

## OSCILLATION CRITERIA FOR EVEN ORDER NEUTRAL DIFFERENCE EQUATIONS

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*Communicated by Marek Galewski*

**Abstract.** In this paper, we present some new sufficient conditions for oscillation of even order nonlinear neutral difference equation of the form

$$\Delta^m(x_n + ax_{n-\tau_1} + bx_{n+\tau_2}) + p_n x_{n-\sigma_1}^\alpha + q_n x_{n+\sigma_2}^\beta = 0, \quad n \geq n_0 > 0,$$

where  $m \geq 2$  is an even integer, using arithmetic-geometric mean inequality. Examples are provided to illustrate the main results.

**Keywords:** even order, neutral difference equation, oscillation, asymptotic behavior, mixed type.

**Mathematics Subject Classification:** 39A10.

### 1. INTRODUCTION

In this paper, we are concerned with the even order mixed type neutral difference equation of the form

$$\Delta^m(x_n + ax_{n-\tau_1} + bx_{n+\tau_2}) + p_n x_{n-\sigma_1}^\alpha + q_n x_{n+\sigma_2}^\beta = 0, \quad (1.1)$$

where  $n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots\}$ , and  $n_0$  is a nonnegative integer, subject to the following conditions:

- (i)  $\{p_n\}$  and  $\{q_n\}$  are positive real sequences for all  $n \in \mathbb{N}(n_0)$ ,
- (ii)  $a$  and  $b$  are nonnegative real numbers,  $\tau_1, \tau_2, \sigma_1$  and  $\sigma_2$  are nonnegative integers,
- (iii)  $\alpha$  and  $\beta$  are ratios of odd positive integers and  $m \geq 2$  is an even integer.

Let  $\theta = \max\{\tau_1, \sigma_1\}$ . By a solution of the equation (1.1), we mean a real sequence  $\{x_n\}$  defined for all  $n \geq n_0 - \theta$ , and satisfying the equation (1.1) for all  $n \geq n_0$ .

A nontrivial solution of the equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

Since the difference equations have important applications in population dynamics, biology, probability theory, computer science and many other fields, there is a permanent interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of various types of even order/odd order difference equations, see references in this paper and their references.

For the oscillation of even order difference equations, see [1–3, 6–8, 12, 13]. Regarding the higher order mixed type neutral difference equations, Agarwal and Grace [4], Agarwal, Bohner, Grace and O'Regan [7], and Grace [9], considered several higher order mixed type neutral difference equations and established sufficient conditions for the oscillation of all solutions.

In [7], Agarwal, Bohner, Grace and O'Regan considered the  $m^{\text{th}}$  order mixed type neutral difference the equation (1.1) with  $\alpha = \beta = 1$ ,  $p_n \equiv p$  and  $q_n \equiv q$ , and established some sufficient conditions for the oscillation of the equation (1.1). Motivated by this observation, in this paper we investigate the oscillatory behavior of solutions of the equation (1.1), and hence the results obtained in this paper complement and generalize that of in [1, 3–6, 8, 9, 12, 13].

In Section 2, we present some basic lemmas which will be used to prove the main results. In Section 3, we obtain sufficient conditions for the oscillation of all solutions of the equation (1.1) by using arithmetic-geometric mean inequality. Examples are provided in Section 4 to illustrate the main results.

## 2. SOME PRELIMINARY LEMMAS

In this section, we present some lemmas, which are useful in proving the main results. We write

$$z_n = x_n + ax_{n-\tau_1} + bx_{n+\tau_2}.$$

**Lemma 2.1.** *Let  $a, b, c$  are positive quantities not all equal. Then*

$$\begin{aligned} a^\alpha + b^\alpha + c^\alpha &\geq \frac{1}{3^{\alpha-1}}(a+b+c)^\alpha \quad \text{if } \alpha \geq 1, \\ a^\alpha + b^\alpha + c^\alpha &\geq (a+b+c)^\alpha \quad \text{if } 0 < \alpha \leq 1. \end{aligned}$$

The proof is elementary and hence it is omitted.

**Lemma 2.2** ([3]). *Let  $\{u_n\}$  be a sequence of positive real numbers with  $\{\Delta^m u_n\}$  be of constant sign eventually and not identically zero eventually. Then there exists integer  $l \in \{0, 1, 2, \dots, m\}$  with  $m+l$  odd for  $\Delta^m u_n \leq 0$ , and  $m+l$  even for  $\Delta^m u_n \geq 0$  and for  $N > 0$  such that*

$$\Delta^j u_n > 0 \quad \text{for } j = 0, 1, 2, 3, \dots, l-1$$

and

$$(-1)^{j+l} \Delta^j u_n > 0 \quad \text{for } j = l, l+1, l+2, \dots, m-1$$

for all  $n \geq N$ .

**Lemma 2.3** ([11]). *Let  $\{u_n\}$  be a sequence of positive real numbers with  $\Delta^m u_n \leq 0$  and not identically zero eventually. Then there exists a large integer  $N$  such that*

$$u_n \geq \frac{(n-N)^{m-1}}{(m-1)!} \Delta^{m-1} u_{2^{m-l-1}n}, \quad \text{for } n \geq N,$$

where  $l$  is defined as in Lemma 2.2. Further if  $\{u_n\}$  is increasing, then

$$u_n \geq \frac{1}{(m-1)!} \left(\frac{n}{2^{m-1}}\right)^{m-1} \Delta^{m-1} u_n \quad \text{for all } n \geq 2^{m-1}N. \quad (2.1)$$

**Lemma 2.4.** *Let  $m$  be an even positive integer, and let  $\{x_n\}$  be a positive solution of the equation (1.1). Then there exists an integer  $n_1 \in \mathbb{N}(n_0)$  such that*

$$z_n > 0, \Delta z_n > 0, \Delta^{m-1} z_n > 0, \text{ and } \Delta^m z_n \leq 0 \text{ for all } n \geq n_1.$$

*Proof.* Since  $\{x_n\}$  is an eventually positive solution of the equation (1.1), there is an integer  $n_1 \in \mathbb{N}(n_0)$  such that  $x_n > 0$ ,  $x_{n-\tau_1} > 0$  and  $x_{n-\sigma_1} > 0$  for all  $n \geq n_1$ . Noting that  $a \geq 0$ ,  $b \geq 0$ , we have  $z_n > 0$  for all  $n \geq n_1$ , and

$$\Delta^m z_n = -p_n x_{n-\sigma_1}^\alpha - q_n x_{n+\sigma_2}^\beta \leq 0, \quad n \geq n_1.$$

