Opuscula Math. 38, no. 4 (2018), 573–590
https://doi.org/10.7494/OpMath.2018.38.4.573 Opuscula Mathematica

TOEPLITZ VERSUS HANKEL: SEMIBOUNDED OPERATORS

Dmitri R. Yafaev

Communicated by P.A. Cojuhari

Abstract. Our goal is to compare various results for Toeplitz T and Hankel H operators. We consider semibounded operators and find necessary and sufficient conditions for their quadratic forms to be closable. This property allows one to define T and H as self-adjoint operators under minimal assumptions on their matrix elements. We also describe domains of the closed Toeplitz and Hankel quadratic forms.

Keywords: semibounded Toeplitz, Hankel and Wiener-Hopf operators, closable and closed quadratic forms.

Mathematics Subject Classification: 47B25, 47B35.

1. INTRODUCTION. BOUNDED OPERATORS

1.1. This is a short survey based on the talk given by the author at the conference "Spectral Theory and Applications" held in May 2017 in Krakow. Our aim is to compare various properties of Hankel and Toeplitz operators. We refer to the books [3, 7, 13, 14] for basic information on these classes of operators.

Formally, Toeplitz T and Hankel H operators are defined in the space $\ell^2(\mathbb{Z}_+)$ of sequences $f = (f_0, f_1, \ldots)$ by the relations

$$(Tf)_n = \sum_{m\in\mathbb{Z}_+} t_{n-m} f_m, \quad n \in \mathbb{Z}_+, \tag{1.1}$$

and

$$(Hf)_n = \sum_{m\in\mathbb{Z}_+} h_{n+m} f_m, \quad n \in \mathbb{Z}_+. \tag{1.2}$$

Let us also introduce discrete convolutions in the space $\ell^2(\mathbb{Z})$ (known also as Laurent operators) acting by the formula

$$(Lg)_n = \sum_{m\in\mathbb{Z}} t_{n-m} g_m, \quad g = \{g_n\}_{n\in\mathbb{Z}}. \tag{1.3}$$

Of course, by the discrete Fourier transform, L reduces to the operator of multiplication by the function (symbol)

$$t(z) = \sum_{n\in\mathbb{Z}} t_n z^n, \tag{1.4}$$

and so its spectral analysis is trivial. If the sequences $\{t_n\}_{n\in\mathbb{Z}}$ in (1.1) and (1.3) are the same, one might expect that properties of the operators T and L are also similar. Surprisingly, this very naive conjecture is not totally wrong.

The precise definitions of the operators T and H require some accuracy. Let $\mathcal{D} \subset \ell^2(\mathbb{Z}_+)$ be the dense set of sequences $f = \{f_n\}_{n\in\mathbb{Z}_+}$ with only a finite number of non-zero components. If the sequences $t = \{t_n\}_{n\in\mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$ and $h = \{h_n\}_{n\in\mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+)$, then for $f \in \mathcal{D}$, the vectors Tf and Hf belong to $\ell^2(\mathbb{Z}_+)$ so that the operators T and H are defined on $\mathcal{D}$. Without such a priori assumptions, only Toeplitz

$$t[f,f] = \sum_{n,m\geq 0} t_{n-m} f_m \bar{f}_n \tag{1.5}$$

and Hankel

$$h[f,f] = \sum_{n,m\geq 0} h_{n+m} f_m \bar{f}_n \tag{1.6}$$

quadratic forms are well defined for all $f \in \mathcal{D}$.

Obviously, a Toeplitz operator T or, respectively, a Hankel operator H is bounded if and only if the estimate

$$|t[f,f]| \leq C\|f\|^2, \quad f \in \mathcal{D}, \tag{1.7}$$

or

$$|h[f,f]| \leq C\|f\|^2, \quad f \in \mathcal{D}, \tag{1.8}$$

is satisfied. Here and below $\|f\|$ is the norm of f in the space $\ell^2(\mathbb{Z}_+)$; C are different positive constants; I is the identity operator.

1.2. Let us recall a necessary and sufficient condition for Toeplitz and Hankel operators to be bounded. In terms of quadratic forms the conditions of boundedness of these operators can be stated without any a priori assumptions on their matrix elements. Below $d\mathbf{m}(z) = (2\pi i z)^{-1} dz$ is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. For $p \geq 1$, we set $L^p(\mathbb{T}) = L^p(\mathbb{T}; d\mathbf{m})$.

Theorem 1.1 (Toeplitz). *Estimate* (1.7) *is true if and only if the t_n are the Fourier coefficients of some bounded function $t(z)$ on $\mathbb{T}$:*

$$t_n = \int_{\mathbb{T}} z^{-n} t(z) d\mathbf{m}(z), \quad n \in \mathbb{Z}, \quad t \in L^\infty(\mathbb{T}). \tag{1.9}$$

Thus a Toeplitz operator T is bounded if and only if the corresponding Laurent operator (1.3) is bounded.

The following result is due to Z. Nehari [12].

Theorem 1.2. *Estimate* (1.8) *is true if and only if there exists a bounded function* $h(z)$ *on* $\mathbb{T}$ *such that*

$$h_n = \int_{\mathbb{T}} z^{-n} h(z) d\mathbf{m}(z), \quad n \in \mathbb{Z}_+, \quad h \in L^\infty(\mathbb{T}). \tag{1.10}$$

Despite a formal similarity, Theorems 1.1 and 1.2 are essentially different because the symbol $t(z)$ of a Toeplitz operator T is uniquely defined by relation (1.9) while (1.10) imposes conditions only on the Fourier coefficients h_n of $h(z)$ with $n \in \mathbb{Z}_+$. So among the functions satisfying (1.10) there may be both bounded and unbounded functions. The following example illustrates this phenomenon.

Example 1.3. Let $h_n = (n+1)^{-1}$ (the corresponding Hankel operator H is known as the Hilbert matrix). The "natural" symbol

$$h(z) = \sum_{n \in \mathbb{Z}_+} (n+1)^{-1} z^n$$

is unbounded on $\mathbb{T}$ (at the single point $z = 1$). However the function

$$\widetilde{h}(z) = 1 + \sum_{n \geq 1} (n+1)^{-1} (z^n - z^{-n})$$

is also a symbol of H and $\widetilde{h} \in L^\infty(\mathbb{T})$. Therefore H is a bounded operator.

Actually, properties of Toeplitz and Hankel operators are quite different. For example, a Toeplitz operator T is never compact unless $T = 0$ (see, e.g., Section 3.1 in the book [14]. On the contrary, a Hankel operator H is compact if its symbol can be chosen as a continuous function (the Hartman theorem, see, e.g., Theorem 5.5 in Chapter 1 of [14]). Properties of compact Hankel operators are very thoroughly studied in [14].

