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ISOTROPIC AND ANISOTROPIC DOUBLE-PHASE PROBLEMS: OLD AND NEW

Vicenţiu D. Rădulescu

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Abstract. We are concerned with the study of two classes of nonlinear problems driven by differential operators with unbalanced growth, which generalize the (p,q)- and $(p(x),q(x))$-Laplace operators. The associated energy is a double-phase functional, either isotropic or anisotropic. The content of this paper is in relationship with pioneering contributions due to P. Marcellini and G. Mingione.

Keywords: differential operator with unbalanced growth, double-phase energy, variable exponent.

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1. INTRODUCTION

This paper complements our recent work [31], which was devoted to nonlinear problems driven by the $p(x)$-Laplace operator. The purpose of this survey paper is to discuss several recent issues related to the analysis of some problems associated to isotropic or anisotropic phenomena that arise in mathematical physics. We are mainly concerned with models described by double-phase variational integrals.

In several pioneering papers, P. Marcellini [23–25] initiated the refined analysis of some lower semicontinuity and regularity properties of minimizers of certain quasi-convex integrals. This study is motivated by models in nonlinear elasticity that are connected with the deformation of an elastic body, cf. J. Ball [5].

We start by briefly describing some real-life phenomena that involve "double phase" energy functionals. Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N\geq 2)$. If $u:\Omega\to\mathbb{R}^N$ is the displacement and if Du is the $N\times N$ matrix of the deformation gradient, then the total energy can be represented by an energy functional of the type

$$I(u)=\int_\Omega f(x,Du(x))dx, \tag{1.1}$$

where the potential $f=f(x,\xi):\Omega\times\mathbb{R}^{N\times N}\to\mathbb{R}$ is quasi-convex with respect to ξ.

Ball [5] considered potentials of the type

$$f(\xi) = g(\xi) + h(\det \xi),$$

where $\det \xi$ is the determinant of the $N \times N$ matrix ξ, and g, h are nonnegative convex functions, which satisfy the growth conditions

$$g(\xi) \geq c_1 \, |\xi|^p, \quad \lim_{t \to +\infty} h(t) = +\infty,$$

with $1 < p \leq N$ and for some positive constant c_1. The hypothesis $p \leq N$ is necessary to study the existence of equilibrium solutions with cavities, that is, minima of the quasi-convex integral (1.1) that are discontinuous at one point where a cavity forms; in fact, every u with finite energy belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$, and thus it is a continuous function if $p > N$.

In strong relationship with the above mentioned model, Marcellini [23–25] considered continuous functions $f = f(x, u)$ with "unbalanced growth", that is, satisfying the growth condition

$$c_1 \, |u|^p \leq |f(x, u)| \leq c_2 \, (1 + |u|^q) \quad \text{for all } (x, u) \in \Omega \times \mathbb{R},$$

where c_1, c_2 are positive constants and $1 \leq p \leq q$.

The study of non-autonomous functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point has been continued in a series of remarkable papers by G. Mingione *et al.* [7–18]. These contributions are also in relationship with the works of V.V. Zhikov [33], who provided models for strongly anisotropic materials in the context of homogenization phenomena. More precisely, Zhikov considered the following three different model functionals:

$$\begin{aligned}
\mathcal{M}(u) &:= \int_\Omega c(x)|Du|^2 dx, \quad 0 < 1/c(\cdot) \in L^t(\Omega), \ t > 1, \\
\mathcal{V}(u) &:= \int_\Omega |Du|^{p(x)} dx, \quad 1 < p(x) < \infty, \\
\mathcal{P}_{p,q}(u) &:= \int_\Omega (|Du|^p + a(x)|Du|^q) dx, \quad 0 \leq a(x) \leq L, \ 1 < p < q.
\end{aligned} \tag{1.2}$$

These functionals revealed to be important also in the study of duality theory and in the context of the Lavrentiev phenomenon [34].

The functional $\mathcal{M}$ generalizes the usual Dirichlet energy and it is characterized by loss of ellipticity on the set $\{x \in \Omega; \ c(x) = 0\}$. The functional $\mathcal{V}$ was studied starting with E. Acerbi and G. Mingione [1] who established gradient estimates and qualitative properties of minimizers of energies involving variable exponents. The energy functional defined by $\mathcal{V}$ was used to illustrate models for strongly anisotropic materials. More precisely, in a material made of different components, the exponent $p(x)$ dictates the geometry of a composite that changes its hardening exponent according to the point.

The functional $\mathcal{P}_{p,q}$ defined in (1.2) appears as un upgraded version of $\mathcal{V}$. Again, in this case, the modulating coefficient $a(x)$ dictates the geometry of the composite made by two differential materials, with hardening exponents p and q, respectively.

The functionals displayed in (1.2) fall in the realm of the so-called functionals with nonstandard growth conditions of (p,q)-type, according to Marcellini's terminology. These are functionals of the type in (1.1), where the energy density satisfies

$$|\xi|^p \le f(x,\xi) \le |\xi|^q + 1, \quad 1 \le p \le q.$$

Another significant model example of a functional with (p,q)-growth studied by Mingione *et al.* [7–18] is given by

$$u \mapsto \int_\Omega |Du|^p \log(1+|Du|)dx, \quad p \ge 1,$$

which is a logarithmic perturbation of the p-Dirichlet energy.

General models with (p,q)-growth in the context of geometrically constrained problems have been recently studied by C. De Filippis [19]. This seems to be the first work dealing with (p,q)-conditions with manifold constraint. Regularity results are also established in [19], by using an approximation technique relying on estimates obtained through a careful use of difference quotients. A key role is played by the method developed by L. Esposito, F. Leonetti and G. Mingione [20] in order to prove the equivalence between the absence of Lavrentiev phenomenon and the extra regularity of the minimizers for unconstrained, non-autonomous variational problems.