It follows that  $\{\Delta^{m-1} z_n\}$  is decreasing and eventually of one sign. We claim that  $\Delta^{m-1} z_n > 0$  for  $n \geq n_1$ . Otherwise, if there is an integer  $n_2 \geq n_1$  such that  $\Delta^{m-1} z_{n_2} \leq 0$  for  $n \geq n_2$ , that is,

$$\Delta^{m-1} z_{n_2} = -c \quad (c > 0),$$

which implies that

$$\Delta^{m-1} z_n \leq -c \quad \text{for } n \geq n_2.$$

Summing the last inequality from  $n_2$  to  $n-1$ , we have

$$\Delta^{m-2} z_n \leq \Delta^{m-2} z_{n_2} - c(n-n_2).$$

Letting  $n \rightarrow \infty$ , we obtain  $\lim_{n \rightarrow \infty} \Delta^{m-2} z_n = -\infty$ , which implies that  $\{z_n\}$  is eventually negative by Lemma 2.2. This contradiction shows that  $\Delta^{m-1} z_n > 0$  for all  $n \geq n_1$ . Again from Lemma 2.2 and noting that  $m$  is even, we have  $\Delta z_n > 0$  for all  $n \geq n_1$ . This completes the proof.  $\square$

### 3. OSCILLATION RESULTS

In this section, we obtain some sufficient conditions for all the solutions of the equation (1.1) to be oscillatory. From the form of the equation (1.1) the assumption of existence of a positive solution leads to contradiction since the proof for the opposite case is similar.

For our convenience, we introduce the following notations:

$$\begin{aligned} P_n &= \min\{p_{n-\tau_1}, p_n, p_{n+\tau_2}\}, \\ Q_n &= \min\{q_{n-\tau_1}, q_n, q_{n+\tau_2}\}, \end{aligned}$$

and

$$R_n = K_1 P_n + K_2 Q_n,$$

where  $K_1$  and  $K_2$  are some positive constants.

**Theorem 3.1.** *Assume that  $\alpha < 1 < \beta$ . If the first order difference inequality*

$$\Delta w_n + \frac{A_n}{(1+d_1+d_2)} \frac{\lambda}{(m-1)!} (n-\sigma_1)^{m-1} w_{n+\tau_1-\sigma_1} \leq 0, \quad (3.1)$$

where

$$\begin{aligned} A_n &= \eta_1^{-\eta_1} \eta_2^{-\eta_2} P_n^{\eta_1} \left( \frac{Q_n}{3^{\beta-1}} \right)^{\eta_2}, \quad \eta_1 = \frac{\beta-1}{\beta-\alpha}, \quad \eta_2 = \frac{1-\alpha}{\beta-\alpha}, \\ d_1 &= \begin{cases} a^\alpha & \text{if } a \leq 1, \\ a^\beta & \text{if } a \geq 1 \end{cases} \quad \text{and} \quad d_2 = \begin{cases} b^\alpha & \text{if } b \leq 1, \\ b^\beta & \text{if } b \geq 1 \end{cases} \end{aligned}$$

has no positive solution for some  $\lambda \in (0, 1)$  and for all  $n \geq n_0$ , then every solution of the equation (1.1) is oscillatory.

*Proof.* Assume that  $\{x_n\}$  is a positive solution of the equation (1.1). Then there exists an integer  $n_1 \geq n_0$  such that  $x_n > 0$ ,  $x_{n-\sigma_1} > 0$  and  $x_{n-\tau_1} > 0$  for all  $n \geq n_1$ . By the definition of  $z_n$  we have  $z_n > 0$  for all  $n \geq n_1$ . Now from the equation (1.1), we obtain

$$\Delta^m z_n = -p_n x_{n-\sigma_1}^\alpha - q_n x_{n+\sigma_2}^\beta \leq 0$$

for all  $n \geq n_1$ . From Lemma 2.4 we have  $\Delta z_n > 0$  for all  $n \geq n_1$ .

Now we discuss the different cases for  $a$  and  $b$ .

*Case 1.* Suppose  $a \leq 1$  and  $b \leq 1$ . Then from the equation (1.1), we have

$$a^\alpha \Delta^m z_{n-\tau_1} + a^\alpha p_{n-\tau_1} x_{n-\sigma_1-\tau_1}^\alpha + a^\alpha q_{n-\tau_1} x_{n+\sigma_2-\tau_1}^\beta = 0, \quad n \geq n_1, \quad (3.2)$$

and

$$b^\alpha \Delta^m z_{n+\tau_2} + b^\alpha p_{n+\tau_2} x_{n-\sigma_1+\tau_2}^\alpha + b^\alpha q_{n+\tau_2} x_{n+\sigma_2+\tau_2}^\beta = 0, \quad n \geq n_1. \quad (3.3)$$

Now combining equations (1.1), (3.2) and (3.3), we obtain

$$\begin{aligned} &\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) \\ &+ P_n (x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ &+ Q_n (x_{n+\sigma_2}^\beta + a^\alpha x_{n+\sigma_2-\tau_1}^\beta + b^\alpha x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Since  $a \leq 1$ ,  $b \leq 1$  and  $\beta > \alpha$ , the last inequality becomes

$$\begin{aligned} &\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) \\ &+ P_n (x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ &+ Q_n (x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1}z_n + a^\alpha \Delta^{m-1}z_{n-\tau_1} + b^\alpha \Delta^{m-1}z_{n+\tau_2}) + P_n z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}} z_{n+\sigma_2}^\beta \leq 0, \quad n \geq n_1. \quad (3.4)$$

*Case 2.* Suppose  $a \geq 1$  and  $b \geq 1$ . Then from the equation (1.1), we have

$$a^\beta \Delta^m z_{n-\tau_1} + a^\beta p_{n-\tau_1} x_{n-\sigma_1-\tau_1}^\alpha + a^\beta q_{n-\tau_1} x_{n+\sigma_2-\tau_1}^\beta = 0, \quad n \geq n_1, \quad (3.5)$$

and

$$b^\beta \Delta^m z_{n+\tau_2} + b^\beta p_{n+\tau_2} x_{n-\sigma_1+\tau_2}^\alpha + b^\beta q_{n+\tau_2} x_{n+\sigma_2+\tau_2}^\beta = 0, \quad n \geq n_1. \quad (3.6)$$

Now combining equations (1.1), (3.5) and (3.6), we obtain

$$\begin{aligned} & \Delta(\Delta^{m-1}z_n + a^\beta \Delta^{m-1}z_{n-\tau_1} + b^\beta \Delta^{m-1}z_{n+\tau_2}) \\ & + P_n(x_{n-\sigma_1}^\alpha + a^\beta x_{n-\sigma_1-\tau_1}^\alpha + b^\beta x_{n-\sigma_1+\tau_2}^\alpha) \\ & + Q_n(x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Since  $a \geq 1$ ,  $b \geq 1$  and  $\beta > \alpha$ , the last inequality becomes

$$\begin{aligned} & \Delta(\Delta^{m-1}z_n + a^\beta \Delta^{m-1}z_{n-\tau_1} + b^\beta \Delta^{m-1}z_{n+\tau_2}) \\ & + P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ & + Q_n(x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1}z_n + a^\beta \Delta^{m-1}z_{n-\tau_1} + b^\beta \Delta^{m-1}z_{n+\tau_2}) + P_n z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}} z_{n+\sigma_2}^\beta \leq 0, \quad n \geq n_1. \quad (3.7)$$

