Opuscula Math. 39, no. 1 (2019), 39–47
https://doi.org/10.7494/OpMath.2019.39.1.39 Opuscula Mathematica

OSCILLATORY BEHAVIOR OF EVEN-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH A SUBLINEAR NEUTRAL TERM

John R. Graef, Said R. Grace, and Ercan Tunç

Communicated by Josef Diblík

Abstract. The authors present a new technique for the linearization of even-order nonlinear differential equations with a sublinear neutral term. They establish some new oscillation criteria via comparison with higher-order linear delay differential inequalities as well as with first-order linear delay differential equations whose oscillatory characters are known. Examples are provided to illustrate the theorems.

Keywords: oscillatory behavior, neutral differential equation, even-order.

Mathematics Subject Classification: 34C10, 34K11.

1. INTRODUCTION

This paper deals with the oscillatory behavior of solutions to a class of even-order neutral differential equations with a sublinear neutral term of the form

$$y^{(n)}(t) + q(t)x^{\beta}(\tau(t)) = 0, \tag{1.1}$$

where $y(t) = x(t) + p(t)x^{\alpha}(\sigma(t))$, $t \geq t_0 > 0$, $n \geq 2$ is an even natural number, and the following conditions are always assumed to hold:

(h_1) α, β are the ratios of positive odd integers with $0 < \alpha < 1$;
(h_2) $p, q : [t_0, \infty) \to (0, \infty)$ are real valued continuous functions with $\lim_{t\to\infty} p(t) = 0$;
(h_3) τ, $\sigma : [t_0, \infty) \to \mathbb{R}$ are real valued continuous functions such that $\tau(t) \leq t$, $\sigma(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

By a *solution* of equation (1.1) we mean a function $x : [t_x, \infty) \to \mathbb{R}$, $t_x \geq t_0$, such that $y \in C^n([t_x, \infty), \mathbb{R})$ and that satisfies equation (1.1) on $[t_x, \infty)$. We consider only those solutions $x(t)$ of (1.1) that satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_x$; moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$, i.e., for any $t_1 \in [t_x, \infty)$ there exists $t_2 \geq t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various differential equations, and we refer the reader to the monographs [7, 14], the papers [1–6, 8–13, 17, 20], and the references contained therein. However, there are few results dealing with the oscillation of differential equations with a sublinear neutral term; see, for example, [3], where second-order differential equations of the type (1.1) are studied.

In this article, we shall present a new technique for the linearization of even-order nonlinear differential equations with a sublinear neutral term of the type (1.1). We establish some new criteria for the oscillation of all solutions via a comparison with higher-order linear delay differential inequalities as well as with first-order linear delay differential equations whose oscillatory characters are known.

2. MAIN RESULTS

We begin with the following lemmas that are essential in the proofs of our theorems. The first one is a well known result that is due to Kiguradze [16].

Lemma 2.1. *Let $f \in C^n([t_0, \infty), (0, \infty))$. If the derivative $f^{(n)}(t)$ is eventually of one sign for all large t, then there exist a $t_x \geq t_0$ and an integer l, $0 \leq l \leq n$, with $n + l$ even for $f^{(n)}(t) \geq 0$, or $n + l$ odd for $f^{(n)}(t) \leq 0$ such that*

$$l > 0 \quad \textit{implies} \quad f^{(k)}(t) > 0 \quad \textit{for} \quad t \geq t_x, \quad k = 0, 1, \ldots, l-1,$$

and

$$l \leq n-1 \quad \textit{implies} \quad (-1)^{l+k} f^{(k)}(t) > 0 \quad \textit{for} \quad t \geq t_x, \quad k = l, l+1, \ldots, n-1.$$

Lemma 2.2 ([7, Lemma 2.2.3])**.** *Let $f \in C^n([t_0, \infty), (0, \infty))$, $f^{(n)}(t) f^{(n-1)}(t) \leq 0$ for $t \geq t_x$, and assume that* $\lim_{t\to\infty} f(t) \neq 0$. *Then for any constant $\theta \in (0, 1)$, there exists a $t_\theta \in [t_x, \infty)$ such that, for all $t \in [t_\theta, \infty)$,*

$$f(t) \geq \frac{\theta}{(n-1)!} t^{n-1} \left| f^{(n-1)}(t) \right|. \tag{2.1}$$

Lemma 2.3 ([15])**.** *If X and Y are nonnegative and $0 < \lambda < 1$, then*

$$X^\lambda - \lambda X Y^{\lambda-1} - (1-\lambda) Y^\lambda \leq 0, \tag{2.2}$$

where equality holds if and only if $X = Y$.

