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ON THE ZEROS OF THE MACDONALD FUNCTIONS

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Abstract. We are concerned with the zeros of the Macdonald functions or the modified Bessel functions of the second kind with real index. By using the explicit expressions for the algebraic equations satisfied by the zeros, we describe the behavior of the zeros when the index moves. Results by numerical computations are also presented.

Keywords: zeros, Macdonald functions, Bessel functions.

Mathematics Subject Classification: 33C10, 30C15, 32A60, 33F05.

1. INTRODUCTION AND OUTLINE

The Bessel and modified Bessel functions appear in various situations in probability theory as well as they do in many situations in mathematical sciences. For example, the probability distributions of the first hitting times of Bessel processes can be explicitly represented by them and their zeros. In particular, when the arrival point is closer to the origin than the starting point, the so-called Macdonald functions K_ν and their zeros play important roles (cf. [1–3]). In addition, it is known that the expectation of the volume of the Wiener sausage for a Brownian motion is also represented by a Macdonald function and its zeros (cf. [4]).

In this article we are concerned with the zeros of the Macdonald function with real index, which we denote by K_ν in the usual notation. It is known that K_ν is a holomorphic function on $\mathcal{D}$ for each $\nu \in \mathbb{C}$ and that $K_\nu(z)$ is an entire function in ν for each $z \in \mathcal{D}$, where $\mathcal{D} = \{z \in \mathbb{C} \setminus \{0\}\,;\, |\arg z| < \pi\}$. The zeros of the Bessel functions J_ν, Y_ν and of the other modified function I_ν are well studied and we can also carry out the numerical computations for them in several ways. However, only a few things are known about the zeros of K_ν. See, for instance, [6, 10–12].

Since $K_{-\nu} = K_\nu$ and there are no zeros when $0 \leq \nu < 3/2$, we assume that $\nu \geq 3/2$ throughout this paper. We write $N(\nu)$ for the number of zeros of K_ν. When $2n-1/2 < \nu < 2n+3/2$ for an integer $n \geq 1$, it is known that K_ν has no real zero and that $N(\nu) = 2n$. For an integer $n \geq 0$, it is also well-known that $z^{2n+3/2}e^z K_{2n+3/2}(z)$

is a polynomial of order $2n+1$, which has a unique negative root, and we regard it as a zero of $K_{2n+3/2}$. Hence $K_{2n+3/2}$ has $2n+1$ zeros. Moreover, the real part of each zero is negative and the non-real zeros are complex conjugate in pairs. For details see [7, 11].

Recently in [4], it has been shown that the zeros of K_ν are obtained as the roots of some polynomials whose coefficients are explicitly given in terms of K_ν and I_ν. For details, see the equations (2.4) and (2.7) below. When $2n-1/2 < \nu < 2n+3/2$, the equations may be taken of order $2n$. Such equations have been already shown in [12] when ν is an integer and they coincide in this special case.

This article deals with the continuity of the zeros of K_ν. We show that each non-real zero of K_μ converges to a non-real zero of K_ν as $\mu \to \nu$. For $\nu > 3/2$ we let ε_ν be the half of the smallest distance between zeros of K_ν, and Ξ_ν be the set of all non-real zeros of K_ν.

Theorem 1.1. *Let $\nu > 3/2$ and $z_\nu \in \Xi_\nu$ be fixed. For any $\varepsilon \in (0, \varepsilon_\nu)$ there exists an open interval $E_\nu \subset \mathbb{R}$ which contains ν such that, for any $\mu \in E_\nu$ we can take $z_\mu \in \Xi_\mu$ uniquely satisfying with $|z_\mu - z_\nu| < \varepsilon$.*

The behavior around the real zero is slightly different from that around non-real zeros. Let $\nu(n) = 2n+3/2$ for an integer $n \geq 0$. Recall that K_ν has a real zero if and only if $\nu = \nu(n)$. In this case, since $\overline{z} \in \Xi_\mu$ if $z \in \Xi_\mu$, the following theorem implies that two of the zeros of K_μ converge to the unique real zero of $K_{\nu(n)}$ as $\mu \downarrow \nu(n)$.

Theorem 1.2. *For an integer $n \geq 0$ let $-x_{\nu(n)}$ be the unique real zero of $K_{\nu(n)}$. For any $\varepsilon \in (0, \varepsilon_{\nu(n)})$ there exists $\delta > 0$ such that, for any $\mu \in (\nu(n), \nu(n)+\delta)$ we can take $z_\mu \in \Xi_\mu$ uniquely satisfying $\operatorname{Im} z_\mu > 0$ and $|z_\mu + x_{\nu(n)}| < \varepsilon$.*

By computing solutions of the algebraic equation whose roots are the zeros of K_ν, we obtain the following graph (see Figure 1), which shows the behavior of the zeros.

The unique zero of $K_{3/2}$ is -1. The two curves from -1 described by the squares give the two zeros in the case of $3/2 < \nu < 7/2$. The endpoints and the negative value between -2 and -3 found in the graph are the three zeros of $K_{7/2}$. The four curves from the zeros of $K_{7/2}$ described by the black circles are the zeros in the case of $7/2 < \nu < 11/2$. The five zeros of $K_{11/2}$ are seen in a similar manner, and so on. See also the table in the last part of this article. It should be mentioned that we find a similar graph for the zeros in [6].

For a given $z_\nu \in \Xi_\nu$ Theorem 1.1 gives a continuous function $g(\,\cdot\,; z_\nu)$ on E_ν such that $g(\mu; z_\nu) = z_\mu$. Here z_μ for $\mu \in E_\nu$ is the unique element in Ξ_μ chosen in Theorem 1.1. It seems that each curve in Figure 1 on the next page is smooth except for at real zeros. Indeed, we show the following.

Theorem 1.3. *For $\nu > 3/2$ let $z_\nu \in \Xi_\nu$ be fixed. The function $g(\,\cdot\,; z_\nu)$ defined on E_ν is analytic at ν.*

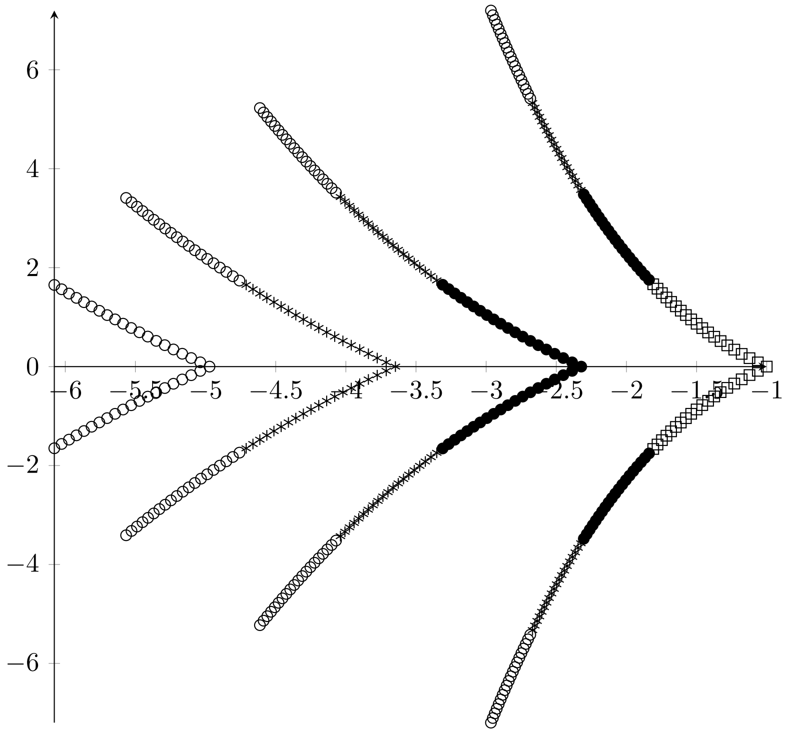

Fig. 1. Zeros of $K_\nu(z)$

This article is organized as follows. Section 2 is devoted to proving the results which have been given in this section and we provide some numerical computations of the zeros by *Mathematica* in Section 5. In Sections 3 and 4, we give proofs of the lemmas needed in Section 2.