*Case 3.* Now suppose  $a \leq 1$ , and  $b \geq 1$ . Then from the equation (1.1), we have

$$a^\alpha \Delta^m z_{n-\tau_1} + a^\alpha p_{n-\tau_1} x_{n-\sigma_1-\tau_1}^\alpha + a^\alpha q_{n-\tau_1} x_{n+\sigma_2-\tau_1}^\beta = 0, \quad n \geq n_1, \quad (3.8)$$

and

$$b^\beta \Delta^m z_{n+\tau_2} + b^\beta p_{n+\tau_2} x_{n-\sigma_1+\tau_2}^\alpha + b^\beta q_{n+\tau_2} x_{n+\sigma_2+\tau_2}^\beta = 0, \quad n \geq n_1. \quad (3.9)$$

Combining equations (1.1), (3.8) and (3.9), we obtain

$$\begin{aligned} & \Delta(\Delta^{m-1}z_n + a^\alpha \Delta^{m-1}z_{n-\tau_1} + b^\beta \Delta^{m-1}z_{n+\tau_2}) \\ & + P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\beta x_{n-\sigma_1+\tau_2}^\alpha) \\ & + Q_n(x_{n+\sigma_2}^\beta + a^\alpha x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Since  $a \leq 1$ ,  $b \geq 1$  and  $\beta > \alpha$ , the last inequality becomes

$$\begin{aligned} & \Delta(\Delta^{m-1}z_n + a^\alpha \Delta^{m-1}z_{n-\tau_1} + b^\beta \Delta^{m-1}z_{n+\tau_2}) \\ & + P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ & + Q_n(x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1}z_n + a^\alpha \Delta^{m-1}z_{n-\tau_1} + b^\beta \Delta^{m-1}z_{n+\tau_2}) + P_n z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}} z_{n+\sigma_2}^\beta \leq 0, \quad n \geq n_1. \quad (3.10)$$

*Case 4.* Suppose  $a \geq 1$ , and  $b \leq 1$ . Then from the equation (1.1), we have

$$a^\beta \Delta^m z_{n-\tau_1} + b^\beta p_{n-\tau_1} x_{n-\sigma_1-\tau_1}^\alpha + a^\beta q_{n-\tau_1} x_{n+\sigma_2-\tau_1}^\beta = 0, \quad n \geq n_1, \quad (3.11)$$

and

$$b^\alpha \Delta^m z_{n+\tau_2} + b^\alpha p_{n+\tau_2} x_{n-\sigma_1+\tau_2}^\alpha + b^\alpha q_{n+\tau_2} x_{n+\sigma_2+\tau_2}^\beta = 0, \quad n \geq n_1. \quad (3.12)$$

Now combining equations (1.1), (3.11), (3.12) and  $\beta > \alpha$ , we obtain

$$\begin{aligned} & \Delta(\Delta^{m-1}z_n + a^\beta \Delta^{m-1}z_{n-\tau_1} + b^\alpha \Delta^{m-1}z_{n+\tau_2}) \\ & + P_n(x_{n-\sigma_1}^\alpha + a^\beta x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ & + Q_n(x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\alpha x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

In view of  $a \geq 1, b \leq 1$  and  $\beta > \alpha$ , the last inequality becomes

$$\begin{aligned} & \Delta(\Delta^{m-1}z_n + a^\beta \Delta^{m-1}z_{n-\tau_1} + b^\alpha \Delta^{m-1}z_{n+\tau_2}) \\ & + P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ & + Q_n(x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1}z_n + a^\beta \Delta^{m-1}z_{n-\tau_1} + b^\alpha \Delta^{m-1}z_{n+\tau_2}) + P_n z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}} z_{n+\sigma_2}^\beta \leq 0, \quad n \geq n_1. \quad (3.13)$$

Now the inequalities (3.4), (3.7), (3.10) and (3.13) can be written as

$$\Delta(\Delta^{m-1}z_n + d_1 \Delta^{m-1}z_{n-\tau_1} + d_2 \Delta^{m-1}z_{n+\tau_2}) + P_n z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}} z_{n+\sigma_2}^\beta \leq 0, \quad n \geq n_1. \quad (3.14)$$

Since  $\{z_n\}$  is increasing, the inequality (3.14) becomes

$$\Delta(\Delta^{m-1}z_n + d_1 \Delta^{m-1}z_{n-\tau_1} + d_2 \Delta^{m-1}z_{n+\tau_2}) + P_n z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}} z_{n-\sigma_1}^\beta \leq 0, \quad n \geq n_1. \quad (3.15)$$

Let  $u_1 \eta_1 = P_n z_{n-\sigma_1}^\alpha$  and  $u_2 \eta_2 = \frac{Q_n}{3^{\beta-1}} z_{n-\sigma_1}^\beta$ . Using the arithmetic-geometric mean inequality

$$\frac{u_1 \eta_1 + u_2 \eta_2}{\eta_1 + \eta_2} \geq (u_1^{\eta_1} u_2^{\eta_2})^{\frac{1}{\eta_1 + \eta_2}},$$

and the fact that  $\eta_1 + \eta_2 = 1$ , we get

$$P_n z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}} z_{n-\sigma_1}^\beta \geq \eta_1^{-\eta_1} \eta_2^{-\eta_2} P_n^{\eta_1} \left( \frac{Q_n}{3^{\beta-1}} \right)^{\eta_2} z_{n-\sigma_1} = A_n z_{n-\sigma_1}, \quad n \geq n_1. \quad (3.16)$$

Now using (3.16) in (3.15), we obtain

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + A_n z_{n-\sigma_1} \leq 0 \quad (3.17)$$

for all  $n \geq n_1$ . Using (2.1) in (3.17), we obtain

$$\begin{aligned} & \Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) \\ & + A_n \frac{\lambda}{(m-1)!} (n-\sigma_1)^{m-1} \Delta^{m-1}z_{n-\sigma_1} \leq 0 \end{aligned} \quad (3.18)$$

for all  $n \geq n_1$ . By setting  $\Delta^{m-1}z_n = y_n$ , we see that  $y_n > 0$  and  $\Delta y_n \leq 0$ , and the inequality (3.18) becomes

$$\Delta(y_n + d_1y_{n-\tau_1} + d_2y_{n+\tau_2}) + A_n \frac{\lambda}{(m-1)!} (n-\sigma_1)^{m-1} y_{n-\sigma_1} \leq 0 \quad (3.19)$$

for all  $n \geq n_1$ . Now by denoting  $y_n + d_1y_{n-\tau_1} + d_2y_{n+\tau_2} = w_n$ , and using the monotonicity of  $y_n$ , we get

$$w_n \leq (1 + d_1 + d_2)y_{n-\tau_1} \quad \text{for all } n \geq n_1.$$

Using the last inequality in the inequality (3.19), we see that  $\{w_n\}$  is a positive solution of the inequality

