Opuscula Math. 38, no. 4 (2018), 557–571
https://doi.org/10.7494/OpMath.2018.38.4.557 Opuscula Mathematica

LINEAR STURM–LIOUVILLE PROBLEMS WITH RIEMANN–STIELTJES INTEGRAL BOUNDARY CONDITIONS

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Communicated by P.A. Cojuhari

Abstract. We study second-order linear Sturm–Liouville problems involving general homogeneous linear Riemann–Stieltjes integral boundary conditions. Conditions are obtained for the existence of a sequence of positive eigenvalues with consecutive zero counts of the eigenfunctions. Additionally, we find interlacing relationships between the eigenvalues of such Sturm–Liouville problems and those of Sturm–Liouville problems with certain two-point separated boundary conditions.

Keywords: nodal solutions, integral boundary value problems, Sturm–Liouville problems, eigenvalues, matching method.

Mathematics Subject Classification: 34B10, 34B15.

1. INTRODUCTION

We study the linear Sturm–Liouville Problem (SLP) consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda w(t)y, \quad t \in (a,b), \tag{1.1}$$

and one of the general homogeneous linear Riemann–Stieltjes integral boundary conditions (BCs)

$$\begin{cases} (py')(c) = 0, \\ \delta_{21}y(b) + \delta_{22}(py')(b) - \int\limits_a^b [y(s)\,d\xi_1(s) + (py')(s)\,d\xi_2(s)] = 0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \delta_{11}y(a) + \delta_{12}(py')(a) - \int\limits_a^b [y(s)\,d\eta_1(s) + y'(s)\,d\eta_2(s)] = 0, \\ \delta_{21}y(b) + \delta_{22}(py')(b) - \int\limits_a^b [y(s)\,d\xi_1(s) + (py')(s)\,d\xi_2(s)] = 0, \end{cases} \tag{1.3}$$

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where $c \in [a,b)$ and the integrals in (1.2) and (1.3) are Riemann–Stieltjes integrals with respect to $\eta_i(s)$, and $\xi_i(s)$, respectively, with $\xi_i(s)$ and $\eta_i(s)$ being functions of bounded variation, for $i = 1, 2$.

In the case that $\xi_1(s) = \xi_2(s) = s$, the Riemann–Stieltjes integrals in BC (1.2) reduce to the Riemann integrals. In the case that $\xi_1(s) = \sum_{i=1}^{g_1} k_{1i}\chi(s - \xi_i^*)$ and $\xi_2(s) = \sum_{i=1}^{g_1} k_{2i}\chi(s - \xi_i^*)$, where $\{\xi_i^*\}_{i=1}^{g_1}$ is a strictly increasing sequence in (a,b), and $\chi(s)$ is the characteristic function on $[0,\infty)$, i.e.,

$$\chi(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0, \end{cases}$$

then BC (1.2) reduces to

$$\begin{cases} (py')(c) = 0, \\ \delta_{21}y(b) + \delta_{22}y'(b) - \sum_{i=1}^{g_1} [k_{1i}y(\xi_i^*) + k_{2i}y'(\xi_i^*)] = 0. \end{cases}$$

A similar comment can be made to BC (1.3).

We assume throughout, and without further mention, that the following conditions hold:

(H1) $p, q, w \in C^1[a,b]$ such that $p(t) > 0$, $w(t) > 0$, and $q'(t) + q^* \leq l(q^* - q(t))$ on $[a,b]$ with

$$q^* := \max_{t\in[a,b]} \{q(t), 0\} \quad \text{and} \quad l^{\pm} := \max_{t\in[a,b]} \left\{ \left(\frac{p'(t) + q^*}{p(t)} \right)_{\mp}, \frac{w'_{\pm}(t)}{w(t)} \right\},$$

where $h_{\pm}(t) := \max\{0, \pm h(t)\}$ for $h : \mathbb{R} \to \mathbb{R}$;

(H2) $\delta_{ij} \in \mathbb{R}$ for $i, j = 1, 2$;

Note that examples of the function classes for q that satisfy (H1) are discussed in Chamberlain and Kong [2, Remark 1].

