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EXISTENCE RESULTS AND A PRIORI ESTIMATES FOR SOLUTIONS OF QUASILINEAR PROBLEMS WITH GRADIENT TERMS

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Abstract. In this paper we establish a priori estimates and then an existence theorem of positive solutions for a Dirichlet problem on a bounded smooth domain in $\mathbb{R}^N$ with a nonlinearity involving gradient terms. The existence result is proved with no use of a Liouville theorem for the limit problem obtained via the usual blow up method, in particular we refer to the modified version by Ruiz. In particular our existence theorem extends a result by Lorca and Ubilla in two directions, namely by considering a nonlinearity which includes in the gradient term a power of u and by removing the growth condition for the nonlinearity f at $u = 0$.

Keywords: existence result, quasilinear problems, a priori estimates.

Mathematics Subject Classification: 35J92, 35J70.

1. INTRODUCTION

We study the existence of positive solutions for the problem

$$\begin{cases} \Delta_p u + f(x, u, Du) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N, N > 1$, is a bounded smooth domain, $1 < p < N$, $f : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative and continuous function and there exist positive exponents r, q, s and ϑ such that

$$p - 1 < r \leq q < p_* - 1, \quad \text{with} \quad p_* = \frac{p(N-1)}{N-p}, \tag{1.2}$$

$$\vartheta \in \left(p - s - 1, \frac{p(r-s)}{r+1}\right), \tag{1.3}$$

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$$s \in \left[0, \min\left\{p-1, r - \frac{\vartheta}{p}(r+1)\right\}\right) \tag{1.4}$$

and positive constants c_0 and M with $c_0 \geq 1$, such that

(F) $\max\{0, u^r - Mu^s|\eta|^\vartheta\} \leq f(x,u,\eta) \leq c_0 u^q + Mu^s|\eta|^\vartheta,$

for all $(x,u,\eta) \in \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N$.

The exponent p_*, known as Serrin's exponent, is the optimal exponent for Liouville theorems for elliptic inequalities, obviously $p_* < p^* = Np/(N-p)$, where p^* is the critical exponent for the Sobolev's embeddings.

Observe that problem (1.1) does not have, in general, a variational structure because of the dependence on the gradient of the nonlinearity. Thus topological methods will be used to prove the existence of solutions. For a priori estimates for solutions of nonvariational problems we refer to [8] as well as to [33] for problems treated in the framework of uniformly elliptic operators.

Similar problems have been intensively studied in literature, especially when $p = 2$, for instance in the classical paper by Brezis and Turner [3] and to [7], while for results relative to boundary Dirichlet problems with a gradient term we refer to the pioneering paper by Ghergu and Radulescu [12] with a linear growth in the gradient, to [13, 14, 25]. Furthermore in [9], the authors study the competition between an anisotropic potential, a convection term $|\nabla u|$ and a singular nonlinearity, see also the recent paper [11] where also a diffusion term depending on u inside the divergence.

For further existence results concerning the p-Laplacian and when gradient terms are involved, we refer to the recent paper by Motreanu and Tanaka, [21], in which the authors develop an approach based on approximate solutions and on a new strong maximum principle, and to the paper of Radulescu, Xiang and Zhang, [26], where existence of nonnegative solutions for a p-Kirchhoff type problem driven by a non-local integro-differential operator with homogeneous Dirichlet boundary data is investigated.

A first natural approach to solve problem (1.1) is to consider radial solutions, this was done by Clèment, Manàsevich and Mitidieri in [4] for the case of a systems of the p-Laplacian case, but without a dependence of the nonlinearity on the gradient. Later in [1], Azizieh and Clement, studied the same problem, not in the radial form, under some additional conditions, precisely they treat the case $1 < p \leq 2$ because of a symmetry result by Damascelli and Pacella in [5], and consider a nonlinear function f not depending on x and on ∇u, and when Ω is a convex domain. In [27], Ruiz removes all these conditions in order to treat problem (1.1), with a nonlinearity f satisfying (F) in the subcase $s = 0$ and $r = q$. The case $q = r$, but $s > 0$ in (F), has been treated in a previous paper [10] in which we extended an existence result by Ruiz.

