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DYNAMIC SYSTEM WITH RANDOM STRUCTURE FOR MODELING SECURITY AND RISK MANAGEMENT IN CYBERSPACE

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Abstract. We deal with the investigation of L_2-stability of the trivial solution to the system of difference equations with coefficients depending on a semi-Markov chain. In our considerations, random transformations of solutions are assumed. Necessary and sufficient conditions for L_2-stability of the trivial solution to such systems are obtained. A method is proposed for constructing Lyapunov functions and the conditions for its existence are justified. The dynamic system and methods discussed in the paper are very well suited for use as models for protecting information in cyberspace.

Keywords: semi-Markov chain, random transformation of solutions, the Lyapunov function, L_2-stability, systems of difference equations, jumps of solutions, cybersecurity.

Mathematics Subject Classification: 34F05, 60J28.

1. INTRODUCTION

The semi-Markov processes theory is very well applicable to many real processes, it is widely applied in biology and medicine for prognosis and the evolution of diseases, in sociology or socioeconomics for a model of the marriage market, in finance for a model of the credit rating and reliability. Semi-Markov processes are also used in the field of computer science and technology. Since cyberspace has become the arena of numerous conflicts, differing in form and method, intensity and degree of threats, the modelling of cybersecurity problems is especially important.

The semi-Markov process was first clearly formulated independently by Lèvy [11] in 1954 and Smith [16] in 1955. The theory was extended and applied to a problem of the reliability theory by many authors, the main development of the theory was proposed, for example, by Çinlar, Korolyuk, Limnios, Turbin, Oprisan [2, 8–10, 12]. The dynamic systems considered in this paper are called systems with random states. They were studied, for example, by Artemiev [1], Katz [7], and others. Some applications of such systems were studied in [3, 4, 6, 14, 15]. Specifically, in [15] the problem of navigation to a target described by differential equation with random parameters is considered. The mathematical model of foreign currency exchange market in the form of a stochastic linear differential equation with coefficients depending on a semi-Markov process is considered in [4]. The dynamic system and methods discussed in this paper are very well suited for use as models for protecting information in cyberspace.

Let $(\Omega, \mathcal{F}, F, P)$ be a filtered probability space consisting of a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $F = \{\mathcal{F}_t, \forall t \geq 0\} \subset \mathcal{F}$. The space Ω is called the sample space, $\mathcal{F}$ is the set of all possible events (the σ-algebra), and P is some probability measure on Ω. A sequence $\xi = \{\xi_i\}_{i=1}^{\infty}$ of random variables $\xi_i \colon \Omega \to S$, $i = 0, 1, 2, \ldots$ is called a discrete-time stochastic chain on the state space S. In our considerations, ξ is a random semi-Markov chain and the state space S is the space of all random variables for which there exists squared mathematical expectation. On such a probability space, we consider system of linear difference equations with semi-Markov switching

$$X_{k+1} = A(k, \xi) X_k, \quad k = 0, 1, 2, \ldots \tag{1.1}$$

$$X_0 = \varphi, \tag{1.2}$$

where A is an $m \times m$ matrix whose elements depend on the semi-Markov chain ξ. The state function X_k is an m-dimensional column vector-function with the initial state $X_0 = \varphi$.

An m-dimensional column vector-function X_k is called a solution to initial value problem (1.1), (1.2) if X_k satisfies (1.1) and initial condition (1.2) whitin the meaning of a strong solution of the initial Cauchy problem.

Our goal in this article is to obtain necessary and sufficient conditions of L_2-stability for systems (1.1) with semi-Markov coefficients and random transformations of solutions.

The basis for most authors in the development of the stability of stochastic systems was the theory of stability of a deterministic system developed by Lyapunov [13]. To study stability in the mean and stability in the mean square, the traditional method of Lyapunov functions was developed by many authors. The method of Lyapunov functions is an effective method for investigation stability of linear or nonlinear difference systems that are explicitly independent of the time. However, the method of Lyapunov functions is often difficult to apply to the study of the stability of non-stationary dynamic systems. This can be explained by the fact that it is inconvenient to use the Lyapunov functions in the sense of the Lyapunov stability concept for this type of systems. The investigation of the Lyapunov stability of differential systems with random parameters becomes even more complicated. For this reason, a modified definition of stability of the trivial solution of non-stationary difference systems is given. This definition of L_2-stability is based on the concept of moments, and it is very well compatible with the Lyapunov functions method.