SLPs have been used to study nonlinear boundary value problems (BVPs) in recent years. For example, researchers have obtained results on the existence of positive solutions and nodal solutions (those with a zero counting property in (a,b)) of the BVP consisting of the equation

$$-y'' + q(t)y = w(t)f(y), \quad t \in (a,b),$$

and the separated BC

$$\begin{cases} \cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, & \alpha \in [0,\pi), \\ \cos\beta\ y(b) - \sin\beta\ (py')(b) = 0, & \beta \in (0,\pi], \end{cases}$$

by comparing $f_0 := \lim_{y\to 0} f(y)/y$ and $f_\infty := \lim_{|y|\to\infty} f(y)/y$ with the eigenvalues of a particular SLP, see Erbe [4] for positive solutions, Kong [9], Kong, and Kong [6], and Naito and Tanaka [17] for nodal solutions.

Nonlinear BVPs with nonlocal BCs, including multi-point BCs, have also received a lot of attention in research, and various conditions are obtained for the existence of positive solutions and nodal solutions. We refer the reader to [1, 3, 5, 7, 10, 15, 16, 18, 19, 21–23] and the references therein for some recent work on this topic.

The study of linear SLPs involving multi-point BCs has become active recently. In particular, the spectra of such problems has been a focus in research due to the fact that the BCs are no longer self-adjoint. Earlier work in this area were given by Ma, Rynne, and Xu in [15, 16, 18, 19, 21–23] for the problem consisting of the equation $-y'' = \lambda y$ and the BC

$$y(0) = 0, \quad y(1) - \sum_{i=1}^{g_1} \alpha_i y(\xi_i) = 0,$$

where $\xi_i \in (0, 1)$. A sequence of real eigenvalues is calculated and is applied to show the existence of nodal solutions for corresponding nonlinear BVPs.

Genoud and Rynne [5] is the first paper dealing with the multi-point SLPs with a variable coefficient function, where an implicit condition is imposed to guarantee the existence of a sequence of real eigenvalues. By a different approach, Kong, Kong, Kwong, and Wong [8] studied the SLPs consisting of Eq. (1.1) with $p(t) \equiv 1$ and $q(t) \equiv 0$, and one of the following BCs

$$\begin{cases} \cos\alpha y(a) - \sin\alpha\ y'(a) = 0, \quad \alpha \in [0, \pi), \\ y(b) - \sum_{i=1}^{g_1} k_i y(\eta_i) = 0 \end{cases}$$

and

$$\begin{cases} y(a) - \sum_{j=1}^{g_2} h_j y(\xi_j) = 0, \\ y(b) - \sum_{i=1}^{g_1} k_i y(\eta_i) = 0, \end{cases}$$

where $\eta_i, \xi_j \in (a, b)$. They obtained explicit conditions for the existence of a sequence of positive eigenvalues and derived zero counts of the corresponding eigenfunctions. They also revealed interlacing relations between the eigenvalues of the above multi-point SLPs and certain two-point SLPs.

The results in [8] have recently been successfully extended by Kong and St. George [11] to the SLPs consisting of the same equation and one of the BCs

$$\begin{cases} \cos\alpha\ y(a) - \sin\alpha\ y'(a) = 0, \quad \alpha \in [0, \pi), \\ \delta_{21} y(b) + \delta_{22} y'(b) - \sum_{i=1}^{g_1} [k_{1i} y(\eta_i) + k_{2i} y'(\eta_i)] = 0, \end{cases}$$

and

$$\begin{cases} \delta_{11} y(a) + \delta_{12} y'(a) - \sum_{j=1}^{g_2} [h_{1j} y(\xi_j) + h_{2j} y'(\xi_j)] = 0, \\ \delta_{21} y(b) + \delta_{22} y'(b) - \sum_{i=1}^{g_1} [k_{1i} y(\eta_i) + k_{2i} y'(\eta_i)] = 0. \end{cases}$$

