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OSCILLATIONS OF EQUATIONS CAUSED BY SEVERAL DEVIATING ARGUMENTS

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Abstract. Linear delay or advanced differential equations with variable coefficients and several not necessarily monotone arguments are considered, and some new oscillation criteria are given. More precisely, sufficient conditions, involving lim sup and lim inf, are obtained, which essentially improve several known criteria existing in the literature. Examples illustrating the results are also given, numerically solved in MATLAB.

Keywords: differential equation, non-monotone argument, oscillatory solution, nonoscillatory solution.

Mathematics Subject Classification: 34K11, 34K06.

1. INTRODUCTION

Consider the first-order linear differential equation with several variable deviating arguments of either delay (DDE)

$$x'(t) + \sum_{i=1}^{m} p_i(t)x\left(\tau_i(t)\right) = 0, \quad t \geq t_0, \tag{E}$$

or advanced type (ADE)

$$x'(t) - \sum_{i=1}^{m} q_i(t)x\left(\sigma_i(t)\right) = 0, \quad t \geq t_0, \tag{E$'$}$$

where p_i, q_i, $1 \leq i \leq m$, are functions of nonnegative real numbers, and τ_i, σ_i, $1 \leq i \leq m$, are functions of positive real numbers such that

$$\tau_i(t) < t, \quad t \geq t_0 \quad \text{and} \quad \lim_{t\to\infty} \tau_i(t) = \infty, \quad 1 \leq i \leq m \tag{1.1}$$

and

$$\sigma_i(t) > t, \quad t \geq t_0, \quad 1 \leq i \leq m, \tag{1.2}$$

respectively.

A *solution* of (E) is an absolutely continuous on $[t_0, \infty)$ function satisfying (E) for almost all $t \geq t_0$. By a solution of (E′) we mean an absolutely continuous on $[t_0, \infty)$ function satisfying (E′) for almost all $t \geq t_0$.

A solution of (E) or (E′) is *oscillatory*, if it is neither eventually positive nor eventually negative. If there exists an eventually positive or an eventually negative solution, the equation is *nonoscillatory*. An equation is *oscillatory* if all its solutions oscillate.

The problem of establishing sufficient conditions for the oscillation of all solutions of equations (E) or (E′) has been the subject of many investigations. The reader is referred to [1−26] and the references cited therein. Most of these papers concern the special case where the arguments are nondecreasing, while a small number of these papers are dealing with the general case where the arguments are not necessarily monotone. See, for example, [1−6, 10, 15] and the references cited therein. The consideration of non-monotone arguments other than the pure mathematical interest, it approximates the natural phenomena described by equation of the type (E) or (E′). That is because there are always natural disturbances (e.g. noise in communication systems) that affect all the parameters of the equation and therefore the fair (from a mathematical point of view) monotone arguments become non-monotone almost always.

Throughout this paper, we are going to use the following notation:

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)ds, \quad \beta := \liminf_{t\to\infty} \int_{t}^{\sigma(t)} \sum_{i=1}^{m} q_i(s)ds$$

and

$$D(\omega) := \begin{cases} 0, & \text{if } \omega > 1/e, \\ \frac{1-\omega-\sqrt{1-2\omega-\omega^2}}{2}, & \text{if } \omega \in [0, 1/e]\,. \end{cases}$$

1.1. DDES

By Remark 2.7.3 in [21], it is clear that if $\tau_i(t)$, $1 \leq i \leq m$ are nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)ds > 1, \tag{1.3}$$

where $\tau(t) = \max_{1\leq i\leq m}\{\tau_i(t)\}$, then all solutions of (E) oscillate. This result is similar to Theorem 2.1.3 [21] which is a special case of Ladas, Lakshmikantham and Papadakis's result [18].

In 1978 Ladde [20] and in 1982 Ladas and Stavroulakis [19] proved that if

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{e}, \tag{1.4}$$

then all solutions of (E) oscillate.

In 1984, Hunt and Yorke [11] proved that if $t - \tau_i(t) \le \tau_0$, $1 \le i \le m$, and

$$\liminf_{t\to\infty} \sum_{i=1}^{m} p_i(t)\,(t - \tau_i(t)) > \frac{1}{e}, \tag{1.5}$$

then all solutions of (E) oscillate.

Assume that $\tau_i(t)$, $1 \le i \le m$ are not necessarily monotone. Set

$$h_i(t) = \sup_{t_0\le s\le t} \tau_i(s), \quad t \ge t_0 \qquad \text{and} \qquad h(t) = \max_{1\le i\le m} h_i(t), \quad t \ge t_0 \tag{1.6}$$

and

$$\begin{aligned} a_1(t,s) &:= \exp\Big\{\int_s^t \sum_{i=1}^{m} p_i(\zeta)d\zeta\Big\},\\ a_{r+1}(t,s) &:= \exp\Big\{\int_s^t \sum_{i=1}^{m} p_i(\zeta)a_r(\zeta,\tau_i(\zeta))d\zeta\Big\}. \end{aligned} \tag{1.7}$$

Clearly, $h_i(t)$, $h(t)$ are nondecreasing and $\tau_i(t) \le h_i(t) \le h(t) < t$ for all $t \ge t_0$.

In 2016, Braverman, Chatzarakis and Stavroulakis [1] proved that if for some $r \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)a_r(h(t),\tau_i(\zeta))d\zeta > 1, \tag{1.8}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)a_r(h(t),\tau_i(\zeta))d\zeta > 1 - D(\alpha), \tag{1.9}$$

or

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)a_r(h(t),\tau_i(\zeta))d\zeta > \frac{1}{e}, \tag{1.10}$$

then all solutions of (E) oscillate.

In 2017, Chatzarakis and Péics [3] proved that if

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)a_r(h(\zeta),\tau_i(\zeta))d\zeta > \frac{1+\ln\lambda_0}{\lambda_0} - D(\alpha), \tag{1.11}$$

where λ_0 is the smaller root of the transcendental equation $e^{\alpha\lambda} = \lambda$, then all solutions of (E) oscillate.

In the same year, Chatzarakis [4] proved that if

$$P_j(t) = P(t)\left[1 + \int_{\tau(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{t} P(u)\exp\left(\int_{\tau(u)}^{u} P_{j-1}(\xi)d\xi\right)du\right)ds\right],$$

with $P_0(t) = P(t) = \sum_{i=1}^{m} p_i(t)$, then for some $j \in \mathbb{N}$ either one of the conditions

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du\right)ds > 1, \tag{1.12}$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du\right)ds > 1 - D(\alpha), \tag{1.13}$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right)ds > \frac{1+\ln\lambda_0}{\lambda_0} - D(\alpha), \tag{1.14}$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{t} P_j(u)du\right)ds > \frac{1}{D(\alpha)} \tag{1.15}$$

and

$$\liminf_{t\to\infty} \int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right)ds > \frac{1}{e}, \tag{1.16}$$

implies that all solutions of (E) are oscillatory.

Several improvements were made to the above condition, see [5, 6] to arrive at the recent form [5]

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P(u)\exp\left(\int_{\tau(u)}^{u} G_\ell(\xi)d\xi\right)du\right)ds > 1, \tag{1.17}$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P(u)\exp\left(\int_{\tau(u)}^{u} G_\ell(\xi)d\xi\right)du\right)ds > 1 - D(\alpha), \tag{1.18}$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{t} P(u)\exp\left(\int_{\tau(u)}^{u} G_\ell(\xi)d\xi\right)du\right)ds > \frac{1}{D(\alpha)} - 1, \tag{1.19}$$

$$\limsup_{t\to\infty}\int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} G_\ell(\xi)d\xi\right)du\right)ds > \frac{1+\ln\lambda_0}{\lambda_0} - D(\alpha), \tag{1.20}$$

$$\liminf_{t\to\infty}\int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} G_\ell(\xi)d\xi\right)du\right)ds > \frac{1}{e}, \tag{1.21}$$

where

$$G_\ell(t) = P(t)\left[1+\int_{\tau(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{t} P(u)\exp\left(\int_{\tau(u)}^{u} G_{\ell-1}(\xi)d\xi\right)du\right)ds\right]$$

and

$$G_0(t) = P(t)\left[1+\int_{\tau(t)}^{t} P(s)\exp\left(\lambda_0\int_{\tau(s)}^{t} P(u)du\right)ds\right].$$

1.2. ADES

For Eq. (E′), the dual condition of (1.3) is

$$\limsup_{t\to\infty}\int_{t}^{\sigma(t)}\sum_{i=1}^{m} q_i(s)ds > 1, \tag{1.22}$$

where $\sigma_i(t)$, $1\le i\le m$ are nondecreasing and $\sigma(t)=\min_{1\le i\le m}\{\sigma_i(t)\}$ (see [21, § 2.7]).