In the next section of this paper we recall some basic properties of the function spaces of variable exponent. Section 3 is devoted to a class of isotropic double phase problems driven by a differential operator introduced by A. Azzollini *et al.* [2,3]. In section 3 we are concerned with the anisotropic abstract setting. In such a way we respond partially to some problems raised by G. Mingione [26], who addressed several research questions concerning double phase problems with variable exponent. We refer to V.D. Rădulescu *et al.* [6,16] for more details concerning the results included in this paper as well as for detailed proofs and perspectives.

2. ABSTRACT FRAMEWORK

In this section, we recall the basic properties of Lebesgue and Sobolev spaces with variable exponent. For more details, we refer to the recent monograph of V.D. Rădulescu and D.D. Repovš [32].

Throughout this paper, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

Define the class

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For every $p \in C_+(\overline{\Omega})$, we let

$$p^+ = \sup_{x\in\Omega} p(x) \qquad \text{and} \qquad p^- = \inf_{x\in\Omega} p(x).$$

For any $p \in C_+(\overline{\Omega})$, the Lebesgue space with variable exponent is defined by

$$L^{p(x)}(\Omega) = \left\{ u;\ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\}.$$

Then $L^{p(x)}(\Omega)$ is a Banach space if it is endowed with the (Luxemburg) norm

$$|u|_{p(x)} = \inf \left\{ \mu > 0;\ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

We also recall that $L^{p(x)}(\Omega)$ is reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions with compact support are dense in $L^{p(x)}(\Omega)$ if $p^+ < \infty$.

The standard embedding between Lebesgue spaces generalizes to the framework of spaces with variable exponent. More precisely, if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ is continuous.

Let $L^{p'(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. Then for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the following Hölder-type inequality holds:

$$\left| \int_\Omega uv\,dx \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)}.$$

Alternatively, in arguments dealing with Lebesgue spaces with variable exponent, an important role is played by the "modular" of $L^{p(x)}(\Omega)$, which is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following properties are true:

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+},$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-},$$

$$|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0.$$

We define the Sobolev space with variable exponent by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) :\ |\nabla u| \in L^{p(x)}(\Omega) \}.$$

On $W^{1,p(x)}(\Omega)$ we may consider one of the following equivalent norms: either

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

or

$$\|u\|_{p(x)} = \inf \left\{ \mu > 0;\ \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \le 1 \right\}.$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure of the set of compactly supported $W^{1,p(x)}$-functions with respect to the norm $\|\cdot\|_{p(x)}$. When smooth functions are dense, we can also use the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. By the Poincaré inequality, the space $W_0^{1,p(x)}(\Omega)$ can be alternatively defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The Banach space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is separable and reflexive. Moreover, if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1 \le p_2$ in Ω then embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega)$ is continuous.

As in the case of Lebesgue spaces, we define

$$\varrho_{p(x)}(u) = \int_\Omega |\nabla u(x)|^{p(x)}\, dx.$$

If (u_n), $u \in W_0^{1,p(x)}(\Omega)$, then the following properties are true:

$$\|u\| > 1 \ \Rightarrow\ \|u\|^{p^-} \le \varrho_{p(x)}(u) \le \|u\|^{p^+},$$

$$\|u\| < 1 \ \Rightarrow\ \|u\|^{p^+} \le \varrho_{p(x)}(u) \le \|u\|^{p^-},$$

$$\|u_n - u\| \to 0 \quad \Leftrightarrow \quad \varrho_{p(x)}(u_n - u) \to 0.$$

Let $p^*(x)$ be the "critical Sobolev exponent", that is,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

We point out that if $q \in C^+(\overline{\Omega})$ and $q(x) \le p^*(x)$ for all $x \in \overline{\Omega}$, then $W^{1,p(\cdot)}(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\Omega)$. This embedding is compact if

$$\inf\{p^*(x) - q(x);\ x \in \Omega\} > 0.$$

As remarked in [32, pp. 8–9], the function spaces with variable exponent have some curious properties. We recall some of them such as:

(i) If $1 < p^- \le p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is smooth, then the formula

$$\int_\Omega |u(x)|^p dx = p \int_0^\infty t^{p-1} \left|\{x \in \Omega;\ |u(x)| > t\}\right| dt$$

has no variable exponent analogue.

(ii) Variable exponent Lebesgue spaces do not have the "mean continuity property". More precisely, if p is continuous and nonconstant in an open ball B, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) The function spaces with variable exponent are never translation invariant. The use of convolution is also limited, for instance the Young inequality

$$|f * g|_{p(x)} \leq C\, |f|_{p(x)}\, \|g\|_{L^1}$$

holds if and only if p is constant.

(iv) Generally, the space of smooth functions with compact support is no longer dense in $W^{1,p(x)}(\Omega)$.

3. ISOTROPIC DOUBLE PHASE PROBLEMS

In [2, 3], A. Azzollini *et al.* introduced a new class of differential operators with a variational structure. They considered nonhomogeneous operators of the type

$$\operatorname{div}[\phi'(|\nabla u|^2)\nabla u],$$

where $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ has a different growth near zero and at infinity. Such a behaviour occurs if $\phi(t) = 2[\sqrt{1+t} - 1]$, which corresponds to the prescribed mean curvature operator (capillary surface operator), which is defined by

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

More generally, $\phi(t)$ behaves like $t^{q/2}$ for small t and $t^{p/2}$ for large t, where $1 < p < q < N$. Such a behaviour is fulfilled if

$$\phi(t) = \frac{2}{p}\,[(1+t^{q/2})^{p/q} - 1],$$

which generates the differential operator

$$\operatorname{div}\left[(1+|\nabla u|^q)^{(p-q)/q}|\nabla u|^{q-2}\nabla u\right].$$

We assume that the potential $\phi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfies the following hypotheses:

(ϕ_1) $\phi(0) = 0$,
(ϕ_2) there exists $c_1 > 0$ such that $\phi(t) \geq c_1 t^{p/2}$ if $t \geq 1$ and $\phi(t) \geq c_1 t^{q/2}$ if $0 \leq t \leq 1$,
(ϕ_3) there exists $c_2 > 0$ such that $\phi(t) \leq c_2 t^{p/2}$ if $t \geq 1$ and $\phi(t) \leq c_2 t^{q/2}$ if $0 \leq t \leq 1$,
(ϕ_4) there exist $0 < \mu < 1$ and $s > 0$ such that $2t\phi'(t) \leq s\mu\phi(t)$ for all $t \geq 0$,
(ϕ_5) the mapping $t \mapsto \phi(t^2)$ is strictly convex.