The idea to introduce an explicit dependence on a power of the solution u^s in the gradient term was motivated by [22] and [2], where the parabolic version of (1.1)–(F) is investigated.

Later, Lorca and Ubilla in [19], allowed f to be bounded from above and from below by different powers of u, actually they assume (F) with $s = 0$ for all $u \geq u_0 > 0$ together with a growth condition on f at $u = 0$ according to which $f \leq Cu^{p-1}$ for u small and uniformly for x and η.

Motivated by [19], we considered problem (1.1) with a nonlinearity satisfying (F) and we remove the growth condition at $u = 0$.

Before stating the main result of our paper, we briefly discuss the technique developed by Ruiz. In [27], Ruiz produces a slight modification of the well known blowup method by Gidas and Spruck (see [15, 16]) using the same technique but centered on a certain fixed point. Indeed, the main tool to obtain existence is to prove an a priori L^∞ estimates for positive solutions of a parametric version of (1.1) and then use the degree theory.

These bounds for positive solutions are obtained by contradiction, that is roughly by assuming the existence of a divergence sequence of solutions u_n of (1.1) attaining their maxima on a point x_n in Ω. Consequently, by using suitable scaling arguments "centered" on x_n, when $r = q$, a positive solution of the limit problem $-\Delta_p u \geq Cu^q$, $C > 0$, either in $\mathbb{R}^N$, if $x_n \to x_0 \in \Omega$ up to subsequences, or in the halfspace if $x_n \to x_0 \in \partial\Omega$ up to subsequences, is produced. Thus, a Liouville theorem gives the required contradiction. One of the difficulties in this procedure is the boundary case, so there are several papers ([1, 4]) in which, for instance, additional geometric conditions are imposed on the domain, in order to avoid the use of a Liouville type result on the halfspace. We mention also the approach of Polacik, Quitter and Souplet based on the doubling lemma, cf. [22].

Ruiz, in [27], proves a new variant of the blowup procedure, which again allows to avoid the boundary case, by centering the blow up function, involved in the scaling argument, on a fixed poin $y_0 \in \Omega$ instead on x_n.

Concerning the case $r < q$, treated first in [19], a new difficulty appears, namely the limit problem obtained via the blow up technique procedure is $-\Delta_p u \geq 0$. Since inequality $-\Delta_p u \geq Cu^q$ is no more in force, then the desired contradiction cannot be reached with a Liouville theorem, for further details we refer to Theorem 3.4. For this reason, Lorca and Ubilla developed a different approach with takes into account both the boundary case $x_0 \in \partial\Omega$ and the case $x_0 \in \Omega$.

In the case we consider, to deal with the factor u^s in the gradient term, we have to adapt the technique by Lorca and Ubilla, combined with a result in [10] which allows us remove the addition growth condition on f at $x = 0$ contained in [19].

The existence result, as in [19] is obtained via a Rabinowitz type theorem, [24], due to Azizieh, Clément in [1].

Precisely, the main result we obtain is the following.

Theorem 1.1. *Let condition* (*F*) *holds under* (1.2)*,* (1.3) *and* (1.4)*. Then, problem* (1.1) *has at least one positive solution.*

Theorem 1.1, when $s = 0$ reduces to Theorem 4.2 in [19], while when $q = r$ it reduces to Theorem 4.2 in [10], finally when $q = r$ and $s = 0$ it is exactly Theorem 4.2 in [27].

The paper is organized as follows. In Section 2, some preliminary lemmas are stated, then Section 3 is devoted to prove the main a priori estimates for solutions of the parametric problem associated to (1.1). Finally, in Section 4 we present the proof of Theorem 1.1, based on the a priori estimates obtained in the previous section together with an extension of the classical Rabinowitz bifurcation theorem.