In 1978, Ladde [21] and in 1982 Ladas and Stavroulakis [19] proved that if

$$\liminf_{t\to\infty}\int_{t}^{\sigma(t)}\sum_{i=1}^{m} q_i(s)ds > \frac{1}{e}, \tag{1.23}$$

then all solutions of (E′) oscillate.

In 1990, Zhou [26] proved that if $\sigma_i(t)-t\le\sigma_0$, $1\le i\le m$, and

$$\liminf_{t\to\infty}\sum_{i=1}^{m} q_i(t)\,(\sigma_i(t)-t) > \frac{1}{e}, \tag{1.24}$$

then all solutions of (E′) oscillate (see also [7, Corollary 2.6.12]).

Assume that $\sigma_i(t)$, $1\le i\le m$ are not necessarily monotone. Set

$$\rho_i(t)=\inf_{s\ge t}\sigma_i(s),\quad t\ge t_0 \qquad\text{and}\qquad \rho(t)=\min_{1\le i\le m}\rho_i(t),\quad t\ge t_0 \tag{1.25}$$

and

$$\begin{aligned} b_1(t,s) &:= \exp\left\{\int_t^s \sum_{i=1}^m q_i(\zeta)d\zeta\right\}, \\ b_{r+1}(t,s) &:= \exp\left\{\int_t^s \sum_{i=1}^m q_i(\zeta)b_r(\zeta,\sigma_i(\zeta))d\zeta\right\}. \end{aligned} \tag{1.26}$$

Clearly, $\rho_i(t)$, $\rho(t)$ are nondecreasing and $\sigma_i(t) \geq \rho_i(t) \geq \rho(t) > t$ for all $t \geq t_0$.

In 2016, Braverman, Chatzarakis and Stavroulakis [1] proved that if for some $r \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^m q_i(\zeta)b_r(\rho(t),\sigma_i(\zeta))\,d\zeta > 1, \tag{1.27}$$

or

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^m q_i(\zeta)b_r(\rho(t),\sigma_i(\zeta))\,d\zeta > 1 - D(\beta), \tag{1.28}$$

or

$$\liminf_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^m q_i(\zeta)b_r(\rho(t),\sigma_i(\zeta))\,d\zeta > \frac{1}{e}, \tag{1.29}$$

then all solutions of (E′) oscillate.

In 2017, Chatzarakis [4] proved that if

$$Q_j(t) = Q(t)\left[1 + \int_t^{\sigma(t)} Q(s)\exp\left(\int_t^{\sigma(s)} Q(u)\exp\left(\int_u^{\sigma(u)} Q_{j-1}(\xi)d\xi\right)du\right)ds\right],$$

with $Q_0(t) = Q(t) = \sum_{i=1}^m q_i(t)$, then, for some $j \in \mathbb{N}$ either one of the conditions

$$\limsup_{t\to\infty} \int_t^{\rho(t)} Q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} Q_j(u)du\right)ds > 1, \tag{1.30}$$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} Q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} Q_j(u)du\right)ds > 1 - D(\beta), \tag{1.31}$$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} Q(s)\exp\left(\int_t^{\sigma(s)} Q_j(u)du\right)ds > \frac{1}{D(\beta)}, \tag{1.32}$$

$$\limsup_{t\to\infty}\int_t^{\rho(t)} Q(s)\exp\left(\int_{\rho(s)}^{\sigma(s)} Q_j(u)du\right)ds > \frac{1+\ln\lambda_0}{\lambda_0} - D(\beta) \tag{1.33}$$

and

$$\liminf_{t\to\infty}\int_t^{\rho(t)} Q(s)\exp\left(\int_{\rho(s)}^{\sigma(s)} Q_j(u)du\right)ds > \frac{1}{e}, \tag{1.34}$$

implies that all solutions of (E′) are oscillatory.

Several improvements were made to the above condition, see [5, 6] to arrive at the recent form [5]

$$\limsup_{t\to\infty}\int_t^{\rho(t)} Q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} Q(u)\exp\left(\int_u^{\sigma(u)} R_\ell(\xi)d\xi\right)du\right)ds > 1, \tag{1.35}$$

$$\limsup_{t\to\infty}\int_t^{\rho(t)} Q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} Q(u)\exp\left(\int_u^{\sigma(u)} R_\ell(\xi)d\xi\right)du\right)ds > 1 - D(\beta), \tag{1.36}$$

$$\limsup_{t\to\infty}\int_t^{\rho(t)} Q(s)\exp\left(\int_{t}^{\sigma(s)} Q(u)\exp\left(\int_u^{\sigma(u)} R_\ell(\xi)d\xi\right)du\right)ds > \frac{1}{D(\beta)} - 1, \tag{1.37}$$

$$\limsup_{t\to\infty}\int_t^{\rho(t)} Q(s)\exp\left(\int_{\rho(s)}^{\sigma(s)} Q(u)\exp\left(\int_u^{\sigma(u)} R_\ell(\xi)d\xi\right)du\right)ds > \frac{1+\ln\lambda_0}{\lambda_0} - D(\beta), \tag{1.38}$$

$$\liminf_{t\to\infty}\int_t^{\rho(t)} Q(s)\exp\left(\int_{\rho(s)}^{\sigma(s)} Q(u)\exp\left(\int_u^{\sigma(u)} R_\ell(\xi)d\xi\right)du\right)ds > \frac{1}{e}, \tag{1.39}$$

where

$$R_\ell(t) = Q(t)\left[1+\int_t^{\sigma(t)} P(s)\exp\left(\int_t^{\sigma(s)} Q(u)\exp\left(\int_u^{\sigma(u)} R_{\ell-1}(\xi)d\xi\right)du\right)ds\right]$$

and

$$R_0(t) = Q(t)\left[1+\int_t^{\sigma(t)} Q(s)\exp\left(\lambda_0\int_t^{\sigma(s)} Q(u)du\right)ds\right].$$

The purpose of this paper is to derive sufficient conditions for all solutions of (E) and (E′) to be oscillatory when the arguments are not necessarily monotone. Our results essentially improve several known criteria existing in the literature.

2. MAIN RESULTS

2.1. DDES

Based on an iterative technique, we further study (E) and derive new sufficient oscillation conditions, involving lim sup and lim inf, which essentially improve several results in the literature.

We now cite three lemmas which will be used in the proof of our next results. The proofs of their are similar to the proofs of Lemmas 2.1.1, 2.1.3 and 2.1.2 in [7], respectively.

Lemma 2.1. *Assume that $h(t)$ is defined by* (1.6)*. Then*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)ds = \liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s)ds. \tag{2.1}$$

Lemma 2.2. *Assume that x is an eventually positive solution of* (E) *and $h(t)$ is defined by* (1.6)*. Then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \geq D(\alpha). \tag{2.2}$$

Lemma 2.3. *Assume that x is an eventually positive solution of* (E) *and $h(t)$ is defined by* (1.6)*. Then*

$$\liminf_{t\to\infty} \frac{x(h(t))}{x(t)} \geq \lambda_0, \tag{2.3}$$

where λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$.

Based on the above lemmas, we establish the following theorems.

Theorem 2.4. *Assume that $h(t)$ is defined by* (1.6) *and for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi)d\xi \right) du \right) ds > 1, \tag{2.4}$$

where

$$W_\ell(t) = P(t)\left[1 + \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_{\ell-1}(\xi)d\xi \right) du \right) ds \right] \tag{2.5}$$

with

$$W_0(t) = P(t)\left[1 + \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(\omega) \exp\left(\lambda_0 \int_{\tau(\omega)}^{\omega} P(u)du \right) d\omega \right) ds \right]$$

and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$. Then all solutions of (E) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (E). Since $-x(t)$ is also a solution of (E), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists a $t_1 > t_0$ such that $x(t)$ and $x(\tau_i(t)) > 0$ for all $t \geq t_1$. Thus, from (E) we have

$$x'(t) = -\sum_{i=1}^{m} p_i(t) x(\tau_i(t)) \leq 0 \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ is an eventually nonincreasing function of positive numbers.