1.3. The results about unbounded operators are very scarce. We can mention only the paper [9] by P. Hartman and the relatively recent survey [16] by D. Sarason; see also references in these articles. These articles are devoted to Toeplitz operators. Note that the theory of general unbounded integral operators was initiated by T. Carleman in [4], but this theory does not practically provide concrete conditions guaranteeing, for example, that a given symmetric operator is essentially self-adjoint. Probably, it is impossible to develop a complete theory for general integral operators, but it is very tempting to do this for Toeplitz and Hankel operators possessing special structures.

Our goal here is to describe exhaustive results in the semibounded case, both for Toeplitz and Hankel operators. It looks instructive to compare the results for these two very different classes. Our approach relies on a certain auxiliary algebraic construction combined with some classical analytical results. The algebraic construction is more or less the same for Toeplitz and Hankel quadratic forms, but the analytic results we use are quite different. In particular, the Riesz Brothers theorem plays the crucial role for Toeplitz operators, while the Paley-Wiener theorem is important for Hankel operators.

In Section 2, we find necessary and sufficient conditions for Toeplitz and Hankel quadratic forms to be closable. In Section 3, we describe their closures. Finally, in Section 4, we very briefly (see [21], for details) discuss Wiener-Hopf operators that are a continuous analogue of Toeplitz operators. To treat Wiener-Hopf operators, we need a continuous version of the Riesz Brothers theorem which was not available earlier in the literature.

2. SEMIBOUNDED OPERATORS AND THEIR QUADRATIC FORMS

2.1. In the semibounded case, it is natural to define operators via their quadratic forms. The corresponding construction has abstract nature. It is due to Friedrichs and is described, for example, in the book [2]. Consider an arbitrary Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|$ and a real quadratic form $b[f,f]$ defined on a set $\mathcal{D}$ dense in $\mathcal{H}$. Assume that the form $b[f,f]$ is semibounded, that is,

$$b[f,f] \geq \gamma \|f\|^2, \quad f \in \mathcal{D}, \tag{2.1}$$

for some $\gamma \in \mathbb{R}$. Suppose first that $\gamma > 0$. Then one can introduce a new norm

$$\|f\|_b = \sqrt{b[f,f]}$$

which is stronger than the initial norm $\|f\|$. If $\mathcal{D}$ is a complete Hilbert space with respect to the norm $\|\cdot\|_b$ (in this case the form b is called closed), then there exists a unique self-adjoint operator B in $\mathcal{H}$ with the domain $\mathcal{D}(B) \subset \mathcal{D}$ such that $B \geq \gamma I$ and

$$b[f,g] = (Bf, g), \quad \forall f \in \mathcal{D}(B),\ \forall g \in \mathcal{D}.$$

Moreover, $\mathcal{D}(\sqrt{B}) = \mathcal{D}$ and

$$b[f,f] = \|\sqrt{B} f\|^2, \quad \forall f \in \mathcal{D}.$$

Thus the self-adjoint operator B is correctly defined although its domain $\mathcal{D}(B)$ does not admit an efficient description.

If $\mathcal{D}$ is not b-complete, then of course one can take its completion $\mathcal{D}[b]$ in the norm $\|\cdot\|_b$, extend $b[f,f]$ by continuity onto $\mathcal{D}[b]$ and then try to apply the construction above to the form $b[f,f]$ defined on $\mathcal{D}[b]$. However this procedure meets with an important obstruction because, in general, $\mathcal{D}[b]$ cannot be realized as a subset of $\mathcal{H}$. One can avoid this problem only for the so-called closable forms. By definition, a form $b[f,f]$ is closable if the conditions

$$\|f^{(k)}\| \to 0 \quad \text{and} \quad \|f^{(k)} - f^{(j)}\|_b \to 0$$

as $k, j \to \infty$ imply that $\|f^{(k)}\|_b \to 0$. In this case $\mathcal{D}[b] \subset \mathcal{H}$, the form $b[f,f]$ is closed on $\mathcal{D}[b]$, and so there exists a self-adjoint operator B corresponding to this form.

If γ in (2.1) is not positive, then one applies the definitions above to a form $b_\beta[f,f] = b[f,f] + \beta\|f\|^2$ for some $\beta > -\gamma$ and defines B by the equality $B = B_\beta - \beta I$. So, we can suppose that the number γ in (2.1) is positive; for definiteness, we choose $\gamma = 1$.

To summarize, semibounded self-adjoint operators are correctly defined if and only if the corresponding quadratic forms are closable. Of course not all forms are closable. On the other hand, it is easy to see that if B_0 is a symmetric semibounded operator with domain $\mathcal{D}(B_0)$, then the form $b[f,f] = (B_0 f, f)$ defined on $\mathcal{D}(B_0)$ is necessarily closable.

2.2. Let us come back to Toeplitz T and Hankel H operators formally defined in the space $\mathcal{H} = \ell^2(\mathbb{Z}_+)$ by equalities (1.1) and (1.2). We now suppose that $t_n = \bar{t}_{-n}$ and $h_n = \bar{h}_n$ for all $n \in \mathbb{Z}_+$, so that the operators T and Hankel H are formally symmetric.

First, we state the conditions for Toeplitz and Hankel quadratic forms defined by equalities (1.1) and (1.2) to be semibounded. Recall that the set $\mathcal{D} \subset \ell^2(\mathbb{Z}_+)$ consists of sequences $f = \{f_n\}_{n\in\mathbb{Z}_+}$ with only a finite number of non-zero components. For Toeplitz quadratic forms, we use the following well known result (see, e.g., §5.1 of the book [1]) that is a consequence of the F. Riesz-Herglotz theorem.

Theorem 2.1. *The condition*

$$\sum_{n,m\geq 0} t_{n-m} f_m \bar{f}_n \geq 0, \quad \forall f \in \mathcal{D},$$

is satisfied if and only if there exists a non-negative (finite) measure $dM(z)$ on the unit circle $\mathbb{T}$ such that the coefficients t_n admit the representations

$$t_n = \int_{\mathbb{T}} z^{-n} dM(z), \quad n \in \mathbb{Z}. \tag{2.2}$$

Equations (2.2) for the measure $dM(z)$ are known as the trigonometric moment problem. Of course their solution is unique. Note that the identity I is the Toeplitz operator (with $t_0 = 1$ and $t_n = 0$ for $n \neq 0$) and the corresponding measure $dM(z)$ in (2.2) is the normalized Lebesgue measure $d\mathbf{m}(z)$. Therefore the measure corresponding to the form $t[g,g] + \beta\|g\|^2$ equals $dM(z) + \beta d\mathbf{m}(z)$. So we have a one-to-one correspondence between Toeplitz quadratic forms satisfying estimate (2.1) and real measures satisfying the condition $M(X) \geq \gamma\mathbf{m}(X)$ for all Borelian sets $X \subset \mathbb{T}$.

