theta_3/p}.\end{aligned} \tag{5.4}$$

Using the relation $\theta_2 + \theta_3 = p/(p+\alpha)$ and the inequality $\|u(t)\|_r \le \|u_0\|_r$ we see from (5.4) that

$$C\|u(t)\|_{p+\alpha}^{p+\alpha} \le Cp^\mu\|u_0\|_r^{2r\alpha/(2r-N\alpha)}\|u(t)\|_p^p + \frac{(p-1)\epsilon_0}{2p^2}\|\nabla|u|^{p/2}\|^2 \tag{5.5}$$

with a certain constant $\mu > 0$. It follows from (5.3) and (5.5) that

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + \frac{\epsilon_0}{p}\|\nabla|u|^{p/2}\|^2 \le C_0p^\mu(\|u\|_p^p + 1), \quad 0 \le t \le T_0, \tag{5.6}$$

where we have changed $\epsilon_0 > 0$.

Now we apply Moser's technique to derive the boundedness of $\|u(t)\|_\infty$ (cf. [1,9]). Setting $p_n = 2p_{n-1}, n = 1, 2, \dots$ with $p_0 = r$, that is, $p_n = 2^n r$ we see by the Gagliardo–Nirenberg inequality (cf. [9,16]),

$$\|u\|_{p_n} \le C^{1/p_n} \|u\|_{p_{n-1}}^{1-\theta} \|\nabla |u|^{p_n/2}\|^{2\theta/p_n} \tag{5.7}$$

with $\theta = N/(N+2)$. Then it follows from (5.6) with $p = p_n$ that

$$\frac{1}{p_n}\frac{d}{dt}\|u(t)\|_{p_n}^{p_n} + \frac{\epsilon_0}{p_n}\|u(t)\|_{p_n}^{p_n/\theta}\|u\|_{p_{n-1}}^{-(1-\theta)p_n/\theta} \le C_0 p_n^\mu (\|u(t)\|_{p_n}^{p_n} + 1) \tag{5.8}$$

with some $\epsilon_0 > 0$.

By induction we can derive

$$\|u(t)\|_{p_n} \le \eta_n, \quad 0 \le t \le T_0, \tag{5.9}$$

where $\eta_0 = \max\{1, \|u_0\|_r, C\|u_0\|_\infty\}$ and η_n is defined by

$$\eta_n = (C_0 p_n)^{C/p_n} \eta_{n-1}.$$

Indeed, if $\|u(t)\|_{p_n} \ge 1$ for some t we have from (5.8) and the assumption of induction,

$$\frac{d}{dt}\|u(t)\|_{p_n} + \left(\frac{\epsilon_0}{p_n}\|u(t)\|_{p_n}^{(1-\theta)p_n/\theta}\eta_{n-1}^{-(1-\theta)p_n/\theta} - 2C_0 p_n^\mu\right)\|u(t)\|_{p_n} \le 0,$$

which implies, for all t,

$$\begin{aligned}\|u(t)\|_{p_n} &\le \max\{1, \|u(0)\|_{p_n}, (2C_0\epsilon_0^{-1}p_n^{\mu+1})^{\theta/p_n(1-\theta)}\eta_{n-1}\} \\ &\le \max\{1, C\|u_0\|_\infty, (C_0 p_n)^{C/p_n}\eta_{n-1}\}\end{aligned}$$

with some $C > 2$, where we have replaced $2C_0\epsilon_0^{-1}$ by C_0 for simplicity of notation.

We may assume $C_0 \ge 1$ and by induction we see that $\eta_{n-1} \ge 1$. Hence, we obtain

$$\|u(t)\|_{p_n} \le \eta_n \equiv (C_0 p_n)^{C/p_n}\eta_{n-1} = \prod_{k=1}^n (C_0 p_k)^{C/p_k}\eta_0 \le C_0\eta_0.$$

Thus, we have

$$\|u(t)\|_{p_n} \le C_0\eta_0 = C_0 \max\{1, \|u_0\|_r, \|u_0\|_\infty\}.$$

Taking the limit as $n \to \infty$, we have the desired estimate

$$\|u(t)\|_\infty \le C_0 \max\{1, \|u_0\|_\infty\}, \quad 0 \le t \le T_0. \tag{5.10}$$

We conclude the following.