2. PRELIMINARES

Before stating and proving the main result, we need to give several preliminary lemmas, which extend the analogous ones in [19], given for the case $s = 0$. In what follows C is a positive constant which may vary from one expression to another, but it is always independent to u.

First we state a result proved in [10], for further comments see the remark at the end of the statement.

Lemma 2.1. *Let u be a positive weak C^1 solution of the inequality*

$$-\Delta_p u \geq u^r - M u^s |Du|^\vartheta \tag{2.1}$$

in a domain $\Omega \subset \mathbb{R}^N$, where $r > p-1$ and

$$s \in \left[0, \min\left\{p-1, r - \frac{\vartheta}{p}(r+1)\right\}\right), \quad p - s - 1 < \vartheta < \frac{p(r-s)}{r+1}.$$

Take

$$\gamma \in (0, r), \quad \sigma \in [0, r) \quad \text{and} \quad \mu \in \left(0, \frac{p(r-\sigma)}{r+1}\right). \tag{2.2}$$

Let $R_0 > 0$ be fixed such that the corresponding ball B_{R_0} of radius R_0 is contained in Ω. Then, there exists a positive constant $C = C(N, p, r, \gamma, R_0)$ such that, for all $0 < R < R_0$,

$$\int_{B_R} u^\gamma dx \leq C R^{N - \frac{p\gamma}{r+1-p}}. \tag{2.3}$$

for all $\gamma \in (0, r)$. Similarly, there exists a positive constant $C = C(N, p, r, \gamma, \sigma, \mu, R_0)$ such that

$$\int_{B_R} u^\sigma |Du|^\mu dx \leq C R^{N - \frac{\mu(r+1-\sigma)+\sigma p}{r+1-p}}. \tag{2.4}$$

Remark 2.2. Lemma 2.1, when $s = 0$ in (2.1) and $\sigma = 0$ in (2.4) reduces to Lemma 2.1 in [27] and furthermore to Lemma 2.4 in [29] when $M = 0, s = 0, \vartheta = 0$ and $\sigma = 0$.

We will also make use of the following weak Harnack inequality, due to Trudinger in [31] (see also [28] for a similar result).

Theorem 2.3. *Let $u \geq 0$ be a weak solution of the inequality*

$$-\Delta_p u \geq 0 \text{ in } \Omega.$$

Take $\gamma \in [1, p_* - 1)$ and $R > 0$ such that $B_{2R} \subset \Omega$. Then there exists $C = C(N, p, \gamma)$ (independent of R) such that

$$\inf_{B_R} u \geq C R^{-N/\gamma} \|u\|_{L^\gamma(B_{2R})}.$$

3. A PRIORI ESTIMATES

In order to prove the existence of positive solutions for (1.1), we will use a degree argument. First we consider the following parametric problem

$$\begin{cases} \Delta_p u + f(x,u,Du) + \lambda = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where $\lambda \geq 0$ positive. Since we are going to use a degree argument, roughly we need nonexistence of solutions of (3.1) for λ large. Indeed, in the next lemma we prove that λ is bounded above by the L^∞ norm of u.

Lemma 3.1. *Let u be a positive solution of Problem* (3.1)*. Then there is a positive constant c depending only on Ω such that*

$$\lambda \leq c\left(\max_{x\in\overline{\Omega}} u\right)^{p-1}. \tag{3.2}$$

Proof. The proof is exactly that of Lemma 3.1 in [19], since it is essentially based on the nonnegativity of f and on a comparison result for the p-Laplacian (cf. [30]) applied to the positive solution v of

$$-\Delta_p v = 1 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega$$

and to $w(x) = (\lambda/2)^{1/(p-1)}u$. □

Theorem 3.2. *Assume that (F) holds. Assume that u is a nonnegative C^1 solution of* (3.1)*, then, there exists $C > 0$ such that $0 \leq u(x) + \lambda \leq C$ for all x in Ω.*

