Opuscula Math. 38, no. 6 (2018), 779–794
https://doi.org/10.7494/OpMath.2018.38.6.779 Opuscula Mathematica

ON SIGNED ARC TOTAL DOMINATION IN DIGRAPHS

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Communicated by Andrzej Żak

Abstract. Let $D = (V, A)$ be a finite simple digraph and $N(uv) = \{u'v' \neq uv \mid u = u'$ or $v = v'\}$ be the open neighbourhood of uv in D. A function $f : A \to \{-1, +1\}$ is said to be a signed arc total dominating function (SATDF) of D if $\sum_{e' \in N(uv)} f(e') \geq 1$ holds for every arc $uv \in A$. The signed arc total domination number $\gamma'_{st}(D)$ is defined as $\gamma'_{st}(D) = \min\{\sum_{e \in A} f(e) \mid f$ is an SATDF of $D\}$. In this paper we initiate the study of the signed arc total domination in digraphs and present some lower bounds for this parameter.

Keywords: signed arc total dominating function, signed arc total domination number, domination in digraphs.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

In this paper we continue the study of signed dominating functions in graphs and digraphs. Let G be a simple graph with edge set $E(G)$ and let $N(e) = N_G(e)$ be the open neighborhood of the edge e. A signed edge total dominating function (SETDF) on a graph G is defined in [6] as a function $f : E(G) \to \{-1, 1\}$ such that $\sum_{e' \in N_G(e)} f(e') \geq 1$ for every $e \in E(G)$. The weight of an SETDF f on a graph G is $\omega(f) = \sum_{e \in E(G)} f(v)$. The signed edge total domination number $\gamma'_{st}(G)$ of G is the minimum weight of an SETDF on G. This concept has been studied by several authors (see, for example, [1, 5, 7]).

Let D be a finite simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. A digraph without directed cycles of length 2 is an *oriented* graph. The order $n = n(D)$ and the size $m = m(D)$ of a digraph D is the number of its vertices and arcs, respectively. We write $d^+_D(v)$ for the out-degree of a vertex v and $d^-_D(v)$ for its in-degree. The minimum and maximum in-degrees and minimum and maximum out-degrees of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If uv is an arc of D, then we also write $u \to v$, and we say that v is an out-neighbor of u and u is an in-neighbor of v. For each vertex $v \in V$, let $N^-_D(v)$ be the in-neighbor set which consists of all vertices of D from which arcs go into v and $N_D^+(v)$ be the out-neighbor set which consists of all vertices of D into which arcs go from v. The *degree* of a vertex u in D is defined by $d_D(u) = d_D^+(u) + d_D^-(u)$ and the minimum degree of D is $\delta(D) = \min\{d_D(u) \mid u \in V\}$. If $d_D(v) = 1$, then we call v a *pendant* vertex in D. If $X \subseteq V$, then $D[X]$ is the subdigraph induced by X. For every $uv \in A$, we define $d_D(uv) = d_D^+(u) + d_D^-(v) - 2$ to be the degree of the arc uv in D. The *minimum* and *maximum* arc degrees of D are denoted by $\delta' = \delta'(D)$ and $\Delta' = \Delta'(D)$, respectively. An arc of D is said to be a *pendant* arc if it is incident with a pendant vertex in D. For $uv \in A$, define $N_D(uv) = N(uv) = \{u'v' \neq uv \mid u = u' \text{ or } v = v'\}$ as the open neighborhood of uv. An *orientation* of a graph G is a digraph obtained from G by replacing every edge of G with a directed edge.

For a real-valued function $f : A(D) \to R$, the *weight* of f is $\omega(f) = \sum_{e\in A(D)} f(e)$, and for $S \subseteq A(D)$, we define $f(S) = \sum_{e\in A(D)} f(e)$, so $\omega(f) = f(A(D))$. Consult [4] for the notation and terminology which are not defined here.

Recently, Meng [2] defined a *signed edge dominating function* (SEDF) on a digraph D as a function $f : A \to \{-1, 1\}$ such that $\sum_{e'\in N[e]} f(e') \geq 1$ for every $e \in A$, where $N[e] = N(e) \cup \{e\}$. The *signed edge domination number* $\gamma_s'(D)$ of D is the minimum weight of a signed edge dominating function on D. Following the ideas in [2] and [6], we initiate the study of signed arc total dominating functions in digraphs.

A function $f : A \to \{-1, +1\}$ is called a *signed arc total dominating function* (SATDF) on a digraph D, if $f(N(uv)) \geq 1$ for each arc $uv \in A$. The minimum of the values of $\omega(f) = f(A)$, taken over all SATDF f of D, is called the *signed arc total domination number* of D and denoted by $\gamma_{st}'(D)$. A $\gamma_{st}'(D)$-function is an SATDF on D of weight $\gamma_{st}'(D)$. Obviously, $\gamma_{st}'(D)$ is defined only for digraphs D with $\delta' \geq 1$. In this note we initiate the study of the signed arc total domination in digraphs and present some (sharp) bounds for this parameter.

A nonempty digraph D with an SATDF f on D, denoted by (D, f), is called a signed arc total digraph. Let (D, f) be a signed arc total digraph and let u be an arbitrary vertex in D, then define

$$\begin{aligned}
A^+(u^+, f) &= \{uv \in A \mid f(uv) = 1\}, & A^-(u^+, f) &= \{uv \in A \mid f(uv) = -1\},\\
A^+(u^-, f) &= \{vu \in A \mid f(vu) = 1\}, & A^-(u^-, f) &= \{vu \in A \mid f(vu) = -1\},\\
A_-(f) &= \{e \in A \mid f(e) = -1\}, & f(u^+) &= |A^+(u^+, f)| - |A^-(u^+, f)|,\\
A_+(f) &= \{e \in A \mid f(e) = 1\}, & f(u^-) &= |A^+(u^-, f)| - |A^-(u^-, f)|.
\end{aligned}$$

We make use of the following observations in this paper.

Observation 1.1. *If f is an SATDF on a digraph D of size m, then*

(a) $\omega(f) = |A_+(f)| - |A_-(f)|$,
(b) $m = |A_+(f)| + |A_-(f)|$,
(c) $\gamma_{st}'(D) \equiv m \pmod 2$.