Proposition 5.1. *In addition to the assumptions in Proposition 4.1 we assume $r > N\alpha/2$ if $N \ge 3$ and $r > \alpha$ if $N = 1, 2$. Then there exists a constant $C_0 = C(\|u_0\|_r)$ such that the estimate* (5.10) *holds on* $[0, T_0]$.

We proceed to the decay estimate for $\|u(t)\|_\infty$ under the same assumptions in Proposition 5.1. We return to the inequality (5.1). We make another device instead of (5.2) as follows.

$$\frac{4}{p^2}\|\nabla|u|^{p/2}\|^2 \le C\left(\int_{\Omega_1}\sigma(|\nabla u|^2)|\nabla u|^2|u|^{p-2}dx\right)^{2/(m+2)} \||u|^{p/2}\|^{2(p-2)m/p(m+2)} + C\int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2|u|^{p-2}dx \tag{5.11}$$

and by the Poincaré and Young inequalities we see that

$$\begin{aligned}\frac{2}{p^2}\|\nabla|u|^{p/2}\|^2 &\le C\left(p^\mu\int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2|u|^{p-2}dx\right)^{p/(p+m)} \\ &\quad + \int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2|u|^{p-2}dx \\ &\le Cp^\mu\left(\int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2|u|^{p-2}dx\right)^{p/(p+m)} \\ &\quad \times\left\{1+\left(\int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2|u|^{p-2}dxdx\right)^{m/(p+m)}\right\}.\end{aligned} \tag{5.12}$$

Here, we know that $\Gamma(t)\le\Gamma(0)$ (Proposition 4.1) and $\|u(t)\|_\infty \le C_0(1+\|u_0\|_\infty)$ (Proposition 5.1), and hence

$$\begin{aligned}&\left(\int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2|u|^{p-2}dx\right)^{m/(p+m)} \\ &\le C\Gamma(0)^{m/(p+m)}\|u(t)\|_\infty^{m(p-2)/(p+m)} \le \tilde{C}_0, \quad 0\le t\le T_0,\end{aligned} \tag{5.13}$$

where $\tilde{C}_0$ denotes a constant which may depend on $\|u_0\|_r, \Gamma(0)$ and $\|u_0\|_\infty$. It follows from (5.1), (5.12) and (5.13) that

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + \frac{1}{\tilde{C}_0 p^\mu}\|\nabla|u|^{p/2}\|^{2(p+m)/p} \le C\|u(t)\|_{p+\alpha}^{p+\alpha} \tag{5.14}$$

with a certain $\mu>0$. We set $w(\tau)=(1+t)^{1/m}u(t)$ and $\tau=\log(1+t)$ (cf. [2]). Then (5.14) is rewritten as

$$\begin{aligned}&\frac{1}{p}\frac{d}{d\tau}\|w(\tau)\|_p^p + \frac{1}{\tilde{C}_0 p^\mu}\|\nabla|w(\tau)|^{p/2}\|^{2(p+m)/p} \\ &\le \frac{1}{m}\|w(\tau)\|_p + C(1+t)^{(m-\alpha)/m}\|w(\tau)\|_{p+\alpha}^{p+\alpha} \le C(\|w(\tau)\|_{p+\alpha}^{p+\alpha}+1).\end{aligned} \tag{5.15}$$

Further we note that by (3.3) and (3.4),

$$\|u(t)\|_r \le C_0(1+t)^{-1/m}, \quad 0 \le t \le T_0,$$

and hence

$$\|w(\tau)\|_r \le C_0 < \infty, \quad 0 \le \tau \le \log(1+T_0).$$

(5.15) is essentially the same inequality as (5.3) and we can repeat the argument deriving (5.10) to obtain

$$\|w(\tau)\|_\infty \le C_0(1, \|u_0\|_\infty)), \quad 0 \le \tau \le \log(1+T_0),$$

that is

$$\|u(t)\|_\infty \le \tilde{C}_0(1+t)^{-1/m}, \quad 0 \le t \le T_0, \tag{5.16}$$

We conclude the following.

Proposition 5.2. *On the same assumptions in Proposition 5.1 we have the decay estimate* (5.16) *for* $\|u(t)\|_\infty$. (*When* $m = 0$ *we should replace the right-hand side of* (5.16) *by* $\tilde{C}_0 e^{-\lambda t}$ *with some* $\lambda > 0$.)