2. CONTINUITY OF ZEROS OF MACDONALD FUNCTIONS

We first observe the function G on $(0,\infty)\times(0,\infty)$ given by

$$G(\mu,x) = K_\mu(x)^2 + \pi^2 I_\mu(x)^2 + 2\pi\sin(\pi\mu)K_\mu(x)I_\mu(x).$$

In particular, we need to consider the zeros of G to prove Theorem 1.2. For $\mu > 0$ and $x > 0$ let

$$H(\mu,x) = K_\mu(x) - \pi I_\mu(x).$$

The following lemma plays an important role in order to give the zeros of G.

Lemma 2.1. *For $\mu > 0$ we have that $H(\mu,\cdot\,)$ is strictly decreasing on $(0,\infty)$ and there exists $x_\mu > 0$ uniquely such that $H(\mu,x_\mu) = 0$. In particular, $H(\mu,\cdot\,)$ is positive on $(0,x_\mu)$ and negative on (x_μ,∞).*

Proof. Note that I_μ and K_μ are strictly increasing and decreasing, respectively. Moreover, it is known that

$$\lim_{x\to\infty} I_\mu(x) = \infty, \quad \lim_{x\to 0} I_\mu(x) = 0,$$
$$\lim_{x\to\infty} K_\mu(x) = 0, \quad \lim_{x\to 0} K_\mu(x) = \infty.$$

These properties can be found in [7, p. 136]. Thus, the claims of this lemma can be obtained by standard arguments. □

Since both K_μ and I_μ are positive on $(0,\infty)$ and

$$G(\mu,x) = H(\mu,x)^2 + 2\pi\{1+\sin(\pi\mu)\}K_\mu(x)I_\mu(x),$$

it follows that $G(\mu,x)\geq 0$. Moreover Lemma 2.1 gives that $G(\mu,x)=0$ if and only if $(\mu,x)=(\nu(n),x_{\nu(n)})$. Hence we obtain the following lemma.

Lemma 2.2. *It follows that $x_{\nu(n)}$ is the unique zero of $G(\nu(n),\cdot)$ for $n\geq 0$ and that $G(\mu,\cdot)$ is strictly positive when $\mu\neq\nu(n)$ for any $n\geq 0$.*

We next give the algebraic equations whose solutions are the zeros of K_ν for each $\nu\geq 3/2$ and the real zero of $K_{\nu(n)}$. For $\mu\geq 0$ and an integer $m\geq 1$ let $(\mu,0)=1$ and

$$(\mu,m) = \frac{(4\mu^2-1^2)(4\mu^2-3^2)\dots(4\mu^2-(2m-1)^2)}{2^{2m}m!}. \tag{2.1}$$

It is easy to see that

$$(\mu,m) = \begin{cases} 0 & \text{if } \mu-m-1/2 \text{ is a negative integer,} \\ \dfrac{\Gamma(\mu+m+1/2)}{m!\Gamma(\mu-m+1/2)} & \text{otherwise.} \end{cases} \tag{2.2}$$

When $\mu\neq\nu(n)$ for any $n\geq 0$ we define the sequence $\{a_m^{(\mu)}\}_{m=0}^{N(\mu)}$ by $a_0^{(\mu)}=1$ and

$$a_m^{(\mu)} = \frac{1}{m}\sum_{k=1}^m a_{m-k}^{(\mu)}\left\{b_{k+1}^{(\mu)} - (-1)^k\cos(\pi\mu)\int_0^\infty \frac{y^{k-1}}{G(\mu,y)}dy\right\} \tag{2.3}$$

for $1\leq m\leq N(\mu)$, where the sequence $\{b_m^{(\mu)}\}_{m=0}^\infty$ of real numbers is determined inductively by

$$\frac{(\mu+1,m)}{2^m} = \sum_{k=0}^m \frac{(\mu,m-k)}{2^{m-k}}b_k^{(\mu)}.$$

Lemma 2.3. *If $\nu \neq \nu(n)$ for any integer $n \geq 0$, each zero of K_ν is the solution of*

$$\sum_{k=0}^{N(\nu)} a^{(\nu)}_{N(\nu)-k} z^k = 0. \tag{2.4}$$

Proof. When $\nu \neq n+1/2$, the claim has been proved in [4]. If $\nu = 2n+1/2$ for an integer $n \geq 1$, we can show that the coefficients are given by $a_0^{(2n+1/2)} = 1$ and

$$a_m^{(2n+1/2)} = \frac{1}{m} \sum_{k=1}^{m} a^{(2n+1/2)}_{m-k} b^{(2n+1/2)}_{k+1} \tag{2.5}$$

for $1 \leq m \leq 2n$. Note that $G(2n+1/2, x) > 0$. This yields that (2.5) is the same as (2.3) for $\mu = 2n+1/2$. □

Since $G(\nu(n), x_{\nu(n)}) = 0$, we need to observe the asymptotic behavior of $G(\nu(n), \cdot\,)$ around $x_{\nu(n)}$ in order to treat the integral of the right hand side of (2.3). We do not have useful information on the asymptotics as we know. However, for an integer $n \geq 0$ it is known that, for $z \in \mathcal{D}$

$$z^{n+1/2} K_{n+1/2}(z) = \sqrt{\frac{\pi}{2}} e^{-z} \sum_{k=0}^{n} \frac{(n+1/2, n-k)}{2^{n-k}} z^k \tag{2.6}$$

(cf. [8, p. 72]). Hence, $z^{n+1/2} K_{n+1/2}(z)$ has holomorphic extension on $\mathbb{C}$ and then we obtain the polynomial of which roots are zeros of $K_{n+1/2}$.

Lemma 2.4. *For an integer $n \geq 1$ the equation for zeros of $K_{n+1/2}$ is*

$$\sum_{k=0}^{n} \frac{(n+1/2, n-k)}{2^{n-k}} z^k = 0. \tag{2.7}$$

We should remark that all solutions of (2.4) and (2.7) are of multiplicity one, which can be immediately obtained by Lemmas 2.3 and 2.4.

Recall that, for $m \in \mathbb{Z}$

$$K_\mu(e^{m\pi i} z) = e^{-\mu m \pi i} K_\mu(z) - \pi i \frac{\sin(\mu m \pi)}{\sin(\mu \pi)} I_\mu(z)$$

(cf. [8, p. 68]). Here $\sin(nm\pi)/\sin(n\pi)$ should be interpreted as $m(-1)^{m(n+1)}$ for each $n \in \mathbb{Z}$. This formula yields that, for $x > 0$ and $m \in \mathbb{Z}$

$$\{e^{(2m+1)\pi i} x\}^{\nu(n)} K_{\nu(n)}(e^{(2m+1)\pi i} x) = x^{\nu(n)} H(\nu(n), x).$$

Lemma 2.4 and (2.6) give that the negative zero of $K_{\nu(n)}$ coincides with the unique zero of $H(\nu(n), \cdot\,)$. Therefore, we deduce from Lemma 2.1 that $-x_{\nu(n)}$ is the unique real zero of $K_{\nu(n)}$.

Before proving theorems in Section 1, we give the result on the continuity of roots of a polynomial with complex coefficients, which is the direct consequence of the Rouché theorem (cf. [9, p. 3]).

Lemma 2.5. *Let $n \geq 1$ be a fixed integer and*

$$P(z) = p_0 + p_1 z + \ldots + p_{n-1} z^{n-1} + p_n z^n \quad (p_n \neq 0),$$
$$Q(z) = (p_0 + q_0) + (p_1 + q_1) z + \ldots + (p_{n-1} + q_{n-1}) z^{n-1} + p_n z^n.$$

Assume that $f(z)$ is factorized in the following way:

$$P(z) = p_n \prod_{k=1}^{p} (z - w_k)^{m_k}.$$

For $k = 1, 2, \ldots, p$ let r_k be a real number with $0 < r_k < \min_{j:j \neq k} |w_k - w_j|$. There exists $\varepsilon > 0$ such that, if $|q_j| < \varepsilon$ for $j = 0, 1, 2, \ldots, n-1$, then Q has precisely m_k zeros in the circle with center at w_k and radius r_k for any $k = 1, 2, \ldots, p$.