Proof. We argue by contradiction, and we suppose that there exist a sequence $(\lambda_n, u_n)_n$, where u_n is a positive solution of (3.1) with λ replaced by $\lambda_n \geq 0$ such that

$$\|u_n\|_\infty + \lambda_n \to \infty.$$

Consequently, being $\lambda_n \leq C\|u_n\|_\infty$ by virtue of Lemma 3.1, we may assume that $\|u_n\|_\infty \to \infty$. By regularity of u_n, there exists a sequence $(x_n)_n$ in Ω such that $\|u_n\|_\infty = u_n(x_n)$. For simplicity, we denote

$$S_n := \|u_n\|_\infty, \qquad \delta_n := d(x_n, \partial\Omega). \tag{3.3}$$

Furthermore, let $x_0 \in \overline{\Omega}$ be such that $x_n \to x_0$ as $n \to \infty$. We consider now two cases either $x_0 \in \Omega$ or $x_0 \in \partial\Omega$.

Case A. Suppose that $x_0 \in \Omega$. Define

$$\tilde{\delta}_n = \sup\{\delta > 0 : x \in B_\delta(x_n) \text{ and } u_n(x) > S_n/2\}.$$

Then, thanks to Lemma 3.3 in [19], there exists $\tilde{x_n} \in \Omega$ such that

$$d(x_n, \tilde{x_n}) = \tilde{\delta_n} \quad \text{and} \quad u_n(\tilde{x_n}) = S_n/2.$$

We now claim that there exist a real constant $c > 0$ and a number $\overline{n} \in \mathbb{N}$ such that for all $n \geq \overline{n}$ we have

$$0 < c < \tilde{\delta_n} S_n^{(q+1-p)/p}. \tag{3.4}$$

In what follows, c will denote a positive constant, which may vary from line to line, but always independent on n.

To prove the claim we follow the proof of Theorem 3.2 in [19] (cf. also the proof of Step 1 in [10]). We make a change of variable and define

$$w_n(x) = S_n^{-1} u_n(y),$$

where

$$y = M_n x + x_n, \quad M_n = S_n^{\frac{p-1-q}{p}}, \tag{3.5}$$

by the contradiction we have $S_n \to \infty$, as $n \to \infty$. The functions w_n are well defined at least in $B\big(0, \delta_n/M_n\big)$ and $w_n(0) = \|w_n\| = 1$. By standard calculations, we have $\nabla w_n(x) = S_n^{-1} M_n D u_n(y)$ so that $\Delta_p w_n(x) = S_n^{-q} \Delta_p u_n(y)$. Hence replacing u_n in (3.1) we get

$$\begin{aligned} -\Delta_p w_n(x) &= S_n^{-q}\big[f\big(M_n x + x_n, S_n w_n(x), S_n M_n^{-1} \nabla w_n(x)\big) + \lambda_n\big] \\ &:= \wp_n(x, w_n, \nabla w_n). \end{aligned} \tag{3.6}$$

By (3.5) and using condition $(F)_2$, we obtain

$$\begin{aligned} \wp_n(x, w_n, \nabla w_n) &\leq S_n^{-q}\big[c_0 S_n^q |w_n(x)|^q + M S_n^s |w_n(x)|^s S_n^{\vartheta\frac{q+1}{p}} |\nabla w_n(x)|^\vartheta + \lambda_n\big] \\ &= c_0 |w_n|^q + M S_n^{\vartheta\frac{q+1}{p}+s-q} |w_n|^s |\nabla w_n|^\vartheta + \lambda_n S_n^{-q} \\ &\leq c_0 |w_n|^q + M S_n^{\vartheta\frac{q+1}{p}+s-q} \|w_n\|^s |\nabla w_n|^\vartheta + \lambda_n S_n^{-q} \end{aligned}$$

and recalling that $\|w_n\| = 1$ we have for n sufficiently large

$$|\wp_n(x, w_n, \nabla w_n)| \leq c_0 |w_n|^q + c_1 |\nabla w_n|^\vartheta + c_2, \quad c_1, c_2 > 0. \tag{3.7}$$

