Opuscula Math. 39, no. 2 (2019), 131–144
https://doi.org/10.7494/OpMath.2019.39.2.131 Opuscula Mathematica

ON UNIQUE SOLVABILITY OF A DIRICHLET PROBLEM WITH NONLINEARITY DEPENDING ON THE DERIVATIVE

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Communicated by Vicentiu D. Radulescu

Abstract. In this work we consider second order Dirichelet boundary value problem with nonlinearity depending on the derivative. Using a global diffeomorphism theorem we propose a new variational approach leading to the existence and uniqueness result for such problems.

Keywords: diffeomorphism, uniqueness, non-potential problems, variational methods, monotone methods, Palais–Smale condition.

Mathematics Subject Classification: 34A12, 47H30, 47J07.

1. INTRODUCTION

We consider the following boundary value problem with nonlinearity depending on the derivative:

$$\begin{cases} -\ddot{x} = f\left(t, x, \dot{x}\right), \\ x\left(0\right) = x\left(1\right) = 0, \end{cases} \tag{1.1}$$

where $f : [0, T] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $m \geqslant 0$ is a nonlinear term subject to some growth conditions which we will provide further in the text. Solutions to (1.1) are obtained in the space $\widetilde{H}_0^2 := H_0^1\left([0,1], \mathbb{R}^m\right) \cap H^2\left([0,1], \mathbb{R}^m\right)$, i.e. are understood as classical a.e. solutions. Due to the presence of term $\dot{x}$ in the nonlinearity f, the problem under consideration is not potential, which means that one cannot derive a classical Euler action functional for it. However, we can propose some variational approach towards (1.1) relying on minimization of the following action functional $\varphi : \widetilde{H}_0^2 \to \mathbb{R}$

$$\varphi\left(x\right) = \int_0^1 \left|\ddot{x}\left(t\right) + f\left(t, x\left(t\right), \dot{x}\left(t\right)\right)\right|^2 dt. \tag{1.2}$$

Direct calculation reveals that critical points to φ need not in general correspond to solutions to (1.1) and also it does not seem possible to impose condition on f which would guarantee that φ is weakly l.s.c. on $\widetilde{H}_0^2$. This suggests that the direct method of the calculus of variations is not applicable and also other variational approaches would not work directly by the first observation. However one has the existence theorem following directly form the Ekeland Variational Principle for which we need to recall some preparatory results.

Let X, Y be Banach spaces, and assume that $U \subset X$ is open. A mapping $f : U \to Y$ is said to be Gâteaux differentiable at $x_0 \in U$ if there exists a continuous linear operator $f'(x_0) : X \to Y$ such that for every $h \in X$

$$\lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t} = f'(x_0)h.$$

Operator $f'(x_0)$ is called the Gâteaux derivative of f at x_0. An operator $f : U \to Y$ is said to be Fréchet-differentiable at $x_0 \in U$ if there exists a continuous linear operator $f'(x_0) : X \to Y$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x_0 + h) - f(x_0) - f'(x_0)h\|_Y}{\|h\|_X} = 0.$$

Operator $f'(x_0)$ is called the Fréchet derivative of operator f at x_0. When f is Fréchet-differentiable, it is continuous and Gâteaux differentiable. A mapping f is continuously Fréchet-differentiable if $f' : X \ni x \mapsto f'(x) \in \mathcal{L}(X, Y)$ is continuous. If f is continuously Gâteaux differentiable then it is also continuously Fréchet-differentiable and thus it is called C^1. We say that $T : X \to Y$ is a diffeomorphism if it is a bijection and both T, T^{-1} are C^1 mappings.

A Gâteaux differentiable functional $J : X \to \mathbb{R}$ satisfies the Palais–Smale condition if every sequence $(u_n)_{n \in \mathbb{N}}$ such that $(J(u_n))_{n \in \mathbb{N}}$ is bounded and $J'(u_n) \to 0$ in X^*, has a convergent subsequence.

Theorem 1.1 ([6]). *Let E be a Banach space and $J : E \to \mathbb{R}$ be a C^1 functional which satisfies the Palais–Smale condition. Suppose in addition that J is bounded from below. Then the infimum of J is achieved at some point $u_0 \in E$ and u_0 is a critical point of J, i.e. $J'(u_0) = 0$.*

The above theorem may serve as a counterpart of the direct method of the calculus of variations (see [12] for detailed description of the direct method) in case when the functional is not weakly l.s.c. We recall that a bounded from below functional which satisfies the Palais–Smale condition is necessarily coercive. Therefore we see, by the Chain Rule and a direct calculation, that minimization of φ may lead to obtaining solutions to (1.1) provided that operator

$$T : \widetilde{H}_0^2 \to L^2$$

defined (a.e. pointwisely on $[0, 1]$) by the following formula

$$T(x) = \ddot{x}(\cdot) + f(\cdot, x(\cdot), \dot{x}(\cdot)) \tag{1.3}$$

has invertible derivative. Such a method is suggested by the following general result.