Hankel quadratic forms are linked to the power moment problem. The following result obtained by Hamburger in [8] plays the role of Theorem 2.1.

Theorem 2.2. *The condition*

$$\sum_{n,m\geq 0} h_{n+m} f_m \bar{f}_n \geq 0, \quad \forall f \in \mathcal{D}, \tag{2.3}$$

is satisfied if and only if there exists a non-negative measure $d\mathsf{M}(x)$ on $\mathbb{R}$ such that the coefficients h_n admit the representations

$$h_n = \int_{-\infty}^{\infty} x^n d\mathsf{M}(x), \quad \forall n = 0, 1, \dots \tag{2.4}$$

2.3. If conditions

$$t[f,f] \geq \gamma \|f\|^2, \quad \forall f \in \mathcal{D}, \tag{2.5}$$

(for some $\gamma \in \mathbb{R}$) or (2.3) are satisfied and

$$\sum_{n\in\mathbb{Z}} |t_n|^2 < \infty \quad \text{or} \quad \sum_{n\in\mathbb{Z}_+} h_n^2 < \infty, \tag{2.6}$$

then the forms $t[f,f]$ or $h[f,f]$ are closable. Indeed, in this case the Toeplitz operator (1.1) or Hankel operator (1.2) are well defined and symmetric on the set $\mathcal{D}$. However, the conditions (2.6) are by no means necessary for the forms $t[f,f]$ or $h[f,f]$ to be closable.

Our main goal is to find *necessary and sufficient* conditions for the forms $t[f,f]$ and $h[f,f]$ to be closable. The answers to these questions are strikingly simple.

We start with Toeplitz forms.

Theorem 2.3 ([20, Theorem 1.3])**.** *Let the form $t[f,f]$ be given by formula* (1.5) *on elements $f \in \mathcal{D}$, and let the condition* (2.5) *be satisfied. Then the form $t[f,f]$ is closable in the space $\ell^2(\mathbb{Z}_+)$ if and only if the measure $dM(z)$ in the equations* (2.2) *is absolutely continuous.*

Example 2.4. If $t_n = 1$ for all $n \in \mathbb{Z}$, then $M(\{1\}) = 1$ and $M(\mathbb{T} \setminus \{1\}) = 0$. This measure is supported by the single point $z = 1$, and the corresponding quadratic form is not closable.

Of course Theorem 2.3 means that $dM(z) = t(z)d\mathbf{m}(z)$ where the function $t \in L^1(\mathbb{T}; d\mathbf{m})$ and $t(z) \geq \gamma$. Thus Theorem 2.3 extends Theorem 1.1 to semibounded operators. The function $t(z)$ is known as the symbol of the Toeplitz operator T. So, Theorem 2.3 shows that for a semibounded Toeplitz operator (even defined via the corresponding quadratic form), the symbol exists and is a semibounded function.

The result for Hankel quadratic forms is stated as follows.

Theorem 2.5 ([19, Theorem 1.2])**.** *Let assumption* (2.3) *be satisfied. Then the following conditions are equivalent:*

(i) *the form $h[f,f]$ defined on $\mathcal{D}$ is closable in the space $\ell^2(\mathbb{Z}_+)$,*
(ii) *the measure $d\mathsf{M}(x)$ defined by equations* (2.4) *satisfies the condition*

$$\mathsf{M}(\mathbb{R} \setminus (-1,1)) = 0 \tag{2.7}$$

(*to put it differently,* $\operatorname{supp} \mathsf{M} \subset [-1,1]$ *and* $\mathsf{M}(\{-1\}) = \mathsf{M}(\{1\}) = 0$),
(iii) *the matrix elements $h_n \to 0$ as $n \to \infty$.*

Remark 2.6. In general, the measure $d\mathsf{M}(x)$ in (2.4) is not unique, but it is unique under the assumptions of Theorem 2.5.

Theorem 2.5 is to a large extent motivated by the following classical result of H. Widom.

Theorem 2.7 ([17, Theorem 3.1]). *Let the matrix elements h_n of the Hankel operator* (1.2) *be given by the equations*

$$h_n = \int_{-1}^{1} x^n d\mathsf{M}(x), \quad \forall n = 0, 1, \dots,$$

with some non-negative measure $d\mathsf{M}(x)$. Then the following conditions are equivalent:

(i) *the operator H is bounded,*
(ii) $\mathsf{M}((1-\varepsilon, 1]) = O(\varepsilon)$ *and* $\mathsf{M}([-1, -1+\varepsilon)) = O(\varepsilon)$ *as* $\varepsilon \to 0$,
(iii) $h_n = O(n^{-1})$ *as* $n \to \infty$.

We emphasize that, in the semibounded case, Theorems 2.3 and 2.5 give optimal conditions for Toeplitz and Hankel operators to be defined as self-adjoint operators. Below we briefly discuss the proofs of these results.

2.4. We start with Theorem 2.3, where we may suppose that $\gamma = 1$ in (2.5). Set

$$(\mathbf{A}f)(z) = \sum_{n=0}^{\infty} f_n z^n. \tag{2.8}$$

Then

$$\|\mathbf{A}f\|_{L^2(\mathbb{T})} = \|f\|_{\ell^2(\mathbb{Z}_+)} \tag{2.9}$$

for all $f \in \ell^2(\mathbb{Z}_+)$. Clearly, $\mathbf{A}$ is a unitary mapping of $\ell^2(\mathbb{Z}_+)$ onto the Hardy space $H^2(\mathbb{T})$ of functions analytic in the unit disc. In view of equations (2.2), we also have

$$\|\mathbf{A}f\|^2_{L^2(\mathbb{T};dM)} = t[f,f], \quad f \in \mathcal{D}. \tag{2.10}$$