6. BOUNDEDNESS OF $\|\nabla u(t)\|_p$ ON $[0, T_0]$, $2 \le p < \infty$

We continue the estimation for the assumed smooth solution $u(t)$. Multiplying the equation by $-\nabla(|\nabla u|^{p-2}\nabla u), p \ge 2$, and integrating it we have

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \int_\Omega \nabla(\sigma(|\nabla u|^2)\nabla u)\nabla(|\nabla u|^{p-2}\nabla u)dx \\ &= \int_\Omega f'(u)|\nabla u|^p dx \le C\int_\Omega |u|^\alpha |\nabla u|^p dx. \end{aligned} \tag{6.1}$$

Here, by integration by parts (cf. [2,13,16] etc.) and using Hyp. A (1), we see (cf. [13,15])

$$\begin{aligned} &\int_\Omega \nabla(\sigma(|\nabla u|^2)\nabla u)\nabla(|\nabla u|^{p-2}\nabla u)dx \\ &\ge k_0 \int_\Omega \sigma|\nabla u|^{p-2}|D^2 u|^2 - (N-1)\int_{\partial\Omega} \sigma|\nabla u|^p H(x)dx \end{aligned} \tag{6.2}$$

where $D^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ and $H(x) = -\nabla \cdot \mathbf{n}(x)/(N-1)$, the mean curvature of $\partial\Omega$ at x. As in [13], we see

$$\int_{\partial\Omega} \sigma|\nabla u|^p H(x)dx \le C_\delta \|\sqrt{\sigma}|\nabla u|^{p/2}\|^2 + \delta\|\nabla(\sqrt{\sigma}|\nabla u|^{p/2})\|^2, \quad 0 < \delta \ll 1, \tag{6.3}$$

and further,

$$\|\nabla(\sqrt{\sigma}|\nabla u|^{p/2})\|^2 \le C \int_\Omega \sigma |\nabla(|\nabla u|^{p/2})|^2 dx, \tag{6.4}$$

where we have used the assumption $|\sigma'(v^2)|v^2 \le k_1 \sigma(v^2)$.

Then, taking δ suficiently small and fixing it, we have from (6.1), (6.2) and (6.3),

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{\epsilon_0}{p^2}\left(\|\sqrt{\sigma}|\nabla u|^{p/2}\|_{H_1}^2 + \|\sqrt{\sigma}\nabla(|\nabla u|^{p/2})\|^2\right) \\ &\le Cp^2\|\sqrt{\sigma}|\nabla u|^{p/2}\|^2 + C\int_\Omega |u|^\alpha |\nabla u|^p dx \end{aligned} \tag{6.5}$$

(cf. [15]). The first term of the right-hand side of (6.5) is treated as follows (see [15])

$$Cp^2\|\sqrt{\sigma}|\nabla u|^{p/2}\|^2 \le \frac{\epsilon_0}{2p^2}\|\sqrt{\sigma}|\nabla u|^{p/2}\|_{H_1}^2 + Cp^\mu\|\sqrt{\sigma}|\nabla u|^{p/2}\|_1^2$$

with $\mu = 2(N+4)/N$, and we see also

$$\|\sqrt{\sigma}|\nabla u|^{p/2}\|_1^2 \le C\Gamma(t)\|\nabla u\|_p^{p-2}.$$

The second term of (6.5) is dominated due to Proposition 5.2 as

$$C\int_\Omega |u(t)|^\alpha |\nabla u(t)|^p dx \le \tilde{C}_0(1+t)^{-\alpha/m}\|\nabla u(t)\|_p^p. \tag{6.6}$$

From (6.5) through (6.6) we obtain

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{\epsilon_0}{2p^2}\left(\|\sqrt{\sigma}|\nabla u|^{p/2}\|_{H_1}^2 + \|\sqrt{\sigma}\nabla(|\nabla u|^{p/2})\|^2\right) \\ &\le Cp^\mu\Gamma(t)\|\nabla u(t)\|_p^{p-2} + C_0(1+t)^{-\alpha/m}\|\nabla u(t)\|_p^p \end{aligned} \tag{6.7}$$

and in particular,

$$\frac{d}{dt}\|\nabla u(t)\|_p^2 \le Cp^\mu\Gamma(t) + C_0(1+t)^{-\alpha/m}\|\nabla u(t)\|_p^2. \tag{6.8}$$