2. PRELIMINARY REMARKS

In our considerations, the random semi-Markov chain ξ can take n possible states $\theta_1, \theta_2, \ldots, \theta_n$. In accordance with the theory, for the discrete random variable we can define particular probability density functions by formula transition intensities $q_{ls}(k)$, $l, s = 1, \ldots, n$, from state θ_s to θ_l at time k satisfy the following conditions:

$$q_{ls}(k) \geqslant 0, \quad \sum_{k=1}^{\infty} q_{ls}(k) = \pi_{ls}, \quad l, s = 1, \ldots, n, \tag{2.1}$$

$$q_s(k) = \sum_{l=1}^{n} q_{ls}(k), \quad s = 1, \ldots, n, \quad \sum_{k=1}^{\infty} q_s(k) = 1.$$

Denote

$$A(k, \xi) = A_s(k), \text{ if } \xi = \theta_s, \quad k = 0, 1, \ldots, \quad s = 1, \ldots, n,$$

and $X_0 = X_{0,s}$, $\varphi = \varphi_s$ if $\xi = \theta_s$, $s = 1, \ldots, n$. Then the initial Cauchy problem (1.1), (1.2) determines the initial Cauchy problem for non-stationary systems of linear equations,

$$X_{k+1,s} = A_s(k) X_{k,s}, \quad k = 0, 1, \ldots, \quad s = 1, \ldots, n, \tag{2.2}$$

$$X_{0,s} = \varphi_s, \tag{2.3}$$

in each of realizations of the semi-Markov chain ξ.

Let $m \times m$ matrices $N_s(k)$, $s = 1, \ldots, n$, be fundamental matrices of solutions to initial Cauchy problem (2.2), (2.3) such that $N_s(0) = \mathcal{I}$, $s = 1, \ldots, n$, where $\mathcal{I}$ is the $m \times m$ identity matrix. Then the solutions to (2.2), (2.3) can be written in the form

$$X_{k,s} = N_s(k) X_{0,s}, \quad k = 0, 1, \ldots, \quad s = 1, \ldots, n.$$

Moreover, for any $m \times m$ regular constant matrix C_{ls}, $l, s = 1, \ldots, n$, the jumps of solutions have the following form:

$$\begin{aligned} X_k &= N_s(k - k_{j-1}) X_{k_{j-1}}, && k_{j-1} \le k \le k_j, \quad j = 1, 2, \ldots, \\ X_{k_j} &= C_{ls} N_s(k - k_{j-1}) X_{k_{j-1}}, && \det C_{ls} \neq 0, \quad\ l, s = 1, \ldots, n, \\ X_k &= N_l(k - k_j) X_{k_j}, && k_j \le k \le k_{j+1}. \end{aligned} \tag{2.4}$$

Hence it is obvious that the solutions to system (1.1), as well as to systems (2.2) are random variables. It is known, the density function $f(k, x, \xi)$ of the random variable X_k depending on the semi-Markov chain ξ with n possible states, can be written as

$$f(k, x, \xi) = \sum_{s=1}^{n} f_s(k, x) \delta(\xi - \theta_s)$$

where δ is the Dirac delta function, and $f_s(k,x)$, $s=1,2,\dots,n$, are the particular density functions of the random variables $X_{k,s}$, corresponding to each of realizations of the semi-Markov chain ξ. They satisfy the following conditions

$$f_s(k,x)\geq 0, \quad \int\limits_{\mathbb{R}^m} f_s(k,x)dx=1, \quad k=1,2,\dots,\ s=1,2,\dots,n,$$
$$f(k,x)=\sum_{s=1}^{n} f_s(k,x).$$

In our considerations it is advisable to use the column-vector function of particular density functions. Let us denote

$$F(k,x)=\big(f_1(k,x),f_2(k,x),\dots,f_n(k,x)\big)^T. \tag{2.5}$$

We use matrix operators to prove our results.

Definition 2.1. Let on the probability space $(\Omega,\,\mathcal{F},F,P)$ be defined two random variables $X\equiv X(\omega)\colon\Omega\to\mathbb{R}^m$ and $Y\equiv Y(\omega)\colon\Omega\to\mathbb{R}^m$ with probability density functions $f_1(x)$ and $f_2(y)$, respectively. Then the operator

$$\mathcal{L}\colon\, f_1(x)\to f_2(y) \quad \text{or} \quad f_2(y)=\mathcal{L}f_1(x)$$

is said to be the *stochastic operator*.