We now prove Theorems 1.1 and 1.2 simultaneously. We first observe the case when $\nu \neq \nu(n)$ for any integer $n \geq 0$. Lemma 2.5 yields that it is sufficient to prove that, for any $m = 1, 2, \ldots, N(\nu)$

$$\lim_{\mu \to \nu} a_m^{(\mu)} = a_m^{(\nu)}. \tag{2.8}$$

In order to see (2.8) we need to prove that, for any $m = 0, 1, 2, \ldots, N(\nu) + 1$

$$\lim_{\mu \to \nu} b_m^{(\mu)} = b_m^{(\nu)} \tag{2.9}$$

and the following lemma.

Lemma 2.6. *Let $n \geq 0$ be an integer. If $\nu(n) < \nu < \nu(n+1)$, we have that*

$$\lim_{\mu \to \nu} \int_0^\infty \frac{x^m}{G(\mu, x)} dx = \int_0^\infty \frac{x^m}{G(\nu, x)} dx$$

for $m = 0, 1, 2, \ldots, 4n + 2$.

We can derive (2.9) easily for any ν. Indeed, (2.9) is the direct consequence of

$$\lim_{\mu \to \nu} (\mu, m) = (\nu, m)$$

for $m \geq 0$, which follows from (2.1). The proof of Lemma 2.6 is given in Section 3. We finish the proof in the case when $\nu \neq \nu(n)$.

We next consider the case when $\nu = \nu(n)$ for an integer $n \geq 0$. Since $G(\nu(n), \cdot)$ has the zero in the interval $(0, \infty)$, the calculation in this case is quite complicated in comparison with other cases. Similarly to Lemma 2.6 the following lemma plays an important role and its proof is also described in Section 3.

Lemma 2.7. *For any $n \geq 0$ we have that, for $m = 0, 1, 2, \dots, 4n+1$*

$$\lim_{\mu \to \nu(n) \pm 0} \cos(\pi\mu) \int_0^\infty \frac{x^m}{G(\mu, x)} dx = \pm x_{\nu(n)}^{m+1}.$$

Let $n \geq 0$ be a given integer. Recall that $N(\mu) = 2n+2$ if $\nu(n) < \mu < \nu(n+1)$. With the help of (2.9) and Lemma 2.7, we obtain by induction that $a_m^{(\mu)}$ converges to a constant $c_m^{(n)}$ as $\mu \downarrow \nu(n)$ for $m = 0, 1, 2, \dots, 2n+2$. Moreover, we deduce that $c_0^{(n)} = 1$ and

$$c_m^{(m)} = \frac{1}{m} \sum_{k=1}^{m} c_{m-k}^{(n)} \left\{ b_{k+1} - (-1)^k x_{\nu(n)}^k \right\}$$

for $m = 1, 2, \dots, 2n+2$, where we have written b_k instead of $b_k^{(\nu(n))}$. The following lemma, of which the proof is postponed to Section 4, gives the behavior of zeros of K_μ as $\mu \downarrow \nu(n)$.

Lemma 2.8. *We have that*

$$\sum_{k=0}^{2n+2} c_{2n+2-k}^{(n)} z^k = (z + x_n) \sum_{k=0}^{2n+1} \frac{(\nu(n), 2n+1-k)}{2^{2n+1-k}} z^k. \tag{2.10}$$

Lemma 2.4 implies that solutions of

$$\sum_{k=0}^{2n+1} \frac{(\nu(n), 2n+1-k)}{2^{2n+1-k}} z^k = 0$$

coincide with zeros of $K_{\nu(n)}$ and hence the roots of the polynomial in the right hand side of (2.10) are also zeros of $K_{\nu(n)}$. We should remark that $-x_{\nu(n)}$ is the double root and other roots are of multiplicity one. Lemma 2.5 yields that, as $\mu \downarrow \nu(n)$, two solutions of the equation

$$\sum_{k=0}^{N(\mu)} a_{N(\mu)-k}^{(\mu)} z^k = 0$$

converge to $-x_{\nu(n)}$ and other solutions to non-real roots of (2.10). Moreover, since $\overline{K_\mu(z)} = K_\mu(\overline{z})$ for $\mu \geq 0$ (cf. [8, pp. 66–67]), two zeros which tend to $-x_{\nu(n)}$ are mutually conjugate.

We observe the case when $\mu \uparrow \nu(n)$. Similarly, to the case when $\mu \downarrow \nu(n)$, we can show that each $a_m^{(\mu)}$ converges to a constant $d_m^{(n)}$ given by $d_0^{(n)} = 1$ and

$$d_m^{(m)} = \frac{1}{m} \sum_{k=1}^{m} d_{m-k}^{(n)} \left\{ b_{k+1} + (-1)^k x_{\nu(n)}^k \right\}$$

for $m = 1, 2, \dots, 2n$.

Lemma 2.9. *For an integer $n \geq 2$ we have that*

$$\sum_{k=0}^{2n+1} \frac{(\nu(n), 2n-k+1)}{2^{2n-k+1}} z^k = (z + x_n) \sum_{k=0}^{2n} d_{2n-k}^{(n)} z^k. \tag{2.11}$$

This lemma will be shown in Section 4. Lemma 2.4 gives that the roots of the left hand side in (2.11) are the zeros of $K_{\nu(n)}$. This means that the solutions of equation

$$\sum_{k=0}^{2n} d_{2n-k}^{(n)} z^k = 0$$

are the non-real zeros of $K_{\nu(n)}$. By virtue of Lemma 2.5 we have that each zero of K_μ converges to one of non-real zeros of $K_{\nu(n)}$ as $\mu \uparrow \nu(n)$. We finish the proofs of Theorems 1.1 and 1.2 in the case when $\nu = \nu(n)$.

The remainder of this section is devoted to the proof of Theorem 1.3. We start to establish the holomorphy of modified Bessel functions.

Lemma 2.10. *For $\zeta \in \mathbb{C}$ and $z \in \mathcal{D}$ let $I(\zeta, z) = I_\zeta(z)$ and $K(\zeta, z) = K_\zeta(z)$. We have that both I and K are holomorphic functions on $\mathbb{C} \times \mathcal{D}$.*

Proof. It is known that $I(\zeta, \cdot\,)$ and $K(\zeta, \cdot\,)$ are holomorphic on $\mathcal{D}$ for each $\zeta \in \mathbb{C}$ and that $I(\,\cdot\,, z)$ and $K(\,\cdot\,, z)$ are holomorphic on $\mathbb{C}$ for each $z \in \mathcal{D}$ (cf. [7, p. 109]). The Hartogs theorem of holomorphy yields that I and K are holomorphic functions on $\mathbb{C} \times \mathcal{D}$ (cf. [5, p. 28]). □

For $\nu > 3/2$ let $z_\nu \in \Xi_\nu$ be fixed. Then we have $K(\nu, z_\nu) = 0$. The uniqueness of the solution of the modified Bessel differential equation gives that $\partial K/\partial z(\nu, z_\nu) \neq 0$. Hence, the implicit function theorem yields that there exists a neighborhood $B_\nu \subset \mathbb{C}$ of ν and a holomorphic function f on B_ν such that $z_\nu = f(\nu)$ and $K(\zeta, f(\zeta)) = 0$ for any $\zeta \in B_\nu$ (cf. [5, p. 24]). Let $D_\nu = B_\nu \cap E_\nu$. By virtue of Theorem 1.1 we can uniquely choose z_μ satisfying with $K(\mu, z_\mu) = 0$ for any $\mu \in D_\nu$ and then have that $f(\mu) = z_\mu$ for $\mu \in D_\nu$. Since $g(\,\cdot\,; z_\nu)$ is the function f restricted on D_ν, we have that $g(\,\cdot\,; z_\nu)$ is analytic at ν. This completes the proof of Theorem 1.3.

3. CONVERGENCES OF INTEGRALS CONTAINING THE FUNCTION G

The purpose of this section is to establish Lemmas 2.6 and 2.7. In order to show Lemma 2.6, we need to observe the continuity of x_μ with respect to μ.