Since

$$\Gamma(t) \le \left(\Gamma^{-m/(m+2)}(0) + mC^{-1}(1+\Gamma(0))^{-m/(m+2)}t\right)^{-(m+2)/m}$$

by Proposition 4.1 and $\alpha > m$, we have from (6.8)

$$\begin{aligned} \|\nabla u(t)\|_p^2 &\le \|\nabla u_0\|_p^2 \exp\left(\tilde{C}_0\int_0^t (1+s)^{-\alpha/m} ds\right) \\ &\quad + Cp^\mu\int_0^t \Gamma(s)\exp\left(\tilde{C}_0\int_s^t (1+\tau)^{-\alpha/m}d\tau\right)ds \\ &\le \tilde{C}_0(p^\mu\Gamma(0) + \|\nabla u_0\|_p^2) \end{aligned}$$

with some $\tilde{C}_0 = C(\|u_0\|_r, \|u_0\|_\infty, \Gamma(0))$.

We summarize the last part.

Proposition 6.1. *We make the same assumptions as in Proposition 5.1. Then we have the estimate for* $p \geq 2$,

$$\|\nabla u(t)\|_p^2 \leq \tilde{C}_0(p^\mu \Gamma(0) + \|\nabla u_0\|_p^2), \quad 0 \leq t \leq T_0, \tag{6.9}$$

with some $\tilde{C}_0 = C(\|u_0\|_r, \|u_0\|_\infty, \Gamma(0))$.

The inequality (6.7) will play an essential role in deriving the estimate for $\|\nabla u(t)\|_\infty$.

7. BOUNDEDNESS AND DECAY ESTIMATES FOR $\|\nabla u(t)\|_\infty$

We shall derive an estimate like $\|\nabla u(t)\|_\infty \leq C_1(K) = C(\|u_0\|_r, \|\nabla u_0\|_\infty, K)$ on $[0, T_0]$ under (3.1). If we could show that $C_1(K) < K$, then we can conclude that the estimate is valid in fact on $[0, \infty)$, and consequently, all of the estimates derived in the previous sections hold for $0 \leq t < \infty$.

We return to the inequality (6.7). Since $\Gamma(t) \leq \Gamma(0)$ and $C\|\sqrt{\sigma}|\nabla u|^{p/2}\|_{H_1} \geq \|\sqrt{\sigma}|\nabla u|^{p/2}\|$ we know from (6.7) and Hyp. A (3),

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{\epsilon_1}{p^2 K^\nu}\||\nabla u|^{(p+m)/2}\|_{H_1}^2 \leq \tilde{C}_0 p^\mu(\|\nabla u(t)\|_p^p + 1), \quad 0 \leq t \leq T_0, \tag{7.1}$$

with some $\epsilon_1 > 0$. This is essentially the same inequality as (5.4) in [13], (6.2) in [15] and also similar to (5.6). Thus we can repeat a similar argument deriving (5.9). For convenience of the reader we sketch an outline.

Setting $p_n = 2p_{n-1} - m$ with $p_0 \geq m + 2$, that is, $p_n = 2^n(p_0 - m) + m$, we see by the Gagliardo–Nirenberg inequality as in (5.7)

$$\|\nabla u\|_{p_n} \leq C^{1/p_n}\|\nabla u\|_{p_{n-1}}^{1-\theta_n}\||\nabla u|^{(p_n+m)/2}\|_{H_1}^{2\theta_n/(p_n+m)} \tag{7.2}$$

with $\theta_n = N(1 - mp_n^{-1})/(N+2)$. It follows from (7.1) with $p = p_n, n \geq 1$, and (7.2) that

$$\frac{1}{p_n}\frac{d}{dt}\|\nabla u(t)\|_{p_n}^{p_n} + \frac{\epsilon_1}{CK^\nu p_n^2}\|\nabla u(t)\|_{p_n}^{\beta_n + p_n}\|\nabla u(t)\|_{p_{n-1}}^{m-\beta_n} \leq \tilde{C}_0 p_n^\mu(\|\nabla u(t)\|_{p_n}^{p_n} + 1), \tag{7.3}$$

where we set

$$\beta_n = ((1 - \theta_n)p_n + m)/\theta_n = 2p_n(p_n + mN + m)/N(p_n - m).$$

Thus we can derive by induction,

$$\|\nabla u(t)\|_{p_n} \leq \eta_n, \quad 0 \leq t \leq T_0, \tag{7.4}$$

where we set $\eta_0 = \max\{1, \sup_{0 \leq t \leq T_0}\|\nabla u(t)\|_{p_0}, C\|\nabla u_0\|_\infty\}$ and

$$\eta_n \equiv \left(\tilde{C}_0 \epsilon_1^{-1} K^\nu p_n^{\mu+2}\right)^{1/\beta_n} \eta_{n-1}.$$