A general form of the stochastic operator is given, for example, in [5]. In the case of a linear transformation $y=Ax$, $\det A\neq 0$, the operator $\mathcal{L}$ has the form

$$\mathcal{L}f(x)=f\big(A^{-1}x\big)\left|\det A^{-1}\right|. \tag{2.6}$$

This implies

$$f(k,x)=\mathcal{L}(k-k_0)f(k_0,x), \tag{2.7}$$

where

$$\mathcal{L}(k)f(k,x)=f\big(k_0,N^{-1}(k)x\big)\left|\det N^{-1}(k)\right|, \tag{2.8}$$

if $N(k)$ is a solution to a initial Cauchy problem of type (1.1), (1.2).

For $s=1,\dots,n$, we define operators

$$\begin{aligned}
&\psi(k):=\operatorname{diag}\big(\psi_1(k),\dots,\psi_n(k)\big),\ \psi_s(k)\equiv\sum_{i=k+1}^{\infty} q_s(i),\\
&S(k):=\big(q_{is}S_{is}\big)_{i,s=1}^{n}, \quad S_{is}f(x)\equiv f(C_{is}^{-1}x)\det C_{is}^{-1}, \quad i=1,\dots,n,\\
&R(k):=\operatorname{diag}\big(R_1(k),\dots,R_n(k)\big),\ R_sf(x)\equiv f(N_s^{-1}(k)x)\det N_s^{-1}(k).
\end{aligned} \tag{2.9}$$

Our task is to obtain reliable and simple method for investigating L_2-stability of solutions to this class of systems, see in [5].

Definition 2.2. The trivial solution to system (1.1) is said to be L_2-stable if for any solution X_k, $k=0,1,\dots$, to system (1.1) the series $\sum_{k=0}^{\infty}E^{(1)}\big\{\|X_k\|^2\big\}$ konverges.

Remark 2.3. It is easy to see that the trivial solution to system (1.1) is L_2-stable if and only if the matrix series $\sum_{k=0}^{\infty}E^{(2)}\big\{X_k\big\}$, or $\sum_{k=0}^{\infty}E^{(1)}\{X_k\,X_k^T\}$ are convergent.

3. MOMENT EQUATIONS FOR DIFFERENCE SYSTEMS WITH RANDOM JUMPS

To obtain the stability conditions, we prove several auxiliary statements.

Lemma 3.1. *Let X_k, $k = 0, 1, 2, \ldots$, be solutions to system* (1.1) *with jumps* (2.4). *If k_j, $j = 0, 1, 2, \ldots$, are moments of jumps of semi-Markov chain ξ, then there exists the stochastic operator $L(k)$,*

$$L(k) = \psi(k)R(k) + \sum_{k_j=k_1}^{k} L(k-k_j)S(k_j)R(k_j), \tag{3.1}$$

such that

$$F(k+k_j, x) = L(k)F(k_j, x), \quad k, j = 0, 1, 2, \ldots, \; k \geq k_j. \tag{3.2}$$

The operator $L(k)$ can be found as a solution to the following system

$$L(k) = \psi(k)R(k) + \sum_{k_j=k_1}^{k} \psi(k-k_j)R(k-k_j)U(k_j), \quad k = 0, 1, 2, \ldots, \tag{3.3}$$

$$U(k) = S(k)R(k) + \sum_{r=1}^{k-1} S(r)R(r)U(k-r), \quad k = 0, 1, 2, \ldots \tag{3.4}$$

Proof. Because at the jumps k_j, $j = 0, 1, 2, \ldots$ the entire history of the random process is “forgotten”, i.e. does not affect the behaviour of the solutions to system (1.1) under the condition $k > k_j$, then there exists a stochastic matrix operator $L(k) := \big(L_{ij}(k)\big)_{i,j=1}^{n}$, $k = 0, 1, 2, \ldots$, such that (3.2) is satisfied. Given that all jump times $k_j, j = 0, 1, 2, \ldots$, are equally probable, we can take $k_0 = 0$ as the initial moment. So, system (3.2) goes into the form

$$f_r(k, x) = \sum_{s=1}^{n} L_{rs}(k) f_s(0, x), \quad r = 1, \ldots, n, \; k > 0. \tag{3.5}$$