indeed, $\vartheta\frac{q+1}{p} + s - q < 0$ thanks to (1.3) and $\lambda_n S_n^{-q} \leq c S_n^{-q+p-1} \to 0$ by virtue of Lemma 3.1. Consequently, the $C^{1,\tau}$ regularity result up to the boundary due to Lieberman ([18, Theorem 1], see also [6]) guarantees that

$$\|\nabla w_n\| \leq C \tag{3.8}$$

in the ball $B_1(0)$ for certain $C > 0$ independent of n. Consequently

$$M S_n^{\vartheta\frac{q+1}{p}-q+s} |\nabla w_n|^\vartheta + \lambda_n S_n^{-q} \to 0 \quad \text{as } n \to \infty \tag{3.9}$$

and

$$B_{\tilde{\delta_n} S_n^{(q+1-p)/p}}(0) \subset B_1(0) \subset B_{\delta_n S_n^{(q+1-p)/p}}(0).$$

Now, by using the mean value theorem we have

$$1/2 = w_n(0) - w_n(M_n^{-1}(\tilde{x_n} - x_n)) \leq \|\nabla w_n(\xi)\| M_n^{-1}\tilde{\delta_n} \leq C M_n^{-1}\tilde{\delta_n},$$

since $w_n(0) = S_n^{-1}u_n(x_n) = 1$ and

$$w_n(M_n^{-1}(\tilde{x_n} - x_n)) = S_n^{-1}u_n(M_n M_n^{-1}(\tilde{x_n} - x_n) + x_n) = S_n^{-1}u_n(\tilde{x_n}) = \frac{1}{2}.$$

Thus, the required bound from below for $S_n^{(q+1-p)/p}\tilde{\delta_n}$ is claimed since $M_n^{-1} = S_n^{(q+1-p)/p}$.

By Lemma 3.5 in [19], there exists an $\varepsilon > 0$ such that, passing to a subsequence, for all $\gamma > N(q+1-p)/p$ we have

$$\int\limits_{B_\varepsilon(x_n)} u_n^\gamma \to \infty. \tag{3.10}$$

Indeed, if $\tilde{\delta_n} \to 0$, then take ε so that for n sufficiently large $\tilde{\delta_n} < \varepsilon$ so that

$$\int\limits_{B_\varepsilon(x_n)} u_n^\gamma \geq \int\limits_{B_{\tilde{\delta_n}}(x_n)} u_n^\gamma \geq \frac{S_n^\gamma}{2}|B_{\tilde{\delta_n}}(x_n)| \geq c S_N^{\gamma - N(q+1-p)/p} \to \infty,$$

by the choice of γ. Otherwise, if $\tilde{\delta_n} \to \tilde{\delta} > 0$ then it is enough to take $\tilde{\delta_n} > \varepsilon$ to obtain (3.10).

Case B. Suppose now that $x_0 \in \partial\Omega$. By Lemma 3.6 in [19], there exist $\varepsilon > 0$ and a sequence $(y_n)_n$ in Ω such that for all $\gamma > N(q+1-p)/p$ passing to a subsequence, we have

$$\int\limits_{B_\varepsilon(y_n)} u_n^\gamma \to \infty. \tag{3.11}$$

In particular the construction of y_n starts from $z_n \in \partial\Omega$ such that $\delta_n = d(x_n, z_n)$. Then, denoting with ν_n the unit exterior normal of $\partial\Omega$, for $\varepsilon > 0$ small it is possible to define $y_n = z_n - \varepsilon\nu_n$. Now the proof proceed by consider two cases: if $u_n(y_n) \geq S_n/2$ or $u_n(y_n) < S_n/2$. The proof Lemma 3.6 in [19] is rather technical.