From this we can derive

$$\log\,\eta_n \le \tilde{C}_0 \frac{\log\,(p_0-m)+\nu\log K}{p_0-m} +\ \log\,\eta_0$$

and hence, by use of (6.9) with $p = p_0$,

$$\begin{aligned}\eta_n &\le \eta_0 \exp\left(\tilde{C}_0(\log(p_0-m)K)/(p_0-m)\right)\\ &\le \tilde{C}_0(\|\nabla u_0\|_\infty + p_0^{\mu/2})\exp\left(\tilde{C}_0(\log((p_0-m)K))/(p_0-m)\right) \equiv C_1(K).\end{aligned} \tag{7.5}$$

Taking the limit as $n \to \infty$, we obtain from (7.4) and (7.6)

$$\|\nabla u(t)\|_\infty \le C_1(K), \quad 0 \le t \le T_0. \tag{7.6}$$

It is easy to see that we can choose $p_0 > m+2$ and $K > 1$ such that $C_1(K) < K$. Indeed, it suffices to choose K and p_0 in such a way that

$$\tilde{C}_0(\log(p_0-m)+\log\,K)) < p_0 - m \text{ and } \tilde{C}_0\left(\|\nabla u_0\|_\infty + p_0^{\mu+2}\right)e < K$$

(cf. [13]).

We state the result at this stage.

Proposition 7.1. *There exists a constant $K > \max\{1, \|\nabla u_0\|_\infty\}$ continuously depending on $\|u_0\|_r, \Gamma(0), \|u_0\|_\infty$ and $\|\nabla u_0\|_\infty$ such that $\|\nabla u(t)\|_\infty < K$, $0 \le t < \infty$, and all of the estimates in Propositions 3.1, 4.1, 5.1, 6.2 and 6.1 are valid for $0 \le t < \infty$ under the assumptions stated there.*

We fix $K > 1$ as in Proposition 7.1 and proceed to the decay estimate for $\|\nabla u(t)\|_\infty$. Once the boundedness of $\|\nabla u(t)\|_\infty$ has been established we see

$$\sigma(|\nabla u(t)|^2) \ge C_1^{-1}|\nabla u(t)|^m$$

with some positive constant C_1 depending on $\|u_0\|_r, \Gamma(0), \|u_0\|_\infty$ and $\|\nabla u_0\|_\infty$. (Of course we can simply say that C_1 depends on $\|\nabla u_0\|_\infty$ continuously.) In what follows, we denote by C_1 various constants depending on such norms of the initial data which may be different from line to line.

Then, we obtain, instead of (6.7),

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{1}{C_1 p^2}\||\nabla u|^{(p+m)/2}\|_{H_1}^2\\ &\le Cp^\mu\Gamma(t)\|\nabla u(t)\|_p^{p-2} + C\int_\Omega |u|^\alpha|\nabla u|^p dx.\end{aligned} \tag{7.7}$$

Since $\|u(t)\|_\infty \le \tilde{C}_0$ and $\alpha > m$, it is easy to see that

$$C\int_\Omega |u|^\alpha|\nabla u|^p dx \le \tilde{C}_0\int_\Omega |u|^m|\nabla u|^p dx \le \tilde{C}_0\|\nabla u\|_{p+m}^{p+m}.$$

We see also, by (4.13), that

$$\Gamma(t)\|\nabla u(t)\|_p^{p-2} \le C_0 p^\mu (1+t)^{-(m+2)/m}\|\nabla u(t)\|_p^{p-2}.$$

Thus, we have from (7.7)

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{1}{C_1 p^2}\||\nabla u|^{(p+m)/2}\|_{H_1}^2 \\ &\le \tilde{C}_0\|\nabla u\|_{p+m}^{p+m} + C_0 p^\mu (1+t)^{-(m+2)/m}\|\nabla u(t)\|_p^{p-2}.\end{aligned} \tag{7.8}$$