The "if" part of Theorem 2.3 is quite easy. Suppose that for a sequence $f^{(k)} \in \mathcal{D}$

$$\|f^{(k)}\|_{\ell^2(\mathbb{Z}_+)} \to 0 \quad \text{and} \quad t[f^{(k)} - f^{(j)}, f^{(k)} - f^{(j)}] \to 0$$

as $k, j \to \infty$. Put $g^{(k)} = \mathbf{A}f^{(k)}$. It follows from (2.9), (2.10) that

$$\|g^{(k)}\|_{L^2(\mathbb{T})} \to 0 \quad \text{and} \quad \|g^{(k)} - g^{(j)}\|^2_{L^2(\mathbb{T};dM)} \to 0 \tag{2.11}$$

as $k, j \to \infty$. Since the space $L^2(\mathbb{T}; dM)$ is complete, there exists a function $g \in L^2(\mathbb{T}; dM)$ such that $g^{(k)} \to g$ in $L^2(\mathbb{T}; dM)$ and hence in $L^2(\mathbb{T})$. The first condition (2.11) implies that $\|g\|_{L^2(\mathbb{T})} = 0$ whence $\|g\|_{L^2(\mathbb{T};dM)} = 0$ because the measure dM is absolutely continuous with respect to the Lebesgue measure. It now follows from (2.10) that

$$t[f^{(k)}, f^{(k)}] = \|g^{(k)}\|_{L^2(\mathbb{T};dM)} \to 0$$

as $k \to \infty$. Thus the form $t[f,f]$ is closable.

The proof of the "only if" part of Theorem 2.3 is less straightforward. Let us define the operator $A\colon \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{T}; dM)$ as the restriction of the operator $\mathbf{A}$ on the set $\mathcal{D} =: \mathcal{D}(A)$. First, we note an assertion which is a direct consequence of the identity (2.10).

Lemma 2.8. *The form $t[f,f]$ defined on $\mathcal{D}$ is closable in the space $\ell^2(\mathbb{Z}_+)$ if and only if the operator A is closable.*

The next step is to construct the adjoint operator $A^*\colon L^2(\mathbb{T}; dM) \to \ell^2(\mathbb{Z}_+)$. Observe that for an arbitrary $u \in L^2(\mathbb{T}; dM)$, the sequences

$$u_n := \int_{\mathbb{T}} u(z) z^{-n} dM(z), \quad n \in \mathbb{Z}_+,$$

are bounded. Let us distinguish a subset $\mathcal{D}_* \subset L^2(\mathbb{T}; dM)$ by the condition $\{u_n\}_{n=0}^\infty \in \ell^2(\mathbb{Z}_+)$ for $u \in \mathcal{D}_*$. For the proof of the following assertion, see Lemma 2.4 in [20].

Lemma 2.9. *The operator A^* is given by the equality*

$$(A^* u)_n = \int_{\mathbb{T}} u(z) z^{-n} dM(z), \quad n \in \mathbb{Z}_+,$$

on the domain $\mathcal{D}(A^*) = \mathcal{D}_*$.

Recall that an operator A is closable if and only if its adjoint operator A^* is densely defined. We use the notation $\operatorname{clos} \mathcal{D}_*$ for the closure of the set $\mathcal{D}_*$ in the space $L^2(\mathbb{T}; dM)$. Lemmas 2.8 and 2.9 yield the following intermediary result.

Lemma 2.10. *The operator A and the form $t[f,f]$ are closable if and only if*

$$\operatorname{clos} \mathcal{D}_* = L^2(\mathbb{T}; dM). \tag{2.12}$$

Recall the Riesz Brothers theorem (see, e.g., Chapter 4 in [10]) that we combine with the Parseval identity.

Theorem 2.11. *For a complex (finite) measure $d\mu(z)$ on the unit circle $\mathbb{T}$, put*

$$\mu_n = \int_{\mathbb{T}} z^{-n} d\mu(z), \quad n \in \mathbb{Z},$$

and suppose that

$$\sum_{n=0}^{\infty} |\mu_n|^2 < \infty.$$

Then the measure $d\mu(z)$ is absolutely continuous.

We also need the following technical assertion (Lemma 2.7 in [20]).

Lemma 2.12. *Suppose that a set $\mathcal{D}_*$ satisfies condition (2.12). Let the measures $u(z)dM(z)$ be absolutely continuous for all $u \in \mathcal{D}_*$. Then the measure $dM(z)$ is also absolutely continuous.*

Now we are in a position to conclude the proof of Theorem 2.3. Suppose that the form $t[f,f]$ is closable. Then by Lemma 2.10 the condition (2.12) is satisfied. By the definition of the set $\mathcal{D}_*$, the Fourier coefficients of the measures $d\mu(z) = u(z)dM(z)$ belong to $\ell^2(\mathbb{Z}_+)$ for all $u \in \mathcal{D}_*$. Therefore it follows from Theorem 2.11 that these measures are absolutely continuous. Hence by Lemma 2.12, the measure $dM(z)$ is also absolutely continuous.

2.5. Next, we sketch the proof of Theorem 2.5. It is almost obvious that the conditions (ii) and (iii) are equivalent. So we discuss only the equivalence of (i) and (ii). Algebraically, we follow the scheme of the previous section, but instead of $L^2(\mathbb{T}; dM)$ we introduce the space $L^2(\mathbb{R}; d\mathsf{M})$ where the measure $d\mathsf{M}$ is defined on $\mathbb{R}$ by equations (2.4). The role of the operator A is now played by the operator $B\colon \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}; d\mathsf{M})$ defined on the set $\mathcal{D}$ by the formula

$$(Bf)(x) = \sum_{n=0}^{\infty} f_n x^n, \quad x \in \mathbb{R}. \tag{2.13}$$

Instead of (2.10), we now have the identity

$$\|Bf\|^2_{L^2(\mathbb{R}; d\mathsf{M})} = h[f,f], \quad f \in \mathcal{D}, \tag{2.14}$$

and the role of Lemma 2.8 is played by the following assertion.

Lemma 2.13. *The form $h[f,f]$ defined on $\mathcal{D}$ is closable in the space $\ell^2(\mathbb{Z}_+)$ if and only if the operator B is closable.*

The adjoint operator $B^*\colon L^2(\mathbb{R}; d\mathsf{M}) \to \ell^2(\mathbb{Z}_+)$ can be constructed similarly to Lemma 2.9.

Lemma 2.14. *Let a subset D_* of $L^2(\mathbb{R}; d\mathsf{M})$ consist of functions $u(x)$ such that the sequence*

$$u_n := \int_{-\infty}^{\infty} u(x) x^n d\mathsf{M}(x)$$

belongs to $\ell^2(\mathbb{Z}_+)$. Then the operator B^* is given by the equality $(B^*u)_n = u_n$ on the domain $\mathcal{D}(B^*) = \mathsf{D}_*$.