$$\Delta w_n + \frac{A_n}{(1 + d_1 + d_2)} \frac{\lambda}{(m-1)!} (n-\sigma_1)^{m-1} w_{n+\tau_1-\sigma_1} \leq 0, \quad n \geq n_1,$$

which is a contradiction to (3.1). This completes the proof.  $\square$

**Theorem 3.2.** Assume that  $\beta < 1 < \alpha$ . If the first order difference inequality

$$\Delta w_n + \frac{B_n}{(1 + d_3 + d_4)} \frac{\lambda}{(m-1)!} (n-\sigma_1)^{m-1} w_{n+\tau_1-\sigma_1} \leq 0, \quad (3.20)$$

where

$$\begin{aligned} B_n &= \eta_1^{-\eta_1} \eta_2^{-\eta_2} \left( \frac{P_n}{3^{\alpha-1}} \right)^{\eta_1} Q_n^{\eta_2}, \quad \eta_1 = \frac{\alpha-1}{\alpha-\beta}, \quad \eta_2 = \frac{1-\beta}{\alpha-\beta}, \\ d_3 &= \begin{cases} a^\alpha & \text{if } a \geq 1, \\ a^\beta & \text{if } a \leq 1 \end{cases} \quad \text{and} \quad d_4 = \begin{cases} b^\alpha & \text{if } b \geq 1, \\ b^\beta & \text{if } b \leq 1 \end{cases} \end{aligned}$$

has no positive solution for some  $\lambda \in (0, 1)$  and for all  $n \geq n_0$ , then every solution of the equation (1.1) is oscillatory.

*Proof.* The proof is similar to that of Theorem 3.1, and hence the details are omitted.  $\square$

**Theorem 3.3.** Assume that  $\alpha < 1 < \beta$  hold. If

$$\sum_{n=n_1}^{\infty} R_n = \infty, \quad (3.21)$$

then every solution of the equation (1.1) is oscillatory.

*Proof.* Assume that  $\{x_n\}$  is a positive solution of the equation (1.1). Then proceeding as in the proof of Theorem 3.1, we obtain (3.15). Since  $\{z_n\}$  is positive increasing there exists  $M > 0$  such that  $z_n \geq M$  for all  $n \geq n_1$ . Therefore from the inequality (3.15), we obtain

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + P_nM^\alpha + \frac{Q_nM^\beta}{3^{\beta-1}} \leq 0, \quad n \geq n_1,$$

that is,

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + R_n \leq 0, \quad n \geq n_1.$$

Now taking summation from  $n_1$  to  $n-1$  and letting  $n \rightarrow \infty$ , we get

$$\sum_{s=n_1}^{\infty} R_s \leq \Delta^{m-1}z_{n_1} + d_1\Delta^{m-1}z_{n_1-\tau_1} + d_2\Delta^{m-1}z_{n_1+\tau_2} < \infty,$$

which is a contradiction to (3.21). This completes the proof.  $\square$

**Theorem 3.4.** Assume that  $1 < \alpha < \beta$ . If the first order difference inequality

$$\Delta w_n + \frac{C_n}{(1+d_1+d_2)} \frac{\lambda}{(m-1)!} (n-\sigma_1)^{m-1} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0, \quad (3.22)$$

where  $C_n = \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{3^{\beta-1}} P_n^{\eta_1} Q_n^{\eta_2}$ ,  $\eta_1 = \frac{\alpha-1}{\beta-1}$ ,  $\eta_2 = \frac{\beta-\alpha}{\beta-1}$ ,  $d_1$  and  $d_2$  are as in Theorem 3.1, has no positive solution for some  $\lambda \in (0, 1)$  and for all  $n \geq n_0$ , then every solution of the equation (1.1) is oscillatory.

*Proof.* Proceeding as in Theorem 3.1, we see that  $z_n > 0$  and  $\Delta z_n > 0$  for all  $n \geq n_1$ . Now we discuss the different cases for  $a$  and  $b$ .

Suppose  $a \leq 1$  and  $b \leq 1$ . Then from the equation (1.1), we get

$$a^\alpha \Delta^m z_{n-\tau_1} + a^\alpha p_{n-\tau_1} x_{n-\sigma_1-\tau_1}^\alpha + a^\alpha q_{n-\tau_1} x_{n+\sigma_2-\tau_1}^\beta = 0, \quad n \geq n_1 \quad (3.23)$$

and

$$b^\alpha \Delta^m z_{n+\tau_2} + b^\alpha p_{n+\tau_2} x_{n-\sigma_1+\tau_2}^\alpha + b^\alpha q_{n+\tau_2} x_{n+\sigma_2+\tau_2}^\beta = 0, \quad n \geq n_1. \quad (3.24)$$

Now combining equations (1.1), (3.23) and (3.24), we obtain

$$\begin{aligned} & \Delta(\Delta^{m-1}z_n + a^\alpha \Delta^{m-1}z_{n-\tau_1} + b^\alpha \Delta^{m-1}z_{n+\tau_2}) \\ & + P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ & + Q_n(x_{n+\sigma_2}^\beta + a^\alpha x_{n+\sigma_2-\tau_1}^\beta + b^\alpha x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Since  $a \leq 1, b \leq 1$  and  $\beta > \alpha$ , the last inequality yields

$$\begin{aligned} & \Delta(\Delta^{m-1}z_n + a^\alpha \Delta^{m-1}z_{n-\tau_1} + b^\alpha \Delta^{m-1}z_{n+\tau_2}) \\ & + P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ & + Q_n(x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$



Now, using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1}z_n + a^\alpha \Delta^{m-1}z_{n-\tau_1} + b^\alpha \Delta^{m-1}z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}}z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}}z_{n+\sigma_2}^\beta \leq 0, \quad n \geq n_1.$$

The proof for the other case of  $a$  and  $b$  are similar to that of in Theorem 3.1. Therefore for all cases of  $a$  and  $b$ , we have the inequality

$$\Delta(\Delta^{m-1}z_n + d_1 \Delta^{m-1}z_{n-\tau_1} + d_2 \Delta^{m-1}z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}}z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}}z_{n+\sigma_2}^\beta \leq 0, \quad n \geq n_1. \quad (3.25)$$

Since  $\{z_n\}$  is increasing, the inequality (3.25) becomes

$$\Delta(\Delta^{m-1}z_n + d_1 \Delta^{m-1}z_{n-\tau_1} + d_2 \Delta^{m-1}z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}}z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}}z_{n-\sigma_1}^\beta \leq 0, \quad n \geq n_1. \quad (3.26)$$