Remark 3.1. Observe that by (ϕ_4) and (ϕ_5), we infer that

$$\phi(t) < 2t\phi'(t) \leq s\mu\phi(t) \quad \text{for all } t \geq 0,$$

hence $s\mu > 1$.

Since our hypotheses allow that ϕ' approaches 0, problems driven by the differential operator $\operatorname{div}[\phi'(|\nabla u|^2)\nabla u]$ are degenerate and no ellipticity condition is assumed.

We observe that our hypotheses imply that

$$\phi(|\nabla u|^2) \simeq \begin{cases} |\nabla u|^p, & \text{if } |\nabla u| \gg 1, \\ |\nabla u|^q, & \text{if } |\nabla u| \ll 1. \end{cases}$$

This different growth at zero and at infinity of the principal part as well as the lack of compactness of the problem advise us not to use classical Sobolev spaces and to introduce a new functional framework.

We denote by $\|\cdot\|_r$ the Lebesgue norm for all $1 \leq r \leq \infty$ and by $C_c^\infty(\mathbb{R}^N)$ the space of all C^∞ functions with a compact support.

Definition 3.2. We define the function space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{L^p+L^q} := \inf\{\|v\|_p + \|w\|_q;\ v \in L^p(\mathbb{R}^N),\ w \in L^q(\mathbb{R}^N),\ u = v + w\}.$$

We observe that the function space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ contains both $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$. Due to standard continuous embeddings, we also observe that spaces of the type $L^p(\Omega) + L^q(\Omega)$ are of interest only if Ω has infinite measure.

For more properties of the Orlicz space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ we refer to M. Badiale, L. Pisani, and S. Rolando [4, Section 2].

3.1. STATEMENT OF THE PROBLEM

We consider the following quasilinear Schrödinger-type equation with lack of compactness:

$$-\operatorname{div}[\phi'(|\nabla u|^2)\nabla u] + a(x)|u|^{\alpha-2}u = f(x,u) \quad \text{in } \mathbb{R}^N \ (N \geq 3). \tag{3.1}$$

We assume that α, p, q are real numbers satisfying the following properties:

$$\begin{cases} 1 < p < q < N, \\ 1 < \alpha < \frac{p^* q'}{p'}, \\ \max\{\alpha, q\} < s < p^* := \frac{pN}{N-p}, \end{cases} \tag{3.2}$$

where s is the same as in hypothesis (ϕ_4) and p' denotes the conjugate exponent of p, that is, $p' = p/(p-1)$.

We suppose that the potential a in problem (3.1) is singular and it satisfies the following hypotheses:

(a_1) $a \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ and $\operatorname{essinf}_{\mathbb{R}^N} a =: a_0 > 0$,
(a_2) $\lim_{x\to 0} a(x) = \lim_{|x|\to\infty} a(x) = +\infty$.

A potential satisfying these conditions is $a(x) = \exp(|x|)/|x|$, for $x \in \mathbb{R}^N \setminus \{0\}$.

We assume that the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function characterized by the following conditions:

(f_1) $f(x,u) = o(u^{\alpha-1})$ as $u \to 0^+$, uniformly for a.e. $x \in \mathbb{R}^N$,
(f_2) $f(x,u) = O(u^{s-1})$ as $u \to \infty$, uniformly for a.e. $x \in \mathbb{R}^N$,
(f_3) there exists $\theta > \alpha$ such that $0 < \theta F(x,u) \le u f(x,u)$ for all $u > 0$, a.e. $x \in \mathbb{R}^N$, where $F(x,u) = \int_0^u f(x,t)dt$,
(f_4) if $\alpha < q$ then $\lim_{u\to+\infty} F(x,u)/u^q = +\infty$ uniformly for a.e. $x \in \mathbb{R}^N$.

We denote

$$\|u\|_{p,q} = \|u\|_{L^p+L^q}.$$

A key role in our arguments is played by the function space

$$\mathcal{X} := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|},$$

where

$$\|u\| := \|\nabla u\|_{p,q} + \left(\int_{\mathbb{R}^N} a(x)|u|^\alpha dx\right)^{1/\alpha}.$$

We notice that $\mathcal{X}$ is continuously embedded in the reflexive Banach space $\mathcal{W}$ defined in [3, p. 202], where $\mathcal{W}$ is the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm $\|u\| = \|\nabla u\|_{p,q} + \|u\|_\alpha$.

Definition 3.3. A weak solution of problem (3.1) is a function $u \in \mathcal{X} \setminus \{0\}$ such that for all $v \in \mathcal{X}$

$$\int_{\mathbb{R}^N} \left[\phi'(|\nabla u|^2)\nabla u \nabla v + a(x)|u|^{\alpha-2}uv - f(x,u)v\right] dx = 0.$$

The energy associated to problem (3.1) is the functional $\mathcal{E} : \mathcal{X} \to \mathbb{R}$ with unbalanced growth defined by

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}^N} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} a(x)|u|^\alpha dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

Then the functional $\mathcal{E}$ is well-defined on $\mathcal{X}$, of class C^1 (see also A. Azzollini [2, Theorem 2.5]). Moreover, for all u, $v \in \mathcal{X}$ its Gâteaux directional derivative is given by

$$\mathcal{E}'(u)(v) = \int_{\mathbb{R}^N} \left[\phi'(|\nabla u|^2)\nabla u \nabla v + a(x)|u|^{\alpha-2}uv - f(x,u)v\right] dx.$$

The following existence property holds.