Let the random process ξ take on state θ_s at $k = 0$. The random process remains in the state θ_s for a time $k > 0$ with probability ψ_s and with probability $q_{rs}(k)$ goes into state θ_r at $t = k_j$. Then, with respect to (2.6), (2.7), and (2.8), for particular density functions we have

$$\begin{aligned} f_s(k, x) = {} & \psi_s(k) f_s(0, N_s^{-1}(k)x) \det N_s^{-1}(k) \\ & + \sum_{k_j=k_1}^{k} q_{rs}(k) L_{sr}(k-k_j) f_s(0, N_s^{-1}(k)C_{rs}^{-1}x) \det N_s^{-1}(k) |\det C_{rs}^{-1}|, \\ f_r(k, x) = {} & \sum_{k_j=k_1}^{k} q_{rs} L_{sr}(k-k_j) f_r(0, N_r^{-1}(k)C_{sk}^{-1}X) \det N_r^{-1}(k) |\det C_{sk}^{-1}|, \\ & s, r = 0, 1, 2, \ldots, n, \; r \neq s. \end{aligned}$$

System for particular density functions can be rewritten into a simpler form, using the Kronecker delta function,

$$
\begin{aligned}
f_r(k,x) &= \delta_{rs}\psi_s(k) f_s(0, N_s^{-1}(k)x) \det N_s^{-1}(k) \\
&\qquad + \sum_{k_j=k_1}^{k} q_{rs}(k) L_{sr}(k-k_j) f_s(0, N_s^{-1}(k) C_{rs}^{-1} x) \det N_s^{-1}(k) |\det C_{rs}^{-1}|, \\
& r = 1,2,\dots,n,
\end{aligned}
$$

then, using (2.9), we have

$$
L(k)F(0,x) = \psi(k)R(k)F(0,x) + \sum_{k_j=k_1}^{k} L(k-k_j)S(k_j)R(k_j)F(0,x),
$$

which proves equation (3.1) for the stochastic operator $L(k)$.

Solution to (3.1) can be found by the method of successive approximations. Let us find the solution to (3.1) in the form (3.3) where U is an unknown matrix operator. If we put $L(k)$ expressed in the form (3.3) into equation (3.1) we obtain

$$
\begin{aligned}
&\sum_{k_j=k_1}^{k} \psi(k-k_j)R(k-k_j)U(k_j) \\
&= \sum_{k_j=k_1}^{k} \psi(k-k_j)R(k-k_j)S(k_j)R(k_j) \\
&\quad + \sum_{k_j=k_1}^{k} \sum_{r=1}^{k-k_j} \psi(k-k_j-r)R(k-k_j-r)U(r)S(k_j)R(k_j),
\end{aligned}
$$

whence after changing the order of summation we get the equation for operator $U(k)$,

$$
U(k) = S(k)R(k) + \sum_{r=1}^{k-1} U(r)S(k-r)R(k-r), \quad k = 0,1,\dots \tag{3.6}
$$

In a similar way, we find solutions to operator equation (3.6) in the form

$$
U(k) = S(k)R(k) + \sum_{r=1}^{k-1} S(r)R(r)V(k-r), \quad k = 0,1,\dots \tag{3.7}
$$

In this case we obtain a difference equation for operator $V(k)$

$$
V(k) = S(k)R(k) + \sum_{r=1}^{k-1} V(r)S(k-r)R(k-r), \quad k = 0,1,\dots \tag{3.8}
$$

Comparing the systems of equations (3.6) and (3.8), we can put

$$
U(k) \equiv V(k), \quad k = 0,1,\dots,
$$

which proves equation (3.4). □

Lemma 3.2. *Let the conditions of Lemma* 3.1 *be satisfied. Then the vector of moments of the first order $E^{(1)}\{X_k\}$ is determined by a system of equations for particular moments of the first order $E_s^{(1)}\{X_k\}$, $s = 1, 2, \dots, n$,*

$$\begin{aligned}
E_s^{(1)}\{X_k\} &= \psi(k)N_s(k)E_s^{(1)}\{X_0\} \\
&\quad + \sum_{k_j=k_1}^{k} \psi_s(k-k_j)N_s(k-k_j)V_s(k_j), \\
V_s(k) &= \sum_{i=1}^{n} q_{si}(k)C_{si}N_i(k)E_i^{(1)}\{X_0\} \\
&\quad + \sum_{k_j=k_1}^{k-k_1}\sum_{i=1}^{n} q_{si}(k-k_j)C_{si}N_i(k-k_j)V_i(k_j).
\end{aligned} \tag{3.9}$$