Theorem 1.2 ([9])**.** *Let B be a Banach space, H a Hilbert space. Let $T : B \to H$ be a C^1-mapping. Assume that:*

(A1) *for every $y \in H$ functional $B \ni x \mapsto \|T(x) - y\|_H^2 \in \mathbb{R}$ satisfies the Palais–Smale condition;*
(A2) *$T'(x) \in \mathrm{Isom}(B, H)$ for every $x \in B$.*

Then T is a diffeomorphism.

The above theorem is proved with the aid of Theorem 1.1 and the Mountain Pass Lemma. For some direct application towards global implicit function theorem we refer to [7, 8]. We would like to note that applications of Theorem 1.2, apart from the work [2] where a second order Dirichlet problem together with its discretization is considered, where provided for integral equations of various types, see [3, 9, 11]. General ideas concerning global invertibility via critical point theory methods are to be found in [5, 10]. For solvability of second order problems by another version of a global diffeomorphism theorem we refer to [14].

The approach proposed by us is different from already developed variational approaches for problems with nonlinearity depending on the derivative. We can mention paper [13] where the approach is based on variational methods combining super- and sub-solution and the existence of critical points via descending flow. On the other hand in [16] monotone interative technique based on the mountain pass geometry is utilized. In [1] results are obtained with the use of Ricerri three critical points theorem from [15].

This paper is organized as follows. In Section 2 we discuss some preparatory material. Section 3 contains main existence result and finally in Section 4 we consider slightly more general problem with non-zero boundary conditions and also provide some examples.

2. AUXILIARY RESULTS

In this section we recall some background results which we believe are needed for the understanding of the paper.

Assume that $X = \times_{i=1}^{k} X_i$, where X_i is a Banach space for $i = 1, \ldots, k$. X is supplied with a standard form, i.e. $\|\cdot\|_X = \sum_{i=1}^{k} \|\cdot\|_{X_i}$. For fixed $u \in X$ we denote $\iota_u : X_i \to X$ by formula $\iota_u(x) = (u_1 \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_n)$. For fixed $i = 1, \ldots, k$ we define $\pi_i : X \to X_i$ by $\pi_i(x) = x_i$. We say that T has an i-th partial derivative at point u if operator $T \circ \iota_u : X_i \to Y$ is Fréchet differentiable. The derivative of an operator $T \circ \iota_u$ at u will be denoted by $\partial^i T(u)$ and we will call it i-th partial derivative of T at u. If $\partial^i T(x)$ exists and it is continuous for every $i = 1, \ldots, n$, then T is Fréchet differentiable at point x and

$$T'(x) = \sum_{i=1}^{n} \partial^i T(x) \circ \pi_i.$$

Let V, X and Y be a Banach spaces. We recall the following result.

Theorem 2.1 ([17], Implicit Function Theorem)**.** *Let $T : V \times X \to Y$ be a C^1-operator. Assume that $T(v_0, x_0) = 0$ for some $(v_0, x_0) \in V \times X$ and $\partial^2 T(v_0, x_0)$ is an isomorphism. Then there exist open neighbourhoods of v_0, x_0, respectively U_v, U_x and unique C^1-function $\gamma : U_{v_0} \to U_{x_0}$ such that $\gamma(v_0) = x_0$ and $T(v, \gamma(v)) = 0$ for all $v \in U_v$.*

The Implicit Function Theorem is true in a more general setting. However, the above formulation is enough for our considerations.

If X is a normed space, $\varphi \in X^*$ and $x \in X$, then by $\langle \varphi, x \rangle_{X^*,X}$ we denote the duality pair, i.e. the action of $\varphi \in X^*$ on an element $x \in X$.

Theorem 2.2 ([17])**.** *Let H be a Hilbert space. If $T : H \to H^*$ is linear, continuous and strongly monotone, i.e. there exists $c > 0$ such that*

$$\langle Tx, x \rangle_{H^*,H} \geqslant c\|x\|_H^2$$

for every $x \in H$, then equation

$$Tx = y^*$$

has a unique solution $x \in H$ for every $y^* \in H^*$. In other words, T is an invertible operator.

For the information on the underlying function space setting we refer to [4]. Let us denote $L^2 := L^2([0,1], \mathbb{R}^m)$ We recall that space $H^1([0,1], \mathbb{R}^m)$ consists of all absolutely continuous functions $x : [0,1] \to \mathbb{R}^m$ such that a weak derivative $\dot{x}$ of x is integrable with square on $[0,1]$. Then

$$H_0^1([0,1], \mathbb{R}^m) = \left\{ x \in H^1([0,1], \mathbb{R}^m) : x(0) = x(1) = 0 \right\}.$$

$H_0^1([0,1], \mathbb{R}^m)$ is a Hilbert space equipped with a standard inner product

$$\langle x, y \rangle_{H_0^1} := \int_0^1 \langle \dot{x}(t) | \dot{y}(t) \rangle \, dt = \langle \dot{x} | \dot{y} \rangle_{L^2},$$

where $\langle \cdot | \cdot \rangle$ denotes an inner product in $\mathbb{R}^m$. The norm in $\mathbb{R}^m$ is denoted as an absolute value.