Now, we present our first oscillation result for Eq. (1.1) in the case where $\beta > 1$.

Theorem 2.4. *Let $\beta > 1$. If the even-order linear delay differential inequality*

$$y^{(n)}(t) + Mq(t)y(\tau(t)) \leq 0 \tag{2.3}$$

has no positive solution for every constant $M > 0$, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. (The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper). From (1.1) and condition (h_2), we have

$$y^{(n)}(t) = -q(t)x^{\beta}(\tau(t)) < 0 \quad \text{for} \quad t \geq t_1. \tag{2.4}$$

Then, in view of Lemma 2.1, there exists a $t_2 \geq t_1$ such that

$$y'(t) > 0 \quad \text{and} \quad y^{(n-1)}(t) > 0 \quad \text{for } t \geq t_2. \tag{2.5}$$

It follows from the definition of $y(t)$ that

$$x(t) = y(t) - [p(t)x^{\alpha}(\sigma(t)) - p(t)x(\sigma(t))] - p(t)x(\sigma(t)). \tag{2.6}$$

Applying Lemma 2.3 with

$$\lambda = \alpha, \; X = p^{1/\alpha}(t)x(\sigma(t)), \quad \text{and} \quad Y = \left(\frac{1}{\alpha}p^{(\alpha-1)/\alpha}(t)\right)^{1/(\alpha-1)},$$

we obtain

$$p(t)x^{\alpha}(\sigma(t)) - p(t)x(\sigma(t)) \leq (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}p(t). \tag{2.7}$$

Substituting (2.7) into (2.6) gives

$$x(t) \geq y(t) - p(t)x(\sigma(t)) - (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}p(t). \tag{2.8}$$

Since $y(t) > 0$ and $y'(t) > 0$ on $[t_2, \infty)$, there exist a $t_3 \geq t_2$ and a constant $c > 0$ such that

$$y(t) \geq c \quad \text{for } t \geq t_3. \tag{2.9}$$

In view of (2.9) and the fact that $x(t) \leq y(t)$, (2.8) yields

$$\begin{aligned} x(t) &\geq y(t) - p(t)y(\sigma(t)) - (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}p(t) \\ &\geq y(t) - p(t)y(t) - (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}p(t) \\ &= \left(1 - p(t) - \frac{1}{y(t)}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}p(t)\right)y(t) \\ &\geq \left(1 - p(t) - \frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}p(t)\right)y(t) \\ &= \left[1 - p(t)\left(1 + \frac{1}{c}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right)\right]y(t) \end{aligned} \tag{2.10}$$

for $t \geq t_3$. From (2.10) and the fact that $\lim_{t\to\infty} p(t) = 0$, for any $\xi \in (0,1)$ there exists $t_\xi \geq t_3$ such that

$$x(t) \geq \xi y(t) \quad \text{for } t \geq t_\xi. \tag{2.11}$$

Fix $\xi \in (0,1)$ and choose t_ξ by (2.11). Since $\lim_{t\to\infty} \tau(t) = \infty$, we can choose $t_5 \geq t_\xi$ such that $\tau(t) \geq t_\xi$ for all $t \geq t_5$. Thus, from (2.11) we have

$$x(\tau(t)) \geq \xi y(\tau(t)) \quad \text{for } t \geq t_5. \tag{2.12}$$

Using (2.12) in (1.1) gives

$$y^{(n)}(t) + \xi^\beta q(t) y^\beta(\tau(t)) \leq 0, \tag{2.13}$$

which can be written as

$$y^{(n)}(t) + \xi^\beta q(t) y^{\beta-1}(\tau(t)) y(\tau(t)) \leq 0 \quad \text{for } t \geq t_5. \tag{2.14}$$

From (2.9), the fact that $y(t)$ is increasing and $\tau(t) \geq t_\xi$, (2.14) yields

$$y^{(n)}(t) + \xi^\beta c^{\beta-1} q(t) y(\tau(t)) \leq 0,$$

or

$$y^{(n)}(t) + M q(t) y(\tau(t)) \leq 0 \quad \text{for } t \geq t_5, \tag{2.15}$$

where $M = \xi^\beta c^{\beta-1} > 0$. That is, (2.3) has a positive solution, which is a contradiction. This completes the proof of the theorem. □

From Theorem 2.4, we immediately have the following oscillation criterion for Eq. (1.1) in the case where $\beta = 1$.