Lemma 3.1. *Let $\nu > 0$ be given. For any $\varepsilon > 0$ there is a constant $\eta_1 > 0$ such that $|x_\mu - x_\nu| < \varepsilon$ if $|\mu - \nu| < \eta_1$.*

Proof. Recall $H(\mu, x) = K_\mu(x) - \pi I_\mu(x)$. We deduce from Lemma 2.10 that H is an analytic function on $(0, \infty) \times (0, \infty)$. Note that $H(\mu, x_\mu) = 0$. The formulae

$$I_{\mu-1}(z) - I_{\mu+1}(z) = \frac{2\mu}{z} I_\mu(z), \quad I_{\mu-1}(z) + I_{\mu+1}(z) = 2I'_\mu(z), \tag{3.1}$$

$$K_{\mu-1}(z) - K_{\mu+1}(z) = -\frac{2\mu}{z} K_\mu(z), \quad K_{\mu-1}(z) + K_{\mu+1}(z) = -2K'_\mu(z) \tag{3.2}$$

(cf. [8, p. 67]) yield that

$$I'_\mu(z) = I_{\mu+1}(z) + \frac{\mu}{z} I_\mu(z), \quad K'_\mu(z) = -K_{\mu+1}(z) + \frac{\mu}{z} K_\mu(z). \tag{3.3}$$

Thus, it follows that

$$\frac{\partial H}{\partial x}(\nu, x_\nu) = K'_\nu(x_\nu) - \pi I'_\nu(x_\nu) = -\left\{K_{\nu+1}(x_\nu) + \pi I_{\nu+1}(x_\nu)\right\} < 0.$$

By the implicit function theorem, we can take the interval $\mathcal{J}$ which contains ν and the continuous function h defined on $\mathcal{J}$ such that $x_\nu = h(\nu)$ and $H(\mu, h(\mu)) = 0$. Lemma 2.1 yields that x_μ is the unique zero of $H(\mu, \cdot\,)$ for any $\mu > 0$ and thus we have $h(\mu) = x_\mu$ if $\mu \in \mathcal{J}$. □

We are ready to show Lemma 2.6. Let $\nu(n) < \nu < \nu(n+1)$ for an integer $n \geq 0$ and m be a non-negative integer which is less than 2ν. Lemma 2.10 gives that G and H are analytic on $(0, \infty) \times (0, \infty)$.

We have by Lemma 2.2 that $x^m / G(\mu, x) > 0$ for $x > 0$ and $\nu(n) < \mu < \nu(n+1)$. The Fatou lemma yields that

$$\int_0^\infty \frac{x^m}{G(\nu, x)} dx \leq \liminf_{\mu \to \nu} \int_0^\infty \frac{x^m}{G(\mu, x)} dx.$$

Let $\eta_2 = \min\{\nu - \nu(n), \nu(n+1) - \nu\}$. By Lemma 3.1 we can take $\delta_\nu \in (0, \eta_2/2)$ such that $|x_\mu - x_\nu| < x_\nu/2$ for any μ with $|\mu - \nu| \leq \delta_\nu$. We choose $\xi \in (0, x_\nu/2)$ and $\eta \in (3x_\nu/2, \infty)$ arbitrary. Note that $\xi < x_\nu/2 < x_\mu$. By Lemma 2.1 we obtain that, for $x \in (0, \xi)$ and $\mu \in \mathcal{J}_\nu$

$$H(\mu, x) \geq H(\mu, x_\nu/2) > 0,$$

where $\mathcal{J}_\nu = [\nu - \delta_\nu, \nu + \delta_\nu]$. Since $G(\mu, x) \geq H(\mu, x)^2$, we have

$$G(\mu, x) \geq H(\mu, x_\nu/2)^2,$$

which immediately yields

$$\limsup_{\mu \to \nu} \int_0^\xi \frac{x^m}{G(\mu, x)} dx \leq \frac{\xi^m}{H(\nu, x_\nu/2)^2}. \tag{3.4}$$

Since $x_\mu < 3x_\nu/2 < \eta$, it follows from Lemma 2.1 that, for $x > \eta$ and $\mu \in \mathcal{J}_\nu$

$$0 < \frac{K_\mu(x)}{\pi I_\mu(x)} < \frac{K_\mu(3x_\nu/2)}{\pi I_\mu(3x_\nu/2)} < 1,$$

which yields that

$$G(\mu, x) \geq \pi^2 I_\nu(x)^2 \left\{1 - \frac{K_\mu(x)}{\pi I_\mu(x)}\right\}^2 \geq \pi^2 I_\nu(x)^2 \left\{1 - \frac{K_\mu(3x_\nu/2)}{\pi I_\mu(3x_\nu/2)}\right\}^2.$$

By virtue of the inequality

$$I_\mu(x) \geq \frac{1}{\Gamma(\mu+1)}\left(\frac{x}{2}\right)^\mu,$$

we obtain that, for $x > \eta$ and $\mu \in \mathcal{J}_\nu$

$$\frac{1}{G(\mu,x)} \leq \frac{1}{\pi^2 I_\mu(x)^2}\frac{\pi^2 I_\mu(3x_\nu/2)^2}{H(\mu,3x_\nu/2)^2} \leq \left\{\frac{2\mu\pi\Gamma(\mu+1)I_\mu(3x_\nu/2)}{H(\mu,3x_\nu/2)}\right\}^2\frac{1}{x^{2\mu}}. \tag{3.5}$$

Hence, it follows that

$$\limsup_{\mu\to\nu}\int_\eta^\infty \frac{x^m}{G(\mu,x)}dx \leq \left\{\frac{2\nu\pi\Gamma(\nu+1)I_\nu(3x_\nu/2)}{(2\nu-m)H(\nu,3x_\nu/2)}\right\}^2\frac{1}{\eta^{2\nu-m-1}}. \tag{3.6}$$

Since G is continuous on $[\xi,\eta]\times\mathcal{J}_\nu$, we can choose $(\hat\mu,\hat x)\in[\xi,\eta]\times\mathcal{J}_\nu$ such that $G(\mu,x)\geq G(\hat\mu,\hat x)$ for any $(\mu,x)\in[\xi,\eta]\times\mathcal{J}_\nu$. Lemma 2.2 yields that $G(\hat\mu,\hat x)>0$ and then we have that

$$0 \leq \frac{x^m}{G(\mu,x)} \leq \frac{\eta^m}{G(\hat\mu,\hat x)}.$$

The bounded convergence theorem gives that

$$\lim_{\mu\to\nu}\int_\xi^\eta \frac{x^m}{G(\mu,x)}dx = \int_\xi^\eta \frac{x^m}{G(\nu,x)}dx. \tag{3.7}$$

By (3.4), (3.6) and (3.7), we can conclude that

$$\begin{aligned}\limsup_{\mu\to\nu}\int_0^\infty \frac{x^m}{G(\mu,x)}dx \leq& \int_\xi^\eta \frac{x^m}{G(\nu,x)}dx + \frac{\xi^m}{H(\nu,x_\nu/2)^2}\\ &+\left\{\frac{2\nu\pi\Gamma(\nu+1)I_\nu(3x_\nu/2)}{(2\nu-m)H(\nu,3x_\nu/2)}\right\}^2\frac{1}{\eta^{2\nu-m-1}}.\end{aligned}$$

Note that $2\nu-m>1$. By letting $\xi\to0$ and $\eta\to\infty$, we obtain that

$$\limsup_{\mu\to\nu}\int_0^\infty \frac{x^m}{G(\mu,x)}dx \leq \int_0^\infty \frac{x^m}{G(\nu,x)}dx.$$

This completes the proof of Lemma 2.6

We next show Lemma 2.7. The calculation is slightly different and complicated in comparison with Lemma 2.6. The following lemma gives a lower bound of the sequence $\{x_{\nu(n)}\}_{n=0}^\infty$.

Lemma 3.2. *We have that $x_{\nu(n)} \geq 1$ for any integer $n \geq 0$.*

Proof. Applying (3.1) and (3.2) for $\mu = \nu(n) + 1$, we have that

$$H(\nu(n), x) - H(\nu(n+1), x) = -\frac{2n+4}{x} \left\{ K_{2n+5/2}(x) + \pi I_{2n+5/2}(x) \right\} < 0$$

for any $x > 0$. This immediately gives that $H(\nu(n), x_{\nu(n+1)}) < 0$. We deduce from Lemma 2.1 that $x_{\nu(n)} < x_{\nu(n+1)}$. Lemma 2.4 gives that $x_{3/2} = 1$ and thus we obtain that $x_{\nu(n)} \geq 1$ for any $n \geq 0$. □

We need to observe the asymptotic behavior of G around $(\nu(n), x_{\nu(n)})$. For an integer $n \geq 0$ let

$$\begin{aligned}
\alpha &= \pi K_{\nu(n)}(x_{\nu(n)}), \\
\beta &= -\frac{\partial K}{\partial \mu}(\nu(n), x_{\nu(n)}) + \pi \frac{\partial I}{\partial \mu}(\nu(n), x_{\nu(n)}), \\
\gamma &= -\frac{\partial K}{\partial x}(\nu(n), x_{\nu(n)}) + \pi \frac{\partial I}{\partial x}(\nu(n), x_{\nu(n)}).
\end{aligned}$$

We remark that (3.3) gives that $\gamma > 0$.