Setting $\mathbf{w}(\tau) = (1+t)^{1/m}\nabla u(t)$ and $\tau = \log(1+t)$, (7.8) is rewritten as

$$\frac{1}{p}\frac{d}{d\tau}\|\mathbf{w}(\tau)\|_p^p + \frac{1}{C_1 p^2}\|\mathbf{w}(\tau)|^{(p+m)/2}\|_{H_1}^2 \le \tilde{C}_0 p^\mu (\|\mathbf{w}(\tau)\|_{p+m}^{p+m} + 1). \tag{7.9}$$

Here we note that

$$\|\nabla u(t)\|_{m+2}^{m+2} \le C_1\Gamma(t) \le C_1(1+t)^{-(m+2)/m}$$

and hence,

$$\|\mathbf{w}(\tau)\|_{m+2} \le (1+t)^{1/m}\|\nabla u(t)\|_{m+2} \le C_1 < \infty.$$

The inequality (7.9) is similar to (5.15) if we replace $u(t)$ by $\mathbf{w}(t)$ and we can derive the boundedness of $\|\mathbf{w}(t)\|_\infty$ (see [15]):

$$\|\mathbf{w}(\tau)\|_\infty \le C_1 < \infty, \quad 0 \le \tau < \infty$$

and consequently,

$$\|\nabla u(t)\|_\infty \le C_1(1+t)^{-1/m}, \quad 0 \le t < \infty. \tag{7.10}$$

Proposition 7.2. *Under the same assumptions as in Proposition 7.1 we have the decay estimate* (7.10). (*When $m = 0$ we have the estimate $\|\nabla u(t)\|_\infty \le C_1 e^{-\lambda t}$ with some $\lambda > 0$.*)

8. PROOF OF THEOREM 2.1

Let $\epsilon > 0$ and let us take a bounded uniformly Lipschitz continuous function $f_\epsilon(u)$ such that

$$|f_\epsilon(u)| \le |f(u)|, \quad |f'_\epsilon(u)| \le |f'(u)|$$

and

$$f_\epsilon(u) \to f(u) \text{ uniformly on each compact interval } [-L, L], L > 0, \text{ as } \epsilon \to 0.$$

We first assume that $u_0 \in C_0^3(\Omega)$ and consider the approximate problem

$$u_t - \operatorname{div}\{\sigma(\epsilon + |\nabla u|^2)\nabla u\} = f_\epsilon(u) \text{ in } \Omega \times (0,\infty), \tag{8.1}$$

with the initial-boundary conditions (1.2). It is easy to show that $\sigma_\epsilon(v^2) \equiv \sigma(\epsilon + v^2)$ belongs to $C^2([0,\infty))$, satisfies Hyp. A (with the same k_0 and k_1) and further $\sigma(\epsilon + v^2) \geq C_\epsilon > 0$. Then the problem (8.1)–(1.2) admits a unique classical solution $u_\epsilon(t) \in C^1([0,\infty); C(\bar{\Omega})) \cap C([0,\infty); C^2(\bar{\Omega}))$ (see [13]). It is clear that all of the arguments deriving the estimates in Propositions 3.1 through 7.2 are applicable to $u_\epsilon(t), 0 < \epsilon \ll 1$, and the resulted estimates are valid under the assumptions made there, where $\Gamma(0)$ in (4.13) and (4.14) should be replaced by

$$\Gamma_\epsilon(0) = \int_\Omega \int_0^{|\nabla u|^2} \sigma(\epsilon + \tau) d\tau dx.$$

All of the estimates are essentially independent of ϵ. Therefore, $u_\epsilon(t)$ is convergent, along a subsequence, to a function $u(t)$ as $\epsilon \to 0$ in the following manner:

$$u_\epsilon(t) \to u(t) \text{ weakly }^* \text{ in } L^\infty_{loc}([0,\infty); L^\infty), \text{ a.e. } (x,t) \in \Omega \times [0,\infty),$$

$$\text{weakly in } W^{1,2}_{loc}([0,\infty); L^2), \text{ weakly in } L^p_{loc}([0,\infty); W^p_0) \text{ for any } p \geq 2,$$

and

$$\nabla u_\epsilon(t) \to \nabla u(t) \text{ weakly }^* \text{ in } L^\infty_{loc}([0,\infty); L^\infty).$$