Theorem 2.5. *Let $\beta = 1$. If the even-order linear delay differential inequality*

$$y^{(n)}(t) + \xi q(t) y(\tau(t)) \leq 0 \tag{2.16}$$

has no positive solution for any $\xi \in (0,1)$, then equation (1.1) *is oscillatory.*

The above theorem follows from (2.13) with $\beta = 1$ and Theorem 2.4; we omit the details of the proof.

Next, we establish an oscillation result for Eq. (1.1) in the case where $0 < \beta < 1$.

Theorem 2.6. *Let $0 < \beta < 1$. If the even-order linear delay differential inequality*

$$y^{(n)}(t) + K\big(\tau^{n-1}(t)\big)^{\beta-1} q(t) y(\tau(t)) \leq 0 \tag{2.17}$$

has no positive solution for every constant $K > 0$, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.4, we again arrive at (2.13) which can be written as

$$y^{(n)}(t) + \frac{\xi^\beta q(t)}{y^{1-\beta}(\tau(t))} y(\tau(t)) \leq 0 \quad \text{for } t \geq t_5. \tag{2.18}$$

Since $y^{(n-1)}(t)$ is positive and decreasing on $[t_5,\infty) \subseteq [t_2,\infty)$, there exist a constant $C>0$ and a $t_6 \geq t_5$ such that

$$y^{(n-1)}(t) \leq C \quad \text{for } t \geq t_6. \tag{2.19}$$

Integrating (2.19) from t_6 to t consecutively $n-1$ times, we deduce that

$$y(t) \leq N t^{n-1}, \quad t \geq t_6, \tag{2.20}$$

for some constant $N>0$, and so,

$$y(\tau(t)) \leq N\tau^{n-1}(t), \quad t \geq t_7 \geq t_6, \tag{2.21}$$

where we assume $\tau(t) \geq t_6$ for $t \geq t_7$. Using (2.21) in (2.18) gives

$$y^{(n)}(t) + \xi^\beta N^{\beta-1}\big(\tau^{n-1}(t)\big)^{\beta-1} q(t) y(\tau(t)) \leq 0,$$

or

$$y^{(n)}(t) + K\big(\tau^{n-1}(t)\big)^{\beta-1} q(t) y(\tau(t)) \leq 0 \quad \text{for } t \geq t_7, \tag{2.22}$$

where $K=\xi^\beta N^{\beta-1}>0$. The remainder of the proof is similar to that of Theorem 2.4 and hence is omitted. □

The following results are concerned with the oscillatory behavior of Eq. (1.1) via comparison with first order equations whose oscillatory characters are known.

Theorem 2.7. *Let $\beta>1$. If there exists $\theta_0 \in (0,1)$ such that the first-order linear delay differential equation*

$$z'(t) + \frac{\theta_0}{(n-1)!} M\tau^{n-1}(t) q(t) z(\tau(t)) = 0 \tag{2.23}$$

is oscillatory for every constant $M>0$, then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t)>0$, $x(\sigma(t))>0$, and $x(\tau(t))>0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.4, we again arrive at (2.15) for $t \geq t_5$. Since $y(t)>0$ and $y'(t)>0$ on $[t_5,\infty) \subseteq [t_2,\infty)$, we have

$$\lim_{t\to\infty} y(t) \neq 0,$$

and so, by Lemma 2.2, there exist θ, with $0<\theta<1$, and $t_6 \geq t_5$ such that

$$y(t) \geq \frac{\theta}{(n-1)!} t^{n-1} y^{(n-1)}(t) \quad \text{for } t \geq t_6. \tag{2.24}$$

Using (2.24) in (2.15) gives

$$y^{(n)}(t) + \frac{\theta}{(n-1)!} M\tau^{n-1}(t) q(t) y^{(n-1)}(\tau(t)) \leq 0, \quad t \geq t_7 \geq t_6. \tag{2.25}$$