Observation 1.2. *Let e be an arc with degree at most 2 in D. If f is an SATDF on D, then f assigns 1 to each arc of $N(e)$.*

For every arc $e \in A$, define

$$A_{odd} = \{e \in A \mid d_D(e) \text{ is odd}\} \quad \text{and} \quad A_{even} = \{e \in A \mid d_D(e) \text{ is even}\}.$$

Denote $m_o = |A_{odd}|$ and $m_e = |A_{even}|$.

Observation 1.3. *Let f be a signed arc total dominating function on D and $e \in A$. If $e \in A_{odd}$, then $\sum_{e' \in N(e)} f(e') \geq 1$ and $\sum_{e' \in N(e)} f(e') \geq 2$, when $e \in A_{even}$.*

A directed graph is called connected if replacing all of its arcs with undirected edges produces a connected (undirected) graph.

Observation 1.4. *If $D_1, D_2, \ldots, D_s$ be the components of D, then*

$$\gamma'_{st}(D) = \sum_{i=1}^{s} \gamma'_{st}(D_i). \tag{1.1}$$

Theorem 1.5. *Let D be a digraph of size m. Then $\gamma'_{st}(D) = m$ if and only if for each arc $e \in A(D)$ there is an arc $e' \in N(e)$ such that $d_D(e') \leq 2$.*

Proof. One side is clear by Observation 1.2. Let $\gamma'_{st}(D) = m$. Assume, to the contrary, there exists an arc $e = uv \in A(D)$ such that for every $e' \in N(e)$, $d_D(e') \geq 3$. It is easy to verify that the function $f : A(D) \to \{-1, 1\}$ that assigns -1 to uv and $+1$ to the remaining arcs, is an SATDF of D of weight $m - 2$, and so $\gamma'_{st}(D) \leq m - 2$, a contradiction. This completes the proof. □

Remark 1.6. We remark that the signed edge total domination and signed arc total domination are not comparable. If D_1 is an orientation of $K_{1,4}$ such that $d^+_{D_1}(w) = d^-_{D_1}(w)$, where w is the central vertex of $K_{1,4}$, then $\gamma'_{st}(D_1) = 4 > \gamma'_{st}(K_{1,4}) = 2$. If D_2 is an orientation of $K_{2,2}$ such that $\delta' \geq 1$, then $\gamma'_{st}(D_2) = \gamma'_{st}(K_{2,2}) = 4$. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3\}$ be the partite sets of $K_{3,3}$ and let D_3 be an orientation of $K_{3,3}$ such that

$$A(D_3) = \{u_1 v_i, v_j u_2, u_3 v_i, u_2 v_1 \mid 1 \leq i \leq 3,\ 2 \leq j \leq 3\}.$$

Define f on $A(D_3)$ by $f(u_1 v_2) = f(u_3 v_2) = -1$ and $f(x) = 1$ otherwise. Obviously, f is an SATDF on D_3 with weight 5. Thus $\gamma'_{st}(D_3) < \gamma'_{st}(K_{3,3}) = 7$.

2. BOUNDS ON THE SIGNED ARC TOTAL DOMINATION NUMBER

In this section, we present some lower bounds for the signed arc total domination number of a digraph D.

Theorem 2.1. For any digraph D of size $m \geq 2$ and $\delta' \geq 1$,

$$\gamma'_{st}(D) \geq \max\{\delta' + 3 - m, \Delta' + 1 - m\}.$$

Furthermore, this bound is sharp.

Proof. Let f be an SATDF on D and let $uv \in A$. Then f assigns 1 to at least $\lceil \frac{\delta'+1}{2} \rceil$ arcs in $N(uv)$. Let $u'v' \in N(uv)$ such that $f(u'v') = 1$. Also f assigns 1 to at least $\lceil \frac{\delta'+1}{2} \rceil$ arcs in $N(u'v')$. Therefore

$$|A_-(f)| \leq m - \frac{\delta' + 1}{2} - 1,$$

which implies that

$$\gamma'_{st}(D) = |A_+(f)| - |A_-(f)| \geq \frac{\delta' + 1}{2} + 1 - \Big(m - \frac{\delta' + 1}{2} - 1\Big) = \delta' + 3 - m,$$

as desired. Now let $uv \in A(D)$ be an arc with maximum arc degree in D, then

$$\frac{m + \gamma'_{st}(D)}{2} \geq |A_+(f)| \geq |A_+(f) \cap N(uv)| \geq \frac{\Delta' + 1}{2},$$

and this leads to $\gamma'_{st}(D) \geq \Delta' + 1 - m$. If D is an orientation of $K_{1,2}$ with central vertex v such that $d^+_D(v) = 2$, then obviously $\gamma'_{st}(D) = 2 = \delta' + 3 - m$. □

Theorem 2.2. *Let D be a digraph with order n and size $m \geq 2$ with $\delta' \geq 1$. Then*

$$\gamma'_{st}(D) \geq \frac{m - (\Delta^+ - \delta^+)(n - \delta^-)(\Delta^- - 1) - (\Delta^- - \delta^-)(n - \delta^+)(\Delta^+ - 1)}{\Delta^+ + \Delta^- - 2}.$$

Proof. Let f be a $\gamma'_{st}(D)$-function. We have

$$\gamma'_{st}(D) = |A_+(f)| - |A_-(f)| = \sum_{u \in V} |A^+(u^+, f)| - \sum_{u \in V} |A^-(u^+, f)| = \sum_{u \in V} f(u^+). \quad (2.1)$$

Similarly, we have

$$\gamma'_{st}(D) = |A_+(f)| - |A_-(f)| = \sum_{u \in V} |A^+(u^-, f)| - \sum_{u \in V} |A^-(u^-, f)| = \sum_{u \in V} f(u^-). \quad (2.2)$$

For an arbitrary $uv \in A$, $f(N(uv)) = f(u^+) + f(v^-) - 2f(uv) \geq 1$. Therefore,

$$\begin{aligned} m + 2\gamma'_{st}(D) &\leq \sum_{uv \in A} (f(u^+) + f(v^-) - 2f(uv)) + 2 \sum_{uv \in A} f(uv) \\ &= \sum_{uv \in A} (f(u^+) + f(v^-)) = \sum_{u \in V} f(u^+) d^+_D(u) + \sum_{v \in V} f(v^-) d^-_D(v). \end{aligned}$$