Motivated by the work in [8] and [11], in this paper, we will study SLPs consisting of Eq. (1.1) with variable coefficient functions $p(t)$, $q(t)$, and $w(t)$ and one of the general homogeneous linear Riemann–Stieltjes integral BCs (1.2) and (1.3). We will establish the existence of a sequence of positive eigenvalues and derive the zero counts of the corresponding eigenfunctions. We will further reveal the interlacing relations between such eigenvalues and the eigenvalues for certain two-point BVPs. For the special case with $p(t) \equiv 1$ and $q(t) \equiv 0$, we will establish the existence of one or more additional eigenvalues whose eigenfunctions have less zero counts.

This paper is structured as follows: the main results are stated in Section 2 and the proofs of the results are given in Section 3.

2. MAIN RESULTS

To study the existence of eigenvalues and zero counting properties of associated eigenfunctions of SLPs for Eq. (1.1), we define the following classes of solutions of Eq. (1.1).

Definition 2.1. Let $n \in \mathbb{N}_0$ and $a \leq a_1 < b_1 \leq b$. A solution y of Eq. (1.1) is said to belong to class $\mathcal{S}_n[a_1, b_1]$ if y has exactly n zeros in (a_1, b_1).

For $c \in [a, b]$ and $d \in [b, d]$, let $\{\mu_m^{[1]}(c)\}_{m=0}^{\infty}$ and $\{\mu_n^{[2]}(d)\}_{n=0}^{\infty}$ be the eigenvalues of the SLPs consisting of Eq. (1.1) and the two point BCs

$$(py')(c) = 0, \quad y(b) = 0 \tag{2.1}$$

and

$$y(a) = 0, \quad (py')(d) = 0, \tag{2.2}$$

respectively. It is well known that

$$-\infty < \mu_0^{[1]}(c) < \mu_1^{[1]}(c) < \cdots < \mu_m^{[1]}(c) < \cdots, \text{ and } \mu_m^{[1]}(c) \to \infty,$$

and

$$-\infty < \mu_0^{[2]}(d) < \mu_1^{[2]}(d) < \cdots < \mu_n^{[2]}(d) < \cdots, \text{ and } \mu_n^{[2]}(d) \to \infty;$$

and any eigenfunction associated with $\mu_i^{[1]}(c)$ or $\mu_i^{[2]}(d)$ has i simple zeros in (a, b) for $i \in \mathbb{N}_0$, see [24, Theorem 4.3.2]. Let $m_0, n_0 \in \mathbb{N}_0$ such that $\mu_{m_0}^{[1]}(c)$ and $\mu_{n_0}^{[2]}(d)$ be the first positive eigenvalues of SLPs (1.1), (2.1) and (1.1), (2.2).

Noting that $\xi(s)$ and $\eta_i(s)$ are of bounded variation, we see that there exist nondecreasing functions $\xi_{ij}(s)$ and $\eta_{ij}(s)$, $i, j = 1, 2$ such that

$$\xi_i(s) = \xi_{i1}(s) - \xi_{i2}(s) \text{ and } \eta_i(s) = \eta_{i1}(s) - \eta_{i2}(s), \quad s \in [a, b]. \tag{2.3}$$

To simplify notation, we denote

$$\xi_i^+(s) := \xi_{i1}(s) + \xi_{i2}(s), \text{ and } \eta_i^+(s) := \eta_{i1}(s) + \eta_{i2}(s), \quad s \in [a, b],$$

where $\xi_{ij}(s), \eta_{ij}(s)$ for $i, j = 1, 2$ are given by (2.3).