With $z(t) = y^{(n-1)}(t)$, we see that $z(t)$ is a positive solution of the first-order linear delay differential inequality

$$z'(t) + \frac{\theta}{(n-1)!} M\tau^{n-1}(t)q(t)z(\tau(t)) \le 0 \quad \text{for } t \ge t_7. \tag{2.26}$$

Integrating inequality (2.26) from $t \ge t_7$ to u and letting $u \to \infty$, we obtain

$$z(t) \ge \int_t^\infty \frac{\theta}{(n-1)!} M\tau^{n-1}(s)q(s)z(\tau(s))ds \quad \text{for } t \ge t_7.$$

The function $z(t)$ is obviously decreasing on $[t_7, \infty)$, and hence, by Theorem 1 in [18], we conclude that there exists a positive solution $z(t)$ of equation (2.23) with $\lim_{t\to\infty} z(t) = 0$, which contradicts the fact that equation (2.23) is oscillatory. This completes the proof of the theorem. □

Similarly, we find the following oscillation results.

Theorem 2.8. *Let $\beta = 1$. If there exists $\theta_0 \in (0,1)$ such that the first-order linear delay differential equation*

$$z'(t) + \frac{\theta_0}{(n-1)!} \xi\tau^{n-1}(t)q(t)z(\tau(t)) = 0 \tag{2.27}$$

is oscillatory for every $\xi \in (0,1)$, then equation (1.1) *is oscillatory.*

The above theorem follows from (2.13) with $\beta = 1$, (2.24), and Theorem 2.7; we omit its proof.

By applying a result of Baculíková and Džurina ([9, Lemma 4]), we have the following result.

Corollary 2.9. *Let $\beta \ge 1$. If*

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} \tau^{n-1}(s)q(s)ds = \infty, \tag{2.28}$$

then equation (1.1) *is oscillatory.*

Proof. Applying Lemma 4 in [9], we see that equations (2.23) and (2.27) are oscillatory, and so from Theorems 2.7 and 2.8, we conclude that equation (1.1) is oscillatory. □

Theorem 2.10. *Let $0 < \beta < 1$. If there exists $\theta_0 \in (0,1)$ such that the first-order linear delay differential equation*

$$z'(t) + \frac{\theta_0}{(n-1)!} K\big(\tau^{n-1}(t)\big)^{\beta} q(t)z(\tau(t)) = 0 \tag{2.29}$$

is oscillatory for every constant $K > 0$, then equation (1.1) *is oscillatory.*

The above theorem follows from (2.22), (2.24), and Theorem 2.7; we omit the details of its proof.

Similar to what we did above, we have the following corollary.

Corollary 2.11. *Let* $0 < \beta < 1$. *If*

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} (\tau^{n-1}(s))^{\beta} q(s)ds = \infty, \tag{2.30}$$

then equation (1.1) *is oscillatory.*

We conclude this paper with the following example to illustrate the above results.

Example 2.12. Consider the nonlinear delay differential equation with a sublinear neutral term

$$\left(x(t) + \frac{1}{(1+t)^{\mu}} x^{\alpha}(t/3)\right)^{(n)} + \frac{l}{t^{\gamma}} x^{3}(t/2) = 0, \quad t \geq t_0 > 0, \tag{2.31}$$

where $\mu > 0$, $0 < \alpha < 1$, $l > 0$, and $0 \leq \gamma < n$. Here $p(t) = 1/(t+1)^{\mu}$, $\sigma(t) = t/3$, $\tau(t) = t/2$, $\beta = 3$, and $q(t) = l/t^{\gamma}$. Then,

$$\begin{aligned} \int_{\tau(t)}^{t} \tau^{n-1}(s)q(s)ds &= \int_{t/2}^{t} \left(\frac{s}{2}\right)^{n-1} \frac{l}{s^{\gamma}} ds \\ &= \frac{l}{2^{n-1}} \int_{t/2}^{t} s^{n-\gamma-1} ds = \frac{l}{2^{n-1}} \frac{(2^{n-\gamma}-1)t^{n-\gamma}}{2^{n-\gamma}(n-\gamma)}. \end{aligned} \tag{2.32}$$

Taking lim as $t \to \infty$ in (2.32), we see that (2.28) holds, and so equation (2.31) is oscillatory by Corollary 2.9.