Since

$$H_0^1([0,1], \mathbb{R}^m) \hookrightarrow C([0,1], \mathbb{R}^m)$$

we see that

$$H_0^1([0,1], \mathbb{R}^m) \hookrightarrow L^\infty([0,1], \mathbb{R}^m).$$

Moreover

$$\|x\|_{L^\infty} \leq \|x\|_{H_0^1} \quad \text{and} \quad \pi \|x\|_{L^2} \leq \|x\|_{H_0^1}$$

for all $x \in H_0^1([0,1], \mathbb{R}^m)$.

By H^2 we denote those elements of $H^1([0,1],\mathbb{R}^m)$ whose second order weak derivative exists and it is integrable with square. One can prove that $\widetilde{H}_0^2$ defined in the Introduction can be equipped with an inner product $\langle x|y\rangle_{\widetilde{H}_0^2} := \langle \ddot{x}|\ddot{y}\rangle_{L^2}$ and it becomes also a Hilbert space.

The following theorem shows that every weak solution to (1.1), i.e. such a function $x \in H_0^1([0,1],\mathbb{R}^m)$ that

$$\int_0^1 \langle \dot{x}(t)|\dot{y}(t)\rangle\, dt = \int_0^1 \langle f(t, x(t), \dot{x}(t))|y(t)\rangle\, dt$$

for all $y \in H_0^1([0,1],\mathbb{R}^m)$ is a classical a.e. solution, i.e. an element of $\widetilde{H}_0^2([0,1],\mathbb{R}^m)$.

Theorem 2.3 ([12], du Bois-Reymond)**.** *Let $x \in L^2$ and $y \in L^1([0,1],\mathbb{R}^m)$ be such that*

$$\int_0^1 \langle x(t)|\dot{\varphi}(t)\rangle\, dt = -\int_0^1 \langle y(t)|\varphi(t)\rangle\, dt$$

for all $\varphi \in H_0^1([0,1],\mathbb{R}^m)$. Then there exists $c \in \mathbb{R}^m$ such that

$$x(t) = \int_0^t y(\tau)\, d\tau + c$$

for a.e. $t \in [0,1]$.

We need also some lemma which is necessary for the proper understanding of the assumptions which we are going to impose and also for obtaining the main result.

Lemma 2.4. *Assume that $A \in L^2([0,1],\mathcal{L}(\mathbb{R}^m))$ and $B \in L^\infty([0,1],\mathcal{L}(\mathbb{R}^m))$ satisfy the following conditions:*

(L1) *$B(t)$ is symmetric or antisymmetric for a.e. $t \in [0,1]$, i.e.*

$$\langle B(t)u|v\rangle = \pm\langle u|B(t)v\rangle$$

for all $u, v \in \mathbb{R}^m$ and a.e. $t \in [0,1]$;

(L2) *there exist $\alpha \in (0,1)$ and $C < 1-\alpha$ such that*

$$\left\langle \left(4\alpha A(t) \pm B^2(t)\right) u\middle|u\right\rangle \le \pi^2 C\,|u|^2$$

for all $u \in \mathbb{R}^m$ and a.e. $t \in [0,1]$.

Then operator $T : H_0^1([0,1],\mathbb{R}^m) \to \left(H_0^1([0,1],\mathbb{R}^m)\right)^*$ defined by the following formula

$$\begin{aligned}\langle Tx, y\rangle_{H_0^1} &:= \langle x|y\rangle_{H_0^1} - \langle B\dot{x}|y\rangle_{L^2} - \langle Ax|y\rangle_{L^2}\\ &= \int_0^1 \langle \dot{x}(t)|\dot{y}(t)\rangle\, dt - \int_0^1 \langle B(t)x(t)|y(t)\rangle\, dt - \int_0^1 \langle A(t)x(t)|y(t)\rangle\, dt\end{aligned}$$

is continuous and strongly monotone.

Remark 2.5. *If $m = 1$, then A and B are real-valued functions and conditions* (L1) *and* (L2) *are equivalent to the following one:*

(L3) *there exists $\alpha \in (0,1)$ such that $A(t) + \frac{1}{4\alpha} B^2(t) < \pi^2 (1-\alpha)$ for a.e. $t \in [0,1]$.*

Proof of Lemma 2.4. First, we observe that T is well defined. Indeed for fixed $x, y \in H_0^1([0,1], \mathbb{R}^m)$

$$\begin{aligned}\left|\langle Tx, y\rangle_{H_0^1}\right| &= \left|\langle x|y\rangle_{H_0^1} - \langle B\dot{x}|y\rangle_{L^2} - \langle Ax|y\rangle_{L^2}\right| \\ &\leq \|x\|_{H_0^1} \|y\|_{H_0^1} + \|B\dot{x}\|_{L^2} \|y\|_{L^2} + \|Ax\|_{L^2} \|y\|_{L^2} \\ &\leq \left(\left(1 + \tfrac{1}{\pi}\|B\|_{L^\infty}\right)\|x\|_{H_0^1} + \tfrac{1}{\pi}\|A\|_{L^2}\|x\|_{L^\infty}\right)\|y\|_{H_0^1}.\end{aligned}$$