Now we divide (E) by $x(t) > 0$ and integrate on $[\tau(t), t]$, so

$$\int_{\tau(t)}^{t} \frac{x'(u)}{x(u)} du = -\int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(u) \frac{x(\tau_i(u))}{x(u)} du,$$

or

$$\int_{\tau(t)}^{t} \frac{x'(u)}{x(u)} du \leq -\int_{\tau(t)}^{t} \left(\sum_{i=1}^{m} p_i(u) \right) \frac{x(\tau(u))}{x(u)} du,$$

or

$$\int_{\tau(t)}^{t} \frac{x'(u)}{x(u)} du \leq -\int_{\tau(t)}^{t} P(u) \frac{x(\tau(u))}{x(u)} du.$$

Therefore

$$\ln \frac{x(t)}{x(\tau(t))} \leq -\int_{\tau(t)}^{t} P(u) \frac{x(\tau(u))}{x(u)} du,$$

i.e.,

$$x(\tau(t)) \geq x(t) \exp\left(\int_{\tau(t)}^{t} P(u) \frac{x(\tau(u))}{x(u)} du \right). \tag{2.6}$$

Combining (E) and (2.6), we have

$$\begin{aligned} 0 &= x'(t) + \sum_{i=1}^{m} p_i(t) x(\tau_i(t)) \geq x'(t) + P(t) x(\tau(t)) \\ &\geq x'(t) + P(t) x(t) \exp\left(\int_{\tau(t)}^{t} P(u) \frac{x(\tau(u))}{x(u)} du \right) \end{aligned} \tag{2.7}$$

Now we divide (2.7) by $x(t) > 0$ and integrate on $[\tau(s), t]$, so

$$\int_{\tau(s)}^{t} \frac{x'(\omega)}{x(\omega)} d\omega \leq -\int_{\tau(s)}^{t} P(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} P(u) \frac{x(\tau(u))}{x(u)} du \right) d\omega,$$

or

$$x(\tau(s)) \geq x(t) \exp\left(\int_{\tau(s)}^{t} P(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} P(u) \frac{x\left(\tau(u)\right)}{x(u)} du\right) d\omega\right). \tag{2.8}$$

Integrating (E) from $\tau(t)$ to t, we have

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s) x\left(\tau_i(s)\right) ds = 0$$

or

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^{t} P(s) x\left(\tau(s)\right) ds \leq 0. \tag{2.9}$$

Combining (2.8) and (2.9), we have

$$x(t) - x(\tau(t)) + x(t) \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} P(u) \frac{x\left(\tau(u)\right)}{x(u)} du\right) d\omega\right) ds \leq 0.$$

Multiplying the last inequality by $P(t)$, we take

$$\begin{aligned} &P(t)x(t) - P(t)x(\tau(t)) \\ &+ P(t)x(t) \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} P(u) \frac{x\left(\tau(u)\right)}{x(u)} du\right) d\omega\right) ds \leq 0. \end{aligned} \tag{2.10}$$

Furthermore,

$$x'(t) = -\sum_{i=1}^{m} p_i(t) x\left(\tau_i(t)\right) \leq -x\left(\tau(t)\right) \sum_{i=1}^{m} p_i(t) = -P(t) x\left(\tau(t)\right). \tag{2.11}$$

Combining the inequalities (2.10) and (2.11), we have

$$\begin{aligned} &x'(t) + P(t)x(t) \\ &+ P(t)x(t) \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} P(u) \frac{x\left(\tau(u)\right)}{x(u)} du\right) d\omega\right) ds \leq 0. \end{aligned}$$

Since $\tau(u) \leq h(u)$, clearly

$$\begin{aligned} &x'(t) + P(t)x(t) \\ &+ P(t)x(t) \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(\omega) \exp\left(\int_{\tau(\omega)}^{\omega} P(u) \frac{x\left(h(u)\right)}{x(u)} du\right) d\omega\right) ds \leq 0. \end{aligned}$$

Taking into account the fact that (2.3) of Lemma 2.3 is satisfied, the last inequality becomes

$$x'(t) + P(t)x(t) + P(t)x(t) \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(\omega) \exp\left((\lambda_0 - \epsilon) \int_{\tau(\omega)}^{\omega} P(u)du \right) d\omega \right) ds \leq 0.$$

Thus,

$$x'(t) + P(t)\left[1 + \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(\omega) \exp\left((\lambda_0 - \epsilon) \int_{\tau(\omega)}^{\omega} P(u)du \right) d\omega \right) ds \right] x(t) \leq 0,$$

or

$$x'(t) + W_0(t, \epsilon)x(t) \leq 0, \tag{2.12}$$

with

$$W_0(t, \epsilon) = P(t)\left[1 + \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(\omega) \exp\left((\lambda_0 - \epsilon) \int_{\tau(\omega)}^{\omega} P(u)du \right) d\omega \right) ds \right].$$

Applying the Grönwall inequality in (2.12), we obtain

$$x(s) \geq x(t) \exp\left(\int_{s}^{t} W_0(\xi, \epsilon)d\xi \right), \qquad t \geq s.$$

Thus

$$x(\tau(u)) \geq x(u) \exp\left(\int_{\tau(u)}^{u} W_0(\xi, \epsilon)d\xi \right). \tag{2.13}$$

Now we divide (E) by $x(t) > 0$ and integrate on $[s, t]$, so

$$-\int_{s}^{t} \frac{x'(u)}{x(u)}du = \int_{s}^{t} \sum_{i=1}^{m} p_i(u) \frac{x(\tau_i(u))}{x(u)}du \geq \int_{s}^{t} P(u) \frac{x(\tau(u))}{x(u)}du,$$

or

$$\ln \frac{x(s)}{x(t)} \geq \int_{s}^{t} P(u) \frac{x(\tau(u))}{x(u)}du \geq \int_{s}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_0(\xi, \epsilon)d\xi \right) du.$$

or

$$x(s) \geq x(t) \exp\left(\int_{s}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_0(\xi,\epsilon)d\xi\right) du\right). \tag{2.14}$$

Setting $s = \tau(s)$ in (2.14) we take

$$x(\tau(s)) \geq x(t) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_0(\xi,\epsilon)d\xi\right) du\right). \tag{2.15}$$

Combining (2.9) and (2.15) we obtain

$$x(t) - x(\tau(t)) + x(t) \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_0(\xi,\epsilon)d\xi\right) du\right) ds \leq 0.$$

Multiplying the last inequality by $P(t)$, we find

$$\begin{aligned} &P(t)x(t) - P(t)x(\tau(t)) \\ &+ P(t)x(t) \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_0(\xi,\epsilon)d\xi\right) du\right) ds \leq 0, \end{aligned}$$

which, in view of (2.11), becomes

$$x'(t)+P(t)x(t)+P(t)x(t) \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_0(\xi,\epsilon)d\xi\right) du\right) ds \leq 0.$$

Hence, for sufficiently large t

$$x'(t) + P(t)\left[1 + \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_0(\xi,\epsilon)d\xi\right) du\right) ds\right] x(t) \leq 0,$$

or

$$x'(t) + W_1(t,\epsilon)x(t) \leq 0, \tag{2.16}$$

where

$$W_1(t,\epsilon) = P(t)\left[1 + \int_{\tau(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_0(\xi,\epsilon)d\xi\right) du\right) ds\right].$$

It becomes apparent, now, that by repeating the above steps, we can build inequalities on $x'(t)$ with progressively higher indices $W_\ell(t)$, $\ell \in \mathbb{N}$. In general, for sufficiently large t, the positive solution $x(t)$ satisfies the inequality

$$x'(t) + W_\ell(t,\epsilon)x(t) \le 0 \quad (\ell \in \mathbb{N}),$$

where

$$W_\ell(t,\epsilon) = P(t)\left[1 + \left[1 + \int_{\tau(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{t} P(u)\exp\left(\int_{\tau(u)}^{u} W_{\ell-1}(\xi,\epsilon)d\xi\right)du\right)ds\right]\right]$$

and

$$x(\tau(s)) \ge x(h(t))\exp\left(\int_{\tau(s)}^{h(t)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right)du\right). \tag{2.17}$$

Integrating (E) from $h(t)$ to t, and using (2.11) and (2.17) we have

$$0 = x(t) - x(h(t)) + \int_{h(t)}^{t}\sum_{i=1}^{m} p_i(s)x(\tau_i(s))ds \ge x(t) - x(h(t)) + \int_{h(t)}^{t} P(s)x(\tau(s))ds$$

or

$$x(t) - x(h(t)) + x(h(t))\int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right)du\right)ds \le 0. \tag{2.18}$$

The inequality is valid if we omit $x(t) > 0$ in the left-hand side. Therefore

$$x(h(t))\left[\int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right)du\right)ds - 1\right] < 0,$$

which means that

$$\limsup_{t\to\infty}\int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right)du\right)ds \le 1.$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.4).