Theorem 3.4. *Assume that hypotheses* (3.2)*,* (a_1)*,* (a_2)*,* (f_1)*–*(f_4)*, and* (ϕ_1)*–*(ϕ_5) *are fulfilled. Then problem* (3.1) *admits at least one weak nontrivial solution.*

Remark 3.5. (i) We point out that a related existence property was established by A. Azzollini, P. d'Avenia, and A. Pomponio [3, Theorem 1.3] but under the assumption that the potential a reduces to a positive constant. The setting described in Theorem 3.4 is different and corresponds to variable potentials that blow-up both at the origin and at infinity. The lack of compactness due to the unboundedness of the domain is handled in [3] by restricting the study to the case of "radially symmetric" weak solutions. In such a setting, a key role in the arguments developed in [3] is played by the compact embedding of a related function space with radial symmetry into a certain class of Lebesgue spaces.

(ii) The abstract setting corresponding to Theorem 3.4 is general and cannot be reduced to radially symmetric solutions, due to the presence of the general potential a. A central role in the arguments developed in [3] is played by the fact that the space $\mathcal{W}$ is continuously embedded in $L^{p^*}(\mathbb{R}^N)$, provided that $1 < p < \min\{q, N\}$, $1 < p^*q'/p'$ and $\alpha \in (1, p^*q'/p')$. By interpolation, the same continuous embedding holds in every Lebesgue space $L^r(\mathbb{R}^N)$ for every $r \in [\alpha, p^*]$.

We point out that with a similar analysis we can treat the case of potentials satisfying

$$\liminf_{|x|\to\infty} a(x) = 0,$$

which is a particular *critical frequency case*, see for details J. Byeon and Z.Q. Wang [13].

The existence of solutions of problem (3.1) in the case of a null potential a was established by H. Berestycki and P.L. Lions [10], where the authors used a "double-power" growth hypothesis on the nonlinearity, that is, $f(x,\cdot)$ has a subcritical behaviour at infinity and a supercritical growth near the origin.

3.2. SKETCH OF THE PROOF OF THEOREM 3.4

A key role in the proof is played by the following version of the mountain pass lemma (see H. Brezis and L. Nirenberg [12]). As pointed out by H. Brezis and F. Browder [11], the mountain pass theorem "extends ideas already present in Poincaré and Birkhoff". We refer to P. Pucci and V.D. Rădulescu [28] for a survey on the mountain pass theorem.

Theorem 3.6. *Let* $\mathcal{X}$ *be a real Banach space and assume that* $\mathcal{E} : \mathcal{X} \to \mathbb{R}$ *is a* C^1*-functional that satisfies the following geometric hypotheses:*

(i) $\mathcal{E}(0) = 0$ *and there exist positive numbers* a *and* r *such that* $\mathcal{E}(u) \geq a$ *for all* $u \in \mathcal{X}$ *with* $\|u\| = r$*,*
(ii) *there exists* $e \in \mathcal{X}$ *with* $\|e\| > r$ *such that* $\mathcal{E}(e) < 0$*.*

Set

$$\mathcal{P} := \{p \in C([0,1];\mathcal{X});\ p(0)=0,\ p(1)=e\}$$

and

$$c := \inf_{p\in\mathcal{P}} \sup_{t\in[0,1]} \mathcal{E}(p(t)).$$

Then there exists a sequence $(u_n) \subset \mathcal{X}$ *such that*

$$\lim_{n\to\infty} \mathcal{E}(u_n) = c \quad \textit{and} \quad \lim_{n\to\infty} \|\mathcal{E}'(u_n)\|_{\mathcal{X}^*} = 0.$$

Moreover, if $\mathcal{E}$ *satisfies the Palais–Smale condition at the level* c*, then* c *is a critical value of* $\mathcal{E}$*.*

Step 1. The energy $\mathcal{E}$ has a mountain pass geometry.

Fix $r \in (0,1)$ and let $u \in \mathcal{X}$ with $\|u\| = r$. Fix $\varepsilon > 0$ small enough. By straightforward computation and using hypotheses (ϕ_1) and (ϕ_2), we deduce that

$$\mathcal{E}(u) \geq c\,\|\nabla u\|^q_{p,q} + \left(\frac{1}{\alpha} - \frac{\varepsilon}{a_0}\right) \int\limits_{\mathbb{R}^N} a(x)|u|^\alpha dx - C\,\|u\|^{p^*}_{p^*}. \tag{3.3}$$

Recall that $\max\{\alpha, q\} < p^*$, see hypothesis (3.2).

Next, for $r \in (0,1)$ small enough, relation (3.3) yields that there exists a positive number a such that

$$\mathcal{E}(u) \geq a \quad \text{for all } u \in \mathcal{X} \text{ with } \|u\| = r. \tag{3.4}$$

This shows the existence of a "mountain" around the origin.

Next, we fix $w \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$. Using hypotheses (ϕ_3) and (3.2) we conclude that $\lim_{t\to+\infty} \mathcal{E}(tw) = -\infty$. Thus, there exists $t_0 > 0$ such that $\mathcal{E}(t_0 w) < 0$. This establishes the existence of a "valley" over the chain of mountains.

Step 2. The associated min-max value given by Theorem 3.6 is positive.