Lemma 3.3. *For any $\varepsilon > 0$ there exists $\eta_3 > 0$ such that*

$$\left| G(\nu(n) + \rho, x_{\nu(n)} + y) - \alpha^2 \rho^2 - (\beta \rho + \gamma y)^2 \right| < \varepsilon(\rho^2 + y^2)$$

for any (ρ, y) with $\rho^2 + y^2 < \eta_3^2$.

Proof. Lemma 2.2 implies $G(\nu(n), x_{\nu(n)}) = 0$. In addition, a standard calculation shows that

$$\frac{\partial G}{\partial \mu}(\nu(n), x_{\nu(n)}) = \frac{\partial G}{\partial x}(\nu(n), x_{\nu(n)}) = 0$$

and that

$$\begin{aligned}
&\frac{\partial^2 G}{\partial \mu^2}(\nu(n), x_{\nu(n)}) = 2\alpha^2 + 2\beta^2, \quad \frac{\partial^2 G}{\partial x \partial \mu}(\nu(n), x_{\nu(n)}) = 2\beta\gamma, \\
&\frac{\partial^2 G}{\partial x^2}(\nu(n), x_{\nu(n)}) = 2\gamma^2.
\end{aligned}$$

We omit the detailed calculation. Here we remark that $K_{\nu(n)}(x_{\nu(n)}) = \pi I_{\nu(n)}(x_{\nu(n)})$, which is obtained by Lemma 2.1, has been applied in order to derive the first equality. The Taylor theorem yields the assertion of this lemma. □

Let

$$\varepsilon_0 = \frac{\alpha^2 \gamma^2}{2(\alpha^2 + \beta^2 + \gamma^2)} < 1$$

and we take $\varepsilon \in (0, \varepsilon_0)$ arbitrary. It is easy to see that we can choose $\eta_4 \in (0, 1)$ such that

$$x_{\nu(n)}^m - \varepsilon < (x_{\nu(n)} - \eta_4)^m < (x_{\nu(n)} + \eta_4)^m < x_{\nu(n)}^m + \varepsilon. \tag{3.8}$$

Lemma 3.2 gives that $x_{\nu(n)}^m - \varepsilon \geq 1 - \varepsilon_0 > 0$. We take η_3 given in Lemma 3.3. Let η be a fixed number satisfying

$$0 < \eta < \min\left\{\frac{\eta_3}{\sqrt{2}}, \eta_4\right\}.$$

Lemma 3.1 yields that there exists $\eta_5 > 0$ such that

$$\left|x_\mu - x_{\nu(n)}\right| < \eta \tag{3.9}$$

if $|\mu - \nu(n)| < \eta_5$. Let $|\rho| < \min\{\eta_5, 1/2\}$. Lemma 2.1 gives that

$$G(\nu(n)+\rho, x) \geq H(\nu(n)+\rho, x)^2 > 0$$

for $0 < x < x_{\nu(n)} - \eta$. Then we obtain that

$$0 \leq \pi|\rho| \int_0^{x_{\nu(n)}-\eta} \frac{x^m}{G(\nu(n)+\rho, x)} dx \leq \pi|\rho| x_{\nu(n)}^m \int_0^{x_{\nu(n)}-\eta} \frac{dx}{H(\nu(n)+\rho, x)^2} dx,$$

which is bounded by

$$\frac{\pi|\rho| x_{\nu(n)}^{m+1}}{H(\nu(n)+\rho, x_{\nu(n)} - \eta)^2}.$$

Since H is analytic, $H(\nu(n)+\rho, x_{\nu(n)} - \eta)$ converges to $H(\nu(n), x_{\nu(n)} - \eta)$ as $\rho \to 0$. Lemma 2.1 implies that $H(\nu(n), x_{\nu(n)} - \eta) > 0$ and hence

$$\lim_{\rho \to 0} \pi\rho \int_0^{x_{\nu(n)}-\eta} \frac{x^m}{G(\nu(n)+\rho, x)} dx = 0.$$

Similarly, to (3.5), it follows that, for $x > x_{\nu(n)} + \eta$

$$\frac{1}{G(\nu(n)+\rho, x)} \leq \left[\frac{2\{\nu(n)+\rho\}\pi\Gamma(\nu(n)+\rho+1) I_{\nu(n)+\rho}(x_{\nu(n)}+\eta)}{H(\nu(n)+\rho, x_{\nu(n)}+\eta)}\right]^2 \frac{1}{x^{2\nu(n)+2\rho-m}}.$$

Since $2\nu(n) + 2\rho - m \geq 2 + 2\rho$ for $m = 0, 1, 2, \ldots, 4n+1$, we have

$$\lim_{\rho \downarrow 0} \int_{x_{\nu(n)}+\eta}^{\infty} \frac{dx}{x^{2\nu(n)+2\rho-m}} = \frac{1}{4n-m+2} \frac{1}{(x_{\nu(n)}+\eta)^{4n-m+2}},$$

which yields that

$$\lim_{\rho \to 0} \pi\rho \int_{x_{\nu(n)+\eta}}^{\infty} \frac{x^m}{G(\nu(n)+\rho, x)} dx = 0.$$

For $0 < |\rho| < \eta$ we deduce from (3.8) and (3.9) that

$$\int_{x_{\nu(n)}-\eta}^{x_{\nu(n)}+\eta} \frac{x^m}{G(\nu(n)+\rho, x)}dx > (x_{\nu(n)}^m - \varepsilon)\int_{-\eta}^{\eta} \frac{dy}{G(\nu(n)+\rho, x_{\nu(n)}+y)}, \tag{3.10}$$

$$\int_{x_{\nu(n)}-\eta}^{x_{\nu(n)}+\eta} \frac{x^m}{G(\nu(n)+\rho, x)}dx < (x_{\nu(n)}^m + \varepsilon)\int_{-\eta}^{\eta} \frac{dy}{G(\nu(n)+\rho, x_{\nu(n)}+y)}. \tag{3.11}$$

We concentrate on considering the integral

$$\pi\rho\int_{-\eta}^{\eta} \frac{dy}{G(\nu(n)+\rho, x_{\nu(n)}+y)}, \tag{3.12}$$

which is equal to

$$\pi\rho\int_{-\eta}^{\eta} \frac{dy}{\alpha^2\rho^2+(\beta\rho+\gamma y)^2} + \pi\rho\int_{-\eta}^{\eta} L(\rho, y)dy. \tag{3.13}$$

Here we have used the following notation:

$$L(\rho, y) = \frac{1}{G(\nu(n)+\rho, x_{\nu(n)}+y)} - \frac{1}{\alpha^2\rho^2+(\beta\rho+\gamma y)^2}.$$

Note that $\rho/|\rho|$ is 1 if $\rho > 0$ and is -1 if $\rho < 0$. A simple calculation shows that the first term of (3.13) is equal to

$$\frac{\pi\rho}{\alpha\gamma|\rho|}\left\{\arctan\left(\frac{\beta\rho}{\alpha|\rho|} + \frac{\gamma\eta}{\alpha|\rho|}\right) - \arctan\left(\frac{\beta\rho}{\alpha|\rho|} - \frac{\gamma\eta}{\alpha|\rho|}\right)\right\}. \tag{3.14}$$

Since α, γ and η are all positive, (3.14) converges to $\pm\pi^2/\alpha\beta$ as $\rho \to \pm 0$. For the computation of the second term of (3.13), we need to derive a lower bound of G on a suitable neighborhood of $(\nu(n), x_{\nu(n)})$. Let $|y| < \eta$. Since $\rho^2 + y^2 < 2\eta^2 < \eta_3^2$, we can apply Lemma 3.3 and have that

$$G(\nu(n)+\rho, x_{\nu(n)}+y) \geq (\alpha^2+\beta^2)\rho^2 + 2\alpha\beta\rho y + \gamma^2 y^2 - \varepsilon(\rho^2+y^2). \tag{3.15}$$