The proof of the theorem is complete. □

Theorem 2.5. *Assume that $h(t)$ is defined by* (1.6) *and for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi) d\xi \right) du \right) ds > 1 - D(\alpha), \quad (2.19)$$

where W_ℓ is defined by (2.5)*. Then all solutions of* (E) *are oscillatory.*

Proof. Let x be an eventually positive solution of (E). Then, as in the proof of Theorem 2.4, (2.18) is satisfied, i.e.,

$$x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds \le 0.$$

That is,

$$\int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds \le 1 - \frac{x(t)}{x(h(t))},$$

which gives

$$\begin{aligned} &\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds \\ &\le 1 - \liminf_{t\to\infty} \frac{x(t)}{x(h(t))}. \end{aligned}$$

By combining Lemmas 2.1 and 2.2, it becomes obvious that inequality (2.2) is fulfilled. So, the above inequality leads to

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds \le 1 - D(\alpha).$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.19).

The proof of the theorem is complete. □

Theorem 2.6. *Assume that $h(t)$ is defined by* (1.6) *and for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi) d\xi \right) du \right) ds > \frac{1}{D(\alpha)} - 1, \quad (2.20)$$

where W_ℓ is defined by (2.5)*. Then all solutions of* (E) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution x of (E) and that x is eventually positive. Then, as in the proof of Theorem 2.4, for sufficiently large t we have

$$x(\tau(s)) \geq x(t) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right) du\right). \tag{2.21}$$

Integrating (E) from $h(t)$ to t and in view of (2.21), we have

$$\begin{aligned}
0 &= x(t) - x(h(t)) + \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s) x(\tau_i(s))\, ds \\
&\geq x(t) - x(h(t)) + \int_{h(t)}^{t} P(s) x(\tau(s))\, ds \\
&\geq x(t) - x(h(t)) \\
&\quad + x(h(t)) \int_{h(t)}^{t} P(s) \frac{x(t)}{x(h(t))} \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right) du\right) ds.
\end{aligned}$$

That is, for all sufficiently large t it holds

$$\int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right) du\right) ds \leq \frac{x(h(t))}{x(t)} - 1$$

and therefore

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right) du\right) ds \leq \limsup_{t\to\infty} \frac{x(h(t))}{x(t)} - 1. \tag{2.22}$$

By combining Lemmas 2.1 and 2.2, it becomes obvious that inequality (2.2) is fulfilled. So, (2.22) leads to

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right) du\right) ds \leq \frac{1}{D(\alpha)} - 1.$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.20).

The proof of the theorem is complete. □

Theorem 2.7. *Assume that $h(t)$ is defined by* (1.6) *and for some $\ell \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(s)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi) d\xi \right) du \right) ds > \frac{1+\ln\lambda_0}{\lambda_0} - D(\alpha), \tag{2.23}$$

where W_ℓ is defined by (2.5) *and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$. Then all solutions of* (E) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution x of (E) and that x is eventually positive. Then, as in the previous theorems, (2.21) holds.

Observe that (2.3) implies that for each $\epsilon > 0$ there exists a t_ϵ such that

$$\lambda_0 - \epsilon < \frac{x(h(t))}{x(t)} \quad \text{for all } t \geq t_\epsilon\,. \tag{2.24}$$

Noting that by nondecreasing nature of the function $\frac{x(h(t))}{x(s)}$ in s, it holds

$$1 = \frac{x(h(t))}{x(h(t))} \leq \frac{x(h(t))}{x(s)} \leq \frac{x(h(t))}{x(t)}, \quad t_\varepsilon \leq h(t) \leq s \leq t,$$

in particular for $\epsilon \in (0, \lambda_0 - 1)$, by continuity we see that there exists a $t^* \in (h(t), t]$ such that

$$1 < \lambda_0 - \epsilon = \frac{x(h(t))}{x(t^*)}. \tag{2.25}$$

By (2.21), it is obvious that

$$x(\tau(s)) \geq x(h(s)) \exp\left(\int_{\tau(s)}^{h(s)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon) d\xi \right) du \right). \tag{2.26}$$

Integrating (E) from t^* to t and using (2.26) we have

$$\begin{aligned}
0 &= x(t) - x(t^*) + \int_{t^*}^{t} \sum_{i=1}^{m} p_i(s) x(\tau_i(s)) ds \\
&\geq x(t) - x(t^*) + \int_{t^*}^{t} \left(\sum_{i=1}^{m} p_i(s) \right) x(\tau(s)) ds \\
&= x(t) - x(t^*) + \int_{t^*}^{t} P(s) x(\tau(s)) ds \\
&\geq x(t) - x(t^*) + x(h(t)) \int_{t^*}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(s)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon) d\xi \right) du \right) ds
\end{aligned}$$

or

$$\int_{t^*}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right)du\right)ds \le \frac{x(t^*)}{x(h(t))} - \frac{x(t)}{x(h(t))}.$$

In view of (2.25) and Lemma 2.2, for the ϵ considered, there exists $t'_\epsilon \ge t_\varepsilon$ such that

$$\int_{t^*}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right)du\right)ds < \frac{1}{\lambda_0-\epsilon} - D(\alpha) + \epsilon, \quad (2.27)$$

for $t \ge t'_\epsilon$.

Dividing (E) by $x(t)$ and integrating from $h(t)$ to t^* we find

$$\begin{aligned}
&-\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)}ds \\
&= \int_{h(t)}^{t^*} \sum_{i=1}^{m} p_i(s)\frac{x(\tau_i(s))}{x(s)}ds \ge \int_{h(t)}^{t^*} \left(\sum_{i=1}^{m} p_i(s)\right)\frac{x(\tau(s))}{x(s)}ds \qquad (2.28)\\
&\ge \int_{h(t)}^{t^*} P(s)\frac{x(h(s))}{x(s)}\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right)du\right)ds.
\end{aligned}$$

By (2.25), for $s \ge h(s) \ge t'_\epsilon$, we have $\frac{x(h(s))}{x(s)} > \lambda_0 - \epsilon$, so from (2.28) we get

$$(\lambda_0-\epsilon)\int_{h(t)}^{t^*} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right)du\right)ds < -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)}ds\,.$$

Hence, for all sufficiently large t we have

$$\begin{aligned}
&\int_{h(t)}^{t^*} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right)du\right)ds \\
&< -\frac{1}{\lambda_0-\epsilon}\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)}ds = \frac{1}{\lambda_0-\epsilon}\ln\frac{x(h(t))}{x(t^*)} = \frac{\ln(\lambda_0-\epsilon)}{\lambda_0-\epsilon},
\end{aligned}$$

i.e.,

$$\int_{h(t)}^{t^*} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon)d\xi\right)du\right)ds < \frac{\ln(\lambda_0-\epsilon)}{\lambda_0-\epsilon}. \qquad (2.29)$$

Adding (2.27) and (2.29), and then taking the limit as $t \to \infty$, we have

$$\limsup_{t\to\infty} \int_{h(t)}^{t^*} P(s) \exp\left(\int_{\tau(s)}^{h(s)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon) d\xi \right) du \right) ds$$
$$\leq \frac{1+\ln(\lambda_0-\epsilon)}{\lambda_0-\epsilon} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} + \epsilon.$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.23).

The proof of the theorem is complete. □

Theorem 2.8. *Assume that $h(t)$ is defined by* (1.6) *and for some $\ell \in \mathbb{N}$*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(s)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi) d\xi \right) du \right) ds > \frac{1}{e}, \tag{2.30}$$

where G_ℓ is defined by (2.5). *Then all solutions of* (E) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (E). Since $-x(t)$ is also a solution of (E), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists $t_1 > t_0$ such that $x(t),\ x(\tau_i(t)) > 0,\ 1 \leq i \leq m$ for all $t \geq t_1$. Thus, from (E) we have

$$x'(t) = -\sum_{i=1}^{m} p_i(t) x(\tau_i(t)) \leq 0, \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ is an eventually nonincreasing function of positive numbers. Furthermore, as in previous theorem, (2.26) is satisfied.