We are now ready to conclude the proof of the theorem. As in [19] we consider only the case when $x_n \to x \in \Omega$, since the case $x_n \to x \in \partial\Omega$ can be proved in the same way. Being $q < N(p-1)/(N-p)$, we can choose γ so that

$$\frac{N(q+1-p)}{p} < \gamma < \frac{N(p-1)}{N-p}.$$

Applying Theorem 2.3 with $R = \varepsilon/2$ and then using (3.10), we get

$$\inf\{u_n(x) : x \in B_{\varepsilon/2}(x_n))\} \geq C\varepsilon^{-N/\gamma}\|u_n\|_{L^\gamma(B_\varepsilon(x_n))} \tag{3.12}$$

On the other hand, by Lemma 2.1,

$$C\varepsilon^{N/\sigma}\inf\{u_n(x) : x \in B_{\varepsilon/2}(x_n))\} \leq \|u_n\|_{L^\sigma(B_{\varepsilon/2}(x_n))} \leq C\varepsilon^{N/\sigma - p/(r+1-p)}$$

for $\sigma \in (0, r)$, so that

$$\|u_n\|_{L^\gamma(B_\varepsilon(x_n))} \leq C\varepsilon^{N/\gamma - p/(r+1-p)}. \tag{3.13}$$

The fact that C can be chosen independent on n is due to the compactness and regularity of $\partial\Omega$. Inequalities (3.12) and (3.13) give the boundness of $\|u_n\|_{L^\gamma(B_\varepsilon(x_n))}$, thus the required contradiction is obtained thanks to (3.10). The a priori estimate is so proved. □

Remark 3.3. In the case when $r < q$ the limit problem obtained via the blow up technique procedure is $-\Delta_p u \geq 0$, indeed

$$\begin{aligned} -\Delta_p w_n(x) &\geq {S_n}^{r-q}{w_n}^r(x) - MS_n^{\vartheta\frac{q+1}{p}+s-q}|Dw_n(x)|^\theta + S_n^{-q}\lambda_n \\ &\geq {S_n}^{r-q}{w_n}^r(x) - MS_n^{\vartheta\frac{q+1}{p}+s-q}|Dw_n(x)|^\theta. \end{aligned}$$

Letting $n \to \infty$, thanks to (3.8), we get the assertion. Since inequality $-\Delta_p u \geq Cu^q$ is not in force, then the desired contradiction cannot be reached with a Liouville theorem. Indeed it holds the following well known result.

Theorem 3.4. *Consider the entire $\mathbb{R}^N$. Then*

- (i) *the inequality $-\Delta_p u \geq 0$ has a noncostant solution positive solution if and only if $N > p$;*
- (ii) *the inequality $-\Delta_p u \geq u^q$ has a positive solution if and only if $q > p_* - 1$ and $1 < N < p$.*

The part (i) is due to Serrin and Zou in [29], while the Liouville theorem (ii) is due to Mitidieri and Pohozaev in [20]. Concerning to positivity of solutions, it is a consequence of the strong maximum principle, cf. Vazquez [32], and for general inequalities see the book [23] by Pucci and Serrin.

Because of Theorem 3.4, Lorca and Ubilla proposed an alternative approach to overcome the problem, as we have seen in the proof of case B of the previous theorem.

4. PROOF OF THE MAIN THEOREM

Now, instead of using a particular version of the classical Krasnoselskii theorem, see [17], we use the following Rabinowitz type theorem, [24], due to Azizieh, Clément in [1], whose statement for completeness we include here.

Lemma 4.1. *Let E be a real Banach space and let $G : \mathbb{R}^+ \times E \to E$ be a continuous function mapping bounded subsets on relatively compact subsets. Suppose moreover that $G(0,0) = 0$ and that there exists $R > 0$ such that*

(i) *if $u \in E$, with $\|u\| \leq R$ and $u = G(0,u)$, then necessarily $u = 0$,*
(ii) *$deg(Id_E - G(0,.), B(0,R), 0) = 1$.*

Let J denote the set of solutions to the problem $u = G(t,u)$ in $\mathbb{R}^+ \times E$. Let $\mathcal{C}$ denote the component (closed connected subset maximal with respect to inclusion) of J to which $(0,0)$ belongs. Then if $\mathcal{C} \cap (\{0\} \times E) = \{(0,0)\}$, then $\mathcal{C}$ is unbounded in $\mathbb{R}^+ \times E$.