For detailed proofs of these assertions see Lemmas 2.1 and 2.2 in [19].

The “if” part of the following result is quite easy, but the converse statement requires the Paley-Wiener theorem.

Theorem 2.15 ([19, Theorem 2.3])**.** *The set D_* is dense in $L^2(\mathbb{R}; d\mathsf{M})$ if and only if condition* (2.7) *is satisfied.*

We only make some comments on the proof of the “only if” part. Actually, only the inclusion

$$\operatorname{supp} \mathsf{M} \subset [-1, 1] \tag{2.15}$$

deserves a special discussion.

For an arbitrary $u \in L^2(\mathbb{R}; d\mathsf{M})$, we put

$$\Psi(z) = \int_{-\infty}^{\infty} e^{izx} u(x) d\mathsf{M}(x). \tag{2.16}$$

Since all functions x^n belong to $L^2(\mathbb{R}; d\mathsf{M})$, we see that $\Psi \in C^\infty(\mathbb{R})$ and

$$\Psi^{(n)}(0) = i^n \int_{-\infty}^{\infty} x^n u(x) d\mathsf{M}(x).$$

If $u \in \mathsf{D}_*$, then this sequence is bounded and hence the function

$$\Psi(z) = \sum_{n=0}^{\infty} \frac{\Psi^{(n)}(0)}{n!} z^n$$

is entire and satisfies the estimate

$$|\Psi(z)| \leq C \sum_{n=0}^{\infty} \frac{1}{n!} |z|^n = Ce^{|z|}, \quad z \in \mathbb{C}.$$

By virtue of the Phragmén-Lindelöf principle, we actually have a stronger estimate

$$|\Psi(z)| \leq Ce^{|\operatorname{Im} z|}, \quad z \in \mathbb{C}. \tag{2.17}$$

According to the Paley-Wiener theorem (see, e.g., Theorem IX.12 in [15]) it follows from estimate (2.17) that the Fourier transform of $\Psi(z)$ (considered as a distribution in the Schwartz class $\mathcal{S}'(\mathbb{R})$) is supported by the interval $[-1, 1]$. Therefore formula (2.16) implies that

$$\int_{-\infty}^{\infty} \varphi(x) u(x) d\mathsf{M}(x) = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R} \setminus [-1, 1]), \tag{2.18}$$

for all $u \in \mathsf{D}_*$. If D_* is dense in $L^2(\mathbb{R}; d\mathsf{M})$, then we can approximate 1 by functions $u \in \mathsf{D}_*$ in this space. Hence equality (2.18) is true with $u(x) = 1$ which implies (2.15).

Finally, we almost repeat the arguments used in the proof of Theorem 2.3. Putting together Lemma 2.14 and Theorem 2.15, we see that the operator B^* is densely defined and hence B is closable if and only if condition (2.7) is satisfied. In view of Lemma 2.13 this proves that the conditions (i) and (ii) of Theorem 2.5 are equivalent. □

2.6. According to Theorem 2.5, the condition $h_n \to 0$ as $n \to \infty$ is necessary and sufficient for a Hankel quadratic form (2.3) to be closable. On the contrary, it is probably impossible to give necessary and sufficient conditions (at least elementary) for a Toeplitz quadratic form (1.5) to be closable in terms of its entries t_n. Indeed, an obvious necessary condition is $t_n \to 0$ as $|n| \to \infty$ because, by Theorem 2.3, the measure $dM(z)$ in the representation (2.2) is absolutely continuous. An obvious sufficient condition is $\{t_n\} \in \ell^2(\mathbb{Z})$ because in this case, by the Parseval identity, the measure $dM(z)$ is absolutely continuous and its derivative $t \in L^2(\mathbb{T})$.

Apparently, this gap between necessary and sufficient conditions cannot be significantly reduced. Indeed, by the Wiener theorem (see, e.g., Theorem XI.114 in [15]), if the Fourier coefficients t_n of some measure $dM(z)$ tend to zero, then this measure is necessarily continuous, but it may be singular with respect to the Lebesgue measure. Thus the condition $t_n \to 0$ as $|n| \to \infty$ does not imply that the measure $dM(z)$ defined by equations (2.2) is absolutely continuous. So in accordance with Theorem 2.3 the corresponding Toeplitz quadratic form $t[f, f]$ need not be closable.

Astonishingly, the condition $\{t_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ guaranteeing the absolute continuity of the measure $dM(z)$ turns out to be very sharp. Indeed, for every $p \in \mathbb{Z}_+$, O.S. Ivašëv-Musatov constructed in [11] a singular measure such that its Fourier coefficients satisfy the estimate

$$t_n = O\big((n(\ln n)(\ln\ln n)\cdots(\ln_{(p)} n))^{-1/2}\big)$$

(here $\ln_{(p)} n$ means that the logarithm is applied p times to n). This sequence "almost belongs" to $\ell^2(\mathbb{Z})$, but, by Theorem 2.3, the corresponding form $t[f, f]$ is not closable.

3. CLOSED QUADRATIC FORMS

Here we will show that closable Toeplitz $t[f, f]$ and Hankel $h[f, f]$ quadratic forms constructed in Theorems 2.3 and 2.5, respectively, are closed on their maximal domains of definition. This yields a description of the domains of the operators $\sqrt{T}$ and $\sqrt{H}$. Again, the algebraic scheme of a study of Toeplitz and Hankel forms is the same, but analytical backgrounds are quite different. As before, it is convenient to use the operators $\mathbf{A}$ and B defined by formulas (2.8) and (2.13).

3.1. Let us start with closable Toeplitz forms when, by Theorem 2.3, the measures $dM(z)$ in equations (2.2) are absolutely continuous, that is, $dM(z) = t(z)d\mathbf{m}(z)$ where $t \in L^1(\mathbb{T})$ and we may suppose $t(z) \geq 1$.

Under the assumptions of Theorem 2.3 the operator A^* adjoint to A (recall that A is the restriction of $\mathbf{A}$ on $\mathcal{D}$) is densely defined so that the second adjoint exists and $A^{**} = \operatorname{clos} A$ (the closure of A). Let us also introduce by formula (2.8) the "maximal" operator $A_{\max}$ on the domain $\mathcal{D}(A_{\max})$ that consists of all $f \in \ell^2(\mathbb{Z}_+)$ such that $\mathbf{A}f \in L^2(\mathbb{T}; dM)$. We will show that

$$\operatorname{clos} A = A_{\max}. \tag{3.1}$$

The first assertion is a direct consequence of the definition of the closure of the operator A.