By the above we see that T is also continuous. Now we show that T is strongly monotone. Using (L1) and (L2) we obtain

$$\begin{aligned}\langle Tx, x\rangle_{H_0^1} &= \langle \dot{x}|\dot{x}\rangle_{L^2} - \langle B\dot{x}|x\rangle_{L^2} - \langle Ax|x\rangle_{L^2} \\ &= (1-\alpha)\langle \dot{x}|\dot{x}\rangle_{L^2} + \alpha\langle \dot{x}|\dot{x}\rangle_{L^2} \pm \langle \dot{x}|Bx\rangle_{L^2} \\ &\quad + \tfrac{1}{4\alpha}\langle Bx|Bx\rangle_{L^2} - \tfrac{1}{4\alpha}\langle Bx|Bx\rangle_{L^2} - \langle Ax|x\rangle_{L^2} \\ &= \left\|\sqrt{\alpha}\dot{x} \pm \tfrac{1}{2\sqrt{\alpha}} Bx\right\|^2 + (1-\alpha)\|x\|_{H_0^1}^2 \mp \tfrac{1}{4\alpha}\left\langle B^2 x\middle|x\right\rangle_{L^2} - \langle Ax|x\rangle_{L^2} \\ &\geqslant (1-\alpha)\|x\|_{H_0^1}^2 - \left\langle Ax \pm \tfrac{1}{4\alpha}B^2 x\middle|x\right\rangle_{L^2} \geqslant (1-\alpha)\|x\|_{H_0^1}^2 - C\pi^2\|x\|_{L^2} \\ &\geqslant (1-\alpha-C)\|x\|_{H_0^1}.\end{aligned}$$

Finally, using Theorem 2.2 we obtain the assertion. □

3. MAIN RESULT

In this section we formulate the main existence result. The proof relies on checking that assumptions of Theorem 1.2 are satisfied, that is we need to define suitable functional suggested by formula (1.2) and then show that it satisfies the Palais–Smale condition and that local invertibility holds for the derivative of the solution operator, i.e. that conditions (A1) and (A2) of Theorem 1.2 are satisfied.

3.1. THE ASSUMPTIONS AND SOME LEMMAS

Let $m \geq 1$ and let $f : [0,1] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$. By f_x' and f_y' we denote partial derivatives of f with respect to the second and third argument. We will assume that

(H1) $f(\cdot, x, y)$ is Lebesgue-measurable on $[0,1]$ for every $x, y \in \mathbb{R}^m$;
(H2) $f(t, \cdot, y)$ is of class C^1 on $\mathbb{R}^m$ for a.e. $t \in [0,1]$ and every $y \in \mathbb{R}^m$;
(H3) $f(t, x, \cdot)$ is of class C^1 on $\mathbb{R}^m$ for a.e. $t \in [0,1]$ and every $x \in \mathbb{R}^m$;

(H4) there exist functions $a, b_0, b_1 \in L^2([0,1], \mathbb{R}_+)$ such that $\|b_0 + \pi b_1\|_{L^2} < \pi$ and

$$|f(t,x,y)| \leq a(t) + b_0(t)|x| + b_1(t)|y|$$

for a.e. $t \in [0,1]$ and every $x, y \in \mathbb{R}^m$;

(H5) there exist functions $\psi_0 \in L^2([0,1], \mathbb{R}_+)$ and $\psi_1 \in L^\infty([0,1], \mathbb{R}_+)$ and $g_0, g_1 \in C(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}_+)$ such that

$$|f'_x(t,x,y)|_{\mathcal{L}(\mathbb{R}^m)} \leq \psi_0(t)\, g_0(x,y),$$
$$|f'_y(t,x,y)|_{\mathcal{L}(\mathbb{R}^m)} \leq \psi_1(t)\, g_1(x,y)$$

for a.e. $t \in [0,1]$ and every $x, y \in \mathbb{R}^m$;

(H6) matrix $f'_y(t,x,y)$ is either symmetric or antisymmetric for a.e. $t \in [0,1]$ and every $x, y \in \mathbb{R}^m$;

(H7) there exist $\alpha \in (0,1)$ and $C < 1 - \alpha$ such that:
 – if $f'_y(t,x,y)$ is symmetric, then

$$\left\langle 4\alpha f'_x(t,x,y)\, u + \left(f'_y(t,x,y)\right)^2 u \middle| u \right\rangle \leq C\pi^2 |u|^2,$$

 – if $f'_y(t,x,y)$ is antisymmetric, then

$$\left\langle 4\alpha f'_x(t,x,y)\, u - \left(f'_y(t,x,y)\right)^2 u \middle| u \right\rangle \leq C\pi^2 |u|^2$$

for a.e. $t \in [0,1]$ and every $x, y, u \in \mathbb{R}^m$.