Dividing (E) by $x(t)$ and integrating from $h(t)$ to t, for some $t_2 \geq t_1$, we have

$$\begin{aligned} \ln\left(\frac{x(h(t))}{x(t)}\right) &= \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s) \frac{x(\tau_i(s))}{x(s)} ds \\ &\geq \int_{h(t)}^{t} \left(\sum_{i=1}^{m} p_i(s) \right) \frac{x(\tau(s))}{x(s)} ds = \int_{h(t)}^{t} P(s) \frac{x(\tau(s))}{x(s)} ds. \end{aligned} \tag{2.31}$$

Combining the inequalities (2.26) and (2.31) we obtain

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_{h(t)}^{t} P(s) \frac{x(h(s))}{x(s)} \exp\left(\int_{\tau(s)}^{h(s)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi,\epsilon) d\xi \right) du \right) ds.$$

From (2.30), it follows that there exists a constant $c > 0$ such that for a sufficiently large t holds

$$\int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(s)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi) d\xi \right) du \right) ds \geq c > \frac{1}{e}.$$

Choose c' such that $c > c' > 1/e$. For every $\epsilon > 0$ such that $c - \epsilon > c'$ we have

$$\int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(s)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds \geq c - \epsilon > c' > \frac{1}{e}. \tag{2.32}$$

Hence,

$$\ln\left(\frac{x(h(t))}{x(t)} \right) \geq c', \quad t \geq t_3.$$

Thus

$$\frac{x(h(t))}{x(t)} \geq e^{c'} \geq ec' > 1,$$

which implies for some $t \geq t_4 \geq t_3$

$$x(h(t)) \geq (ec')x(t).$$

Repeating the above procedure, it follows by induction that for any positive integer k,

$$\frac{x(h(t))}{x(t)} \geq (ec')^k, \quad \text{for sufficiently large } t.$$

Since $ec' > 1$, there is $k \in \mathbb{N}$ satisfying $k > 2(\ln(2) - \ln(c'))/(1 + \ln(c'))$ such that for t sufficiently large

$$\frac{x(h(t))}{x(t)} \geq (ec')^k > \left(\frac{2}{c'} \right)^2. \tag{2.33}$$

Taking the integral on $[h(t), t]$, which is not less than c', we split the interval into two parts where integrals are not less than $c'/2$, let $t_m \in (h(t), t)$ be the splitting point:

$$\begin{aligned} &\int_{h(t)}^{t_m} P(s) \exp\left(\int_{\tau(s)}^{h(s)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds \geq \frac{c'}{2}, \\ &\int_{t_m}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(s)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds \geq \frac{c'}{2}. \end{aligned} \tag{2.34}$$

Integrating (E) from t_m to t, gives

$$x(t) - x(t_m) + \int_{t_m}^{t} \sum_{i=1}^{m} p_i(s) x(\tau_i(s)) = 0,$$

or

$$x(t) - x(t_m) + \int_{t_m}^{t} \left(\sum_{i=1}^{m} p_i(s) \right) x(\tau(s)) \leq 0.$$

Thus

$$x(t) - x(t_m) + \int_{t_m}^{t} P(s)x(\tau(s)) \leq 0,$$

or

$$x(t) - x(t_m) + x(h(t)) \int_{t_m}^{t} P(s) \exp \left(\int_{\tau(s)}^{h(s)} P(u) \exp \left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 0.$$

The strict inequality is valid if we omit $x(t) > 0$ in the left-hand side:

$$-x(t_m) + x(h(t)) \int_{t_m}^{t} P(s) \exp \left(\int_{\tau(s)}^{h(s)} P(u) \exp \left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds < 0.$$

Together with the second inequality in (2.34), implies

$$x(t_m) > \frac{c'}{2} x(h(t)). \tag{2.35}$$

Similarly, integration of (E) from $h(t)$ to t_m with a later application of (2.26) leads to

$$x(t_m) - x(h(t)) + x(h(t_m)) \int_{h(t)}^{t_m} P(s) \exp \left(\int_{\tau(s)}^{h(s)} P(u) \exp \left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds \leq 0.$$

The strict inequality is valid if we omit $x(t_m) > 0$ in the left-hand side:

$$-x(h(t)) + x(h(t_m)) \int_{h(t)}^{t_m} P(s) \exp \left(\int_{\tau(s)}^{h(s)} P(u) \exp \left(\int_{\tau(u)}^{u} W_\ell(\xi, \epsilon) d\xi \right) du \right) ds < 0.$$

Together with the first inequality in (2.34) implies

$$x(h(t)) > \frac{c'}{2} x(h(t_m)). \tag{2.36}$$

Combining the inequalities (2.35) and (2.36), we obtain

$$x(h(t_m)) < \frac{2}{c} x(h(t)) < \left(\frac{2}{c'} \right)^2 x(t_m),$$

which contradicts (2.33).

The proof of the theorem is complete. □

2.2. ADES

Similar oscillation conditions for the (dual) advanced differential equation (E′) can be derived easily. The proofs are omitted, since they are quite similar to the delay equation.

Theorem 2.9. *Assume that* $\rho(t)$ *is defined by* (1.25) *and for some* $\ell \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} Q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} Q(u) \exp\left(\int_u^{\sigma(u)} S_\ell(\xi)d\xi \right) du \right) ds > 1, \tag{2.37}$$

where

$$S_\ell(t) = Q(t)\left[1 + \int_t^{\sigma(t)} P(s) \exp\left(\int_t^{\sigma(s)} Q(u) \exp\left(\int_u^{\sigma(u)} S_{\ell-1}(\xi)d\xi \right) du \right) ds \right], \tag{2.38}$$

with

$$S_0(t) = Q(t)\left[1 + \int_t^{\sigma(t)} Q(s) \exp\left(\int_t^{\sigma(s)} Q(\omega) \exp\left(\lambda_0 \int_\omega^{\sigma(\omega)} Q(u)du \right) d\omega \right) ds \right]$$

and λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\beta\lambda}$. *Then all solutions of* (E′) *are oscillatory.*

Theorem 2.10. *Assume that* $\rho(t)$ *is defined by* (1.25) *and for some* $\ell \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} Q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} Q(u) \exp\left(\int_u^{\sigma(u)} S_\ell(\xi)d\xi \right) du \right) ds > 1 - D(\beta), \tag{2.39}$$

where S_ℓ *is defined by* (2.38) *Then all solutions of* (E′) *are oscillatory.*

Theorem 2.11. *Assume that* $\rho(t)$ *is defined by* (1.25) *and for some* $\ell \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} Q(s) \exp\left(\int_t^{\sigma(s)} Q(u) \exp\left(\int_u^{\sigma(u)} S_\ell(\xi)d\xi \right) du \right) ds > \frac{1}{D(\beta)} - 1, \tag{2.40}$$

where S_ℓ *is defined by* (2.38) *Then all solutions of* (E′) *are oscillatory.*

Theorem 2.12. *Assume that* $\rho(t)$ *is defined by* (1.25) *and for some* $\ell \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} Q(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} Q(u) \exp\left(\int_u^{\sigma(u)} S_\ell(\xi)d\xi \right) du \right) ds > \frac{1 + \ln\lambda_0}{\lambda_0} - D(\beta), \tag{2.41}$$

where S_ℓ *is defined by* (2.38) *and* λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\beta\lambda}$. *Then all solutions of* (E′) *are oscillatory.*

Theorem 2.13. *Assume that* $\rho(t)$ *is defined by* (1.25) *and for some* $\ell \in \mathbb{N}$

$$\liminf_{t\to\infty} \int_t^{\rho(t)} Q(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} Q(u) \exp\left(\int_u^{\sigma(u)} S_\ell(\xi) d\xi \right) du \right) ds > \frac{1}{e}, \tag{2.42}$$

where S_ℓ *is defined by* (2.38). *Then all solutions of* (E′) *are oscillatory.*

2.3. DIFFERENTIAL INEQUALITIES

A slight modification in the proofs of Theorems 2.4–2.8 (resp. 2.9–2.13) leads to the following results about differential inequalities.

Theorem 2.14. *Assume that all the conditions of Theorem 2.4 or 2.5 or 2.6 or 2.7 or 2.8* (*resp. 2.9 or 2.10 or 2.11 or 2.12 or 2.13*) *hold. Then*

(i) *the delay* (*resp. advanced*) *differential inequality*

$$x'(t) + \sum_{i=1}^{m} p_i(t) x\left(\tau_i(t)\right) \le 0 \quad \left(x'(t) - \sum_{i=1}^{m} q_i(t) x\left(\sigma_i(t)\right) \ge 0 \right), \ \forall t \ge t_0,$$

has no eventually positive solutions;

(ii) *the delay* (*resp. advanced*) *differential inequality*

$$x'(t) + \sum_{i=1}^{m} p_i(t) x\left(\tau_i(t)\right) \ge 0 \quad \left(x'(t) - \sum_{i=1}^{m} q_i(t) x\left(\sigma_i(t)\right) \le 0 \right), \ \forall t \ge t_0,$$

has no eventually negative solutions.

3. EXAMPLES

In this section, examples illustrate cases when the results of the present paper imply oscillation while previously known results fail. Also, these examples illustrate the strength of the obtained conditions over known ones. The calculations were made by the use of MATLAB software.