Set $c := \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} \mathcal{E}(p(t))$, where

$$\mathcal{P} := \{p \in C([0,1];\mathcal{X});\ p(0)=0,\ p(1)=t_0 w\}.$$

We observe that for all $p \in \mathcal{P}$ we have $c \geq \mathcal{E}(p(0)) = \mathcal{E}(0) = 0$. In fact, we claim that

$$c > 0. \tag{3.5}$$

Arguing by contradiction, we assume that $c = 0$. In particular, this means that for all $\varepsilon > 0$ there exists $q \in \mathcal{P}$ such that $0 \leq \max_{t\in[0,1]} \mathcal{E}(q(t)) < \varepsilon$. Fix $\varepsilon < a$, where a is given by (3.4). Then $q(0) = 0$ and $q(1) = t_0 w$, hence $\|q(0)\| = 0$ and $\|q(1)\| > r$. Using the continuity of q, there exists $t_1 \in (0,1)$ such that $\|q(t_1)\| = r$, hence $\|\mathcal{E}(q(t_1))\| = a > \varepsilon$, which is a contradiction. This shows that our claim (3.5) is true.

Applying Theorem 3.6, we find a Palais–Smale sequence for the level $c > 0$, that is, a sequence $(u_n) \subset \mathcal{X}$ such that

$$\lim_{n\to\infty} \mathcal{E}(u_n) = c \quad \text{and} \quad \lim_{n\to\infty} \|\mathcal{E}'(u_n)\|_{\mathcal{X}^*} = 0. \tag{3.6}$$

Step 3. The associated Palais–Smale sequence is bounded.

By hypothesis (f_3), we deduce that

$$\begin{aligned} c + O(1) + o(\|u_n\|) &= \mathcal{E}(u_n) - \frac{1}{\theta}\mathcal{E}'(u_n)u_n \\ &\geq \int\limits_{\mathbb{R}^N} \left[\frac{1}{2}\phi(|\nabla u_n|^2) - \frac{1}{\theta}\phi'(|\nabla u_n|^2)|\nabla u_n|^2\right] dx \\ &\quad + \left(\frac{1}{\alpha} - \frac{1}{\theta}\right) \int\limits_{\mathbb{R}^N} a(x)|u_n|^\alpha dx. \end{aligned} \tag{3.7}$$

Next, by hypothesis (ϕ_4), we deduce that for all $t \geq 0$

$$\frac{1}{2}\phi(t) - \frac{1}{\theta}\phi'(t)t \geq \frac{1-\mu s}{2}\phi(t),$$

where $\mu s \in (0,1)$. Hypothesis (f_3) yields that $\theta > \alpha$. Thus, returning to (3.7) we deduce that there exists $c_0 > 0$ such that for all $n \in \mathbb{N}$

$$\mathcal{E}(u_n) - \frac{1}{\theta}\mathcal{E}'(u_n)u_n \geq c_0 \left[\min\{\|\nabla u_n\|_{p,q}^q, \|\nabla u_n\|_{p,q}^p\} + \int\limits_{\mathbb{R}^N} a(x)|u_n|^\alpha dx\right]. \tag{3.8}$$

Combining relations (3.7) and (3.8), we deduce that the sequence $(u_n) \subset \mathcal{X}$ is bounded.

Since $\mathcal{X}$ is a closed subset of $\mathcal{W}$, using Proposition 2.5 in [3] we deduce that the sequence (u_n) converges weakly (up to a subsequence) in $\mathcal{X}$ and strongly in $L^s_{loc}(\mathbb{R}^N)$ to some u_0. We show in what follows that u_0 is a solution of problem (3.1).

Fix $\zeta \in C_c^\infty(\mathbb{R}^N)$ and set $\Omega := \operatorname{supp}(\zeta)$. Define

$$A(u) = \frac{1}{2}\int\limits_{\Omega} \phi(|\nabla u|^2)dx + \frac{1}{\alpha}\int\limits_{\Omega} a(x)|u|^\alpha dx$$

and $B(u) = \int_\Omega F(x,u)dx$. Using (3.6) we have

$$A'(u_n)(\zeta) - B'(u_n)(\zeta) \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

Since $u_n \to u_0$ in $L^s(\Omega)$ and the mapping $u \mapsto F(x,u)$ is compact from $\mathcal{X}$ into L^1, it follows that

$$B(u_n) \to B(u_0) \text{ and } B'(u_n)(\zeta) \to B'(u_0)(\zeta) \quad \text{as } n \to \infty. \tag{3.10}$$

Combining relations (3.9) and (3.10) we deduce that

$$A'(u_n)(\zeta) \to B'(u_0)(\zeta) \quad \text{as } n \to \infty. \tag{3.11}$$

Using hypothesis (ϕ_5), we obtain that the nonlinear mapping A is convex. Therefore

$$A(u_n) \leq A(u_0) + A'(u_n)(u_n - u_0) \quad \text{for all } n \in \mathbb{N}. \tag{3.12}$$

Using (3.11) in combination with $u_n \rightharpoonup u_0$ in $\mathcal{X}$, relation (3.12) yields $\limsup_{n\to\infty} A(u_n) \leq A(u_0)$. But A is lower semicontinuous, since it is convex and continuous. It follows that $A(u_0) \leq \liminf_{n\to\infty} A(u_n)$. We conclude that

$$A(u_n) \to A(u_0) \quad \text{as } n \to \infty.$$

Next, we deduce that

$$\nabla u_n \to \nabla u_0 \quad \text{as } n \to \infty \text{ in } L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx \to \int_{\mathbb{R}^N} a(x)|u_0|^\alpha dx \quad \text{as } n \to \infty.$$

Therefore

$$\int_\Omega \phi'(|\nabla u_0|^2)\nabla u_0 \nabla\zeta dx + \int_\Omega a(x)|u_0|^{\alpha-2}u_0\zeta dx - \int_\Omega f(x,u_0)\zeta dx = 0.$$

By density, we obtain that this identity holds for all $\zeta \in \mathcal{X}$, hence u_0 is a solution of problem (3.1).