Lemma 3.1. *Suppose that $f : [0,1] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies assumptions* (H1)–(H3). *If $x : [0,1] \to \mathbb{R}^m$ is an a.e. differentiable function with the Lebesgue-measurable derivative $\dot{x}$, then $f(\cdot, x(\cdot), \dot{x}(\cdot))$ is Lebesgue-measurable.*

Proof. Let x and $\dot{x}$ be as in the assumptions. There exist sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ of simple functions such that $x_n \to x$ and $y_n \to \dot{x}$ a.e. on $[0,1]$. Then $f(\cdot, x_n(\cdot), y_n(\cdot))$ is obviously Lebesgue-measurable for every $n \in \mathbb{N}$ and by continuity of f we see that $f(t, x_n(t), y_n(t)) \to f(t, x(t), \dot{x}(t))$ for a.e. $t \in [0,1]$ which means that $f(\cdot, x(\cdot), \dot{x}(\cdot))$ is Lebesgue-measurable. $\square$

We define operator $F : \widetilde{H}_0^2 \to L^2$ (a.e. pointwisely on $[0,1]$) by

$$F(x) = f(\cdot, x(\cdot), \dot{x}(\cdot)) \tag{3.1}$$

and for simplicity of notation for a fixed $x \in \widetilde{H}_0^2$ we denote

$$F_0(x) = f'_x(\cdot, x(\cdot), \dot{x}(\cdot)),$$

$$F_1(x) = f'_y(\cdot, x(\cdot), \dot{x}(\cdot)).$$

Lemma 3.2. *Assume that* (H1)–(H3) *and* (H5) *hold. Then operator F defined by* (3.1) *is a C^1-mapping with a derivative at any fixed $x \in \widetilde{H}_0^2$ given by the formula*

$$F'(x)\,\xi = F_0(x)\,\xi + F_1(x)\,\dot{\xi}$$

for all $\xi \in \widetilde{H}_0^2$.

Proof. We could treat F as a composition of mappings

$$\widetilde{H}_0^2 \ni x \mapsto (x, \dot{x}) \in \widetilde{H}_0^2 \times H^1([0,1], \mathbb{R}^m)$$

and

$$\widetilde{H}_0^2 \times H^1([0,1], \mathbb{R}^m) \ni (x, y) \mapsto f(\cdot, x(\cdot), y(\cdot)) \in L^2$$

and then use the Chain Rule. □

3.2. PALAIS–SMALE CONDITION AND LOCAL INVERTIBILITY

We fix $y \in L^2$ and define similarly to (1.2) functional $\varphi : \widetilde{H}_0^2 \to \mathbb{R}$ by

$$\varphi(x) := \|T(x) - y\|_{L^2}^2 = \int_0^1 |\ddot{x}(t) + f(t, x(t), \dot{x}(t)) - y(t)|^2 \, dt. \tag{3.2}$$

Lemma 3.3. *Assume that* (H1)–(H5) *hold. Then functional* φ *given by* (3.2) *is coercive on* $\widetilde{H}_0^2$.

Proof. Fix any $x \in \widetilde{H}_0^2$. Using assumption (H4) and

$$\|\dot{x}\|_\infty \leq \|x\|_{\widetilde{H}_0^2} \text{ and } \|x\|_\infty \leq \frac{1}{\pi} \|x\|_{\widetilde{H}_0^2}$$

we obtain that

$$\begin{aligned}
\|F(x)\|_{L^2} &= \left(\int_0^1 |f(t, x(t), \dot{x}(t))|^2 \, dt \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^1 |a(t) + b_0(t) |x(t)| + b_1(t) |\dot{x}(t)||^2 \, dt \right)^{\frac{1}{2}} \\
&\leq \|a\|_{L^2} + \left(\int_0^1 |b_0(t) \|x\|_\infty + b_1(t) \|\dot{x}\|_\infty|^2 \, dt \right)^{\frac{1}{2}} \\
&= \|a\|_{L^2} + \|x\|_{\widetilde{H}_0^2} \left\| \tfrac{b_0}{\pi} + b_1 \right\|_{L^2}.
\end{aligned}$$

Hence, for $\gamma := \left\| \frac{b_0}{\pi} + b_1 \right\|_{L^2} < 1$ and $\delta := \|a\|_{L^2}$ we see that

$$\|F(x)\|_{L^2} \leq \gamma \|x\|_{\widetilde{H}_0^2} + \delta.$$

Therefore, for every $x \in \widetilde{H}_0^2$ we obtain

$$\|\ddot{x} + F(x) - y\|_{L^2} \geqslant \|\ddot{x}\|_{L^2} - \|F(x)\|_{L^2} - \|y\|_{L^2} \geqslant (1 - \gamma) \|x\|_{\widetilde{H}_0^2} - \delta - \|y\|_{L^2}.$$

Consequently, the assertion holds. □

From the above Lemma we see that any Palais–Smale sequence $(x_n)_{n\in\mathbb{N}} \subset \widetilde{H}_0^2$ for functional φ, that is such a sequence for which

(a) $(\varphi(x_n))_{n\in\mathbb{N}}$ is bounded;
(b) $\lim_{n\to\infty} \varphi'(x_n) = \mathbf{0}_{\left(\widetilde{H}_0^2\right)^*}$

is necessarily bounded. Then $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence which we denote by $(x_n)_{n\in\mathbb{N}}$ and its limit by x_0. Thus $x_n \to x_0$ and $\dot{x}_n \to \dot{x}_0$ strongly in $C([0,1],\mathbb{R}^m)$, possibly for a subsequence. This observation simplifies checking that the Palais–Smale condition is satisfied.

Lemma 3.4. *Assume that* (H1)–(H5) *are satisfied. Then* φ *given by* (3.2) *satisfies the Palais–Smale condition.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset \widetilde{H}_0^2$ be a Palais–Smale sequence. Therefore, $(x_n)_{n\in\mathbb{N}}$ can be assumed to have the above mentioned properties.