Let  $u_1\eta_1 = \frac{P_n}{3^{\beta-1}}z_{n-\sigma_1}^\alpha$  and  $u_2\eta_2 = \frac{Q_n}{3^{\beta-1}}z_{n-\sigma_1}^\beta$ . Using the arithmetic-geometric mean inequality, and the fact  $\eta_1 + \eta_2 = 1$ , we get

$$\frac{P_n}{3^{\beta-1}}z_{n-\sigma_1}^\alpha + \frac{Q_n}{3^{\beta-1}}z_{n-\sigma_1}^\beta \geq \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{3^{\beta-1}}P_n^{\eta_1}Q_n^{\eta_2}z_{n-\sigma_1}^{\alpha\eta_1+\beta\eta_2} = C_n z_{n-\sigma_1}^{\alpha\eta_1+\beta\eta_2}. \quad (3.27)$$

Now using (3.27) in (3.26), we obtain

$$\Delta(\Delta^{m-1}z_n + d_1 \Delta^{m-1}z_{n-\tau_1} + d_2 \Delta^{m-1}z_{n+\tau_2}) + C_n z_{n-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0 \quad (3.28)$$

for all  $n \geq n_1$ . From (2.1) and (3.28), we obtain

$$\begin{aligned} & \Delta(\Delta^{m-1}z_n + d_1 \Delta^{m-1}z_{n-\tau_1} + d_2 \Delta^{m-1}z_{n+\tau_2}) \\ & + C_n \frac{\lambda}{(m-1)!} (n-\sigma_1)^{m-1} \Delta^{m-1}z_{n-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0 \end{aligned} \quad (3.29)$$

for all  $n \geq n_1$ . By setting  $\Delta^{m-1}z_n = y_n$ , we see that  $y_n > 0$  and  $\Delta y_n \leq 0$ , and the inequality (3.29) becomes

$$\Delta(y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2}) + C_n \frac{\lambda}{(m-1)!} (n-\sigma_1)^{m-1} y_{n-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0 \quad (3.30)$$

for all  $n \geq n_1$ . Now by denoting  $y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2} = w_n$ , and using the monotonicity of  $y_n$ , we get

$$w_n \leq (1 + d_1 + d_2)y_{n-\tau_1} \quad \text{for all } n \geq n_1.$$

From the last inequality and (3.30), we see that  $\{w_n\}$  is a positive solution of the inequality

$$\Delta w_n + \frac{C_n}{(1 + d_1 + d_2)} \frac{\lambda}{(m-1)!} (n-\sigma_1)^{m-1} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0, \quad n \geq n_1,$$

which is a contradiction to (3.22). This completes the proof.  $\square$

**Theorem 3.5.** Assume that  $1 < \beta < \alpha$ . If the first order difference inequality

$$\Delta w_n + \frac{D_n}{(1 + d_3 + d_4)(m-1)!} (n - \sigma_1)^{m-1} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0, \quad (3.31)$$

where  $D_n = \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{3^{\alpha-1}} P_n^{\eta_1} Q_n^{\eta_2}$ ,  $\eta_1 = \frac{\beta-1}{\alpha-1}$ , and  $\eta_2 = \frac{\alpha-\beta}{\alpha-1}$ ,  $d_3$  and  $d_4$  are as in Theorem 3.2, has no positive solution for some  $\lambda \in (0, 1)$  and for all  $n \geq n_0$ , then every solution of the equation (1.1) is oscillatory.

*Proof.* The proof is similar to that of Theorem 3.4, and hence the details are omitted.  $\square$

**Theorem 3.6.** Assume that  $1 < \alpha < \beta$  holds. If

$$\sum_{n=n_1}^{\infty} R_n = \infty, \quad (3.32)$$

then every solution of the equation (1.1) is oscillatory.

*Proof.* Assume that  $\{x_n\}$  is a positive solution of the equation (1.1). Then proceeding as in the proof of Theorem 3.4, we obtain (3.26). Since  $\{z_n\}$  is positive increasing there exists  $M > 0$  such that  $z_n \geq M$  for all  $n \geq n_1$ . Then from the inequality (3.26), we obtain

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}}M^\alpha + \frac{Q_n}{3^{\beta-1}}M^\beta \leq 0, \quad n \geq n_1,$$

that is,

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + R_n \leq 0, \quad n \geq n_1.$$

Now taking summation from  $n_1$  to  $n-1$  and letting  $n \rightarrow \infty$ , we get

$$\sum_{s=n_1}^{\infty} R_s \leq \Delta^{m-1}z_{n_1} + d_1\Delta^{m-1}z_{n_1-\tau_1} + d_2\Delta^{m-1}z_{n_1+\tau_2} < \infty,$$

which is a contradiction to (3.32). This completes the proof.  $\square$

**Theorem 3.7.** Assume that  $\alpha < \beta < 1$ . If the first order difference inequality

$$\Delta w_n + \frac{E_n}{(1 + d_1 + d_2)} \frac{\lambda}{(m-1)!} (n - \sigma_1)^{m-1} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0, \quad (3.33)$$

where  $E_n = \eta_1^{-\eta_1}\eta_2^{-\eta_2} P_n^{\eta_1} Q_n^{\eta_2}$ ,  $\eta_1 = \frac{\beta-\alpha}{1-\alpha}$ ,  $\eta_2 = \frac{1-\beta}{1-\alpha}$ ,  $d_1$  and  $d_2$  are as in Theorem 3.1, has no positive solution for some  $\lambda \in (0, 1)$  and for all  $n \geq n_0$ , then every solution of equation (1.1) is oscillatory.

*Proof.* Let  $\{x_n\}$  be a positive solution of the equation (1.1). Then proceeding as in Theorem 3.1, we have  $z_n > 0$  and  $\Delta z_n \geq 0$  for all  $n \geq n_1$ . Now we discuss the different cases for  $a$  and  $b$ .

From the equation (1.1), we get

$$a^\alpha \Delta^m z_{n-\tau_1} + a^\alpha p_{n-\tau_1} x_{n-\sigma_1-\tau_1}^\alpha + a^\alpha q_{n-\tau_1} x_{n+\sigma_2-\tau_1}^\beta = 0, \quad n \geq n_1, \quad (3.34)$$

and

$$b^\alpha \Delta^m z_{n+\tau_2} + b^\alpha p_{n+\tau_2} x_{n-\sigma_1+\tau_2}^\alpha + b^\alpha q_{n+\tau_2} x_{n+\sigma_2+\tau_2}^\beta = 0, \quad n \geq n_1. \quad (3.35)$$

Now combining equations (1.1), (3.34) and (3.35), we obtain

$$\begin{aligned} & \Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) \\ & + P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ & + Q_n(x_{n+\sigma_2}^\beta + a^\alpha x_{n+\sigma_2-\tau_1}^\beta + b^\alpha x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Since  $a \leq 1, b \leq 1$  and  $\alpha < \beta < 1$ , the last inequality becomes

$$\begin{aligned} & \Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) \\ & + P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ & + Q_n(x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \leq 0, \quad n \geq n_1. \end{aligned}$$

Now, using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) + P_n z_{n-\sigma_1}^\alpha + Q_n z_{n+\sigma_2}^\beta \leq 0, \quad n \geq n_1.$$