Example 3.1. Consider the delay differential equation

$$x'(t) + \frac{127}{1250} x(\tau_1(t)) + \frac{127}{5000} x(\tau_2(t)) = 0, \quad t \ge 0, \tag{3.1}$$

with (see Fig. 1(a))

$$\tau_1(t) = \begin{cases} -t + 12k - 1, & \text{if } t \in [6k, 6k+1], \\ 6k - 2, & \text{if } t \in [6k+1, 6k+2], \\ 3t - 12k - 8, & \text{if } t \in [6k+2, 6k+3], \\ -0.5t + 9k + 2.5, & \text{if } t \in [6k+3, 6k+5], \\ 5t - 24k - 25, & \text{if } t \in [6k+5, 6k+6] \end{cases} \quad \text{and} \quad \tau_2(t) = \tau_1(t) - 0.5$$

where $k \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

By (1.6), we see (Fig. 1, (b)) that

$$h_1(t) = \begin{cases} 6k-1, & \text{if } t \in [6k, 6k+7/3]\,, \\ 3t-12k-8, & \text{if } t \in [6k+7/3, 6k+3]\,, \\ 6k+1, & \text{if } t \in 6k+3, 6k+26/5], \\ 5t-24k-25, & \text{if } t \in [6k+26/5, 6k+6] \end{cases} \quad \text{and} \quad h_2(t) = h_1(t) - 0.5$$

and consequently

$$h(t) = \max_{1\le i\le 2}\{h_i(t)\} = h_1(t) \quad \text{and} \quad \tau(t) = \max_{1\le i\le 2}\tau_i(t) = \tau_1(t).$$

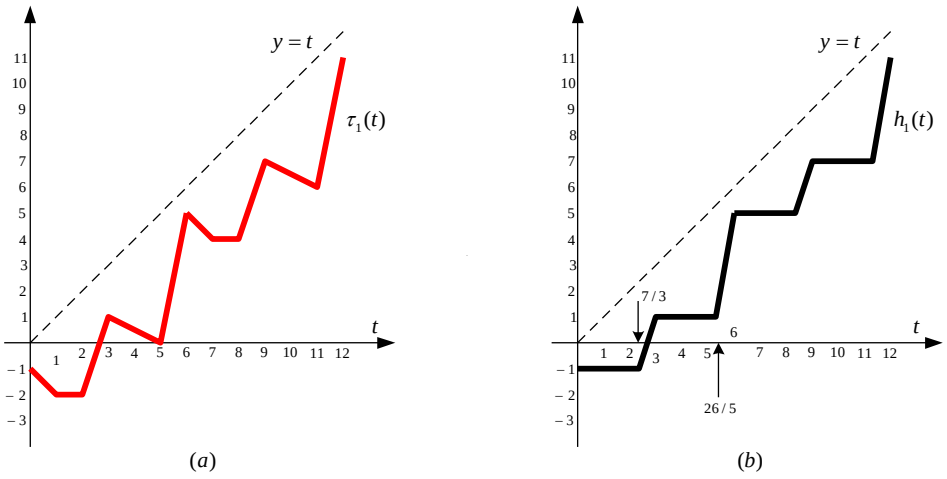

Fig. 1. The graphs of $\tau_1(t)$ and $h_1(t)$

It is easy to see that

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{2} p_i(s)ds = \liminf_{k\to\infty} \int_{6k+5}^{6k+6} \left(\frac{127}{1250} + \frac{127}{5000}\right) ds = 0.127$$

and therefore, the smaller root of $e^{0.127\lambda} = \lambda$ is $\lambda_0 = 1.1585$.

Observe that the function $F_\ell : \mathbb{R}_0 \to \mathbb{R}_+$ defined as

$$F_\ell(t) = \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P(u) \exp\left(\int_{\tau(u)}^{u} W_\ell(\xi)d\xi\right) du\right) ds,$$

attains its maximum at $t = 6k + 26/5$, $k \in \mathbb{N}_0$, for every $\ell \in \mathbb{N}$.

Specifically

$$F_1(t=6k+26/5)=\int\limits_{6k+1}^{6k+26/5}P(s)\exp\left(\int\limits_{\tau(s)}^{6k+1}P(u)\exp\left(\int\limits_{\tau(u)}^{u}W_1(\xi)d\xi\right)du\right)ds,$$

where

$$W_1(\xi)=P(\xi)\left[1+\int\limits_{\tau(\xi)}^{\xi}P(v)\exp\left(\int\limits_{\tau(v)}^{\xi}P(w)\exp\left(\int\limits_{\tau(w)}^{w}W_0(z)dz\right)dw\right)dv\right]$$

and

$$W_0(z)=P(z)\left[1+\int\limits_{\tau(z)}^{z}P(\omega)\exp\left(\int\limits_{\tau(\omega)}^{z}P(\varphi)\exp\left(\lambda_0\int\limits_{\tau(\varphi)}^{\varphi}P(r)dr\right)d\varphi\right)d\omega\right].$$

By using an algorithm on MATLAB software, we obtain

$$F_1(t=6k+26/5)\simeq 1.0673$$

and therefore

$$\limsup_{t\to\infty}F_1(t)\simeq 1.0673>1.$$

That is, condition (2.4) of Theorem 2.4 is satisfied for $\ell=1$, and therefore all solutions of (3.1) are oscillatory.

Observe, however, that

$$\limsup_{k\to\infty}\int\limits_{6k+1}^{6k+26/5}\left(\frac{127}{1250}+\frac{127}{5000}\right)ds\simeq 0.5334<1,$$

$$\alpha=0.127<\frac{1}{e}$$

and

$$\begin{aligned}
&\liminf_{t\to\infty}\sum_{i=1}^{2}p_i(t)\left(t-\tau_i(t)\right)\\
&=\liminf_{t\to\infty}\left[\frac{127}{1250}\left(t-\tau_1(t)\right)+\frac{127}{5000}\left(t-(\tau_1(t)-0.5)\right)\right]\\
&=\liminf_{t\to\infty}\left[0.127\left(t-\tau_1(t)\right)+0.0127\right]=\liminf_{t\to\infty}\left[0.127\left(t-\tau_1(t)\right)\right]+0.0127\\
&=0.127\cdot\liminf_{t\to\infty}\left(t-\tau_1(t)\right)+0.0127=0.127\cdot 1+0.0127=0.1397<\frac{1}{e}.
\end{aligned}$$

Also, observe that the function $D_r : [0,\infty) \to \mathbb{R}_+$ defined as

$$D_r(t) = \int_{h(t)}^{t} \sum_{i=1}^{2} p_i(\zeta) a_r(h(t), \tau_i(\zeta)) d\zeta$$

attains its maximum at $t = 6k + 26/5$ and its minimum at $t = 6k + 6$, $k \in \mathbb{N}_0$, for every $r \in \mathbb{N}$. Specifically,

$$\begin{aligned} D_1(t = 6k + 26/5) &= \int_{6k+1}^{6k+26/5} \sum_{i=1}^{2} p_i(\zeta) a_1(6k+1, \tau_i(\zeta)) d\zeta \\ &= \int_{6k+1}^{6k+2} [p_1(\zeta) a_1(6k+1, \tau_1(\zeta)) + p_2(\zeta) a_1(6k+1, \tau_2(\zeta))]\, d\zeta \\ &\quad + \int_{6k+2}^{6k+3} [p_1(\zeta) a_1(6k+1, \tau_1(\zeta)) + p_2(\zeta) a_1(6k+1, \tau_2(\zeta))]\, d\zeta \\ &\quad + \int_{6k+3}^{6k+5} [p_1(\zeta) a_1(6k+1, \tau_1(\zeta)) + p_2(\zeta) a_1(6k+1, \tau_2(\zeta))]\, d\zeta \\ &\quad + \int_{6k+5}^{6k+26/5} [p_1(\zeta) a_1(6k+1, \tau_1(\zeta)) + p_2(\zeta) a_1(6k+1, \tau_2(\zeta))]\, d\zeta \\ &\simeq 0.6468 \end{aligned}$$

and

$$\begin{aligned} D_1(t = 6k + 6) &= \int_{6k+5}^{6k+6} \sum_{i=1}^{2} p_i(\zeta) a_1(6k+5, \tau_i(\zeta)) d\zeta \\ &= \int_{6k+5}^{6k+6} [p_1(\zeta) a_1(6k+5, \tau_1(\zeta)) + p_2(\zeta) a_1(6k+5, \tau_2(\zeta))]\, d\zeta \\ &\simeq 0.1797. \end{aligned}$$

Thus,

$$\limsup_{t\to\infty} D_1(t) \simeq 0.6468 < 1,$$
$$\liminf_{t\to\infty} D_1(t) \simeq 0.1797 < 1/e$$

and

$$0.6468 < 1 - D(\alpha) \simeq 0.9907.$$

Clearly

$$\int_{6k+1}^{6k+26/5} \sum_{i=1}^{2} p_i(\zeta) a_1(h(\zeta), \tau_i(\zeta)) d\zeta \le D_1(t = 6k + 26/5) \simeq 0.6468.$$