The matrix of moments of the second order $E^{(2)}\{X_k\}$ is determined by a system for particular second-order moments $E_s^{(2)}\{X_k\}$, $s = 1, 2, \dots, n$,

$$\begin{aligned}
E_s^{(2)}\{X_k\} &= \psi_s(k)N_s(k)E_s^{(2)}\{X_0\}N_s^T(k) \\
&\quad + \sum_{k_j=k_1}^{k} \psi_s(k-k_j)N_s(k-k_j)W_s(k_j)N_s^T(k-k_j), \\
W_s(k) &= \sum_{i=1}^{n} q_{si}(k)C_{si}N_i(k)E_i^{(2)}\{X_0\}N_i^T(k)C_{si}^T \\
&\quad + \sum_{k_j=k_1}^{k-k_1}\sum_{i=1}^{n} q_{si}(k-k_j)C_{si}N_i(k-k_j)W_i(k_j)N_i^T(k-k_j)C_{si}^T, \\
&\quad s = 1, 2, \dots, n, \ \ k = 1, 2, \dots
\end{aligned} \tag{3.10}$$

Proof. We multiply the operator equations (3.3) and (3.4) from the right to the vector $F(0, x)$. Denoting

$$F(k, x) = L(k)F(0, x), \quad H(k, x) = U(k)F(0, x), \quad k = 1, 2, \dots$$

we obtain the system of equations

$$F(k, x) = \psi(k)R(k)F(0, x) + \sum_{k_j=k_1}^{k} \psi(k-k_j)R(k-k_j)H(k_j, x), \tag{3.11}$$

$$H(k, x) = \psi(k)R(k)F(0, x) + \sum_{k_j=k_1}^{k-k_1} S(k_j)R(k_j)H(k-k_j, x). \tag{3.12}$$

Using notation (2.5) and $H(k,x) = \big(h_1(k,x),\dots,h_n(k,x)\big)^T$ systems of equations (3.11) and (3.12) can be written in the scalar form

$$f_s(k,x) = \psi_s(k)R_s(k)f_s(0,x) + \sum_{k_j=k_1}^{k} \psi_s(k-k_j)R_s(k-k_j)h_s(k_j,x), \tag{3.13}$$

$$\begin{aligned} h_s(k,x) &= \sum_{i=1}^{n} q_{si}(k)S_{si}R_i(k)f_1(0,x) \\ &\quad + \sum_{k_j=k_1}^{k-k_1}\sum_{i=1}^{n} q_{si}R_i(k-k_j)h_i(k_j,x), \quad s=1,2,\dots n. \end{aligned} \tag{3.14}$$

We multiply the system of equations (3.13) and (3.14) by the vector $x \in \mathbb{R}^m$ and integrate troughout the space $\mathbb{R}^m$. By this we immediately get system (3.9) for moments of the first order. In the same way, if we multiply the system (3.13) and (3.14) by the matrix xx^T, we get system (3.10) for moments of the second order. □

4. NECESSARY AND SUFFICIENT CONDITIONS FOR L_2-STABILITY OF SOLUTIONS TO SYSTEM (1.1)

In order to study the mean stability of solutions one can use system of moment equations (3.9). To the mean square stability of solutions, the system of equations (3.10) can be used. We use the moment equations to prove L_2-stability of the trivial solution to system (1.1). It should be noted that if the trivial solution to system (1.1) is L_2-stable, then it is asymptotically stable in the mean square.

Theorem 4.1. *Let the sums*

$$I_s = \sum_{k=0}^{\infty} \psi_s(k)N_s(k)N_s^T(k), \quad s=1,\dots,n$$

converge and $I_s > 0$, $s=1,\dots,n$. Then the trivial solution to system (1.1) *is L_2-stable if and only if the symmetric matrices*

$$W_s \equiv \sum_{k=0}^{\infty} W_s(k), \quad s=1,\dots,n \tag{4.1}$$

are bounded.

Proof. 1. Let the trivial solution to system (1.1) is L_2-stable. Then, in view of Remark 2.3, matrices $E_s^{(2)} = \sum_{k=0}^{\infty} E_s^{(2)}\{X_k\}$, $s=1,\dots,n$, are bounded. This is, there exist constants ρ_s, $s=1,\dots,n$, such that $E_s^{(2)} \le \rho_s I_s$. So, taken into account $E_s^{(2)} \ge 0$, $W_s \ge 0$, (3.10), and

$$E_s^{(2)} \ge \sum_{k=0}^{\infty} \psi_s(k)N_s(k)W_sN_s^T(k), \quad s=1,\dots,n,$$

we get

$$\sum_{k=0}^{\infty} \psi_s(k) N_s(k) (\rho_s I_s - W_s) N_s^T(k) \geq 0, \quad s = 1, \dots, n,$$

from where

$$W_s \leq \rho_s I_s, \quad s = 1, \dots, n, \tag{4.2}$$

which means that W_s, $s = 1, \dots, n$ are bounded.