Step 4. The proof of Theorem 3.4 completed.

We show that $u_0 \neq 0$.

Using the fact that (u_n) is a Palais–Smale sequence, relation (3.6) implies that if n is a positive integer sufficiently large then

$$\begin{aligned}
\frac{c}{2} &\leq E(u_n) - \frac{1}{2}E'(u_n)u_n \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left[\phi(|\nabla u_n|^2) - \phi'(|\nabla u_n|^2)|\nabla u_n|^2\right] dx \\
&\quad + \left(\frac{1}{\alpha} - \frac{1}{2}\right)\int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx + \int_{\mathbb{R}^N} \left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right] dx.
\end{aligned}$$

Using hypothesis (ϕ_5) that concerns the convexity of the map $t \mapsto \phi(t^2)$, we deduce that $\phi(t^2) - \phi(0) \leq \phi'(t^2)t^2$. Using now ($\phi_1$) we obtain $\phi(t^2) \leq \phi'(t^2)t^2$, hence

$$\phi(|\nabla u_n|^2) \leq \phi'(|\nabla u_n|^2)|\nabla u_n|^2. \tag{3.13}$$

We first assume that $\alpha \geq 2$. Thus, relations (3.2) and (3.13) combined with hypothesis (f_3) imply that for all n large enough we have

$$\frac{c}{2} \leq \int_{\mathbb{R}^N} \left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right] dx \leq \frac{1}{2}\int_{\mathbb{R}^N} f(x,u_n)u_n dx. \tag{3.14}$$

Hypotheses (f_1) and (f_2) show that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \leq \varepsilon |u|^{p^*-1} + C_\varepsilon |u|^{\alpha-1}, \quad \text{for all } u \in \mathbb{R}, \text{ a.e. } x \in \mathbb{R}^N.$$

Returning to (3.14) we obtain for all n large enough

$$\frac{c}{2} \leq \frac{\varepsilon}{2} \|u_n\|_{p^*}^{p^*} + C_\varepsilon \|u_n\|_\alpha^\alpha.$$

Since (u_n) is bounded in $L^{p^*}(\mathbb{R}^N)$, we fix $\varepsilon > 0$ small enough such that

$$\frac{\varepsilon}{2} \sup_n \|u_n\|_{p^*}^{p^*} \leq \frac{c}{4}.$$

It follows that for all $n \geq n_0$ we have $c \leq 4C_0 \|u_n\|_\alpha^\alpha$, where C_0 is a positive constant.

In order to show that $u_0 \neq 0$ we argue by contradiction. Assume that $u_0 = 0$. In particular, this implies that

$$u_n \to 0 \quad \text{in } L^\alpha_{loc}(\mathbb{R}^N). \tag{3.15}$$

Let k be a positive integer and set

$$\omega := \{x \in \mathbb{R}^N;\ 1/k < |x| < k\}. \tag{3.16}$$

By (3.15), it follows that if k is large enough, then for all $n \geq n_0$, we have $C_0 \int_\omega |u_n|^\alpha dx \leq c/8$. Therefore,

$$\begin{aligned} \frac{c}{8} &\leq C_0 \int\limits_{\mathbb{R}^N \setminus \omega} |u_n|^\alpha dx \\ &\leq \frac{C_0}{\inf_{|x| \leq 1/k} a(x)} \int\limits_{|x| \leq 1/k} a(x)|u_n|^\alpha dx + \frac{C_0}{\inf_{|x| \geq k} a(x)} \int\limits_{|x| \geq k} a(x)|u_n|^\alpha dx \\ &\leq C_0 M \left[\frac{1}{\inf_{|x| \leq 1/k} a(x)} + \frac{1}{\inf_{|x| \geq k} a(x)} \right], \end{aligned} \tag{3.17}$$

where $M = \sup_n \int_{\mathbb{R}^N} a(x)|u_n|^\alpha dx$. Choosing k large enough and using hypothesis (a_2), relation (3.17) implies that $c = 0$, a contradiction.

It remains to study the case $1 < \alpha < 2$. Relations (3.2) and (3.13) imply that for all n large enough we have

$$\frac{c}{2} \leq E(u_n) - \frac{1}{2} E'(u_n) u_n \leq \left(\frac{1}{\alpha} - \frac{1}{2} \right) \int\limits_{\mathbb{R}^N} a(x)|u_n|^\alpha dx + \frac{1}{2} \int\limits_{\mathbb{R}^N} f(x, u_n) u_n dx. \tag{3.18}$$

We argue again by contradiction and assume that $u_0 = 0$. With the same choice of ω as in (3.16) and with similar estimates in (3.18) as above, we obtain a contradiction. The proof is now complete. □

3.3. COMMENTS AND PERSPECTIVES

(i) We expect that new and interesting results can be established if the nonhomogeneous operator in problem (3.1) is a replaced by a differential operator with two competing potentials ϕ_1 and ϕ_2. We refer to operators of the type

$$\operatorname{div}\left[(\phi_1'(|\nabla u|^2)+\phi_2'(|\nabla u|^2))|\nabla u|^2\right],$$

where ϕ_1 and ϕ_2 have different growth decay. This new abstract framework is inspired by the analysis developed in [32, Chapter 3.3] in the framework of nonlinear problems with variable exponents.

(ii) A new research direction in strong relationship with several relevant applications is the study of problems described by the nonlocal term

$$M\left[\int_{\mathbb{R}^N}\phi(|\nabla u|^2)|\nabla u|^2\right].$$

We refer here to the pioneering papers by P. Pucci *et al.* [29,30] related to Kirchhoff problems involving nonlocal operators associated to the standard Laplace, p-Laplace or $p(x)$-Laplace operators.

(iii) We refer to N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš [27] for the study of a class of double-phase problems with reaction of arbitrary growth.