Thus,

$$\limsup_{k\to\infty} \int_{6k+1}^{6k+26/5} \sum_{i=1}^{2} p_i(\zeta) a_1(h(\zeta), \tau_i(\zeta)) d\zeta \le 0.6468 < \frac{1 + \ln \lambda_0}{\lambda_0} - D(\alpha) \simeq 0.9808.$$

Finally, using algorithms on MATLAB software, we obtain

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P_1(u) du \right) ds \simeq 0.7739 < 1,$$

$$0.7739 < 1 - D(\alpha) \simeq 0.9907,$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(s)} P_1(u) du \right) ds \le 0.7739 < \frac{1 + \ln \lambda_0}{\lambda_0} - D(\alpha) \simeq 0.9808,$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P_1(u) du \right) ds \simeq 2.4233 < \frac{1}{D(\alpha)} \simeq 107.09,$$

$$\liminf_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(s)} P_1(u) du \right) ds \simeq 0.129 < \frac{1}{e},$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{h(t)} P(u) \exp\left(\int_{\tau(u)}^{u} G_1(\xi) d\xi \right) du \right) ds \simeq 0.9309 < 1,$$

$$0.9309 < 1 - D(\alpha) \simeq 0.9907,$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} P(s) \exp\left(\int_{\tau(s)}^{t} P(u) \exp\left(\int_{\tau(u)}^{u} G_1(\xi) d\xi \right) du \right) ds \simeq 4.8645 < \frac{1}{D(\alpha)} - 1 \simeq 106.09,$$

$$\limsup_{t\to\infty}\int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} G_1(\xi)d\xi\right)du\right)ds$$
$$\leq 0.9309 < \frac{1+\ln\lambda_0}{\lambda_0} - D(\alpha) \simeq 0.9808,$$
$$\liminf_{t\to\infty}\int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} G_1(\xi)d\xi\right)du\right)ds \simeq 0.131 < \frac{1}{e}.$$

That is, none of the conditions (1.3)–(1.5), (1.8)–(1.11) (for $r = 1$), (1.12)–(1.16) (for $j = 1$) and (1.17)–(1.21) (for $\ell = 1$) is satisfied.

It is worth noting that the improvement of condition (2.4) to the corresponding condition (1.3) is significant, approximately 100.09%, if we compare the values on the left-side of these conditions. Also, the improvement compared to conditions (1.8), (1.12) and (1.17) is very satisfactory, around 65.01%, 37.91% and 14.65%, respectively. In addition, observe that conditions (1.8)–(1.21) do not lead to oscillation for first iteration. On the contrary, condition (2.4) is satisfied from the first iteration. This means that our condition is better and much faster than (1.8)–(1.21).

Example 3.2. Consider the advanced differential equation

$$x'(t) - \frac{6}{25}x(\sigma_1(t)) - \frac{3}{50}x(\sigma_2(t)) = 0, \quad t \geq 0, \tag{3.2}$$

with (see Fig. 2(a))

$$\sigma_1(t) = \begin{cases} t+1, & \text{if } t \in [3.5k, 3.5k+1], \\ 3t-7k-1, & \text{if } t \in [3.5k+1, 3.5k+1.5], \\ -t+7k+5, & \text{if } t \in [3.5k+1.5, 3.5k+2], \\ t+1, & \text{if } t \in [3.5k+2, 3.5k+2.5], \\ 3t-7k-4, & \text{if } t \in [3.5k+2.5, 3.5k+3], \\ -t+7k+8, & \text{if } t \in [3.5k+3, 3.5k+3.5] \end{cases} \quad \text{and} \quad \sigma_2(t) = \sigma_1(t) + 0.5$$

where $k \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

By (1.25), we see (Fig. 2(b)) that

$$\rho_1(t) = \begin{cases} t+1, & \text{if } t \in [3.5k, 3.5k+1], \\ 3t-7k-1, & \text{if } t \in [3.5k+1, 3.5k+4/3], \\ 3.5k+3, & \text{if } t \in [3.5k+4/3, 3.5k+2], \\ t+1, & \text{if } t \in [3.5k+2, 3.5k+2.5], \\ 3t-7k-4, & \text{if } t \in [3.5k+2.5, 3.5k+17/6], \\ 3.5k+4.5, & \text{if } t \in [3.5k+17/6, 3.5k+3.5] \end{cases} \quad \text{and} \quad \rho_2(t) = \rho_1(t) + 0.5$$

and consequently

$$\rho(t) = \min_{1\leq i\leq 2}\{\rho_i(t)\} = \rho_1(t) \quad \text{and} \quad \sigma(t) = \min_{1\leq i\leq 2}\{\sigma_i(t)\} = \sigma_1(t).$$

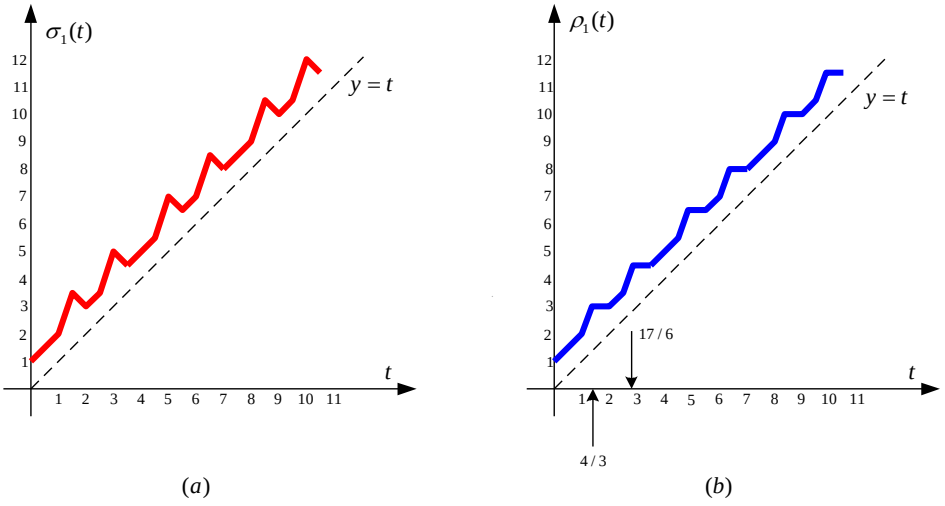

Fig. 2. The graphs of $\sigma_1(t)$ and $\rho_1(t)$

It is easy to see that

$$\beta = \liminf_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^{m} q_i(s)ds = \liminf_{k\to\infty} \int_{3.5k+1}^{3.5k+2} 0.3ds = 0.3$$

and therefore, the smaller root of $e^{0.3\lambda} = \lambda$ is $\lambda_0 = 1.63134$.

Observe, that the function $F_\ell : \mathbb{R}_0 \to \mathbb{R}_+$ defined as

$$F_\ell(t) = \int_t^{\rho(t)} Q(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} Q(u) \exp\left(\int_u^{\sigma(u)} S_\ell(\xi)d\xi \right) du \right) ds,$$

attains its minimum at $t = 3.5k + 1$, $k \in \mathbb{N}_0$, for every $\ell \in \mathbb{N}$. Specifically, by using an algorithm on MATLAB software and taking into account the fact that $Q(t) = \sum_{i=1}^{2} q_i(t) = 0.3$, we obtain

$$F_1(t = 3.5k + 1) = \int_{3.5k+1}^{3.5k+2} Q(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} Q(u) \exp\left(\int_u^{\sigma(u)} S_1(\xi)d\xi \right) du \right) ds \simeq 0.375$$

and therefore

$$\liminf_{t\to\infty} F_1(t) \simeq 0.375 > \frac{1}{e}.$$

That is, condition (2.42) of Theorem 2.13 is satisfied for $\ell = 1$, and therefore all solutions of (3.2) oscillate.