Proof of Theorem 1.1. As in [19], we apply Lemma 4.1 with $E = C^1(\overline{\Omega})$ as a Banach space with the uniform norm $\|\cdot\|$. Then in $\mathbb{R}^+ \times C^1(\overline{\Omega})$ we consider the map $G(t,u) = T(S(u)+t)$, $t \in [0,1]$, where the operator $S : C^1(\overline{\Omega}) \to C(\overline{\Omega})$ is defined by $S(u) = f(x,u,Du)$, while $T(v)\big(\in C^{1,\tau}(\Omega)\big)$ denotes the unique weak solution u of the problem

$$-\Delta_p u = v \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega.$$

The operator T is compact (see for instance Lemma 1.1 in [1]), consequently G is compact. In particular $G(0,0) = 0$ by (F).

Now, we show that G verifies assumption (i) of Lemma 4.1. To this aim, let u be a nonnegative nontrivial solution of (1.1), thus u solves $u = G(0,u)$. Now, multiplying the equation in (1.1) and integrating, arguing as in the proof of Theorem 4.2 in [10], we get, since $q < p^* - 1$

$$\begin{aligned}\int_\Omega |\nabla u|^p dx &\leq \Big[c_0 \int_\Omega u^{q+1} dx + M \int_\Omega u^{s+1} |\nabla u|^\vartheta dx\Big] \\ &\leq C\Big(\int_\Omega u^{p^*} dx\Big)^{\frac{q+1}{p^*}} + \Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{\vartheta}{p}} \Big(\int_\Omega u^{p^*} dx\Big)^{\frac{s+1}{p^*}}.\end{aligned}$$

Now, applying Sobolev inequality, we obtain that

$$\int_\Omega |\nabla u|^p dx \leq C\Big[\Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{q+1}{p}} + \Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{\vartheta+s+1}{p}}\Big].$$

Thus, since $q > p-1$ and $\theta > p-s-1$, we can conclude that

$$\int_\Omega |\nabla u|^p dx > c > 0.$$

By the compactness of G we can choose $R < c$, such that $\|u\|_{C^1} < R$, so that $\int_\Omega |\nabla u|^p < R < c$. This contradiction proves that u has to be trivial, in turn (i) is proved.

On the other hand condition (ii) follows from homotopy properties of degree. Now if we define $\mathcal{C} = \{u \in C^1(\overline{\Omega}), u \text{ is nonnegative}\}$, then $\mathcal{C}$ is bounded by virtue of Theorem 3.2, hence thanks to Lemma 4.1, condition $\mathcal{C} \cap (\{0\} \times E) = \{(0,0)\}$ cannot occur, thus the existence of a positive solution to problem (1.1) for $t = 0$ is guaranteed and the theorem is so proved. □

Remark 4.2. We point out that in our proof, the growth condition (4.8) assumed on f in [19] has been removed, thanks to an iteration of Hölder and Sobolev inequalities, according to an argument used in [27] for the subcase $s = 0$ and then adapted to the general case (F) in [10].

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REFERENCES

[1] C. Azizieh, P. Clément, *A priori estimates and continuation methods for positive solutions of p-Laplace equations*, J. Diff. Eq. **179** (2002), 213–245.

[2] J.P. Bartier, *Global behavior of solutions of a reaction-diffusion equation with gradient absorption in unbounded domains*, Asymptot. Anal. **46** (2006), 325–347.

[3] H. Brezis, R.E.L. Turner, *On a class of superlinear elliptic problems*, Comm. Partial Differential Equations **2** (1977), 601–614.

[4] P. Clément, R. Manásevich, E. Mitidieri, *Positive solutions for a quasilinear system via blow-up*, Comm. Partial Differential Equations **18** (1993), 2071–2106.

[5] L. Damascelli, F. Pascella, *Monotonicity and symmetry of solutions of p-Laplace equations, $1 < p \leq 2$, via the moving plane method*, Ann. Scuola Norm. Pisa, Cl. Sci. **26** (1998), 689–707.