The proof for the other case of  $a$  and  $b$  are similar to that of Theorem 3.1. Therefore for all cases of  $a$  and  $b$ , we have the inequality

$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + P_n z_{n-\sigma_1}^\alpha + Q_n z_{n+\sigma_2}^\beta \leq 0, \quad n \geq n_1. \quad (3.36)$$

Since  $\{z_n\}$  is increasing, the inequality (3.36) becomes

$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + P_n z_{n-\sigma_1}^\alpha + Q_n z_{n-\sigma_1}^\beta \leq 0, \quad n \geq n_1. \quad (3.37)$$

Now set  $u_1 \eta_1 = P_n z_{n-\sigma_1}^\alpha$ ,  $u_2 \eta_2 = Q_n z_{n-\sigma_1}^\beta$ ,  $\eta_1 = \frac{\beta-\alpha}{1-\alpha}$  and  $\eta_2 = \frac{1-\beta}{1-\alpha}$ . Then by the arithmetic-geometric mean inequality

$$\frac{u_1 \eta_1 + u_2 \eta_2}{\eta_1 + \eta_2} \geq (u_1^{\eta_1} u_2^{\eta_2})^{\frac{1}{\eta_1 + \eta_2}}$$

implies that

$$P_n z_{n-\sigma_1}^\alpha + Q_n z_{n-\sigma_1}^\beta \geq \eta_1^{-\eta_1} \eta_2^{-\eta_2} P_n^{\eta_1} Q_n^{\eta_2} z_{n-\sigma_1}^{\alpha \eta_1 + \beta \eta_2} = E_n z_{n-\sigma_1}^{\alpha \eta_1 + \beta \eta_2}, \quad n \geq n_1. \quad (3.38)$$

Combining (3.37) and (3.38), we obtain

$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + E_n z_{n-\sigma_1}^{\alpha \eta_1 + \beta \eta_2} \leq 0.$$

From the last inequality by taking  $\Delta^{m-1}z_n = y_n$ , we see that  $y_n > 0$  and  $\Delta y_n \leq 0$ , and

$$\Delta(y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2}) + E_n z_{n-\sigma_1}^{\alpha\eta_1 + \beta\eta_2} \leq 0. \quad (3.39)$$

Now let  $y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2} = w_n$ . Then  $w_n > 0$  and using  $\Delta y_n \leq 0$ , we get

$$w_n \leq (1 + d_1 + d_2)y_{n-\tau_1} \quad \text{for all } n \geq n_1. \quad (3.40)$$

Combining (3.39) and (3.40), we see that  $\{w_n\}$  is a positive solution of the inequality

$$\Delta w_n + E_n \frac{\lambda(n - \sigma_1)^{m-1}}{(m-1)!(1 + d_1 + d_2)} w_{n-\sigma_1+\tau_1}^{\alpha\eta_1 + \beta\eta_2} \leq 0, \quad n \geq n_1,$$

which is a contradiction to (3.33). This completes the proof of the theorem.  $\square$

**Theorem 3.8.** Assume that  $\beta < \alpha < 1$ . If the first order difference inequality

$$\Delta w_n + \frac{E_n}{(1 + d_3 + d_4)} \frac{\lambda}{(m-1)!} (n - \sigma_1)^{m-1} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1 + \beta\eta_2} \leq 0, \quad (3.41)$$

where  $\eta_1 = \frac{\alpha-\beta}{1-\beta}$ ,  $\eta_2 = \frac{1-\alpha}{1-\beta}$ ,  $d_3$  and  $d_4$  are as in Theorem 3.2, and  $E_n$  is as defined in Theorem 3.7, has no positive solution, then every solution of the equation (1.1) is oscillatory.

*Proof.* The proof is similar to Theorem 3.7 and hence it is omitted.  $\square$

**Theorem 3.9.** Assume that  $\alpha < \beta < 1$  holds. If

$$\sum_{n=n_1}^{\infty} R_n = \infty \quad (3.42)$$

holds, then every solution of the equation (1.1) is oscillatory.

*Proof.* Let  $\{x_n\}$  be a positive solution of the equation (1.1). Then proceeding as in the proof of Theorem 3.7, we deduce the inequality (3.37). Since  $\{z_n\}$  is positive increasing there exists  $M > 0$  such that  $z_n \geq M$  for all  $n \geq n_1$ . Then from the inequality (3.37), we obtain

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + P_n M^\alpha + Q_n M^\beta \leq 0, \quad n \geq n_1,$$

that is,

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + R_n \leq 0, \quad n \geq n_1.$$

Now taking summation from  $n_1$  to  $n-1$  and letting  $n \rightarrow \infty$ , we get

$$\sum_{s=n_1}^{\infty} R_s \leq \Delta^{m-1}z_{n_1} + d_1\Delta^{m-1}z_{n_1-\tau_1} + d_2\Delta^{m-1}z_{n_1+\tau_2} < \infty,$$

which is a contradiction to (3.42). This completes the proof.  $\square$

**Corollary 3.10.** Assume that  $\alpha < 1 < \beta$  and  $\sigma_1 > \tau_1$  hold. If

$$\liminf_{n \rightarrow \infty} \sum_{s=n-(\sigma_1-\tau_1)}^{n-1} A_s(s-\sigma_1)^{m-1} > \frac{(1+d_1+d_2)(m-1)!}{\lambda} \left( \frac{\sigma_1-\tau_1}{\sigma_1-\tau_1+1} \right)^{\sigma_1-\tau_1+1}, \quad (3.43)$$

then every solution of the equation (1.1) is oscillatory.

*Proof.* By Theorem 7.5.1 of [10], the condition (3.43) guarantees that the first order difference inequality (3.1) has no positive solution. Now the result follows from Theorem 3.1.  $\square$

**Corollary 3.11.** Assume that  $\beta < 1 < \alpha$  and  $\sigma_1 > \tau_1$  hold. If

$$\liminf_{n \rightarrow \infty} \sum_{s=n-(\sigma_1-\tau_1)}^{n-1} B_s(s-\sigma_1)^{m-1} > \frac{(1+d_3+d_4)(m-1)!}{\lambda} \left( \frac{\sigma_1-\tau_1}{\sigma_1-\tau_1+1} \right)^{\sigma_1-\tau_1+1}, \quad (3.44)$$

then every solution of the equation (1.1) is oscillatory.

*Proof.* By Theorem 7.5.1 of [10], the condition (3.44) guarantees that the first order difference inequality (3.20) has no positive solution. Now the result follows from Theorem 3.2.  $\square$

Note that for  $\beta > \alpha > 1$ ,  $\eta_1 = \frac{\alpha-1}{\beta-1}$  and  $\eta_2 = \frac{\beta-\alpha}{\beta-1}$ , imply  $\alpha\eta_2 + \beta\eta_2 > 1$ . Now using Theorem 3.4, we have the following corollary.