Let

$$B^+_+ = \{u \in V \mid f(u^+) \geq 1\}, \quad B^+_0 = \{u \in V \mid f(u^+) = 0\}, \quad B^+_- = \{u \in V \mid f(u^+) \leq -1\},$$
$$B^-_+ = \{u \in V \mid f(u^-) \geq 1\}, \quad B^-_0 = \{u \in V \mid f(u^-) = 0\}, \quad B^-_- = \{u \in V \mid f(u^-) \leq -1\}.$$

Then by (2.1)–(2.3), we have

$$\begin{aligned}
m + 2\gamma'_{st}(D) &\leq \sum_{u\in V} f(u^+)d_D^+(u) + \sum_{v\in V} f(v^-)d_D^-(v)\\
&= \sum_{u\in B_+^+} f(u^+)d_D^+(u) + \sum_{u\in B_-^+} f(u^+)d_D^+(u)\\
&\quad + \sum_{v\in B_+^-} f(v^-)d_D^-(v) + \sum_{v\in B_-^-} f(v^-)d_D^-(v)\\
&\leq \Delta^+ \sum_{u\in B_+^+} f(u^+) + \delta^+ \sum_{u\in B_-^+} f(u^+)\\
&\quad + \Delta^- \sum_{v\in B_+^-} f(v^-) + \delta^- \sum_{v\in B_-^-} f(v^-)\\
&= \Delta^+ \sum_{u\in V} f(u^+) + (\delta^+ - \Delta^+) \sum_{u\in B_-^+} f(u^+)\\
&\quad + \Delta^- \sum_{v\in V} f(v^-) + (\delta^- - \Delta^-) \sum_{v\in B_-^-} f(v^-)\\
&= \Delta^+\gamma'_{st}(D) + (\delta^+ - \Delta^+) \sum_{u\in B_-^+} f(u^+)\\
&\quad + \Delta^-\gamma'_{st}(D) + (\delta^- - \Delta^-) \sum_{v\in B_-^-} f(v^-).
\end{aligned}$$

Hence

$$(\Delta^+ + \Delta^- - 2)\gamma'_{st}(D) \geq m + (\Delta^+ - \delta^+) \sum_{u\in B_-^+} f(u^+) + (\Delta^- - \delta^-) \sum_{v\in B_-^-} f(v^-). \tag{2.3}$$

For each $u \in B_-^+$ and $v \in N^+(u)$, we have $v \in B_+^- \cup B_0^-$. Since

$$f(u^+) + f(v^-) - 2f(uv) \geq 1,$$

it follows that

$$\delta^+ \leq |N^+(u)| \leq |B_+^-| + |B_0^-| = n - |B_-^-|.$$

Therefore

$$|B_-^-| \leq n - \delta^+. \tag{2.4}$$

Similarly, for each $v \in B_-^-$ and $u \in N^-(v)$, we have $u \in B_+^+ \cup B_0^+$, which implies

$$|B_-^+| \leq n - \delta^-. \tag{2.5}$$

On the other hand, for each $u \in B_-^+$, there must be a vertex $v \in N^+(u)$ such that $f(uv) = -1$. Using this and the fact that $f(u^+) + f(v^-) - 2f(uv) \geq 1$, we get $f(u^+) + f(v^-) \geq -1$. Since $f(v^-) \leq \Delta^- - 2$, we have

$$f(u^+) \geq 1 - \Delta^-. \tag{2.6}$$

Similarly, for each $v \in B_-^-$, we have

$$f(v^-) \geq 1 - \Delta^+. \tag{2.7}$$

Applying (2.3)–(2.7), we obtain

$$\begin{aligned}(\Delta^+ + \Delta^- - 2)\gamma'_{st}(D) \geq m &- (\Delta^+ - \delta^+)(n - \delta^-)(\Delta^- - 1)\\ &- (\Delta^- - \delta^-)(n - \delta^+)(\Delta^+ - 1)\end{aligned}$$

as desired. □

A digraph D is regular if $\Delta^+ = \delta^+ = \Delta^- = \delta^-$. As an application of Proposition 2.2, we obtain a lower bound on the signed arc total domination number for r-regular digraphs.

Corollary 2.3. *If D is an r-regular digraph of size m with $r \geq 2$, then*

$$\gamma'_{st}(D) \geq \left\lceil \frac{m}{2r-2} \right\rceil.$$

Theorem 2.4. *For any digraph D of order n and size m,*

$$\gamma'_{st}(D) \geq 2\left\lceil \frac{m^2}{n(\Delta^+ + \Delta^- - 2)} - \frac{m_o}{2(\Delta^+ + \Delta^- - 2)} \right\rceil - m.$$

Proof. Let f be a $\gamma'_{st}(D)$-function and let $e = uv$ be an arc in D. If e is an arc of odd degree, then

$$|N(e) \cap A_+(f)| \geq \frac{1}{2}(d_D^+(u) + d_D^-(v) - 1)$$

and if e is an arc of even degree, then

$$|N(e) \cap A_+(f)| \geq \frac{1}{2}(d_D^+(u) + d_D^-(v)).$$

Thus

$$\begin{aligned}\sum_{e\in A} |N(e) \cap A_+(f)| &\geq \frac{1}{2}\sum_{uv\in A}(d_D^+(u) + d_D^-(v)) - \frac{1}{2}m_o\\ &= \frac{1}{2}\left(\sum_{u\in V}(d_D^+(u))^2 + \sum_{v\in V}(d_D^-(v))^2\right) - \frac{1}{2}m_o\\ &\geq \frac{1}{2n}\left[\left(\sum_{u\in V} d_D^+(u)\right)^2 + \left(\sum_{v\in V} d_D^-(v)\right)^2\right] - \frac{1}{2}m_o = \frac{m^2}{n} - \frac{m_o}{2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\Delta^+ + \Delta^- - 2)|A_+(f)| &\geq \sum_{e\in A_+(f)} |N(e)| \\ &= \sum_{e\in A_+(f)} (|N(e)\cap A_+(f)| + |N(e)\cap A_-(f)|) \\ &= \sum_{e\in A_+(f)} |N(e)\cap A_+(f)| + \sum_{e\in A_+(f)} |N(e)\cap A_-(f)| \\ &= \sum_{e\in A_+(f)} |N(e)\cap A_+(f)| + \sum_{e\in A_-(f)} |N(e)\cap A_+(f)| \\ &= \sum_{e\in A} |N(e)\cap A_+(f)| \geq \frac{m^2}{n} - \frac{m_o}{2}.\end{aligned}$$