[6] E. Di Benedetto, *$C^{1,\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), 827–850.

[7] D. De Figueiredo, P.L. Lions, R.D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures Appl. **61** (1982), 41–63.

[8] D. De Figueiredo, J. Yang, *A priori bounds for positive solutions of a non-variational elliptic system*, Comm. Partial Differential Equations **26** (2001), 2305–2321.

[9] L. Dupaigne, M. Ghergu, V. Rădulescu, *Lane-Emden-Fowler equations with convection and singular potential*, J. Math. Pures Appl. **87** (2007), 563–581.

[10] R. Filippucci, C. Lini, *Existence of solutions for quasilinear Dirichlet problems with gradient terms*, Discrete Contin. Dyn. Syst. Ser. S, Special Issue on the occasion of the 60th birthday of Vicentiu D. Radulescu, **12** (2019), 267–286.

[11] L. Gasinski, N.S. Papageorgiou, *Positive solutions for nonlinear elliptic problems with dependence on the gradient*, J. Differential Equations **263** (2017), 1451–1476.

[12] M. Ghergu, V. Rădulescu, *Nonradial blow-up solutions of sublinear elliptic equations with gradient terms*, Comm. Pure Appl. An. **3** (2004), 465–474.

[13] M. Ghergu, V. Rădulescu, *On a class of sublinear elliptic problems with convection term*, J. Math. Anal. Appl. **311** (2005), 635–646.

[14] M. Ghergu, V. Rădulescu, *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis*, vol. 37, Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, 2008.

[15] B. Gidas, J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), 525–598.

[16] B. Gidas, J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations **6** (1981), 883–901.

[17] M.A. Krasnoselskii, *Fixed point of cone-compressing or cone-extending operators*, Soviet. Math. Dokl. **1** (1960), 1285–1288.

[18] G.M. Lieberman, *Boundary regulary for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), 1203–1219.

[19] S. Lorca, P. Ubilla, *A priori estimate for a quasilinear problem depending on the gradient*, J. Math. Anal. Appl. **367** (2010), 60–74.

[20] E. Mitidieri, S.I. Pohozaev, *The absence of global positive solutions to quasilinear elliptic inequalities*, Dokl. Math. **57** (1998), 250–253.

[21] D. Motreanu, M. Tanaka, *Existence of positive solutions for nonlinear elliptic equations with convection terms*, Nonlinear Anal. **152** (2017) 38.

[22] P. Polacik, P. Quitter, P. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems, I: Elliptic equations and systems*, Duke Math. J. **139** (2007), 1203–1219.

[23] P. Pucci, J. Serrin, *The strong maximum principle*, Progress in nonlinear differential equations and their applications, vol. 73, Birkhäuser, 2007.

[24] P.H. Rabinowitz, *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain J. Math. **3** (1973), 161–202.

[25] V. Rădulescu, *Bifurcation and asymptotics for elliptic problems with singular nonlinearity. Elliptic and parabolic problems*, Progr. Nonlinear Differential Equations Appl., Birkhäuser, Basel **63** (2005), 389–401.

[26] V. Rădulescu, M. Xiang, B. Zhang, *Existence of solutions for a bi-nonlocal fractional p-Kirchhoff type problem*, Comput. Math. Appl. **71** (2016), 255–266.

[27] D. Ruiz, *A priori estimates and existence of positive solutions for strongly nonlinear problems*, J. Differential Equations **199** (2004), 96–114.

[28] J. Serrin, *Local behavior of solutions of quasilinear equations*, Acta Math. **111** (1964), 247–302.

[29] J. Serrin, H. Zou, *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189** (2002), 79–142.

[30] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150.

[31] N. Trudinger, *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

[32] J.L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.

[33] X.-J. Wang, Y.-B. Deng, *Existence of multiple solutions to nonlinear elliptic equations of nondivergence form*, J. Math. Anal. Appl. **189** (1995), 617–630.

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