2. Let the matrices W_s be bounded, that is, inequalities of the form (4.2) are fulfilled. Since

$$E_s^{(2)} = \sum_{k=0}^{\infty} \psi_s(k) N_s(k) \Big(E_s^{(2)} \{X_0\} + W_s \Big) N_s^T(k), \quad s = 1, \dots, n,$$

then, taken into acount (4.2), we have

$$E_s^{(2)} \leqslant \sum_{k=0}^{\infty} \psi_s(k) N_s(k) \Big(E_s^{(2)} \{X_0\} + \rho_s I_s \Big) N_s^T(k), \quad s = 1, \dots, n. \tag{4.3}$$

Therefore, in view of the convergence I_s and Remark 2.3, formula (4.3) means that the trivial solution to (1.1) is L_2-stable. □

Theorem 4.2. *Let the conditions of Theorem* 4.1 *be satisfied. Then in order for the trivial solution to system* (1.1) *with jumps of solutions* (2.4) *to be L_2-stable, it is necessary and sufficient that one of the following equivalent conditions holds:*

1) *there exists a solution $B_s = E_s^{(2)}\{X_0\} + W_s > 0$ to system of matrix equations* (3.10) *under condition $E_s^{(2)}\{X_0\} > 0$,*
2) *successive approximations*

$$B_s^{(j+1)} = E_s^{(2)} \{X_0\} + \sum_{l=1}^{n} \sum_{k=1}^{\infty} q_{sl}(k) C_{sl} N_l(k) B_l^{(j)} N_j^T(k) C_{sl}^T, \tag{4.4}$$
$$B_s^{(0)} = 0, \qquad s = 1, \dots, n, \ j = 0, 1, 2, \dots,$$

are convergent.

Proof. We introduce monotone operators L_{ks}, $k, s = 1, \dots, n$, and write the system (4.1) in the operator form

$$L_{sl} B_l = \sum_{k=1}^{\infty} q_{sl}(k) C_{sl} N_l(k) B_l N_l^T(k) C_{sl}^T, \quad l, s = 1, \dots, n. \tag{4.5}$$

From Theorem 4.1 it follows that the system of equations (3.7) has a bounded positive definite solution $B_s > 0$, $s = 1, \dots, n$, under condition $E_s^{(2)}\{X_0\} > 0$, $s = 1, \dots, n$, if and only if when the successive approximations (4.4) converge. If the conditions of Theorem 4.1 are satisfied, then from the boundedness of the matrices B_s, $s = 1, \dots, n$, and from formulas (4.5) the boundedness of the matrices $E_s^{(2)}$, $s = 1, \dots, n$, follows and, consequently, L_2-stability of solutions to system of difference equations (1.1). □

5. CONSTRUCTION OF LYAPUNOV FUNCTIONS

An effective method for studying solutions of a system of difference equations (1.1) with random semi-Markov coefficients is the method of Lyapunov functions. We describe the basic idea of constructing the Lyapunov function and give the main result.

We introduce a positive definite quadratic form

$$w(k, X_k, \xi) = X_k^T B(k, \xi) X_k, \quad B(k, \xi) > 0, \quad k = 0, 1, 2, \dots \tag{5.1}$$

where the elements of the matrix $B(k, \xi)$ are semi-Markov functions.

To define the matrix $B(k, \xi)$, we introduce n different symmetric matrices $B_s(k)$ for $k = 0, 1, 2, \dots, s = 1, \dots, n$. Let k_j, $j = 0, 1, 2, \dots$, be moments of jumps of the semi-Markov chain ξ. Assume that the process ξ is in state θ_s, this is $\xi = \theta_s$, for $k_j \le k < k_{j-1}$. Then we set

$$B(k, \xi = \theta_s) = B_s(k - k_j),$$

$$w_s(k, X_k) = X_k^T B_s(k) X_k, \quad s = 1, \dots, n, \ k, j = 0, 1, 2, \dots$$

where $B_s(k)$, $s = 1, \dots, n$, are different symmetric matrices. We define a quadratic functional v using the mean values of functions (5.1)

$$v = \sum_{k=0}^{\infty} E^{(1)} \big\{ X_k^T B(k, \xi) X_k \big\} = \sum_{k=0}^{\infty} E^{(1)} \big\{ w(k, X_k, \xi) \big\}. \tag{5.2}$$