The following conditions are stated here in order to shorten the statement of our results: for $c \in [a, b)$ and $d \in (a, b]$,

$$\int_a^b \frac{e^{(b-a)l^-/2}\, d\xi_1^+(s)}{\sqrt{p(b)(q^* - q(s) + \mu_m^{[1]}(c)w(s))}} + e^{(b-a)l^-/2} \int_a^b \sqrt{\frac{p(s)}{p(b)}}\, d\xi_2^+(s) < \delta_{22} \tag{2.4}$$

and

$$\begin{aligned} &\int_a^b \frac{e^{(b-a)l^+/2}\, d\eta_1^+(a+b-s)}{\sqrt{p(a)(q^* - q(a+b-s) + \mu_n^{[2]}(d)w(a+b-s))}} \\ &+ e^{(b-a)l^+/2} \int_a^b \sqrt{\frac{p(s)}{p(a)}}\, d\eta_2^+(a+b-s) < \delta_{12}. \end{aligned} \tag{2.5}$$

The first result is for the existence of eigenvalues and the zero counts of eigenfunctions of SLP (1.1), (1.2).

Theorem 2.2. *Assume* (2.4) *holds for some integer* $m \geq m_0$. *Then SLP* (1.1), (1.2) *has an infinite number of positive eigenvalues* $\{\lambda_i(c)\}_{i=m+1}^{\infty}$ *which satisfies the interlacing relation with* $\{\mu_i^{[1]}(c)\}_{i=m}^{\infty}$

$$\mu_m^{[1]}(c) < \lambda_{m+1}(c) < \mu_{m+1}^{[1]}(c) < \lambda_{m+2}(c) < \cdots < \mu_i^{[1]}(c) < \lambda_{i+1}(c) < \cdots. \tag{2.6}$$

Moreover, the eigenfunction y_i *associated with* $\lambda_i(c)$ *belongs to* $\mathcal{S}_i[c, b]$ *for* $i \geq m+1$.

Since $\lim_{m\to\infty} \mu_m^{[1]}(c) = \infty$, the following result follows immediately from Theorem 2.2.

Corollary 2.3. *Assume* $e^{(b-a)l^-/2} \int_a^b \sqrt{\frac{p(s)}{p(b)}}\, d\xi_2^+(s) < \delta_{22}$. *Then SLP* (1.1), (1.2) *has an infinite number of positive eigenvalues tending towards infinity.*

The next result is for the existence of eigenvalues and the zero counts of eigenfunctions of SLP (1.1), (1.3). Here, for $r \in \mathbb{N}_0$, we denote λ_r^D as the r-th eigenvalue of the SLP consisting of Eq. (1.1) and the Dirichlet BC $y(a) = y(b) = 0$.

Theorem 2.4. *Assume* (2.4) *with* $c = a$ *and* (2.5) *with* $d = b$ *hold for some integers* $m \geq m_0$ *and* $n \geq n_0$, *respectively. Then SLP* (1.1), (1.3) *has a sequence of positive eigenvalues* $\{\lambda_r\}_{r=m+n+2}^{\infty}$ *which satisfies the interlacing relation with* $\{\lambda_r^D\}_{r=m+n}^{\infty}$

$$\lambda_{r-2}^D < \lambda_r < \lambda_r^D \quad \text{for } r \geq m+n+2,$$

and the eigenfunction y_r *associated with* λ_r *belongs to* $\mathcal{S}_r[a, b]$ *for* $r \geq m+n+2$.

The following corollary is an immediate consequence of Theorem 2.4 due to the fact that both $\lim_{m\to\infty} \mu_m^{[1]}(a) = \infty$ and $\lim_{n\to\infty} \mu_n^{[2]}(b) = \infty$.

Corollary 2.5. *Assume that*

$$e^{(b-a)l^-/2}\int_a^b \sqrt{\frac{p(s)}{p(b)}}\,d\xi_2^+(s) < \delta_{22} \quad \textit{and} \quad e^{(b-a)l^+/2}\int_a^b \sqrt{\frac{p(s)}{p(a)}}\,d\eta_2^+(a+b-s) < \delta_{12}.$$

Then SLP (1.1), (1.3) *has an infinite number of positive eigenvalues tending towards infinity.*