**Corollary 3.12.** Assume that  $1 < \alpha < \beta$  and  $\sigma_1 > \tau_1$  hold. If there exists a  $\mu > 0$  such that  $\mu > \frac{1}{\sigma_1-\tau_1} \ln(\alpha\eta_1 + \beta\eta_2)$ , and

$$\liminf_{n \rightarrow \infty} C_n(n-\sigma_1)^{m-1} \exp(-e^{\mu n}) > 0, \quad (3.45)$$

then every solution of the equation (1.1) is oscillatory.

*Proof.* By Theorem 2 of [14], condition (3.45) guarantees that the first order difference inequality (3.22) has no positive solution. Now the result follows from Theorem 3.4.  $\square$

**Corollary 3.13.** Assume that  $1 < \beta < \alpha$  and  $\sigma_1 > \tau_1$  hold. If there exists a  $\mu > 0$  such that  $\mu > \frac{1}{\sigma_1-\tau_1} \ln(\alpha\eta_1 + \beta\eta_2)$ , and

$$\liminf_{n \rightarrow \infty} D_n(n-\sigma_1)^{m-1} \exp(-e^{\mu n}) > 0, \quad (3.46)$$

then every solution of the equation (1.1) is oscillatory.

*Proof.* By Theorem 2 of [14], condition (3.46) guarantees that the first order difference inequality (3.31) has no positive solution. Now the result follows from Theorem 3.5.  $\square$

Note that for  $\alpha < \beta < 1$ ,  $\eta_1 = \frac{\beta-\alpha}{1-\alpha}$  and  $\eta_2 = \frac{1-\beta}{1-\alpha}$ , we have  $\alpha\eta_2 + \beta\eta_2 < 1$ . Now using Theorem 3.7, we have the following corollary.

**Corollary 3.14.** *Assume that  $\alpha < \beta < 1$  hold. If*

$$\liminf_{n \rightarrow \infty} \sum_{s=n_0}^{\infty} E_s (s - \sigma_1)^{m-1} = \infty, \quad (3.47)$$

*then every solution of the equation (1.1) is oscillatory.*

*Proof.* By Theorem 1 of [14], condition (3.47) guarantees that the first order difference inequality (3.33) has no positive solution. Now the result follows from Theorem 3.7.  $\square$

Note that for  $\beta < \alpha < 1$ ,  $\eta_1 = \frac{\alpha - \beta}{1 - \beta}$  and  $\eta_2 = \frac{1 - \alpha}{1 - \beta}$ , we have  $\alpha\eta_2 + \beta\eta_2 < 1$ . Now using Theorem 3.8, we have the following Corollary.

**Corollary 3.15.** *Assume that  $\beta < \alpha < 1$  hold. If*

$$\liminf_{n \rightarrow \infty} \sum_{s=n_0}^{\infty} E_s (s - \sigma_1)^{m-1} = \infty, \quad (3.48)$$

*then every solution of the equation (1.1) is oscillatory.*

*Proof.* By Theorem 1 of [14], the condition (3.48) guarantees that the first order difference inequality (3.41) has no positive solution. Now the result follows from Theorem 3.8.  $\square$

**Theorem 3.16.** *Assume that conditions  $\alpha < 1 < \beta$  and  $\sigma_1 \leq \tau_1$  hold. Further assume that there exists real valued function  $H : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{R}$  such that*

$$\begin{aligned} H_{n,n} &= 0 \quad \text{for } n \geq n_0 > 0, \\ H_{n,s} &> 0 \quad \text{for } n > s \geq n_0, \\ \Delta_2 H_{n,s} &\leq 0 \quad \text{for } n > s \geq n_0, \end{aligned}$$

where

$$\Delta_2 H_{n,s} = H_{n,s+1} - H_{n,s}.$$

If

$$\limsup_{n \rightarrow \infty} \frac{1}{H_{n,n_1}} \sum_{s=n_1}^{n-1} A_s H_{n,s} = \infty, \quad n \geq n_1 \geq n_0, \quad (3.49)$$

*then every solution of (1.1) is oscillatory.*

*Proof.* Assume that  $\{x_n\}$  is a positive solution of the equation (1.1). Then there exists an integer  $n_1 \geq n_0$  such that  $x_n > 0$ ,  $x_{n-\sigma_1} > 0$  and  $x_{n-\tau_1} > 0$  for all  $n \geq n_1$ . Then by Lemma 2.4, we have  $\Delta z_n > 0$  for all  $n \geq n_1 \geq n_0$ . Now define a function

$$w_n = \frac{\Delta^{m-1} z_n}{z_{n-\tau_1}}$$

for all  $n \geq n_1$ . Then  $w_n > 0$  for all  $n \geq n_1$ , and

$$\Delta w_n = \frac{\Delta^m z_n}{z_{n-\tau_1}} - \frac{\Delta^{m-1} z_{n+1}}{z_{n-\tau_1} z_{n+1-\tau_1}} \Delta z_{n-\tau_1} \leq \frac{\Delta^m z_n}{z_{n-\tau_1}}, \quad n \geq n_1.$$

Similarly by defining  $v_n$  and  $u_n$  for all  $n \geq n_1$ , respectively, by

$$v_n = \frac{\Delta^{m-1} z_{n-\tau_1}}{z_{n-\tau_1}}, \quad n \geq n_1,$$

and

$$u_n = \frac{\Delta^{m-1} z_{n+\tau_2}}{z_{n-\tau_1}}, \quad n \geq n_1,$$

we obtain  $v_n > 0$  and  $u_n > 0$  for all  $n \geq n_1$ , and

$$\Delta v_n \leq \frac{\Delta^m z_{n-\tau_1}}{z_{n-\tau_1}},$$

and

$$\Delta u_n \leq \rho_n \frac{\Delta^m z_{n+\tau_2}}{z_{n-\tau_1}}$$

for all  $n \geq n_1$ . Now combining these inequalities, we obtain

$$\Delta w_n + a^\beta \Delta v_n + b^\beta \Delta u_n \leq \frac{1}{z_{n-\tau_1}} [\Delta^m z_n + a^\beta \Delta^m z_{n-\tau_1} + b^\beta \Delta^m z_{n+\tau_2}]$$

for all  $n \geq n_1$ . Now using (3.17) and the monotonicity of  $z_n$ , the last inequality becomes

$$\Delta w_n + a^\beta \Delta v_n + b^\beta \Delta u_n \leq -A_n.$$

Replacing  $n$  by  $s$  and multiplying the last inequality by  $H_{n,s}$  and then summing the resulting inequality from  $n_1$  to  $n-1$ , we have

$$\sum_{s=n_1}^{n-1} A_s H_{n,s} \leq - \sum_{s=n_1}^{n-1} [\Delta w_s + a^\beta \Delta v_s + b^\beta \Delta u_s] H_{n,s}.$$