To calculate the functional v defined in (5.2), we introduce the so-called basic Lyapunov functions using the mean values of functions $w_s(k, X_k)$, that is,

$$v_s = \sum_{k=0}^{\infty} E^{(1)} \big\{ w_s(k, X_k, \xi) \big\}, \quad s = 1, \dots, n, \tag{5.3}$$

where $X_0 = \varphi$, and $\xi = \theta_s$ for $k = 0$. The basic Lyapunov functions (5.3) can be found in the form

$$v_s(X) = X^T C_s X, \quad s = 1, \dots, n,$$

where

$$C_s = \sum_{k=0}^{\infty} U_s(k),$$

and

$$U_s(k) = \psi_s(k) N_s^T(k) B_s(k) N_s(k) + \sum_{l=1}^{n} \sum_{k_j=k_1}^{k} q_{ls}(k_j) N_s^T(k_j) C_{ls} U_l(k-k_j) C_{ls}^T N_s(k_j), \quad s=1,\ldots,n. \tag{5.4}$$

We sum systems (5.4) with respect to the index k in order to obtain a system of matrix equations for the matrices C_s, $s=1,\ldots,n$,

$$C_s = \sum_{k=0}^{\infty} \psi_s(k) N_s^T(k) B_s(k) N_s(k) + \sum_{l=1}^{n} \sum_{k=1}^{\infty} q_{ls}(k) N_s^T(k) C_{ls}^T C_l C_{ls} N_s(k), \quad s=1,\ldots,n. \tag{5.5}$$

We introduce the notation

$$H_s = \sum_{k=0}^{\infty} \psi_s(k) N_s^T(k) B_s(k) N_s(k), \quad s=1,\ldots,n.$$

Then system (5.5) can be rewritten into the form

$$C_s = H_s + \sum_{l=1}^{n} \sum_{k=1}^{\infty} q_{ls}(k) N_s^T(k) C_{ls}^T C_l C_{ls} N_s(s), \quad s=1,\ldots,n. \tag{5.6}$$

System (5.6) is conjugate to system of equations (3.10), and it can be written in operator form

$$C_s = H_s + \sum_{l=1}^{n} L_{sl}^T C_l, \quad s=1,\ldots,n, \tag{5.7}$$

where operators L_{sl} for $s,l=1,\ldots,n$ are defined in (3.5).

The existence of a positive solution $C_s > 0$, $s=1,\ldots,n$ to system (5.5) is equivalent to the existence of a positive definite solution $B_s > 0$ to system (5.7), given that $E_s^{(2)}\{X_0\} > 0$, $s=1,\ldots,n$. Therefore, this is equivalent to L_2-stability of the trivial solution to system (1.1).

Suppose that the condition

$$\lambda_1 \mathcal{I} \le B_s(k) \le \lambda_2 \mathcal{I}, \quad k=0,1,2,\ldots,\ s=1,\ldots,n,$$

with $\lambda_1 > 0$, is fulfilled for matrices $B_s(k)$. Then the existence of functions $v_s(X)$ implies the convergence of the series in Remark 2.3 that is, L_2-stability of the trivial solution to system (1.1).

We formulate obtained results in the following two theorems.

Theorem 5.1. *Suppose that the conditions of Lemmas* 3.1 *and* 3.2 *are satisfied. Then in order for the trivial solution to system* (1.1) *to be* L_2*-stable, it is necessary and sufficient that one of the following equivalent conditions holds:*

1) *there exists solution* $C_s > 0$, $s = 1, \dots, n$ *to system* (5.7) *for any matrices* $H_s > 0$, $s = 1, \dots, n$,
2) *successive approximations*

$$
\begin{aligned}
&C_s^{(j+1)} = H_s + \sum_{j=1}^{n} \sum_{k=1}^{\infty} q_{ls}(k) N_s^T(k) C_{ls}^T C_l^{(j)} C_{ls} N_s(k) > 0,\\
&C_s^{(0)} = \Theta, \quad j = 0, 1, 2, \dots, \ s = 1, \dots, n,
\end{aligned}
$$

converge.