Observe, however, that

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^{2} q_i(s)ds = \limsup_{k\to\infty} \int_{3.5k+4/3}^{3.5k+3} 0.3ds = 0.5 < 1,$$

$$\beta = 0.3 < \frac{1}{e},$$

$$\begin{aligned}
\liminf_{t\to\infty} \sum_{i=1}^{2} q_i(t)\left(\sigma_i(t) - t\right) &= \liminf_{t\to\infty} \sum_{i=1}^{2} q_i(t)\left(\rho_i(t) - t\right) \\
&= \liminf_{t\to\infty} \left[\frac{6}{25}\left(\rho_1(t) - t\right) + \frac{3}{50}\left(\rho_1(t) + 0.5 - t\right)\right] \\
&= \liminf_{t\to\infty} \left[0.3\left(\rho_1(t) - t\right) + 0.03\right] \\
&= \liminf_{t\to\infty} \left[0.3\left(\rho_1(t) - t\right)\right] + 0.03 \\
&= 0.3 \cdot 1 + 0.03 = 0.33 < \frac{1}{e}.
\end{aligned}$$

Finally, by a similar procedure as in previous example and by using algorithms on MATLAB software, we obtain

Condition	**Value**	**Conclusion**
(1.27) for $r = 1$	$\simeq 0.6171 < 1$	is not satisfied
(1.28) for $r = 1$	$0.6171 < 1 - D(\beta) \simeq 0.9284$	"
(1.29) for $r = 1$	$\simeq 0.312 < 1/e$	"
(1.30) for $j = 1$	$\simeq 0.723 < 1$	"
(1.31) for $j = 1$	$0.723 < 1 - D(\beta) \simeq 0.9284$	"
(1.32) for $j = 1$	$\simeq 2.0476 < 1/D(\beta) \simeq 13.9642$	"
(1.33) for $j = 1$	$\simeq 0.546 < \frac{1+\ln\lambda_0}{\lambda_0} - D(\beta) \simeq 0.8414$	"
(1.34) for $j = 1$	$\simeq 0.3374 < 1/e$	"
(1.35) for $\ell = 1$	$\simeq 0.8663 < 1$	"
(1.36) for $\ell = 1$	$0.8663 < 1 - D(\beta) \simeq 0.9284$	"
(1.37) for $\ell = 1$	$\simeq 4.02 < 1/D(\beta) - 1 \simeq 12.9642$	"
(1.38) for $\ell = 1$	$\simeq 0.573 < \frac{1+\ln\lambda_0}{\lambda_0} - D(\beta) \simeq 0.8414$	"
(1.39) for $\ell = 1$	$\simeq 0.3584 < 1/e$	"

That is, none of the conditions (1.22)–(1.24), (1.27)–(1.29) (for $r = 1$), (1.30)–(1.34) (for $j = 1$) and (1.35)–(1.39) (for $\ell = 1$) is satisfied.

It is worth noting that the improvement of condition (2.42) to the corresponding condition (1.23) is significant, approximately 25%, if we compare the values on the left-side of these conditions. In addition, observe that conditions (1.27)–(1.39) do not lead to oscillation for first iteration. On the contrary, condition (2.42) is satisfied from the first iteration. This means that our condition is better and much faster than (1.27)–(1.39).

Remark 3.3. Similarly, one can construct examples to illustrate the other main results.

4. ASSESSMENT OF THE MAIN RESULTS

4.1. DDES

Since

$$\exp\left(\int_{\tau(s)}^{h(t)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi)d\xi\right)du\right) \geq 1$$

and

$$\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi)d\xi\right)du\right)ds \geq 1,$$

it is clear that

$$\int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi)d\xi\right)du\right)ds \geq \int_{h(t)}^{t}\sum_{i=1}^{m} p_i(s)ds$$

and

$$\int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(s)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi)d\xi\right)du\right)ds \geq \int_{h(t)}^{t}\sum_{i=1}^{m} p_i(s)ds,$$

respectively. Therefore the condition (2.4) and (2.30) are weaker than the conditions (1.3) and (1.4), respectively.

Also, since

$$W_0(t) \geq G_0(t) > P_0(t),$$

clearly

$$\begin{aligned}
&\int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P(u)\exp\left(\int_{\tau(u)}^{u} W_\ell(\xi)d\xi\right)du\right)ds\\
&\geq \int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P(u)\exp\left(\int_{\tau(u)}^{u} G_\ell(\xi)d\xi\right)du\right)ds\\
&\geq \int_{h(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du\right)ds.
\end{aligned}$$

Therefore, the condition (2.4) is weaker than the conditions (1.17) and (1.12).

4.2. ADES

Similar comments as the above, can be made for Theorems 2.9 and 2.13, concerning equation (E′). Hence, the conditions (2.37) and (2.42) are weaker than the conditions (1.22) and (1.23), respectively. Also, the condition (2.37) is weaker than the conditions (1.35) and (1.30).

REFERENCES

[1] E. Braverman, G.E. Chatzarakis, I.P. Stavroulakis, *Iterative oscillation tests for differential equations with several non-monotone arguments*, Adv. Difference Equ. (2016), 2016:87, 18 pp.

[2] G.E. Chatzarakis, *Differential equations with non-monotone arguments: Iterative Oscillation results*, J. Math. Comput. Sci. **6** (2016) 5, 953–964.

[3] G.E. Chatzarakis, H. Péics, *Differential equations with several non-monotone arguments: An oscillation result*, Appl. Math. Lett. **68** (2017), 20–26.

[4] G.E. Chatzarakis, *Oscillations caused by several non-monotone deviating arguments*, Differ. Equ. Appl. **9** (2017) 3, 285–310.

[5] G.E. Chatzarakis, I. Jadlovská, *Explicit criteria for the oscillation of differential equations with several arguments*, Dynamic Systems and Applications **28** (2019) 2, 217–242.

[6] G.E. Chatzarakis, Tongxing Li, *Oscillations of differential equations generated by several deviating arguments*, Adv. Difference Equ. (2017) 2017:292, 24 pp.

[7] L.H. Erbe, Q. Kong, B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, 1995.

[8] L.H. Erbe, B.G. Zhang, *Oscillation of first order linear differential equations with deviating arguments*, Differential Integral Equations **1** (1988), 305–314.

[9] N. Fukagai, T. Kusano, *Oscillation theory of first order functional-differential equations with deviating arguments*, Ann. Mat. Pura Appl. **136** (1984), 95–117.

[10] M.K. Grammatikopoulos, M.R.S. Kulenovic, *First order functional differential inequalities with oscillating coefficients*, Nonlinear Analysis TMA **8** (1984) 9, 1043–1054.

[11] B.R. Hunt, J.A. Yorke, *When all solutions of* $x'(t) = -\sum q_i(t)x(t - T_i(t))$ *oscillate*, J. Differential Equations **53** (1984), 139–145.

[12] J. Jaroš, I.P. Stavroulakis, *Oscillation tests for delay equations*, Rocky Mountain J. Math. **45** (2000), 2989–2997.

[13] M. Kon, Y.G. Sficas, I.P. Stavroulakis, *Oscillation criteria for delay equations*, Proc. Amer. Math. Soc. **128** (1994), 675–685.

[14] R.G. Koplatadze, T.A. Chanturija, *Oscillating and monotone solutions of first-order differential equations with deviating argument*, Differentsial'nye Uravneniya **18** (1982), 1463–1465, 1472 [in Russian].

[15] R.G. Koplatadze, G. Kvinikadze, *On the oscillation of solutions of first order delay differential inequalities and equations*, Georgian Math. J. **3** (1994), 675–685.

[16] T. Kusano, *On even-order functional-differential equations with advanced and retarded arguments*, J. Differential Equations **45** (1982), 75–84.

[17] M.K. Kwong, *Oscillation of first-order delay equations*, J. Math. Anal. Appl. **156** (1991), 274–286.

[18] G. Ladas, V. Lakshmikantham, L.S. Papadakis, *Oscillations of higher-order retarded differential equations generated by the retarded arguments*, *Delay and Functional Differential Equations and their Applications*, Academic Press, New York, 1972, 219–231.

[19] G. Ladas, I.P. Stavroulakis, *Oscillations caused by several retarded and advanced arguments*, J. Differential Equations **44** (1982), 134–152.

[20] G.S. Ladde, *Oscillations caused by retarded perturbations of first order linear ordinary differential equations*, Atti Acad. Naz. Lincei Rendiconti **63** (1978), 351–359.

[21] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 110, Marcel Dekker, Inc., New York, 1987.

[22] X. Li, D. Zhu, *Oscillation and nonoscillation of advanced differential equations with variable coefficients*, J. Math. Anal. Appl. **269** (2002), 462–488.

[23] A.D. Myshkis, *Linear homogeneous differential equations of first order with deviating arguments*, Uspekhi Mat. Nauk **5** (1950), 160–162 [in Russian].

[24] J.S. Yu, Z.C. Wang, B.G. Zhang, X.Z. Qian, *Oscillations of differential equations with deviating arguments*, Panamer. Math. J. **2** (1992) 2, 59–78.

[25] B.G. Zhang, *Oscillation of solutions of the first-order advanced type differential equations*, Science Exploration **2** (1982), 79–82.

[26] D. Zhou, *On some problems on oscillation of functional differential equations of first order*, J. Shandong University **25** (1990), 434–442.

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