Further, there exists a function (operator)

$$\chi(t) \in L^{p_0/(p_0-1)}_{loc}([0,\infty); W^{-1,p_0/(p_0-1)}) \equiv (L^{p_0}_{loc}([0,\infty); W^{1,p_0}_0)^*, \quad p_0 \geq 2,$$

such that

$$\int_0^T \int_\Omega \sigma(\epsilon + |\nabla u_\epsilon|^2) \nabla u_\epsilon \cdot \nabla \phi(x,t) dx dt \to \langle \chi(t), \phi(t) \rangle_T$$

for all $T > 0$ and all $\phi(t) \in L^{p_0}_{loc}([0,\infty); W^{1,p_0}_0)$, where $\langle \cdot, \cdot \rangle_T$ denotes the pairing of $L^{p_0/(p_0-1)}([0,T]; W^{-1,p_0/(p_0-1)})$ and $L^{p_0}([0,T]; W^{1,p_0}_0)$. Thus, the limit function $u(t)$ satisfies

$$\int_0^T (u_t(t), \phi(t)) dt + \langle \chi(t), \phi(t) \rangle_T = \int_0^T (f(u(t)), \phi(t)) dt \tag{8.2}$$

for all $T > 0$ and all $\phi(t) \in L^{p_0}_{loc}([0,\infty); W^{1,p_0}_0)$. It is also easy from $u_\epsilon(t) - u_0 = \int_0^t u_{\epsilon,t}(s) ds$ to show that $u(0) = u_0$. Further, we see by Hyp. A (1),

$$\begin{aligned} &(\sigma_\epsilon(|\nabla u|^2)\nabla u - \sigma_\epsilon(|\nabla v|^2)\nabla v, \nabla u - \nabla v) \\ &\geq (\sigma_\epsilon(|\nabla u|^2)|\nabla u| - \sigma_\epsilon(|\nabla v|^2)|\nabla v|, |\nabla u| - |\nabla v|) \geq 0 \end{aligned}$$

and we can apply a standard monotonicity argument to show that

$$\langle \chi(t), \phi(t) \rangle_T = \int_0^T \int_\Omega \sigma(|\nabla u|^2) \nabla u \cdot \nabla \phi(x,t) dx dt \tag{8.3}$$

for all $T > 0$ and all $\phi(t) \in L^{p_0}_{loc}([0,\infty); W_0^{1,p_0})$. Thus, $u(t)$ is the desired (weak) solution of the original problem (1.1)–(1.2). All of the estimates established for the approximate solutions u_ϵ are valid for $u(t)$ (with $\epsilon = 0$).

Finally, we consider the case $u_0 \in W_0^{1,\infty}$. In this case, we can construct a sequence $u_{n,0} \in C_0^3(\Omega)$ by using a mollifier such that

$$u_{n,0} \to u_0 \text{ in } W_0^{1,p} \text{ for any } p \geq 1 \text{ and } \|\nabla u_{0,n}\|_\infty \leq \tilde{C}\|\nabla u_0\|_\infty$$

with a constant $\tilde{C} \geq 1$ independent of $\|\nabla u_0\|_\infty$ (cf. [8, Section 5, Chapter 3]). (If supp $u_0 \subset \Omega$, we can take $\tilde{C} = 1$.)

If u_0 satisfies the conditions in (4.15) in Proposition 4.1, $u_{n,0}$ also satisfy them, though K may be changed according to the change $\|\nabla u_0\|_\infty$ by $\tilde{C}\|\nabla u_0\|_\infty$. From the above argument we know that the problem (1.1)–(1.2) admits a solution $u_n(t) \in L^\infty([0,\infty; W_0^{1,\infty}) \cap W^{1,2}([0,\infty); L^2)$. All of the assertions of Propositions in previous sections are applied to the solutions $u_n(t)$ with $u_n(0) = u_{n,0}$, and we can repeat the above argument to see that $u_n(t)$ converges, along a subsequence, to a solution $u(t)$ of the problem with $u(0) = u_0$.

Since $u(t) \in L^\infty([0,\infty); W_0^{1,\infty}) \subset L^\infty([0,\infty); L^\infty)$ the uniqueness of solution is easily shown by use of the monotonicity of the operator $-\text{div}(\sigma(|\nabla u|^2)\nabla u)$ and the Lipschitz continuity of $f(u)$. The proof of Theorem 2.1 is completed.

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Received: May 28, 2018.
Accepted: August 14, 2018.