Lemma 3.1. *The inclusion* $\operatorname{clos} A \subset A_{\max}$ *holds.*

Indeed, if $f \in \mathcal{D}(\operatorname{clos} A)$, then there exists a sequence $f^{(k)} \in \mathcal{D}(A)$ such that $f^{(k)} \to f$ in $\ell^2(\mathbb{Z}_+)$ or, in view of (2.9), $\mathbf{A}f^{(k)} \to \mathbf{A}f$ in $L^2(\mathbb{T})$ as $k \to \infty$. Moreover,

$\mathbf{A}f^{(k)}$ converges to some g in $L^2(\mathbb{T}; dM)$ (of course $g = (\operatorname{clos} A)f$). It follows that $\mathbf{A}f = g \in L^2(\mathbb{T}; dM)$, that is, $f \in \mathcal{D}(A_{\max})$.

The opposite inclusion is less trivial.

Lemma 3.2. *The inclusion* $A_{\max} \subset \operatorname{clos} A$ *holds.*

Pick $f \in \mathcal{D}(A_{\max})$. Then $f \in \ell^2(\mathbb{Z}_+)$ and $u := \mathbf{A}f \in H^2(\mathbb{T}) \cap L^2(\mathbb{T}; dM)$. Since $t \in L^1(\mathbb{T})$ and $t(\zeta) \geq 1$,

$$t_{\rm out}(z) := \exp\Big(\frac{1}{2}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \ln t(\zeta) d\mathbf{m}(\zeta)\Big), \quad |z| < 1,$$

is an outer function, $t_{\rm out} \in H^2(\mathbb{T})$ and the angular limits of $t_{\rm out}(z)$ as $z \to \zeta \in \mathbb{T}$ equal $\sqrt{t(\zeta)}$. Note also that $v := u t_{\rm out} \in H^2(\mathbb{T})$ because $u\sqrt{t} \in L^2(\mathbb{T})$. By the V. Smirnov theorem (see, e.g., [13], Section 1.7), every function in $H^2(\mathbb{T})$ can be approximated by linear combinations of functions $z^n t_{\rm out}(z)$, and hence there exists a sequence of polynomials $\mathcal{P}^{(k)}(z)$ such that

$$\|v - \mathcal{P}^{(k)} t_{\rm out}\|^2_{H^2(\mathbb{T})} = \int_{\mathbb{T}} |u(z) - \mathcal{P}^{(k)}(z)|^2 t(z) d\mathbf{m}(z) \to 0$$

as $k \to \infty$. This means that

$$\lim_{k\to\infty} \|\mathbf{A}f - \mathbf{A}f^{(k)}\|_{L^2(\mathbb{T};dM)} = 0 \tag{3.2}$$

for $f^{(k)} \in \mathcal{D}$ such that $\mathbf{A}f^{(k)} = \mathcal{P}^{(k)}$. Since the convergence in $L^2(\mathbb{T}; dM)$ is stronger than in $L^2(\mathbb{T})$, we see that $\mathbf{A}f^{(k)} \to \mathbf{A}f$ in $L^2(\mathbb{T})$ as $k \to \infty$. According to (2.9) this implies that $f^{(k)} \to f$ in $\ell^2(\mathbb{Z}_+)$ as $k \to \infty$. Combining this relation with (3.2), we see that $f \in \mathcal{D}(\operatorname{clos} A)$ and $\mathbf{A}f = (\operatorname{clos} A)f$.

In view of the identities (2.9) and (2.10), equality (3.1) can be reformulated in terms of Toeplitz quadratic forms $t[f, f]$.

Theorem 3.3 ([20, Theorem 2.10])**.** *Under the assumptions of Theorem* 2.3 *the closure of the form* $t[f, f]$ *is given by the equality*

$$t[f, f] = \int_{\mathbb{T}} |(\mathbf{A}f)(z)|^2 dM(z) \tag{3.3}$$

on the set $\mathcal{D}[t] = \mathcal{D}(A_{\max})$ *of all* $f \in \ell^2(\mathbb{Z}_+)$ *such that the right-hand side of* (3.3) *is finite.*

3.2. For Hankel quadratic forms, we proceed from Theorem 2.5. We suppose that condition (2.7) is satisfied, and hence the form $h[f, f]$ is closable. Let us now define the operator $\mathbf{B}$ by formula (2.13) on all $f \in \ell^2(\mathbb{Z}_+)$. The series in the right-hand side of (2.13) converges for each $x \in (-1, 1)$, but only the estimate

$$|(\mathbf{B}f)(x)| \leq \sum_{n=0}^{\infty} |f_n||x|^n \leq (1 - x^2)^{-1/2} \, \|f\|_{\ell^2(\mathbb{Z}_+)}$$

holds. So it is of course possible that $\mathbf{B}f \notin L^2((-1,1);d\mathsf{M})$. Therefore, we also introduce the "maximal" operator $B_{\max}$ as the restriction of $\mathbf{B}$ on the domain $\mathcal{D}(B_{\max})$ that consists of all $f \in \ell^2(\mathbb{Z}_+)$ such that $\mathbf{B}f \in L^2((-1,1);d\mathsf{M})$. The following result plays the role of equality (3.1).

Lemma 3.4 ([19, Lemma 3.3])**.** *Let one of equivalent conditions of Theorem* 2.5 *be satisfied. Then equality*

$$\operatorname{clos} B = B_{\max}. \tag{3.4}$$

is true.

Similarly to Lemma 3.1, the inclusion $\operatorname{clos} B \subset B_{\max}$ is a direct consequence of the definition of the closure of the operator B. Surprisingly, the proof of the opposite inclusion $B_{\max} \subset \operatorname{clos} B$ turns out to be rather tricky although it does not require any deep analytical results. Since $\operatorname{clos} B = B^{**}$, it suffices to check that

$$(B_{\max} f, u)_{L^2((-1,1);d\mathsf{M})} = (f, B^* u)_{\ell^2(\mathbb{Z}_+)}$$

for all $f \in \mathcal{D}(B_{\max})$ and all $u \in \mathcal{D}(B^*) = \mathsf{D}_*$. In the detailed notation, this relation means that

$$\int_{-1}^{1} \Big(\sum_{n=0}^{\infty} f_n x^n \Big) \overline{u(x)} d\mathsf{M}(x) = \sum_{n=0}^{\infty} f_n \Big(\int_{-1}^{1} x^n \overline{u(x)} d\mathsf{M}(x) \Big). \tag{3.5}$$

The problem is that these integrals do not converge absolutely. So the Fubini theorem cannot be applied, and we have not found a direct proof of relation (3.5).