Note that

$$0 < \varepsilon < \varepsilon_0 < \frac{1}{2}\gamma^2. \tag{3.16}$$

Then the right hand side of (3.15) is not larger than

$$(\gamma^2-\varepsilon_0)\left(y + \frac{\beta\gamma}{\gamma^2-\varepsilon_0}\rho\right)^2 + \frac{(\alpha^2+\beta^2-\varepsilon_0)(\gamma^2-\varepsilon_0)-\beta^2\gamma^2}{\gamma^2-\varepsilon_0}\rho^2. \tag{3.17}$$

It follows that the numerator of the second term of (3.17) is equal to

$$\alpha^2\gamma^2 - \varepsilon_0(\alpha^2+\beta^2+\gamma^2) = \frac{1}{2}\alpha^2\gamma^2.$$

Hence, we deduce from (3.16) that

$$G(\nu(n)+\rho, x_{\nu(n)}+y) \geq \frac{1}{2}\gamma^2(y+\kappa_1\rho)^2 + \frac{1}{2}\alpha^2\rho^2,$$

where $\kappa_1 = \beta\gamma/(\gamma^2-\varepsilon_0)$. In addition, we put $\kappa_2 = \beta/\gamma$ and $\kappa_3 = \alpha/\gamma$. The absolute value of the second term of (3.13) is dominated by

$$2\varepsilon\pi|\rho| \int_{-\eta}^{\eta} \frac{y^2+\rho^2}{\{\gamma^2(y+\kappa_1\rho)^2+\alpha^2\rho^2\}\{\gamma^2(y+\kappa_2\rho)^2+\alpha^2\rho^2\}} dy. \tag{3.18}$$

A change of variables from y to w given by $w = y/\rho$ gives that (3.18) in not larger than

$$\frac{2\varepsilon\pi}{\gamma^2}\int_{\mathbb{R}} \frac{w^2+1}{\{(w+\kappa_1)^2+\kappa_3^2\}\{(w+\kappa_2)^2+\kappa_3^2\}} dy,$$

which implies that

$$\lim_{\rho\to 0} \pi\rho \int_{-\eta}^{\eta} L(\rho, y) dy = 0.$$

Therefore, we conclude that (3.12) converges to $\pm\pi^2/\alpha\gamma$ as $\rho \to \pm 0$. It follows from (3.10) and (3.11) that

$$\lim_{\rho\to\pm 0} \pi\rho \int_{x_{\nu(n)}-\eta}^{x_{\nu(n)}+\eta} \frac{x^m}{G(\nu(n)+\rho, x)} dx = \pm\frac{\pi^2 x_{\nu(n)}^m}{\alpha\gamma}.$$

Since

$$\lim_{\rho\to 0} \frac{\cos\pi(\nu(n)+\rho)}{\pi\rho} = 1,$$

we consequently obtain that

$$\lim_{\mu\to\nu(n)\pm 0} \cos(\pi\mu) \int_0^\infty \frac{x^m}{G(\mu, x)} dx = \pm\frac{\pi^2 x_{\nu(n)}^m}{\alpha\gamma}.$$

If we succeed to prove that

$$x_{\nu(n)} = \frac{\pi^2}{\alpha\gamma}, \tag{3.19}$$

we obtain the assertion of Lemma 2.7. The definitions of α and γ shows that

$$\frac{\alpha\gamma}{\pi} = \pi\{I'_{\nu(n)}(x_{\nu(n)})K_{\nu(n)}(x_{\nu(n)}) - I_{\nu(n)}(x_{\nu(n)})K'_{\nu(n)}(x_{\nu(n)})\} \\ + K'_{\nu(n)}(x_{\nu(n)})\{\pi I_{\nu(n)}(x_{\nu(n)}) - K_{\nu(n)}(x_{\nu(n)})\}.$$

By Lemma 2.1 and the formula

$$I'_\mu(z)K_\mu(z) - I_\mu(z)K'_\mu(z) = \frac{1}{z}$$

(cf. [8, p. 68]) we have that $\alpha\gamma/\pi = \pi x_{\nu(n)}$, which implies (3.19). We finish the proof of Lemma 2.7.

4. TWO REPRESENTATIONS OF THE POLYNOMIAL FOR ROOTS OF $K_{\nu(n)}$

Our goal of this section is to prove Lemmas 2.8 and 2.9 concerned the equation for zeros of $K_{\nu(n)}$.

We first show Lemma 2.8. Recall that $c_0^{(n)} = 1$ and

$$c_m^{(n)} = \frac{1}{m}\sum_{k=1}^{m} c_{m-k}^{(n)}\left\{b_{k+1} - (-1)^k x_{\nu(n)}^k\right\} \tag{4.1}$$

for $m = 1, 2, \ldots, 2n+2$, where the sequence $\{b_m\}_{m=0}^\infty$ is defined by

$$\frac{(\nu(n)+1, m)}{2^m} = \sum_{k=0}^{m} \frac{(\nu(n), m-k)}{2^{m-k}} b_k. \tag{4.2}$$

This formula gives the explicit from of $c_m^{(n)}$.

Lemma 4.1. *We have that*

$$c_m^{(n)} = \frac{(\nu(n), m)}{2^m} + \frac{(\nu(n), m-1)}{2^{m-1}} x_{\nu(n)} \tag{4.3}$$

for $m = 1, 2, \ldots, 2n+2$.

We shall complete the proof of Lemma 2.8 before showing Lemma 4.1. Note that $(\nu(n), 2n+2) = 0$. Lemma 4.1 yields that

$$\sum_{k=0}^{2n+2} c_{2n-k+2}^{(n)} z^k = z^{2n+2} + \sum_{k=0}^{2n} \frac{(\nu(n), 2n-k+1)}{2^{2n-k+1}} z^{k+1} \\ + x_{\nu(n)} \sum_{k=0}^{2n+1} \frac{(\nu(n), 2n-k+1)}{2^{2n-k+1}} z^k,$$

which is equal to the right hand side of (2.10).

We prove (4.3) by induction. It is easy to show that

$$b_0 = 1, \quad b_1 = \nu(n) + \frac{1}{2}, \quad b_2 = \frac{1}{2}\left(\nu(n)^2 - \frac{1}{4}\right)$$

(cf. [4]). If $m = 1$, the right hand sides of (4.1) and (4.3) are

$$c_0^{(n)} \left\{ b_2 - (-1)x_{\nu(n)} \right\} = \frac{1}{2}\left(\nu(n)^2 - \frac{1}{4}\right) + x_{\nu(n)}$$

and

$$\frac{(\nu(n), 1)}{2} + (\nu(n), 0)x_{\nu(n)} = \frac{1}{2}\frac{4\nu(n)^2 - 1}{4} + x_{\nu(n)},$$

respectively. Thus, it follows (4.3) for $m = 1$.

Assume that (4.3) holds for all integers which are less than m. We have by (4.1) that $c_m^{(n)}$ is the sum of the followings:

$$\frac{1}{m}\sum_{k=1}^{m} \frac{(\nu(n), m-k)}{2^{m-k}} b_{k+1}, \tag{4.4}$$

$$\frac{x_{\nu(n)}}{m}\sum_{k=1}^{m-1} \frac{(\nu(n), m-k-1)}{2^{m-k-1}} b_{k+1}, \tag{4.5}$$

$$-\frac{1}{m}\sum_{k=1}^{m}(-1)^k \frac{(\nu(n), m-k)}{2^{m-k}} x_{\nu(n)}^k, \tag{4.6}$$

$$-\frac{1}{m}\sum_{k=1}^{m-1}(-1)^k \frac{(\nu(n), m-k-1)}{2^{m-k-1}} x_{\nu(n)}^{k+1}. \tag{4.7}$$

The formula (4.2) yields that (4.4) is

$$\begin{aligned}
&\frac{1}{m}\sum_{k=2}^{m+1} \frac{(\nu(n), m-k+1)}{2^{m-k+1}} b_k \\
&= \frac{1}{m}\left\{ \frac{(\nu(n), m+1)}{2^{m+1}} - \frac{(\nu(n), m)}{2^m} b_1 - \frac{(\nu(n), m+1)}{2^{m+1}} b_0 \right\},
\end{aligned}$$

which is equal to

$$\frac{(\nu(n)+1, m+1) - (2\nu(n)+1)(\nu(n), m) - (\nu(n), m+1)}{m 2^{m+1}}. \tag{4.8}$$