4. ANISOTROPIC DOUBLE PHASE PROBLEMS

In this section we are concerned with the study of a nonlinear problem whose associated energy is a double-phase anisotropic functional. The main result establishes the existence of an unbounded continuous spectrum in an abstract setting introduced by I.H. Kim and Y.H. Kim [22].

We study the following nonlinear problem:

$$\begin{cases} -\operatorname{div}(\phi(x,|\nabla u|)\nabla u)-\operatorname{div}(\psi(x,|\nabla u|)\nabla u)=\lambda f(x,u) & \text{in } \Omega\,, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\phi,\psi:\Omega\times[0,\infty)\to[0,\infty)$ satisfying the following hypotheses:

(H1) the mappings $\phi(\cdot,\xi)$ and $\psi(\cdot,\xi)$ are measurable on Ω for all $\xi\geq 0$ and $\phi(x,\cdot)$, $\psi(x,\cdot)$ are locally absolutely continuous on $[0,\infty)$ for almost all $x\in\Omega$,

(H2) for some $p_1,p_2\in C_+(\overline{\Omega})$, there exist $a_1\in L^{p_1'}(\Omega)$ and $a_2\in L^{p_2'}(\Omega)$ and $b>0$ such that

$$|\phi(x,|v|)v|\leq a_1(x)+b|v|^{p_1(x)-1},\quad |\psi(x,|v|)v|\leq a_2(x)+b|v|^{p_2(x)-1}$$

for almost all $x\in\Omega$ and for all $v\in\mathbb{R}^N$, (H3) there exists $c > 0$ such that

$$\phi(x,\xi) \geq c\xi^{p_1(x)-2}, \quad \phi(x,\xi) + \xi\frac{\partial\phi}{\partial\xi}(x,\xi) \geq c\xi^{p_1(x)-2}$$

and

$$\psi(x,\xi) \geq c\xi^{p_2(x)-2}, \quad \psi(x,\xi) + \xi\frac{\partial\psi}{\partial\xi}(x,\xi) \geq c\xi^{p_2(x)-2}$$

for almost all $x \in \Omega$ and for all $\xi > 0$.

Assume that $q \in C_+(\overline{\Omega})$ and

(Q) $p_1(x) < q^- \leq q^+ < p_2(x) < p_1^*(x)$ for all $x \in \overline{\Omega}$.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that the following assumptions are fulfilled:

(f1) we have $tf(x,t) \geq 0$ for a.a. $(x,t) \in \Omega \times \mathbb{R}$ and there exists $m \in L^\infty(\Omega)_+ \setminus \{0\}$ such that

$$|f(x,t)| \leq m(x)|t|^{q(x)-1} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R},$$

(f2) there exist $M > 0$ and $\theta > p_1^+$ such that

$$0 < \theta F(x,t) \leq tf(x,t) \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \setminus \{0\},$$

where $F(x,t) := \int_0^t f(x,s)ds$.

For ϕ and ψ described in hypotheses (H1)–(H3) we set

$$A_0(x,t) := \int_0^t [\phi(x,s) + \psi(x,s)]s\,ds. \tag{4.2}$$

We also consider the following technical assumption.

(H4) For all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^N$, the following estimate holds:

$$0 \leq [\phi(x,|\xi|) + \psi(x,|\xi|)]\,|\xi|^2 \leq p_1^+ A_0(x,|\xi|).$$

We notice that our hypothesis (f1) implies that

$$0 \leq F(x,t) \leq \frac{m(x)}{q(x)}\,|t|^{q(x)} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{4.3}$$

Define the double-phase energy $A : W_0^{1,p_2(x)}(\Omega) \to \mathbb{R}$ by

$$A(u) := \int_\Omega A_0(x,|\nabla u|)dx,$$

where A_0 is defined in (4.2). Then $A \in C^1(W_0^{1,p_2(x)}(\Omega), \mathbb{R})$ and for all $u, v \in W_0^{1,p_2(x)}(\Omega)$

$$A'(u)(v) = \int_\Omega [\phi(x, |\nabla u|) + \psi(x, |\nabla v|)] \nabla u \cdot \nabla v dx.$$

Moreover, the operator $A : W_0^{1,p_2(x)}(\Omega) \to (W_0^{1,p_2(x)}(\Omega))^*$ is strictly monotone, weakly lower semicontinuous, and is a mapping of type (S_+), that is, if

$$u_n \rightharpoonup u \text{ in } W_0^{1,p_2(x)}(\Omega) \text{ as } n \to \infty \text{ and } \limsup_{n\to\infty} \langle A'(u_n) - A'(u), u_n - u\rangle \leq 0,$$

then

$$u_n \to u \text{ in } W_0^{1,p_2(x)}(\Omega) \text{ as } n \to \infty.$$

Set

$$B(u) := \int_\Omega F(x, u) dx, \quad u \in W_0^{1,p_2(x)}(\Omega).$$

According to S. Fučik, J. Nečas, J. Souček, and V. Souček [21, p. 117], the function $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ is a solution of problem (4.1) if and only if $A'(u) = \lambda B'(u)$.

We define the following Rayleigh-type quotients:

$$\lambda^* := \inf_{u \in W_0^{1,p_2(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega A_0(x, |\nabla u|) dx}{\int_\Omega F(x, u) dx}$$

and

$$\lambda_* := \inf_{u \in W_0^{1,p_2(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega (\phi(x, |\nabla u|) + \psi(x, |\nabla u|)) |\nabla u|^2 dx}{\int_\Omega u f(x, u) dx}.$$

The main result of this section establishes the following qualitative property.

Theorem 4.1. *Assume that hypotheses* (H1)–(H4), (f1), (f2), (Q) *are fulfilled. Then the following properties hold:*

(i) *problem* (4.1) *has a solution for all* $\lambda \geq \lambda^*$,
(ii) *problem* (4.1) *does not have any solution, provided that* $\lambda < \lambda_*$.