Since $\gamma'_{st}(D) = 2|A^+(f)| - m$, we get

$$\gamma'_{st}(D) \geq 2\left\lceil \frac{m^2}{n(\Delta^+ + \Delta^- - 2)} - \frac{m_o}{2(\Delta^+ + \Delta^- - 2)} \right\rceil - m. \qquad \square$$

Theorem 2.5. *Let D be a digraph of size m. Then*

$$\gamma'_{st}(D) \geq \frac{(2+\delta'-\Delta')m + 2m_e}{\delta' + \Delta'}.$$

Proof. Let f be a $\gamma'_{st}(D)$-function and $\sum_{e\in A} d_D(e) = L$. By Observation 1.3, we have

$$\begin{aligned}\sum_{e\in A}\sum_{e'\in N(e)} f(e') &= \sum_{e\in A_{even}}\sum_{e'\in N(e)} f(e') + \sum_{e\in A_{odd}}\sum_{e'\in N(e)} f(e') \\ &\geq 2|A_{even}| + |A_{odd}| = m_e + m.\end{aligned} \tag{2.8}$$

On the other hand,

$$\begin{aligned}\sum_{e\in A}\sum_{e'\in N(e)} f(e') &= \sum_{e\in A} d_D(e)f(e) = \sum_{e\in A_+(f)} d_D(e)f(e) + \sum_{e\in A_-(f)} d_D(e)f(e) \\ &= \sum_{e\in A_+(f)} d_D(e) - \sum_{e\in A_-(f)} d_D(e) = 2\sum_{e\in A_+(f)} d_D(e) - \sum_{e\in A} d_D(e) \\ &\leq 2\Delta'|A_+(f)| - L.\end{aligned} \tag{2.9}$$

Similarly, we have

$$\begin{aligned}\sum_{e\in A}\sum_{e'\in N(e)} f(e') &= \sum_{e\in A} d_D(e)f(e) = \sum_{e\in A_+(f)} d_D(e)f(e) + \sum_{e\in A_-(f)} d_D(e)f(e) \\ &= \sum_{e\in A_+(f)} d_D(e) - \sum_{e\in A_-(f)} d_D(e) = \sum_{e\in A} d_D(e) - 2\sum_{e\in A_-(f)} d_D(e) \\ &\leq \sum_{e\in A} d_D(e) - 2|A_-(f)|\delta' = L - 2(m - |A_+(f)|)\delta'.\end{aligned} \tag{2.10}$$

By (2.8)–(2.10), we deduce the following inequalities:

$$m + m_e + L \leq 2\Delta'|A_+(f)| \quad \text{and} \quad m + 2m\delta' + m_e - L \leq 2\delta'|A_+(f)|. \tag{2.11}$$

Summing the inequalities in (2.11), we have

$$|A_+(f)| \geq \frac{(1+\delta')m + m_e}{\delta' + \Delta'},$$

and hence

$$\gamma'_{st}(D) = 2|A_+(f)| - m \geq \frac{(2+\delta'-\Delta')m + 2m_e}{\delta' + \Delta'}.$$

□

Theorem 2.6. *Let D be a digraph of size m with the arc degree sequence $d'_1 \geq d'_2 \geq \ldots \geq d'_m$. Then*

$$\gamma'_{st}(D) \geq 2\left\lceil \frac{m + m_e + L - 2L_t + 2td'_{t+1}}{2d'_{t+1}} \right\rceil - m,$$

where $t = \max\{\left\lceil \frac{m(1+2\delta')-L+m_e}{2\delta'} \right\rceil, \left\lceil \frac{m+L+m_e}{2\Delta'} \right\rceil\}$, $L_t = \sum_{i=1}^{t} d'_i$ *and* $L = \sum_{e \in A} d_D(e)$.

Proof. Let f be a $\gamma'_{st}(D)$-function on D. From (2.11), we have

$$|A_+(f)| \geq \frac{m + L + m_e}{2\Delta'}, \quad |A_+(f)| \geq \frac{m(1+2\delta') - L + m_e}{2\delta'}.$$

So

$$|A_+(f)| \geq t = \max\left\{ \left\lceil \frac{m(1+2\delta') - L + m_e}{2\delta'} \right\rceil, \left\lceil \frac{m + L + m_e}{2\Delta'} \right\rceil \right\}.$$

It follows from inequality (2.8) and the inequality chain (2.9) that

$$\begin{aligned} m + m_e &\leq 2 \sum_{e \in A_+(f)} d_D(e) - \sum_{e \in A} d_D(e) \\ &\leq 2\left(\sum_{i=1}^{t} d'_i + (|A_+(f)| - t)d'_{t+1} \right) - L \\ &= 2\left(L_t + (|A_+(f)| - t)d'_{t+1}\right) - L. \end{aligned}$$

Therefore

$$|A_+(f)| \geq \left\lceil \frac{m + m_e + L - 2L_t + 2td'_{t+1}}{2d'_{t+1}} \right\rceil$$

and hence

$$\gamma'_{st}(D) = 2|A_+(f)| - m \geq 2\left\lceil \frac{m + m_e + L - 2L_t + 2td'_{t+1}}{2d'_{t+1}} \right\rceil - m. \qquad \square$$

Theorem 2.7. *For every simple connected digraph D with $2 \le \delta' \le \Delta' \le 6$, $\gamma'_{st}(D) \ge 0$.*

Proof. Let f be a $\gamma'_{st}(D)$-function. Since $2 \le \delta' \le \Delta' \le 6$, we have $|N_D(e) \cap A_+(f)| \ge 2$ and $|N_D(e) \cap A_-(f)| \le 2$. Now it is clear that

$$2|A_-(f)| \le \sum_{e \in A_-(f)} |N_D(e) \cap A_+(f)| = \sum_{e \in A_+(f)} |N_D(e) \cap A_-(f)| \le 2|A_+(f)|.$$

Thus $|A_-(f)| \le |A_+(f)|$ and hence, $\gamma'_{st}(D) = |A_+(f)| - |A_-(f)| \ge 0$. □