Since $x_n \rightharpoonup x_0$ in $\widetilde{H}_0^2$, then $x_n \to x_0$ and $\dot{x}_n \to \dot{x}_0$ strongly in $C([0,1],\mathbb{R}^m)$. A direct calculation yields

$$\varphi'(x_n)(x_n - x_0) - \varphi'(x_0)(x_n - x_0) = \int_0^1 |\ddot{x}_n(t) - \ddot{x}_0(t)|^2 \, dt + \sum_{k=1}^{7} r_k(x_n),$$

where

$$r_1(x_n) = \int_0^1 (\ddot{x}_0(t) - f(t, x_0(t), \dot{x}_0(t))) f'_x(t, x_0(t), \dot{x}_0(t)) (x_n(t) - x_0(t)) \, dt,$$

$$r_2(x_n) = \int_0^1 (\ddot{x}_0(t) - f(t, x_0(t), \dot{x}_0(t))) f'_y(t, x_0(t), \dot{x}_0(t)) (\dot{x}_n(t) - \dot{x}_0(t)) \, dt,$$

$$r_3(x_n) = \int_0^1 (f(t, x_n(t), \dot{x}_n(t)) - \ddot{x}_n(t)) f'_x(t, x_n(t), \dot{x}_n(t)) (x_n(t) - x_0(t)) \, dt,$$

$$r_4(x_n) = \int_0^1 (f(t, x_n(t), \dot{x}_n(t)) - \ddot{x}_n(t)) f'_y(t, x(t), \dot{x}(t)) (\dot{x}_n(t) - \dot{x}_0(t)) \, dt,$$

$$r_5(x_n) = \int_0^1 (f(t, x_0(t), \dot{x}_0(t)) - f(t, x_n(t), \dot{x}_n(t))) (\ddot{x}_n(t) - \ddot{x}_0(t)) \, dt,$$

$$r_6(x_n) = \int_0^1 y(t) (f'_x(t, x_0(t), \dot{x}_0(t)) - f'_x(t, x_n(t), \dot{x}_n(t))) (x_n(t) - x_0(t)) \, dt,$$

$$r_7(x_n) = \int_0^1 y(t) \left(f'_y(t, x_0(t), \dot{x}_0(t)) - f'_y(t, x_n(t), \dot{x}_n(t))\right) (\dot{x}_n(t) - \dot{x}_0(t)) \, dt.$$

Following the proof of Lemma 3.5 from [2] it is easy to observe that $r_i(x_n) \to 0$ for $i = 1, \ldots, 7$ whenever $n \to \infty$. $\square$

Lemma 3.5. *Assume that* (H1)–(H7) *hold. Then $T'(x)$ is invertible at every fixed $x \in \widetilde{H}_0^2$.*

Proof. Let us fix any $x \in \widetilde{H}_0^2$. We recall that

$$T'(x)\,\xi = \ddot{\xi} + F'(x)\,\xi = \ddot{\xi} + F_0(x)\,\xi + F_1(x)\,\dot{\xi}$$

for any $\xi \in \widetilde{H}_0^2$. Then $T'(x)$ is invertible at x if and only if the following Dirichlet problem

$$\begin{cases} \ddot{\xi} + f'_x(t, x(t), \dot{x}(t))\,\xi + f'_y(t, x(t), \dot{x}(t))\,\dot{\xi} = y(t), \\ \xi(0) = \xi(1) = 0, \end{cases} \tag{3.3}$$

has a unique solution for every fixed $y \in L^2$. By Theorem 2.3 we can consider an equivalent weak formulation

$$\begin{aligned} \int_0^1 \dot{\xi}(t)\dot{\varphi}(t)dt - \int_0^1 f'_x(t, x(t), \dot{x}(t))\,\xi(t)\varphi(t)dt& \\ - \int_0^1 f'_y(t, x(t), \dot{x}(t))\,\dot{\xi}(t)\varphi(t)dt &= \int_0^1 y(t)\varphi(t)dt \end{aligned} \tag{3.4}$$

for all $\varphi \in H_0^1([0,1], \mathbb{R}^m)$.

For a fixed $x \in H_0^1([0,1], \mathbb{R}^m)$ we define operator

$$T_x : H_0^1([0,1], \mathbb{R}^m) \to \left(H_0^1([0,1], \mathbb{R}^m)\right)^*$$

by

$$\begin{aligned} \langle T_x\xi, \varphi\rangle_{H_0^1} &= \int_0^1 \dot{\xi}(t)\dot{\varphi}(t)dt - \int_0^1 f'_x(t, x(t), \dot{x}(t))\,\xi(t)\varphi(t)dt \\ &\quad - \int_0^1 f'_y(t, x(t), \dot{x}(t))\,\dot{\xi}(t)\varphi(t)dt \\ &= \langle \xi|\varphi\rangle_{H_0^1} - \langle F_0(x)\,\xi|\varphi\rangle_{L^2} - \left\langle F_1(x)\,\dot{\xi}\middle|\varphi\right\rangle_{L^2} \end{aligned}$$

for all $\xi, \varphi \in H_0^1([0,1], \mathbb{R}^m)$. Let us observe that T_x is well defined, i.e. $T_x\xi \in \left(H_0^1([0,1], \mathbb{R}^m)\right)^*$ for all $\xi \in H_0^1([0,1], \mathbb{R}^m)$.