By some, rather mysterious reasons, it appears to be more convenient to treat this problem in the realization of Hankel operators as integral operators in the space $L^2(\mathbb{R}_+)$. To put it differently, instead of the operator $\mathbf{B}$ defined by formula (2.13) on all $f \in \ell^2(\mathbb{Z}_+)$, we now consider the operator (the Laplace transform) defined by the formula

$$(\mathbf{G}f)(\lambda) = \int_0^{\infty} e^{-t\lambda} f(t) dt \tag{3.6}$$

on all $f \in L^2(\mathbb{R}_+)$. The role of $L^2((-1,1);d\mathsf{M})$ is played by the space $L^2(\mathbb{R}_+;d\Sigma)$ where the non-negative measure $d\Sigma(\lambda)$ on $\mathbb{R}_+$ satisfies the condition

$$\int_0^{\infty} (\lambda+1)^{-k} d\Sigma(\lambda) < \infty \tag{3.7}$$

for $k=2$. The integral (3.6) converges for all $f \in L^2(\mathbb{R}_+)$ and $\lambda > 0$, but the estimate

$$|(\mathbf{G}f)(\lambda)| \leq (2\lambda)^{-1/2} \, \|f\|_{L^2(\mathbb{R}_+)}$$

does not of course guarantee that $\mathbf{G}f \in L^2(\mathbb{R}_+;d\Sigma)$.

Properties of the operators $\mathbf{B}$ and $\mathbf{G}$ are basically the same. We first define the restriction G of the operator $\mathbf{G}$ on domain $\mathcal{D}(G)$ that consists of functions compactly supported in $\mathbb{R}_+$. Evidently, $Gf \in L^2(\mathbb{R}_+; d\Sigma)$ if $f \in \mathcal{D}(G)$. It is easy to show (see [18], for details) that the operator G^* is given by the formula

$$(G^* v)(t) = \int_0^\infty e^{-t\lambda} v(\lambda) d\Sigma(\lambda),$$

and $v \in \mathcal{D}(G^*)$ if and only if $v \in L^2(\mathbb{R}_+; d\Sigma)$ and $G^* v \in L^2(\mathbb{R}_+)$. Obviously, this condition is satisfied if v is compactly supported in $\mathbb{R}_+$. Since the set of such v is dense in $L^2(\mathbb{R}_+; d\Sigma)$, the operator G^* is densely defined. Thus G is closable and $\operatorname{clos} G = G^{**}$. Let us now define the operator $G_{\max}$ as the restriction of the operator $\mathbf{G}$ on the domain $\mathcal{D}(G_{\max})$ that consists of all $f \in L^2(\mathbb{R}_+)$ such that $\mathbf{G} f \in L^2(\mathbb{R}_+; d\Sigma)$.

The following assertion plays the central role.

Lemma 3.5 ([18, Theorem 3.9])**.** *Let $d\Sigma(\lambda)$ be a measure on $\mathbb{R}_+$ such that the condition* (3.7) *is satisfied for some $k > 0$. Then*

$$\operatorname{clos} G = G_{\max}. \tag{3.8}$$

We will not comment on a rather complicated proof of this result, but explain the equivalence of relations (3.4) and (3.8) (for the particular case $k = 2$). Suppose that the measures $d\Sigma(\lambda)$ and $d\mathsf{M}(x)$ are linked by the equality

$$d\mathsf{M}(x) = (\lambda + 1/2)^{-2} d\Sigma(\lambda), \quad x = \frac{2\lambda - 1}{2\lambda + 1}.$$

Thus, $\mathsf{M}((-1,1)) < \infty$ if and only if the condition (3.7) holds for $k = 2$. Let us also set

$$(Vu)(\lambda) = \frac{1}{\lambda + 1/2} u\Big(\frac{2\lambda - 1}{2\lambda + 1}\Big).$$

Obviously, $V : L^2((-1,1); d\mathsf{M}) \to L^2(\mathbb{R}_+; d\Sigma)$ is a unitary operator.

We need the identity (see formula (10.12.32) in [5])

$$\int_0^\infty \mathsf{L}_n(t) e^{-(1/2+\lambda)t} dt = \frac{1}{\lambda + 1/2} \Big(\frac{2\lambda - 1}{2\lambda + 1}\Big)^n, \quad \lambda > -1/2, \tag{3.9}$$

for the Laguerre polynomials $\mathsf{L}_n(t)$ (see, for example, the book [5], Chapter 10.12, for their definition). It can be deduced from this fact that the functions $\mathsf{L}_n(t) e^{-t/2}$, $n = 0, 1, \ldots$, form an orthonormal basis in the space $L^2(\mathbb{R}_+)$, and hence the operator $U \colon l^2(\mathbb{Z}_+) \to L^2(\mathbb{R}_+)$ defined by the formula

$$(Uf)(t) = \sum_{n=0}^\infty f_n \mathsf{L}_n(t) e^{-t/2}, \quad f = (f_0, f_1, \ldots), \tag{3.10}$$

is unitary.

A link of the operators $\mathbf{B}$ and $\mathbf{G}$ and hence of B and G is stated in the following assertion which can be easily derived from (3.6), (3.9) and (3.10); see Lemma 3.2 in [19], for details.

Lemma 3.6. *For all $f \in \mathcal{D}$, the identity holds*

$$V\mathbf{B}f = \mathbf{G}Uf.$$

Combining Lemmas 3.5 and 3.6, we arrive at the following result.

Lemma 3.7. *Let $d\mathsf{M}(x)$ be a finite measure on $(-1,1)$. Then equality (3.4) holds.*

In view of identity (2.14), equality (3.4) leads to the following result which plays the role of Theorem 3.3.

Theorem 3.8 ([19, Theorem 3.4])**.** *Let the form $h[f,f]$ be defined on the set $\mathcal{D}$ by formula (1.6), and let assumption (2.3) be true. Suppose that one of three equivalent conditions (i), (ii) or (iii) of Theorem 2.5 is satisfied. Then the closure of $h[f,f]$ is given by the equality*

$$h[f,f] = \int_{-1}^{1} \Big| \sum_{n=0}^{\infty} f_n x^n \Big|^2 d\mathsf{M}(x) \tag{3.11}$$

on the set of all $f \in \ell^2(\mathbb{Z}_+)$ such that the right-hand side of (3.11) is finite.