Theorem 5.2. *Suppose that the conditions of Lemmas* 3.1 *and* 3.2 *are satisfied. Then in order for the trivial solution to system* (1.1) *to be* L_2*-stable, it is sufficien that for some symmetric positive definite matrices* $C_s, s = 1, \dots, n$, *the matrix inequalities*

$$
C_s = \sum_{l=1}^{n} \sum_{k=1}^{\infty} q_{ls}(k) N_s^T(k) C_{ls}^T C_l^{(j)} C_{ls} N_s(k) > 0, \quad s = 1, \dots, n
$$

hold.

6. EXAMPLE

Let us investigate the stability of solutions of the difference equation

$$
x_{k+1} = a(\xi) x_k, \quad k = 0, 1, \dots,
$$

with jumps of solutions

$$
x_{k+1} = c\, x_k, \quad k = 0, 1, \dots,
$$

where ξ is a semi-Markov chain that can take three possible states $\theta_1, \theta_2, \theta_3$. Denote

$$
a(\xi = \theta_s) \equiv a_s, \quad s = 1, 2, 3,
$$

and suppose that the transition intensities are given as

$$
\begin{aligned}
&q_{12}(1) = a, \quad q_{12}(2) = b, \quad q_{12}(3) = 1 - (a + b),\\
&q_{13}(1) = e, \quad q_{13}(2) = d, \quad q_{13}(3) = 1 - (e + d),\\
&q_{23}(1) = k, \quad q_{23}(2) = l, \quad q_{23}(3) = 1 - (k + l).
\end{aligned}
$$

Then system (4.4) takes the form

$$
\begin{aligned}
b_1 = E_1^{(2)}\{X_0\} + c^2\big(a\,a_2^2 + ba_2^4 + (1-(a+b))a_2^6\big)b_2 \\
+ c^2\big(ea_3^2 + da_3^4 + (1-(e+d)a_3^6)\big)b_3, \\
b_2 = E_2^{(2)}\{X_0\} + c^2(a\,a_1^2 + ba_1^4 + (1-(a+b))a_1^6)b_1 \\
+ c^2(ka_3^2 + la_3^4 + (1-(k+l)a_3^6))b_3, \\
b_3 = E_3^{(2)}\{X_0\} + c^2(e\,a_1^2 + da_1^4 + (1-(e+d))a_2^6)b_2.
\end{aligned}
$$

Then, conditions of L_2-stability can be expressed in the form

$$
\begin{aligned}
&c^4\,a_2^2\,a_1^2\big(a + ba_2^2 + \big(a + ba_2^2 + (1-a-b)\big)a_2^4\big)\big(a + ba_1^2 + (1-a-b)a_1^4\big) < 1, \\
&c^6 a_1^2 a_2^2 a_3^2\big(a + ba_1^2 + (1-a-b)a_1^4\big) \\
&\quad\cdot\big(k + ba_2^2 + (1-k-l)a_2^4\big)\big(e + da_3^2 + (1-e-d)a_3^4\big) \\
&+ c^6 a_1^2 a_2^2 a_3^2\big(e + da_1^2 + (1-e-d)a_1^4\big) \\
&\quad\cdot\big(a + ba_2^2 + (1-a-b)a_2^4\big)\big(k + la_3^2 + (1-k-l)a_3^4\big) \\
&+ c^4 a_1^2 a_3^2\big(e + da_1^2 + (1-e-d)a_1^4\big)\big(e + da_3^2 + (1-e-d)a_3^4\big) \\
&+ c^4 a_2^2 a_3^2\big(k + la_2^2 + (1-k-l)a_2^4\big)\big(k + la_3^2 + (1-k-l)a_3^4\big) \\
&+ c^4 a_1^2 a_2^2\big(a + ba_1^2 + (1-a-b)a_2^4\big)(a + ba_1^2 + (1-a-b)a_1^4) < 1.
\end{aligned}
$$

7. CONCLUSION

This work develops and solves the problem of L_2-stability of the trivial solution to the system of difference equations with coefficients depending on a semi-Markov chain. Dynamic systems of this kind can be very well used for modeling security and risk management in cyberspace. In particular, as a subspace, one can consider health, the environment, economic cyberspace and others. Cyberspace has become the arena of numerous conflicts, differing in form and method, intensity and degree of threats, which they carry. Cyberattacks and data violations are growing in various industries. For example, since healthcare data is unique, it makes the privacy and security so important. New challenges and threats in cyberspace force us to look for new solutions and, accordingly, new models that allow us to predict the situation in advance. This caused the investigation in the field of difference equations with random parameters that can serve as a mathematical model of the problems under discussion. In our next works, we propose to develop specific models using the above material and provide computer implementation with a user-friendly interface.

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