Indeed, let $\xi \in H_0^1([0,1],\mathbb{R}^m)$ be fixed. Since $T_x\xi$ is linear it is enough to show that it is also bounded. Using the Poincaré inequality we obtain

$$\begin{aligned}\left|\langle T_x\xi,\varphi\rangle_{H_0^1}\right| &\le \left|\langle \xi|\varphi\rangle_{H_0^1}\right| + \left|\langle F_0(x)\,\xi|\varphi\rangle_{L^2}\right| + \left|\left\langle F_1(x)\,\dot\xi\middle|\varphi\right\rangle_{L^2}\right| \\ &\le \|\xi\|_{H_0^1}\|\varphi\|_{H_0^1} + \|F_0(x)\,\xi\|_{L^2}\|\varphi\|_{L^2} + \left\|F_1(x)\,\dot\xi\right\|_{L^2}\|\varphi\|_{L^2} \\ &\le \left(\|\xi\|_{H_0^1} + \|F_0(x)\,\xi\|_{L^2} + \left\|F_1(x)\,\dot\xi\right\|_{L^2}\right)\|\varphi\|_{H_0^1}.\end{aligned}$$

Fix any $y \in L^2$. Observe that functional $y^* : H_0^1([0,1],\mathbb{R}^m) \to \mathbb{R}$ given by

$$y^*(\varphi) = \int_0^1 \varphi(t)y(t)\,dt$$

is linear and by the Poincaré inequality it is also continuous on $H_0^1([0,1],\mathbb{R}^m)$. Hence, if we show that problem $T_x\xi = y^*$ has a unique solution for every $y^* \in \left(H_0^1([0,1],\mathbb{R}^m)\right)^*$, then we know that problem (3.4) has a unique solution for every $y \in L^2$. In order to prove this assertion by Theorem 2.2 it is enough to show that T_x is strongly monotone. We see that by (H6)–(H7) the assumptions (L1) and (L2) of Lemma 2.4 are satisfied. This observation shows that $T'(x)$ is strongly monotone. Since it is also continuous by Theorem 2.2 we see that equation (3.4) is uniquely solvable. Thus, by Theorem 2.3, problem (3.3) is uniquely solvable. □

3.3. THE EXISTENCE RESULT

Finally, using Theorem 1.2 and Lemmas 3.4 and 3.5 we obtain

Theorem 3.6. *Assume that* (H1)–(H7) *hold. Then problem* (1.1) *has a unique solution.*

We see also that in fact a more general theorem can be obtained

Theorem 3.7. *Assume that* (H1)–(H7) *hold. Then operator* T *defined by* (1.3) *is a diffeomorphism.*

4. FINAL COMMENTS AND EXAMPLES

Now let us fix $v, w \in \mathbb{R}^m$ and consider, under assumptions (H1)–(H7), the following problem

$$\begin{cases} -\ddot x = f(t,x,\dot x), \\ x(0) = v,\ x(1) = w. \end{cases} \tag{4.1}$$

Defining $h_{vw} : [0,1] \to \mathbb{R}^m$ by the formula

$$h_{vw}(t) := (w-v)\,t + v$$

we observe that

$$\begin{cases} -\ddot{x} = f\left(t, x + h_{vw}\left(t\right), \dot{x} + w - v\right), \\ x\left(0\right) = x\left(1\right) = 0 \end{cases} \tag{4.2}$$

is uniquely solvable if and only if problem (4.1) has an unique solution. Moreover, if ϕ is solution to (4.2), then $\phi + h_{vw}$ is solution to (4.1).

See that if f satisfies assumptions (H1)–(H7), then function

$$[0,1] \times \mathbb{R}^m \times \mathbb{R}^m \ni (t,x,y) \mapsto f\left(t, x + h_{vw}\left(t\right), y + w - v\right) \in \mathbb{R}^m$$

satisfies this assumptions as well for every fixed $v, w \in \mathbb{R}^m$. We define a C^1-mapping $S : \mathbb{R}^m \times \mathbb{R}^m \times \widetilde{H}_0^2 \to L^2$ by formula

$$S\left(v,w,x\right) := \ddot{x}\left(\cdot\right) - f\left(\cdot, x\left(\cdot\right) + h_{vw}\left(\cdot\right), \dot{x}\left(\cdot\right) + w - v\right).$$

Reasoning exactly as previously we obtain a result which guarantees solvability of (4.2).

Lemma 4.1. *Fix any $v, w \in \mathbb{R}^m$. If f satisfies* (H1)–(H7)*, then $S(v, w, \cdot)$ is a diffeomorphism.*

Since for every $v, w \in \mathbb{R}^m$ problem (4.2) is uniquely solvable, there exists an global implicit function γ for S, i.e. there is

$$\gamma : \mathbb{R}^m \times \mathbb{R}^m \ni (v,w) \mapsto \left(S(v,w,\cdot)\right)^{-1}(0) \in \widetilde{H}_0^2 \tag{4.3}$$

such that $S\left(v, w, \gamma(v,w)\right) = 0$ for every $v, w \in \mathbb{R}^m$. Using Lemma 4.1 and the Implicit Function Theorem we obtain the following lemma.