We conclude this paper with the following observations. There are many results in the literature on the oscillation of first and higher order linear differential equations, and so it would be possible to formulate many criteria for the oscillation of equation (1.1) based on the results in this paper. Also, it would be of interest to study equation (1.1) with $\alpha > 1$, i.e., equation (1.1) with a superlinear neutral term.

REFERENCES

[1] R.P. Agarwal, S.R. Grace, *The oscillation of higher-order differential equations with deviating arguments*, Comput. Math. Appl. **38** (1999), 185–199.

[2] R.P. Agarwal, S.R. Grace, *Oscillation theorems for certain functional differential equations of higher order*, Math. Comput. Model. **39** (2004), 1185–1194.

[3] R.P. Agarwal, M. Bohner, T. Li, C. Zhang, *Oscillation of second-order differential equations with a sublinear neutral term*, Carpathian J. Math. **30** (2014), 1–6.

[4] R.P. Agarwal, S.R. Grace, I.T. Kiguradze, D. O'Regan, *Oscillation of functional differential equations*, Math. Comput. Model. **41** (2005), 417–461.

[5] R.P. Agarwal, S.R. Grace, D. O'Regan, *Oscillation criteria for certain nth order differential equations with deviating arguments*, J. Math. Anal. Appl. **262** (2001), 601–622.

[6] R.P. Agarwal, S.R. Grace, D. O'Regan, *Oscillation of certain fourth-order functional differential equations*, Ukrainian Math. J. **59** (2007), 315–342.

[7] R.P. Agarwal, S.R. Grace, D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 2000.

[8] R.P. Agarwal, S.R. Grace, P.J.Y. Wong, *On the bounded oscillation of certain fourth order functional differential equations*, Nonlinear Dyn. Syst. Theory **5** (2005), 215–227.

[9] B. Baculíková, J. Džurina, *Oscillation theorems for second-order nonlinear neutral differential equations*, Comput. Math. Appl. **62** (2011), 4472–4478.

[10] S.R. Grace, *Oscillation theorems for nth-order differential equations with deviating arguments*, J. Math. Anal. Appl. **101** (1984), 268–296.

[11] S.R. Grace, B.S. Lalli, *On oscillation and nonoscillation of general functional differential equations*, J. Math. Anal. Appl. **109** (1985), 522–533.

[12] S.R. Grace, B.S. Lalli, *A comparison theorem for general nonlinear ordinary differential equations*, J. Math. Anal. Appl. **120** (1986), 39–43.

[13] S.R. Grace, R.P. Agarwal, M. Bohner, D. O'Regan, *Oscillation of second-order strongly superlinear and strongly sublinear dynamic equations*, Commun. Nonlinear Sci. Numer. Simul. **14** (2009), 3463–3471.

[14] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[15] G.H. Hardy, I.E. Littlewood, G. Polya, *Inequalities*, Reprint of the 1952 edition, Cambridge University Press, Cambridge, 1988.

[16] I.T. Kiguradze, *On the oscillatory character of solutions of the equation* $d^m u/dt^m + a(t)|u|^n sgnu = 0$, Mat. Sb. (N.S.) **65** (1964), 172–187.

[17] T. Li, Y.V. Rogovchenko, *Oscillation criteria for even-order neutral differential equations*, Appl. Math. Lett. **61** (2016), 35–41.

[18] Ch.G. Philos, *On the existence of nonoscillatory solutions tending to zero at* ∞ *for differential equations with positive delays*, Arch. Math. (Basel) **36** (1981), 168–178.

[19] Ch.G. Philos, *A new criterion for the oscillatory and asymptotic behavior of delay differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Mat. **30** (1981), 61–64.

[20] Q. Zhang, J. Yan, L. Gao, *Oscillation behavior of even-order nonlinear neutral differential equations with variable coefficients*, Comput. Math. Appl. **59** (2010), 426–430.

John R. Graef
John-Graef@utc.edu

University of Tennessee at Chattanooga
Department of Mathematics
Chattanooga, TN 37403, USA

Said R. Grace
saidgrace@yahoo.com

Cairo University
Department of Engineering Mathematics
Faculty of Engineering, Orman, Giza 12221, Egypt

Ercan Tunç
ercantunc72@yahoo.com

Gaziosmanpasa University
Department of Mathematics
Faculty of Arts and Sciences, 60240, Tokat, Turkey

Received: February 8, 2018.
Accepted: April 1, 2018.