Note that that domains $\mathcal{D}(T)$ and $\mathcal{D}(H)$ of Toeplitz T and Hankel H do not admit an explicit description, but Theorems 3.3 and 3.8 characterize the domains $\mathcal{D}(\sqrt{T})$ and $\mathcal{D}(\sqrt{H})$ of their square roots.

4. WIENER-HOPF SEMIBOUNDED OPERATORS

4.1. Wiener-Hopf operators W are formally defined in the space $L^2(\mathbb{R}_+)$ of functions $f(x)$ by the formula

$$(Wf)(x) = \int_{\mathbb{R}_+} w(x-y)f(y)dy.$$

These operators are continuous analogues of the Toeplitz operators defined by (1.1). However optimal results on Wiener-Hopf operators are not direct consequences of the corresponding results for Toeplitz operators and, in some sense, they are more general. One of the differences is that Wiener-Hopf operators require a consistent work with distributions.

To be precise, we define the operator W via its quadratic form

$$w[f,f] = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} w(x-y)f(y)\overline{f(x)}dxdy. \tag{4.1}$$

With respect to w, we a priori only assume that it is a distribution in the class $C_0^\infty(\mathbb{R})'$ dual to $C_0^\infty(\mathbb{R})$. Then the quadratic form is correctly defined for all $f \in C_0^\infty(\mathbb{R}_+)$.

We always suppose that $w(x) = \overline{w(-x)}$ so that the operator W is formally symmetric and the quadratic form (4.1) is real. We also assume that it is semibounded from below, that is,

$$w[f,f] \geq \gamma \|f\|^2, \quad f \in C_0^\infty(\mathbb{R}_+), \quad \|f\| = \|f\|_{L^2(\mathbb{R}_+)}, \tag{4.2}$$

for some $\gamma \in \mathbb{R}$.

Here we follow basically the scheme we used before for semibounded Toeplitz operators. However the analytical basis is rather different. The role of Theorem 2.1 is now played by the Bochner-Schwartz theorem (see, e.g., Theorem 3 in §3 of Chapter II of the book [6]).

Theorem 4.1. *Let the form $w[f,f]$ be defined by the relation* (1.5) *where the distribution $w \in C_0^\infty(\mathbb{R})'$. Then the condition*

$$w[f,f] \geq 0, \quad \forall f \in C_0^\infty(\mathbb{R}_+),$$

is satisfied if and only if there exists a non-negative measure $d\mathrm{M}(\lambda)$ on the line $\mathbb{R}$ such that

$$w(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\lambda} d\mathrm{M}(\lambda). \tag{4.3}$$

Here the measure obeys the condition

$$\int_{\mathbb{R}} (1+\lambda^2)^{-p} d\mathrm{M}(\lambda) < \infty \tag{4.4}$$

for some p (*that is, it has at most a polynomial growth at infinity*).

Observe that the Lebesgue measure $d\mathrm{M}(\lambda) = d\lambda$ satisfies the condition (4.4) with $p > 1/2$. For the Lebesgue measure, relation (4.3) yields $w(x) = \delta(x)$ (the delta-function) so that $W = I$ and $w[f,f] = \|f\|^2$. Therefore the measure corresponding to the form $w[f,f] + \beta\|f\|^2$ equals $d\mathrm{M}(\lambda) + \beta d\lambda$, and relation (4.3) extends to all semibounded Wiener-Hopf quadratic forms. Thus we have the one-to-one correspondence between Wiener-Hopf quadratic forms satisfying estimate (4.2) and real measures satisfying the condition $\mathrm{M}(X) \geq \gamma |X|$ ($|X|$ is the Lebesgue measure of X) for all Borelian sets $X \subset \mathbb{R}$.

4.2. Our goal is to find necessary and sufficient conditions for the form $w[f,f]$ to be closable. The following result plays the role of Theorem 2.3.

Theorem 4.2 ([21, Theorem 1.3])**.** *Let the form $w[f,f]$ be given by formula* (4.1) *on elements $f \in C_0^\infty(\mathbb{R}_+)$, and let the condition* (4.2) *be satisfied for some $\gamma \in \mathbb{R}$. Then the form $w[f,f]$ is closable in the space $L^2(\mathbb{R}_+)$ if and only if the measure $d\mathrm{M}(\lambda)$ in the equation* (4.3) *is absolutely continuous.*

We always understand the absolute continuity with respect to the Lebesgue measure. Therefore Theorem 4.2 means that $d\mathrm{M}(\lambda) = \varphi(\lambda) d\lambda$ where $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R})$,

$$\int_{\mathbb{R}} (1+\lambda^2)^{-p} |\varphi(\lambda)| d\lambda < \infty$$

and $\varphi(\lambda) \geq \gamma$. The function $\varphi(\lambda)$ is known as the symbol of the Wiener-Hopf operator W. Thus Theorem 4.2 shows that in the semibounded case, the symbol of a Wiener-Hopf operator can be correctly defined if and only if the corresponding quadratic form is closable.

Our proof of Theorem 4.2 requires a continuous analogue of the classical Riesz Brothers theorem. Let us state this result here. For a measure $d\mathrm{M}(\lambda)$ on $\mathbb{R}$, we denote by $d|\mathrm{M}|(\lambda)$ its variation.

Theorem 4.3 ([21, Theorem 1.4])**.** *Let $d\mathrm{M}(\lambda)$ be a complex measure on the line $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} (1+\lambda^2)^{-p} d|\mathrm{M}|(\lambda) < \infty$$

for some p. Put

$$\sigma(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\lambda} d\mathrm{M}(\lambda)$$

and suppose that $\sigma \in L^2(a,\infty)$ for some $a \in \mathbb{R}$. Then the measure $d\mathrm{M}(\lambda)$ is absolutely continuous.

We allow $a \in \mathbb{R}$ in Theorem 4.3 to be arbitrary since, for example, the function $\sigma(x) = \delta(x - x_0)$ for any $x_0 \in \mathbb{R}$ does not belong to $L^2_{\mathrm{loc}}(\mathbb{R})$, but the corresponding measure $d\mathrm{M}(\lambda) = e^{ix_0\lambda} d\lambda$ is of course absolutely continuous.

Acknowledgements

Supported by project Russian Science Foundation 17-11-01126.
The author thanks N.K. Nikolski for a useful discussion.

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Dmitri R. Yafaev
yafaev@univ-rennes1.fr

Université de Rennes I
IRMAR
Campus de Beaulieu, 35042 Rennes Cedex, France

SPGU
Univ. Nab. 7/9, Saint Petersburg
199034 Russia

Received: November 7, 2017.
Accepted: December 10, 2017.