Lemma 4.2. *Assume that* (H1)–(H7) *hold. Then function γ given by* (4.3) *is of class C^1.*

Define $\Lambda : \mathbb{R}^m \times \mathbb{R}^m \to H^2$ by formula $\Lambda(v,w) = \gamma(v,w) + h_{vw}$. See that operator Λ maps pair (v,w) into solution of (4.1). Using Lemma 4.2 we easily obtain

Theorem 4.3. *Assume that* (H1)–(H7) *hold. Then operator Λ is of class C^1.*

We give examples of functions which satisfies assumptions (H1)–(H7).

Example 4.4. *For $m = 1$, $f_1 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by*

$$f_1\left(t,x,y\right) = x + \tfrac{y}{2} + \cos(x)\cos(y)$$

satisfies assumptions (H1)–(H7)*.*

For $m = 2$ we define $f_2 : [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f_2(x_1,x_2,y_1,y_2) = \left(\tfrac{y_1}{2} + \sin(x_2+y_1), x_2 + \cos(x_1+y_2)\right).$$

We see that f_2 satisfies assumptions (H1)–(H7)*. See that derivative of f_2 with respect to (y_1,y_2) reads*

$$f'_{2|(y_1,y_2)}(x_1,x_2,y_1,y_2) = \begin{bmatrix} \frac{1}{2} + \cos(x_2+y_1) & 0 \\ 0 & -\sin(x_1+y_2) \end{bmatrix}.$$

and it is symmetric.

Taking $f_3 : [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ *given by*

$$f_3(x_1, x_2, y_1, y_2) = \left(\tfrac{y_2}{2} + \sin(x_2 + y_1), -\tfrac{y_1}{2} + \cos(x_1 + y_2)\right).$$

we have that derivative of f_3 *with respect to* (y_1, y_2) *is antisymmetric. Moreover,* f_3 *satisfies assumptions* (H1)–(H7).

REFERENCES

[1] G.A. Afrouzi, A. Hadjian, V.D. Rădulescu, *A variational approach of Sturm–Liouville problems with the nonlinearity depending on the derivative*, Bound. Value Probl. **2015** (2015) 81.

[2] M. Bełdziński, M. Galewski, *Global diffeomorphism theorem applied to the solvability of discrete and continuous boundary value problems*, J. Diff. Equ. Appl **24** (2018) 2, 277–290.

[3] D. Bors, A. Skowron, S. Walczak, *System described by Volterra type integral operators*, Dys. Cont. Dyn System B **19** (2014), 2401–2416.

[4] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010.

[5] I. Ekeland, *An inverse function theorem in Fréchet spaces*, Ann. Inst. Henri Poincaré, Anal. Non Lineaire **28** (2011) 1, 91–105.

[6] D.G. Figueredo, *Lectures on the Ekeland Variational Principle with Applications and Detours*, Preliminary Lecture Notes, SISSA, 1988.

[7] D. Idczak, *A global implicit function theorem and its applications to functional equations*, Discrete Contin. Dyn. Syst., Ser. B **19** (2014) 8, 2549–2556.

[8] D. Idczak, *On a generalization of a global implicit function theorem*, Adv. Nonlinear Stud. **16** (2016) 1, 87–94.

[9] D. Idczak, A. Skowron, S. Walczak, *On the diffeomorphisms between Banach and Hilbert spaces*, Adv. Nonlinear Stud. **12** (2012) 1, 89–100.

[10] G. Katriel, *Mountain pass theorems and global homeomorphism theorems*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **11** (1994) 2, 189–209.

[11] M. Majewski, *Control system defined by some integral operator*, Opuscula Math. **37** (2017) 2, 313–325.

[12] J. Mawhin, *Problemes de Dirichlet Variationnels Non Linéaires*, Séminaire de Mathématiques Supérieures, vol. 104, Montreal, 1987.

[13] D. Motreanu, M. Tanaka, *Multiple existence results of solutions for quasilinear elliptic equations with a nonlinearity depending on a parameter*. Ann. Mat. Pura Appl. **193** (2014) 5, 1255–1282.

[14] M. Rădulescu, S. Rădulescu, *Local inversion theorems without assuming continuous differentiability*, J. Math. Anal. Appl. **138** (1989) 2, 581–590.

[15] B. Ricceri, *A three critical points theorem revisited*, Nonlinear Anal. **70** (2009), 3084–3089.

[16] C. Torres Ledesma, *Existence of solutions for fractional Hamiltonian systems with nonlinear derivative dependence in* $\mathbb{R}$, J. Fract. Calc. Appl. **7** (2016) 2, 74–87.

[17] E. Zeidler, *Applied Functional Analysis. Main Principles and Their Applications*, Applied Mathematical Sciences, vol. 109, New York, Springer-Verlag, 1995.

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Received: April 27, 2018.
Revised: November 5, 2018.
Accepted: November 6, 2018.