3. SIGNED ARC TOTAL DOMINATION IN ORIENTED GRAPHS

Let G be the complete bipartite graph $K_{2,3}$ with bipartite sets $V = \{v_1, v_2\}$ and $U = \{u_1, u_2, u_3\}$. Let D_1 be an orientation of G such that all arcs go from V into U and let D_2 be an orientation of G such that $A(D_2) = \{(v_1, u_j), (u_j, v_2) \mid j = 1, 2, 3\}$. It is easy to see that $\gamma'_{st}(D_1) = 2$ and $\gamma'_{st}(D_2) = 6$. Therefore, two distinct orientations of a graph can have different signed total arc domination numbers. Motivated by this observation, we define lower orientable signed total arc domination number $\mathrm{dom}'_{st}(G)$ and upper orientable signed total arc domination number $\mathrm{Dom}'_{st}(G)$ of a graph G as follows:

$$\mathrm{dom}'_{st}(G) = \min\{\gamma'_{st}(D) \mid D \text{ is an orientation of } G \text{ with } \delta' \ge 1\},$$

and

$$\mathrm{Dom}'_{st}(G) = \max\{\gamma'_{st}(D) \mid D \text{ is an orientation of } G \text{ with } \delta' \ge 1\}.$$

An immediate consequence of Proposition 1.5 now follows.

Corollary 3.1. *For $n \ge 3$, $\mathrm{dom}'_{st}(P_n) = n - 1$, $\mathrm{dom}'_{st}(C_n) = n$.*

Proposition 3.2. *If $G = K_{1,m}$ is a star, then* $\mathrm{dom}'_{st}(K_{1,m}) = \begin{cases} 3 & m \text{ is odd,} \\ 2 & m \text{ is even.} \end{cases}$

Proof. Consider the graph $K_{1,m}$ with bipartite sets $\{v_1\}$ and $\{u_1, u_2, \ldots, u_m\}$. Let D be an orientation of $K_{1,m}$ and let f be a $\gamma'_{st}(D)$-function. If $d^+_D(v_1) = 0$ or $d^-_D(v_1) = 0$, then $|A_-(f)| = (m-2)/2$ if m is even and $|A_-(f)| = (m-3)/2$ if m is odd. Hence, $\gamma'_{st}(D) = 2$ if m is even and $\gamma'_{st}(D) = 3$ if m is odd. Suppose that $d^+_D(v_1)$ and $d^-_D(v_1) \ge 1$. If either $d^+_D(v_1) = 1$ or $d^-_D(v_1) = 1$, then there is an arc $e = v_1u_i$ with $d_D(e) = 0$, a contradiction. So $d^+_D(v_1)$ and $d^-_D(v_1) \ge 2$. Let, without loss of generality, that $u_1 \in N^+(v_1)$ and $u_2 \in N^-(v_1)$. If m is odd, then either $f(N(v_1u_1)) \ge 2$ or $f(N(u_2v_1)) \ge 2$. Thus $\gamma'_{st}(D) \ge 3$. If m is even, since $f(N(v_1u_1)) \ge 1$ and $f(N(u_2v_1)) \ge 1$, it follows that $\gamma'_{st}(D) \ge 2$. This completes the proof. □

Lemma 3.3. *For $m \ge 2$,* $\gamma'_{st}(K_{2,m}) = \begin{cases} 4 & \text{if } m \text{ is even,} \\ 2 & \text{if } m \text{ is odd.} \end{cases}$

Proof. Let $X = \{u_1, u_2\}$ and $Y = \{v_1, v_2, \dots, v_m\}$ be the partite sets of $K_{2,m}$ and let f be a $\gamma'_{st}(K_{2,m})$-function. We consider two cases.

Case 1. m is odd.
Since

$$f(N(u_1v_1)) = f(u_2v_1) + \sum_{i=2}^{m} f(u_1v_i) \geq 1$$

and

$$f(N(u_2v_1)) = f(u_1v_1) + \sum_{i=2}^{m} f(u_2v_i) \geq 1,$$

we have

$$\omega(f) = \sum_{i=1}^{m} f(u_1v_i) + \sum_{i=1}^{m} f(u_2v_i) = f(N(u_1v_1)) + f(N(u_2v_1)) \geq 2.$$

Define $g : E(K_{2,m}) \to \{-1, 1\}$ by $g(u_1v_1) = g(u_2v_1) = 1$ and $g(u_1v_i) = g(u_2v_i) = (-1)^i$ for $2 \leq i \leq m$. Obviously, g is an SETDF of $K_{2,m}$ of weight 2 and so $\gamma'_{st}(K_{2,m}) \leq 2$. Therefore $\gamma'_{st}(K_{2,m}) = 2$.

Case 2. m is even.
Define $g : E(K_{2,m}) \to \{-1, 1\}$ by $g(u_iv_1) = g(u_iv_2) = 1$ for $i = 1, 2$ and $g(u_1v_i) = g(u_2v_i) = (-1)^i$ for $3 \leq i \leq m$. Obviously g is an SETDF of $K_{2,m}$ of weight 4 and hence $\gamma'_{st}(K_{2,m}) \leq 4$. Now we show that $\gamma'_{st}(K_{2,m}) = 4$. Since m is even, $f(N(u_1v_1)) \geq 2$ and $f(N(u_2v_1)) \geq 2$. Hence,

$$\omega(f) = f(N(u_1v_1)) + f(N(u_2v_1)) \geq 4.$$

Therefore $\gamma'_{st}(K_{2,m}) = 4$. □

Proposition 3.4. *For* $m \geq 2$, $\mathrm{dom}'_{st}(K_{2,m}) = \begin{cases} 2 & \textit{if } m \textit{ is odd,} \\ 4 & \textit{if } m \textit{ is even.} \end{cases}$

Proof. Let $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, \dots, v_m\}$ be the partite sets of $K_{2,m}$, D be an orientation on $K_{2,m}$ and f be a $\gamma'_{st}(D)$-function. If $d^+_D(v_i) = 2$ (or $d^-_D(v_i) = 2$) for each $1 \leq i \leq m$, then we are done by Lemma 3.3. Without loss of generality, suppose that $d^+_D(u_1) \geq d^-_D(u_1)$. We distinguish two cases.