Now using summation by parts we get

$$\begin{aligned} \sum_{s=n_1}^{n-1} A_s H_{n,s} &\leq H_{n,n_1} [w_{n_1} + a^\beta v_{n_1} + b^\beta u_{n_1}] + \sum_{s=n_1}^{n-1} [w_{s+1} + a^\beta v_{s+1} + b^\beta u_{s+1}] \Delta_2 H_{n,s} \\ &\leq [w_{n_1} + a^\beta v_{n_1} + b^\beta u_{n_1}] H_{n,n_1} \end{aligned}$$

or

$$\frac{1}{H(n, n_1)} \sum_{s=n_1}^{n-1} A_s H_{n,s} \leq [w_{n_1} + a^\beta v_{n_1} + b^\beta u_{n_1}].$$

Taking  $\limsup$  as  $n \rightarrow \infty$ , in the last inequality, we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{H_{n,n_1}} \sum_{s=n_1}^{n-1} A_s H_{n,s} < \infty,$$

which is a contradiction to (3.49). The theorem is now proved.  $\square$

## 4. EXAMPLES

In this section, we provide some examples to illustrate the main results.

**Example 4.1.** Consider an even order neutral difference equation

$$\Delta^m \left( x_n + \frac{1}{2}x_{n-1} + \frac{1}{3}x_{n+2} \right) + nx_{n-2}^{\frac{1}{3}} + \frac{1}{n}x_{n+1}^3 = 0, \quad n \geq 1, \quad (4.1)$$

where  $m \geq 2$  is an even integer.

Here  $a = \frac{1}{2}$ ,  $b = \frac{1}{3}$ ,  $\tau_1 = 1$ ,  $\tau_2 = 2$ ,  $\sigma_1 = 2$ ,  $\sigma_2 = 1$ ,  $p_n = n$ ,  $q_n = \frac{1}{n}$ ,  $\alpha = \frac{1}{3}$  and  $\beta = 3$ . A simple calculation shows that  $P_n = (n-1)$ ,  $Q_n = \frac{1}{n+2}$ ,  $\eta_1 = \frac{3}{4}$ ,  $\eta_2 = \frac{1}{4}$ ,  $d_1 = \left(\frac{1}{2}\right)^{1/3}$ ,  $d_2 = \left(\frac{1}{3}\right)^{1/3}$  and  $A_n = \frac{4}{3^{5/4}} \frac{(n-1)^{3/4}}{(n+2)^{1/4}}$ . Further calculation shows that

$$\liminf_{n \rightarrow \infty} \sum_{s=n-1}^{n-1} \frac{4}{3^{5/4}} \frac{(s-1)^{3/4}}{(s+2)^{1/4}} (s-2)^{(m-1)} = \liminf_{n \rightarrow \infty} \frac{4}{3^{5/4}} \frac{(n-2)^{3/4}}{(n+1)^{1/4}} (n-3)^{(m-1)} = \infty$$

for all  $m \geq 2$ . Hence all conditions of Corollary 3.10 are satisfied and therefore every solution of the equation (4.1) is oscillatory.

**Example 4.2.** Consider an even order neutral difference equation

$$\Delta^m (x_n + 2x_{n-1} + 3x_{n+2}) + nx_{n-2}^3 + \frac{1}{n}x_{n+1}^{1/3} = 0, \quad n \geq 1. \quad (4.2)$$

Here  $a = 2$ ,  $b = 3$ ,  $\tau_1 = 1$ ,  $\tau_2 = 2$ ,  $\sigma_1 = 2$ ,  $\sigma_2 = 1$ ,  $p_n = n$ ,  $q_n = \frac{1}{n}$ ,  $\alpha = 3$  and  $\beta = \frac{1}{3}$ . A simple calculation shows that  $P_n = (n-1)$ ,  $Q_n = \frac{1}{n+2}$ ,  $\eta_1 = \frac{3}{4}$ ,  $\eta_2 = \frac{1}{4}$ ,  $d_3 = 8$ ,  $d_4 = 27$  and  $B_n = \frac{4}{3^{5/4}} (n-1)^{3/4} \frac{1}{(n+2)^{1/4}}$ . Also we see that

$$\liminf_{n \rightarrow \infty} \sum_{s=n-1}^{n-1} \frac{4}{3^{5/4}} \frac{(s-1)^{3/4}}{(s+2)^{1/4}} (s-2)^{(m-1)} = \liminf_{n \rightarrow \infty} \frac{4}{3^{5/4}} \frac{(n-2)^{3/4}}{(n+1)^{1/4}} (n-3)^{(m-1)} = \infty$$

for all  $m \geq 2$ . Therefore all conditions of Corollary 3.11 are satisfied and therefore every solution of the equation (4.2) is oscillatory.

**Example 4.3.** Consider an even order neutral difference equation

$$\Delta^m (x_n + 3x_{n-1} + 3x_{n+2}) + \frac{2^m}{n}x_{n-2}^3 + 2^m \left( \frac{n+1}{n} \right) x_{n+3}^5 = 0, \quad n \geq 1, \quad (4.3)$$

where  $m \geq 2$  is an even integer.

Here  $a = b = 3$ ,  $\tau_1 = 1$ ,  $\tau_2 = 2$ ,  $\sigma_1 = 2$ ,  $\sigma_3 = 3$ ,  $p_n = \frac{2^n}{n}$ ,  $q_n = 2^m \left( \frac{n+1}{n} \right)$ ,  $\alpha = 3$  and  $\beta = 5$ . Further  $P_n = \frac{2^m}{n+2}$ ,  $Q_n = 2^m \left( \frac{n+3}{n+2} \right)$ ,  $R_n = \frac{2^m}{(n+2)} (k_1 + k_2(n+3))$  and

$$\sum_{n=1}^{\infty} R_n = \sum_{n=1}^{\infty} \frac{2^m}{n+2} (k_1 + k_2(n+3)) = \infty.$$

Therefore, by Theorem 3.6, every solution of the equation (4.3) is oscillatory. In fact  $\{x_n\} = \{(-1)^n\}$  is one such oscillatory solution of equation (4.3).



**Remark 4.4.**

1. The established results are presented in a form which is essentially new and include some of the existing results as special cases.
2. The existing results [4, 7, 9] cannot to be applied to equations (4.1), (4.2) and (4.3) since  $\alpha \neq 1$  and  $\beta \neq 1$ .
3. The results of this paper may be extended to equation of the form

$$\Delta \left( a_n (\Delta^{m-1} (x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2})) \right) + q_n x_{n-\sigma_1}^\alpha + p_n x_{n+\sigma_2}^\beta = 0,$$

when  $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$  or  $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$ , and the details are left to the reader.

**Acknowledgements**

*The publication was supported by the Ministry of Education and Science of the Russian Federation (the agreement number N.02.a03.21.0008).*

*The authors thank the referee for his/her suggestions and corrections that improved the content of the paper.*

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*Received: February 11, 2018.*

*Revised: July 7, 2018.*

*Accepted: July 7, 2018.*