The formula (2.2) is useful for the calculation of (4.8). In the case when $m \leq 2n$, we have that the numerator of (4.8) is

$$\frac{\Gamma(\nu(n)+m+1/2)}{(m+1)!\Gamma(\nu(n)-m+1/2)}L_m,$$

where

$$L_m = \left(\nu(n)+m+\frac{3}{2}\right)\left(\nu(n)+m+\frac{1}{2}\right) - (2\nu(n)+1)(m+1)$$
$$- \left(\nu(n)+m+\frac{1}{2}\right)\left(\nu(n)-m-\frac{1}{2}\right).$$

Since $L_m = 2m(m+1)$, the numerator of (4.8) is equal to $2m(\nu(n), m)$, which yields that (4.8) and also (4.4) are $(\nu(n), m)/2^m$. Recall that $(\mu, k) = 0$ if $\mu - k - 1/2$ is a negative integer. If $m = 2n+1$, the numerator of (4.8) is equal to

$$\left(2n+\frac{5}{2}, 2n+2\right) - 4(n+1)\left(2n+\frac{3}{2}, 2n+1\right) = 2(2n+1)\left(2n+\frac{3}{2}, 2n+1\right).$$

Moreover, the numerator of (4.8) is equal to 0 if $m = 2n+2$. This implies that (4.8) and also (4.4) are $(\nu(n), m)/2^m$.

Similarly, to (4.4), we obtain by (4.2) that (4.5) is represented by

$$\frac{x_{\nu(n)}}{m}\left\{\frac{(\nu(n)+1, m)}{2^m} - \frac{(\nu(n), m-1)}{2^{m-1}}b_1 - \frac{(\nu(n), m)}{2^m}b_0\right\}.$$

In the case when $m \leq 2n+1$, since (4.7) is

$$\frac{1}{m}\sum_{k=2}^{m}(-1)^k\frac{(\nu(n), m-k)}{2^{m-k}}x_{\nu(n)}^k,$$

the sum of (4.6) and (4.7) is equal to $(\nu(n), m-1)x_{\nu(n)}/m2^{m-1}$. This yields that the sum of (4.5), (4.6) and (4.7) is equal to

$$\frac{x_{\nu(n)}}{m}\left\{\frac{(\nu(n)+1, m)}{2^m} - \frac{(\nu(n), m-1)}{2^{m-1}}\left(\nu(n)-\frac{1}{2}\right) - \frac{(\nu(n), m)}{2^m}\right\}. \tag{4.9}$$

We can easily obtain by (2.2) that (4.9) coincides with $(\nu(n), m-1)x_{\nu(n)}/2^{m-1}$ for $m \leq 2n+1$. in addition, we have the same relation for $m = 2n+2$. The calculations are left to the reader.

Hence, we accordingly have (4.3) and then finish the proof of Lemma 4.1.

We next show Lemma 2.9. Recall that $d_0^{(n)} = 1$ and

$$d_m^{(m)} = \frac{1}{m}\sum_{k=1}^{m} d_{m-k}^{(n)}\left\{b_{k+1} + (-1)^k x_{\nu(n)}^k\right\} \tag{4.10}$$

for $m = 1, 2, \ldots, 2n$. The explicit form of $d_m^{(n)}$ can be derived.

Lemma 4.2. *We have that*

$$d_m^{(n)} = \sum_{k=0}^{m} (-1)^k \frac{(\nu(n), m-k)}{2^{m-k}} x_{\nu(n)}^k \tag{4.11}$$

for $m = 0, 1, 2, \ldots, 2n$.

The proof of this lemma is postponed to the last of this section. Recall that $x_{\nu(0)} = 1$. Lemma 4.2 gives that the right hand side of (2.11) is equal to

$$\begin{aligned} &\sum_{k=1}^{2n+1} d_{2n-k+1}^{(n)} z^k + x_{\nu(n)} \sum_{k=0}^{2n} d_{2n-k}^{(n)} z^k \\ &= z^{2n+1} + \sum_{k=1}^{2n} \left\{ d_{2n-k+1}^{(n)} + d_{2n-k}^{(n)} x_{\nu(n)} \right\} z^k + d_{2n}^{(n)}. \end{aligned} \tag{4.12}$$

It follows from Lemma 4.2 that, for $m = 1, 2, \ldots, 2n$

$$d_m^{(n)} + d_{m-1}^{(n)} x_{\nu(n)} = \frac{(\nu(n), m)}{2^m}. \tag{4.13}$$

Since $-x_{\nu(n)}$ is the real zero of $K_{\nu(n)}$, we deduce from Lemma 2.4 that

$$\sum_{k=0}^{2n+1} (-1)^k \frac{(\nu(n), 2n-k+1)}{2^{2n-k+1}} x_{\nu(n)}^k = 0.$$

Combining this formula with Lemma 4.2, we obtain that

$$d_{2n}^{(n)} = \sum_{k=1}^{2n+1} (-1)^{k-1} \frac{(\nu(n), 2n-k+1)}{2^{2n-k+1}} x_{\nu(n)}^k = \frac{(\nu(n), 2n+1)}{2^{2n+1}}. \tag{4.14}$$

By virtue of (4.13) and (4.14), we can conclude that the right hand side of (4.12) coincides with the left hand side of (2.11). This completes the proof of Lemma 2.9.

The remainder of this section is devoted to proving Lemma 4.2 by induction. It is trivial that (4.11) holds for $m = 0$. Assume that (4.11) holds for all integers which is less than m. It follows from (4.10) that $d_m^{(n)}$ is the sum of the following two summations:

$$\frac{1}{m}\sum_{h=0}^{m-1}\sum_{k=1}^{m-h}(-1)^h\frac{(\nu(n), m-k-h)}{2^{m-k-h}}x_{\nu(n)}^h b_{k+1}, \tag{4.15}$$

$$\frac{1}{m}\sum_{h=0}^{m-1}\sum_{k=1}^{m-h}(-1)^{h+k}\frac{(\nu(n), m-k-h)}{2^{m-k-h}}x_{\nu(n)}^{h+k}. \tag{4.16}$$

The same calculation as (4.4) gives that

$$\frac{1}{m-h}\sum_{k=1}^{m-h}\frac{(\nu(n), m-h-k)}{2^{m-h-k}}b_{k+1} = \frac{(\nu(n)+1, m-h)}{2^{m-h}}$$

and hence we obtain by induction assumption that (4.15) is

$$\sum_{h=0}^{m}(-1)^h\frac{(\nu(n), m-h)}{2^{m-h}}x_{\nu(n)}^h - \frac{1}{m}\sum_{h=0}^{m}(-1)^h h\frac{(\nu(n), m-h)}{2^{m-h}}x_{\nu(n)}^h. \tag{4.17}$$

Moreover, it is easy to show that (4.16) is equal to

$$\frac{1}{m}\sum_{h=0}^{m-1}\sum_{k=h+1}^{m}(-1)^k\frac{(\nu(n), m-k)}{2^{m-k}}x_{\nu(n)}^k = \frac{1}{m}\sum_{k=1}^{m}(-1)^k k\frac{(\nu(n), m-k)}{2^{m-k}}x_{\nu(n)}^k,$$

which is the same as the second term of (4.17). Hence, we have (4.11) and this means the end of the proof of Lemma 4.2.

5. TABLE OF THE ZEROS OF MACDONALD FUNCTIONS

Finally, we give a table of the approximate values of the zeros of K_ν, which are numerically computed by "*Mathematica*" (see Table 1). As is mentioned above, the algebraic equations for the reciprocals of the zeros are also given in [4]. The numerical results for the zeros of the two algebraic equations coincide and it gives a good check for our results. Moreover, we find in [10] a table of the zeros of $K_\nu(z)$ when the index ν is an integer and see a perfect agreement.