Case 1. $d^+_D(v_i) = d^-_D(v_i) = 1$, for some i, say $i = 1$.
Without loss of generality, suppose that $u_1v_1, v_1u_2 \in A(D)$. Since $f(N(u_1v_1)) \geq 1$, there is at least one arc $e' \in N(u_1v_1)$ such that $f(e') = 1$. Similarly, there is an arc $e'' \in N(v_1u_2)$ such that $f(e'') = 1$. Since

$$|N(e') \cap (\{u_2v_i \mid u_2v_i \in A(D)\})| \leq 1$$

and

$$|N(e'') \cap (\{v_iu_1 \mid v_iu_1 \in A(D)\})| \leq 1,$$

we have

$$\begin{aligned}\gamma'_{st}(D) &\geq \sum_{u_2v_i\in A(D)} f(u_2v_i) + f(e') + f(N(e')) + f(e'') + f(N(e''))\\ &\quad + \sum_{v_iu_1\in A(D)} f(v_iu_1) - 2\\ &\geq 4-2=2.\end{aligned}$$

Hence, if m is odd, then the statement is true. Assume that m is even. If either $|N(e')|$ and $|N(e'')|$ are even or

$$|N(e') \cap (\{u_2v_i \mid u_2v_i \in A(D)\})| = |N(e'') \cap (\{v_iu_1 \mid v_iu_1 \in A(D)\})| = 0,$$

then by an argument similar to that described above we get $\gamma'_{st}(D) \geq 4$. We consider two subcases.

Subcase 1.1. $|N(e')|$ is odd and $|N(e') \cap (\{u_2v_i \mid u_2v_i \in A(D)\})| = 1$ (the case $|N(e'')|$ is odd and $|N(e'') \cap (\{v_iu_1 \mid v_iu_1 \in A(D)\})| = 1$ is similar).
Then $|N(u_1v_1)|$ is even. Let

$$\{x\} = N(e') \cap (\{u_2v_i \mid u_2v_i \in A(D)\}).$$

If $f(x) = -1$, then $\sum_{u_1v_i\in A(D)} f(u_1v_i) \geq 3$ and $\sum_{u_2v_i\in A(D)} f(u_2v_i) \geq -1$ and if $f(x) = 1$, then $\sum_{u_1v_i\in A(D)} f(u_1v_i) \geq 1$ and $\sum_{u_2v_i\in A(D)} f(u_2v_i) \geq 1$. Consequently, $\sum_{u_1v_i\in A(D)} f(u_1v_i) + \sum_{u_2v_i\in A(D)} f(u_2v_i) \geq 2$. Moreover, since $f(N(e'')) \geq 1$, we have $\sum_{v_iu_2\in A(D)} f(v_iu_2) \geq 1$. If there is an arc $y = v_iu_1$ (note that since m and $|N(u_1v_1)|$ are even, there is at least one arc v_iu_1 in $A(D)$) such that $f(y) = 1$, then $\sum_{v_iu_1\in A(D)} f(v_iu_1) \geq 1$. Therefore

$$\begin{aligned}\gamma'_{st}(D) &= \sum_{u_1v_i\in A(D)} f(u_1v_i) + \sum_{u_2v_i\in A(D)} f(u_2v_i)\\ &\quad + \sum_{v_iu_1\in A(D)} f(v_iu_1) + \sum_{v_iu_2\in A(D)} f(v_iu_2)\\ &\geq 4.\end{aligned}$$

Suppose that $f(v_iu_1) = -1$ for each $v_iu_1 \in A(D)$. Then $d^-_D(u_1) = 1$. Without loss of generality, suppose that $\{v_m\} = N^-_D(u_1)$. Since $\sum_{e\in N(v_mu_1)} f(e) \geq 1$, we have $f(v_mu_2) = 1$ and since $f(N(v_mu_2)) \geq 1$, we have $\sum_{v_iu_2\in A(D)} f(v_iu_2) \geq 3$. Therefore,

$$\gamma'_{st}(D) = \sum_{u_1v_i\in A(D)} f(u_1v_i) + \sum_{u_2v_i\in A(D)} f(u_2v_i) + f(v_mu_1) + \sum_{v_iu_2\in A(D)} f(v_iu_2) \geq 4.$$

Subcase 1.2. $|N(e')|$ is odd and $|N(e'') \cap (\{v_iu_1 \mid v_iu_1 \in A(D)\})| = 1$ (the case $|N(e'')|$ is odd and $|N(e') \cap (\{u_2v_i \mid u_2v_i \in A(D)\})| = 1$ is similar).

Let $\{z\} = N(e'') \cap (\{v_iu_1 \mid v_iu_1 \in A(D)\})$. If $f(z) = -1$, then $\sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 3$ and $\sum_{v_iu_1 \in A(D)} f(v_iu_1) \geq -1$ and if $f(z) = 1$, then $\sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 1$ and $\sum_{v_iu_1 \in A(D)} f(v_iu_1) \geq 1$. Hence, $\sum_{v_iu_1 \in A(D)} f(v_iu_1) + \sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 2$. If $d_D^+(u_2) = 0$, since $f(N(e')) \geq 1$, then $\sum_{u_1v_i \in A(D)} f(u_1v_i) \geq 2$ and if there is an arc $y = u_2v_i$ such that $f(y) = 1$, then $\sum_{u_2v_i \in A(D)} f(u_2v_i) \geq 1$. Therefore

$$\gamma'_{st}(D) = \sum_{u_1v_i \in A(D)} f(u_1v_i) + \sum_{u_2v_i \in A(D)} f(u_2v_i) + \sum_{v_iu_1 \in A(D)} f(v_iu_1) + \sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 4.$$

Suppose that $f(u_2v_i) = -1$ for each $u_2v_i \in A(D)$. Then $d_D^+(u_2) = 1$. Without loss of generality, suppose that $\{v_m\} = N_D^+(u_2)$. Then $f(u_1v_m) = 1$ and $\sum_{u_1v_i \in A(D)} f(u_1v_i) \geq 3$. Therefore,

$$\gamma'_{st}(D) = \sum_{u_1v_i \in A(D)} f(u_1v_i) + f(u_2v_m) + \sum_{v_iu_1 \in A(D)} f(v_iu_1) + \sum_{v_iu_2 \in A(D)} f(v_iu_2)) \geq 4.$$

Case 2. $d_D^+(v_i) = 2$ and $d_D^-(v_j) = 2$, for some i, j.
Without loss of generality, suppose that $d_D^+(v_i) = 2$ for $1 \leq i \leq t$ and $d_D^-(v_j) = 2$ for $t + 1 \leq j \leq m$. Then by Lemma 3.3,

$$\gamma'_{st}(D) = \gamma'_{st}(K_{2,t}) + \gamma'_{st}(K_{2,m-t}) \geq 2 + 2 = 4.$$

This completes the proof. □

Theorem 3.5. *For any integer t, there is a graph G with* $\mathrm{dom}'_{st}(G) = -t$.