We now describe the main steps in the proof of Theorem 4.1.

Step 1. We have $\lambda^* > \lambda_* > 0$.

The proof combines hypothesis (H4) with related energy estimates.

Step. 2 We have

$$\lim_{u\to 0} \frac{A(u)}{B(u)} = \lim_{\|u\|\to\infty} \frac{A(u)}{B(u)} = +\infty.$$

This step describes the growth of the Rayleigh quotient $A(u)/B(u)$ near the origin and at infinity. The proof is based on hypotheses (H3), (H4) and relation (4.3).

The next step establishes that the infimum in $W_0^{1,p_2(x)}(\Omega)$ of the Rayleigh quotient $A(u)/B(u)$ is attained.

Step 3. There exists $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ such that

$$\lambda^* = \frac{A(u)}{B(u)}.$$

Step 4. The minimizer u found at Step 3 is a solution of problem (4.1) for $\lambda = \lambda^*$.

The basic idea is that

$$\lambda^* = \frac{A(u)}{B(u)} = \inf_{v \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}} \frac{A(v)}{B(v)}.$$

Fix arbitrarily $v \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ and consider the map

$$t \mapsto h(t) := \frac{A(u+tv)}{B(u+tv)},$$

which is defined in a neighbourhood of the origin. It follows that $h'(0) = 0$, hence

$$[A'(u+tv)B(u+tv) - A(u+tv)B'(u+tv)]_{|t=0} = 0.$$

Therefore

$$B(u) \int_\Omega [\phi(x, |\nabla u|) + \psi(x, |\nabla u|)] \nabla u \cdot \nabla v dx - A(u) \int_\Omega f(x,u) v dx = 0.$$

Since $A(u) = \lambda^* B(u)$, we conclude that u solves (4.1), hence λ^* is an eigenvalue of this problem.

Step 5. Problem (4.1) admits a solution for all $\lambda > \lambda^*$.

Step 6. For all $\lambda < \lambda_*$, problem (4.1) does not have a solution.

Combining these steps we obtain the conclusion of Theorem 4.1.

4.1. COMMENTS AND PERSPECTIVES

(i) We do not have any information about the contribution of real parameters satisfying $\lambda \in [\lambda_*, \lambda^*)$ even in simple cases, for instance if Ω is a ball or for particular values of ϕ, ψ and f.

(ii) The variable exponents $p_1(x)$ and $p_2(x)$ dictate the geometry of a composite that changes its hardening exponent according to the point.

(iii) A very interesting research direction is to extend the approach developed in this section to the abstract setting recently studied by Mingione *et al.* [7–9, 26], namely non-autonomous problems with associated energies of the type

$$u \mapsto \int_\Omega \left[|\nabla u|^{p_1(x)} + a(x) |\nabla u|^{p_2(x)} \right] dx \tag{4.4}$$

and

$$u \mapsto \int_{\Omega} \Big[|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)} \log(e + |x|) \Big]\, dx, \tag{4.5}$$

where $p_1(x) \le p_2(x)$, $p_1 \neq p_2$, and $a(x) \ge 0$. Considering two different materials with power hardening exponents $p_1(x)$ and $p_2(x)$ respectively, the coefficient $a(x)$ dictates the geometry of a composite of the two materials. When $a(x) > 0$ then $p_2(x)$-material is present, otherwise the $p_1(x)$-material is the only one making the composite. On the other hand, since the integral functional defined in (4.5) is degenerate on the zero set of the gradient, it is natural to ask us what happens if we modify the integrand in such a way that, also when $|\nabla u|$ is small, there is an unbalance between the two terms of the integrand. For instance, we can consider the functional

$$u \mapsto \int_{\Omega} \Big[|\nabla u|^{p_1(x)} + a(x)|\nabla u|^{p_2(x)} \log(1 + |x|) \Big]\, dx.$$

For the isotropic case, we refer for further comments to P. Baroni, M. Colombo and G. Mingione [7, pp. 376–377], including for remarks on degeneracy phenomena at the phase transition.

(iv) The study of the integral functionals defined in (4.4) and (4.5) corresponds to the analysis of the differential operators

$$-\mathrm{div}\,(\phi(x, |\nabla u|)\nabla u) - \mathrm{div}\,(a(x)\psi(x, |\nabla u|)\nabla u)$$

and

$$-\mathrm{div}\,(\phi(x, |\nabla u|)\nabla u) - \mathrm{div}\,(a(x)\psi(x, |\nabla u|)\log(e + |x|)\nabla u).$$

This approach can be developed not only in Sobolev spaces with variable exponents (like in the present work) but also in the more general framework of Musielak–Orlicz spaces (see [32, Chapter 4] for a collection of stationary problems studied in these function spaces).

(v) The problem analyzed in this paper corresponds to a subcritical setting, as described in hypothesis (Q). We appreciate that valuable research directions correspond either to the critical or to the supercritical framework (in the sense of Sobolev variable exponents). No results are known even for the "almost critical" case with lack of compactness, namely assuming that hypothesis (Q) is replaced with

(Q') $p_1(x) < q^- \le q^+ < p_2(x) \preceq p_1^*(x)$ for all $x \in \overline{\Omega}$,

where $p_2(x) \preceq p_1^*(x)$ means that there exists $z \in \Omega$ such that $p_2(z) = p_1^*(z)$ and $p_2(x) < p_1^*(x)$ for all $x \in \overline{\Omega} \setminus \{z\}$.

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Vicenţiu D. Rădulescu
vicentiu.radulescu@imar.ro

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Kraków, Poland

Institute of Mathematics
Physics and Mechanics
Jadranska 19, 1000 Ljubljana, Slovenia

Institute of Mathematics "Simion Stoilow"
Romanian Academy of Sciences
P.O. Box 1-764, 014700 Bucharest, Romania

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