Table 1. Table of zeros of K_ν

ν	zeros$_1$	zeros$_2$	zeros$_3$	zeros$_4$	zeros$_5$
1.5	−1				
1.6	−1.06356 ± 0.0852232i				
1.7	−1.12292 ± 0.170806i				
1.8	−1.1787 ± 0.256725i				
1.9	−1.23139 ± 0.342957i				
2.0	−1.28137 ± 0.429485i				
2.1	−1.32896 ± 0.516291i				
2.2	−1.37442 ± 0.603361i				
2.3	−1.41795 ± 0.690682i				
2.4	−1.45976 ± 0.77824i				
2.5	−1.5 ± 0.866025i				
2.6	−1.5388 ± 0.954027i				
2.7	−1.57628 ± 1.04223i				
2.8	−1.61255 ± 1.13064i				
2.9	−1.64769 ± 1.21924i				
3.0	−1.68179 ± 1.30801i				
3.1	−1.71492 ± 1.39696i				
3.2	−1.74714 ± 1.48608i				
3.3	−1.77851 ± 1.57536i				
3.4	−1.80908 ± 1.6648i				
3.5	−1.83891 ± 1.75438i	−2.32219			
3.6	−1.86802 ± 1.84411i	−2.3873 ± 0.0864217i			
3.7	−1.89647 ± 1.93398i	−2.45036 ± 0.172895i			
3.8	−1.92429 ± 2.02398i	−2.51152 ± 0.259427i			
3.9	−1.95151 ± 2.11411i	−2.57092 ± 0.346026i			
4.0	−1.97816 ± 2.20437i	−2.62867 ± 0.432697i			
4.1	−2.00427 ± 2.29475i	−2.68489 ± 0.519443i			
4.2	−2.02987 ± 2.38525i	−2.73967 ± 0.606267i			
4.3	−2.05497 ± 2.47586i	−2.79309 ± 0.693173i			
4.4	−2.0796 ± 2.56659i	−2.84525 ± 0.780161i			
4.5	−2.10379 ± 2.65742i	−2.89621 ± 0.867234i			
4.6	−2.12755 ± 2.74835i	−2.94604 ± 0.954392i			
4.7	−2.15089 ± 2.83939i	−2.99479 ± 1.04164i			
4.8	−2.17384 ± 2.93053i	−3.04252 ± 1.12897i			
4.9	−2.19642 ± 3.02176i	−3.08929 ± 1.21638i			
5.0	−2.21863 ± 3.11308i	−3.13513 ± 1.30388i			
5.1	−2.24049 ± 3.2045i	−3.1801 ± 1.39147i			
5.2	−2.26201 ± 3.296i	−3.22423 ± 1.47914i			
5.3	−2.28321 ± 3.38759i	−3.26756 ± 1.5669i			
5.4	−2.30409 ± 3.47927i	−3.31013 ± 1.65474i			
5.5	−2.32467 ± 3.57102i	−3.35196 ± 1.74266i	−3.64674		
5.6	−2.34497 ± 3.66286i	−3.39308 ± 1.83067i	−3.71228 ± 0.0866256i		
5.7	−2.36498 ± 3.75477i	−3.43354 ± 1.91875i	−3.77647 ± 0.173268i		
5.8	−2.38471 ± 3.84676i	−3.47334 ± 2.00692i	−3.83937 ± 0.259933i		
5.9	−2.40419 ± 3.93883i	−3.51252 ± 2.09517i	−3.90105 ± 0.346624i		
6.0	−2.4234 ± 4.03096i	−3.5511 ± 2.1835i	−3.96156 ± 0.433345i		
6.1	−2.44238 ± 4.12317i	−3.5891 ± 2.2719i	−4.02096 ± 0.520101i		
6.2	−2.46111 ± 4.21544i	−3.62654 ± 2.36038i	−4.07929 ± 0.606892i		
6.3	−2.47961 ± 4.30779i	−3.66344 ± 2.44894i	−4.13661 ± 0.693723i		
6.4	−2.49788 ± 4.4002i	−3.69983 ± 2.53757i	−4.19295 ± 0.780595i		
6.5	−2.51593 ± 4.49267i	−3.73571 ± 2.62627i	−4.24836 ± 0.86751i		
6.6	−2.53377 ± 4.58521i	−3.7711 ± 2.71505i	−4.30287 ± 0.954469i		
6.7	−2.55141 ± 4.67781i	−3.80603 ± 2.8039i	−4.35652 ± 1.04147i		
6.8	−2.56884 ± 4.77047i	−3.8405 ± 2.89282i	−4.40935 ± 1.12853i		
6.9	−2.58608 ± 4.86319i	−3.87453 ± 2.98181i	−4.46137 ± 1.21563i		
7.0	−2.60313 ± 4.95597i	−3.90813 ± 3.07087i	−4.51263 ± 1.30278i		
7.1	−2.61999 ± 5.0488i	−3.94131 ± 3.16i	−4.56314 ± 1.38998i		
7.2	−2.63667 ± 5.1417i	−3.97409 ± 3.24919i	−4.61294 ± 1.47723i		
7.3	−2.65318 ± 5.23464i	−4.00649 ± 3.33846i	−4.66205 ± 1.56453i		
7.4	−2.66951 ± 5.32764i	−4.0385 ± 3.42778i	−4.71049 ± 1.65188i		
7.5	−2.68568 ± 5.42069i	−4.07014 ± 3.51717i	−4.75829 ± 1.73929i	−4.97179	
7.6	−2.70168 ± 5.5138i	−4.10142 ± 3.60663i	−4.80546 ± 1.82674i	−5.03753 ± 0.0866936i	
7.7	−2.71753 ± 5.60695i	−4.13236 ± 3.69615i	−4.85203 ± 1.91425i	−5.10226 ± 0.173395i	
7.8	−2.73322 ± 5.70016i	−4.16295 ± 3.78573i	−4.89801 ± 2.00181i	−5.16602 ± 0.260108i	
7.9	−2.74875 ± 5.79341i	−4.19321 ± 3.87537i	−4.94342 ± 2.08942i	−5.22884 ± 0.346835i	
8.0	−2.76414 ± 5.88671i	−4.22315 ± 3.96507i	−4.98828 ± 2.17708i	−5.29076 ± 0.433578i	
8.1	−2.77939 ± 5.98006i	−4.25278 ± 4.05483i	−5.0326 ± 2.2648i	−5.35181 ± 0.52034i	
8.2	−2.79449 ± 6.07346i	−4.2821 ± 4.14464i	−5.07641 ± 2.35256i	−5.41202 ± 0.607123i	
8.3	−2.80946 ± 6.1669i	−4.31112 ± 4.23452i	−5.1197 ± 2.44038i	−5.47142 ± 0.693928i	
8.4	−2.82429 ± 6.26038i	−4.33985 ± 4.32445i	−5.16251 ± 2.52825i	−5.53003 ± 0.780758i	
8.5	−2.83898 ± 6.35391i	−4.36829 ± 4.41444i	−5.20484 ± 2.61618i	−5.58789 ± 0.867614i	
8.6	−2.85355 ± 6.44748i	−4.39646 ± 4.50449i	−5.2467 ± 2.70415i	−5.645 ± 0.954498i	
8.7	−2.86799 ± 6.5411i	−4.42435 ± 4.59459i	−5.28812 ± 2.79217i	−5.70141 ± 1.04141i	
8.8	−2.88231 ± 6.63475i	−4.45198 ± 4.68474i	−5.32909 ± 2.88025i	−5.75712 ± 1.12835i	
8.9	−2.89651 ± 6.72845i	−4.47935 ± 4.77495i	−5.36963 ± 2.96837i	−5.81216 ± 1.21532i	
9.0	−2.91058 ± 6.82219i	−4.50647 ± 4.86521i	−5.40975 ± 3.05654i	−5.86655 ± 1.30233i	
9.1	−2.92454 ± 6.91597i	−4.53334 ± 4.95552i	−5.44946 ± 3.14477i	−5.92031 ± 1.38936i	
9.2	−2.93839 ± 7.00978i	−4.55996 ± 5.04588i	−5.48877 ± 3.23304i	−5.97345 ± 1.47643i	
9.3	−2.95212 ± 7.10364i	−4.58635 ± 5.1363i	−5.5277 ± 3.32137i	−6.026 ± 1.56354i	
9.4	−2.96574 ± 7.19753i	−4.61251 ± 5.22676i	−5.56625 ± 3.40974i	−6.07797 ± 1.65068i	
9.5	−2.97926 ± 7.29146i	−4.63844 ± 5.31727i	−5.60442 ± 3.49816i	−6.12937 ± 1.73785i	−6.29702

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