Proof. For a given positive integer $r \geq 4$, let T be a graph that obtained from a star $K_{1,r}$ by subdividing all of its edges once and let G be the graph obtained from $t+1$ copies of T with central vertices $v_1, v_2, \ldots, v_{t+1}$ by adding the edges $v_1v_2, v_2v_3, \ldots, v_tv_{t+1}$ (see Figure 1).

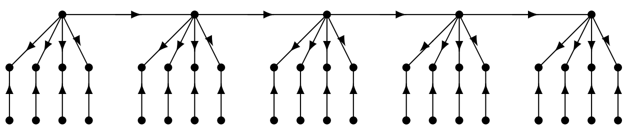

Fig. 1. A digraph with $\gamma'_{st}(D) = -4$

Let $\{v_j, v_{i,j}, u_{i,j} \mid 1 \leq i \leq t\}$ be the vertex set of jth copy of T, where $N(v_{i,j}) = \{v_j, u_{i,j}\}$ and $u_{i,j}$ are leaves for each i. Let D be an arbitrary orientation of G and let f be a $\gamma'_{st}(D)$-function. Clearly, either $d_D^+(v_{i,j}) = 2$ or $d_D^-(v_{i,j}) = 2$ for each i, j because $\delta' \geq 1$. In both cases, f assigns $+1$ to each non-pendant arc of each copy of T.

Since the least possible weight for f will be achieved if $f(e) = -1$ for each other arcs, we have $\omega(f) \geq (t+1)r - (t+1)r - t = -t$. In order to show that $\text{dom}'_{st}(G) \leq -t$, let D be an orientation of G such that

$$A(D) = \{(v_j, v_{j+1}), (v_j, v_{i,j}), (u_{i,j}, v_{i,j}) : \ 1 \leq i \leq r, \ 1 \leq j \leq t\},$$

as illustrated in Figure 1 for $t = 4$. Define $f : A(D) \to \{-1, 1\}$ by $f(v_j v_{i,j}) = +1$ and $f(v_j, v_{j+1}) = f(u_{i,j} v_{i,j}) = -1$ for $1 \leq i \leq r$ and $1 \leq j \leq t$. Obviously, f is an SATDF on D of weight $-t$. Therefore, $\text{dom}'_{st}(G) = -t$. □

Theorem 3.6. *If T is a tree of order $n \geq 3$, then*

$$\text{dom}'_{st}(T) \geq \frac{7-n}{3}.$$

Furthermore, this bound is sharp.

Proof. The proof is by induction on n. The statement holds for all trees of order $n = 3, 4, 5$. Assume T is a tree of order $n \geq 6$ and that the statement holds for all trees with smaller orders. Let D be an arbitrary orientation of T with $\delta' \geq 1$ and let f be a $\gamma'_{st}(D)$-function. We consider two cases.

Case 1. There is a non-pendant arc, say $e = uv \in A(D)$, for which $f(e) = -1$.
Let D_1 and D_2 be the components of $D - e$ with $u \in D_1$ and $v \in D_2$. Obviously, the order of D_1 and D_2 are greater than 3 and $\gamma'_{st}(D) = f(A(D_1)) - 1 + f(A(D_2))$. For $i = 1, 2$, the function f, restricted to D_i, is an SATDF of D_i, and so $\gamma'_{st}(D_i) \leq f(A(D_i))$. By the inductive hypothesis,

$$\gamma'_{st}(D_i) \geq \frac{7 - |A(D_i)|}{3}.$$

Thus

$$\gamma'_{st}(D) \geq -1 + \frac{7 - |A(D_1)|}{3} + \frac{7 - |A(D_2)|}{3} = \frac{11-n}{3} > \frac{7-n}{3}.$$

Case 2. The only arcs e for which $f(e) = -1$ are pendant arcs.
Then $f(v^+) \geq 0$ for each $v \in V(D)$ with $d^+_D(v) \geq 2$ and $f(v^-) \geq 0$ for each $v \in V(D)$ with $d^-_D(v) \geq 2$. Let

$$P^+_D = \{v \in V(D) \mid d^+_D(v) \geq 2 \text{ and } f(v^+) = 0\} \text{ and}$$
$$P^-_D = \{v \in V(D) \mid d^-_D(v) \geq 2 \text{ and } f(v^-) = 0\}.$$

First, let $P^+_D = P^-_D = \emptyset$. Then f is an SEDF of D. Hence, $\gamma'_s(D) \geq |V(D)| - |A(D)|$ (see [2]). Since $n \geq 6$ and $|V(D)| = |A(D)| + 1$, it follows that

$$\gamma'_{st}(D) = f(A(D)) \geq \gamma'_s(D) \geq 1 > \frac{7-n}{3}.$$

Without loss of generality, suppose that $P^+_D \neq \emptyset$. Let $P^+_D = \{u_1, u_2, \ldots, u_k\}$. Obviously, there is no $+1$ pendant arc out from u_i for each i. Let

$$M^+_D(u_i) = \{u \in N^+_D(u_i) \mid d^-_D(u) \geq 2\}.$$

Let first $|M^+_D(u_i)| \geq 2$ for some i. Without loss of generality we may assume $|M^+_D(u_1)| \geq 2$ and $v_1, v_2 \in M^+_D(u_1)$. Let D_1 and D_2 be the connected components of $D - u_1v_1$ for which $v_1 \in V(D_1)$. Let D'_1 be obtained from D_1 by adding a new pendant arc w_1v_1 and let D'_2 be obtained from D_2 by deleting one of the -1 pendant arcs out from u_1. Now define $g : A(D'_1) \to \{-1, +1\}$ by $g(w_1v_1) = +1$ and $g(e) = f(e)$ if $e \in A(D_1)$. Obviously, g is an SATDF of D'_1 and $f|_{D'_2}$ is an SATDF of D'_2. By the inductive hypothesis,

$$\gamma'_{st}(D'_i) \geq \frac{7 - |A(D'_i)|}{3}.$$

Thus

$$\begin{aligned}\gamma'_{st}(D) &= f(A(D)) = g(A(D'_1)) + f|_{D'_2}(A(D'_2)) - 1\\ &\geq -1 + \frac{7 - |A(D'_1)|}{3} + \frac{7 - |A(D'_2)|}{3} > \frac{7-n}{3}.\end{aligned}$$

Now let $M^+_D(u_i) = \{v_i\}$ for each $1 \leq i \leq k$. Since $f(N(u_iv_i)) \geq 1$, we have $f(v_i^-) \geq 3$ for each i. Let D' be obtained from D by deleting all pendant vertices and the vertices of P^+_D. We distinguish three subcases.

Subcase 2.1. $d^-_{D'}(v_1) \geq 1$, $e = vv_1 \in A(D')$ and $f(v^+) = 1$ in D.
By the construction of D' we have $d^+_D(v) \geq 3$. Since $f(v^+) = 1$ and all arcs in D' are $+1$ arcs, there exists a pendant arc e' out from v in D, say $e' = vz$. Let D_1 and D_2 be the connected components of $D - e$ containing v_1 and v, respectively. Let D'_1 be obtained from D_1 by adding a new pendant arc $v'v_1$ at v_1 and $D'_2 = D_2 - z$. It is easy to see that the order of D'_1 and D'_2 are greater than 3. Define $g : A(D'_1) \to \{-1, +1\}$ by $g(v'v_1) = 1$ and $g(e) = f(e)$ if $e \in A(D_1)$. Obviously, g and $f|_{D'_2}$ are SATDFs of D'_1 and D'_2, respectively. By the inductive hypothesis,

$$\gamma'_{st}(D'_i) \geq \frac{7 - |V(D_i)|}{3}.$$

Thus

$$\begin{aligned}\gamma'_{st}(D) &= f(A(D)) = g(A(D'_1)) + f|_{D'_2}(A(D'_2)) - 1\\ &\geq -1 + \frac{7 - |V(D'_1)|}{3} + \frac{7 - |V(D'_2)|}{3} > \frac{7-n}{3}.\end{aligned}$$

Subcase 2.2. $d^-_{D'}(v_1) \geq 1$, $e = vv_1 \in A(D')$ and $f(v^+) \geq 2$ in D.
Let D_1 and D_2 be the connected components of $D - e$. Let D'_1 and D'_2 be obtained from D_1 and D_2 by adding new pendant arcs $v'v_1$ and vv'', respectively. Define $g_1 : A(D'_1) \to \{-1, +1\}$ by $g_1(v'v_1) = 1$ and $g(e) = f(e)$ if $e \in A(D_1)$, and $g_2 : A(D'_2) \to \{-1, +1\}$ by $g(vv'') = 1$ and $g(e) = f(e)$ if $e \in A(D_2)$. Obviously, g_i is an SATDF of D'_i for $i = 1, 2$. In addition, we have $|V(D'_1)| + |V(D'_2)| = n + 2$. By the inductive hypothesis,

$$\gamma'_{st}(D) = f(A(T)) = g_1(A(D'_1)) + g_2(A(D'_2)) - 1 > \frac{7-n}{3}.$$

Subcase 2.3. $d^-_{D'}(v_1) = 0$.
This implies that $u_iv_1 \in A(D)$ for each $1 \le i \le k$. If there exist two pendant arcs at v_1, say $e' = xv_1$, $e'' = yv_1$, such that $f(e') = -1$ and $f(e'') = 1$, then using the inductive hypothesis on $D - \{x, y\}$ we have

$$\gamma'_{st}(D) \ge \frac{7-(n-2)}{3} > \frac{7-n}{3}.$$

Let r be the number of pendant in-neighbors of v_1. By assumption $k - r = f(v_1^-) \ge 3$. Furthermore, since $f(u_i^+) = 0$, there exists a pendant arc u_iw_i for each i. Therefore, $n \ge 2k+r+1$ and hence, $r \le \frac{n-7}{3}$. If D_1 is the subdigraph induced by $(\cup_{i=1}^k N_D^+(u_i)) \cup N_D^-(v_1)$, then $\omega(f|_{D_1}) = -r$. Now let D_2 be the digraph obtained from D by deleting all arcs of D_1 and all the isolated vertices. If $|V(D_2)| = 0$, then $D = D_1$ and we are done. Let $|V(D_2)| \ne 0$. Since D is an oriented tree, it is easy to verify that D_2 has t components, where $t = |V(D_1) \cap V(D_2)|$. Since the order of each component of D_2 is greater than 2, by the induction hypothesis and Observation 1.4, we have

$$\gamma'_{st}(D_2) \ge \frac{7t - |V(D_2)|}{3}.$$

Therefore

$$\begin{aligned}\gamma'_{st}(D) \ge \gamma'_{st}(D_1) + \gamma'_{st}(D_2) &\ge \frac{7-|V(D_1)|}{3} + \frac{7t-|V(D_2)|}{3}\\ &= \frac{7(t+1)-(n+t)}{3} > \frac{7-n}{3}.\end{aligned}$$

In order to show the sharpness of the lower bound, let D be a digraph with vertex set

$$V(D) = \{w, u_i, v_i, w_j \mid 1 \le i \le k,\ k \ge 3 \text{ and } 1 \le j \le k-3\},$$

and arc set

$$A(D) = \{ww_j, wu_i, v_iu_i \mid 1 \le i \le k \text{ and } 1 \le j \le k-3\}$$

(see Figure 2).

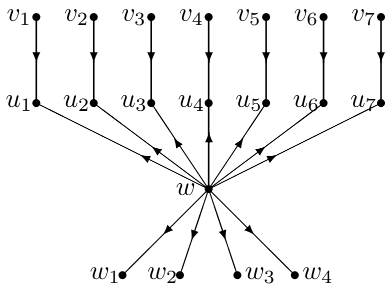

Fig. 2. Digraph D with $k = 7$

Define $f : A(D) \to \{-1, 1\}$ by $f(ww_j) = f(v_i u_i) = -1$ and $f(wu_i) = 1$ for each $1 \le i \le k$ and $1 \le j \le k-3$. Clearly, f is an SATDF of D with weight $\frac{7-n}{3}$. This completes the proof. □

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Received: February 25, 2016.
Revised: May 21, 2018.
Accepted: May 22, 2018.