When $p(t) \equiv 1$ and $q(t) \equiv 0$, Eq. (1.1) becomes the equation

$$y'' + \lambda w(t) y = 0. \tag{2.7}$$

In this case, apart from the results in Theorems 2.2 and 2.4, we show that SLPs (2.7), (1.2) and (2.7), (1.3) may have one or more additional positive eigenvalues. Note that in this case, $m_0 = n_0 = 0$ and conditions (2.4) and (2.5) become

$$\int_a^b \frac{e^{(b-a)l^-/2}\,d\xi_1^+(s)}{\sqrt{\mu_m^{[1]}(c)w(s)}} + e^{(b-a)l^-/2}\int_a^b d\xi_2^+(s) < \delta_{22} \tag{2.8}$$

and

$$\int_a^b \frac{e^{(b-a)l^+/2}\,d\eta_1^+(a+b-s)}{\sqrt{\mu_n^{[2]}(d)w(a+b-s)}} + e^{(b-a)l^+/2}\int_a^b d\eta_2^+(a+b-s) < \delta_{12}, \tag{2.9}$$

where $l^{\pm}$ are given by (H1) with $p(t) \equiv 1$ and $q(t) \equiv 0$. The following conditions are also stated here to simplify the statements of the results.

$$\int_a^b d\xi_1^+(s) < \delta_{21}, \tag{2.10}$$

$$\int_a^b d\eta_1^+(a+b-s) < \delta_{11}, \tag{2.11}$$

$$\left(\sqrt{\mu_{n+1}^{[2]}(b)}\int_a^b d\xi_2^+(s) + \int_a^b \frac{d\xi_1^+(s)}{\sqrt{w(s)}}\right) e^{(b-a)l^-/2} < \frac{\delta_{21}}{\sqrt{w(b)}}, \tag{2.12}$$

$$\left(\sqrt{\mu_{m+1}^{[1]}(a)}\int_a^b d\eta_2^+(s) + \int_a^b \frac{d\eta_1^+(a+b-s)}{\sqrt{w(a+b-s)}}\right) e^{(b-a)l^+/2} < \frac{\delta_{11}}{\sqrt{w(a)}}. \tag{2.13}$$

Theorem 2.6. *Assume* (2.8) *holds for some even integer* $m \geq 0$ *and* (2.10) *holds. Then in addition to the conclusion in Theorem 2.2, SLP* (2.7), (1.2) *also has a positive eigenvalue* $\lambda_m(c)$ *which satisfies that* $\lambda_m(c) < \mu_m^{[1]}(c)$*. Moreover, the eigenfunction* y_m *associated with* $\lambda_m(c)$ *belongs to* $\mathcal{S}_i[c,b]$ *for some* $0 \leq i \leq m$*.*

The following corollary follows directly from Theorem 2.6.

Corollary 2.7. *Assume* (2.8) *holds for* $m = 0$ *and* (2.10) *holds. Then SLP* (2.7), (1.2) *has an infinite number of positive eigenvalues* $\{\lambda_i(c)\}_{i=0}^{\infty}$ *which satisfies the interlacing relation with* $\{\mu_i^{[1]}(c)\}_{i=0}^{\infty}$

$$\lambda_0(c) < \mu_0^{[1]}(c) < \lambda_1(c) < \mu_1^{[1]}(c) < \lambda_2(c) < \cdots < \mu_i^{[1]}(c) < \lambda_{i+1}(c) < \cdots .$$

Moreover, the eigenfunction y_i *associated with* λ_i *belongs to* $\mathcal{S}_i[c, b]$ *for* $i \in \mathbb{N}$.

Theorem 2.8.

(a) *Assume* (2.8) *with* $c = a$ *holds for some even integer* $m \geq 0$, (2.9) *with* $d = b$ *and* (2.12) *hold for some integer* $n \geq 0$, *and* (2.10) *holds. Then in addition to the conclusion in Theorem 2.4, SLP* (2.7), (1.3) *also has a positive eigenvalue* λ_{m+n+1} *which satisfies that* $\lambda_{m+n+1} < \lambda_{m+n+1}^D$. *Moreover, the eigenfunction* y_{m+n+1} *associated with* λ_{m+n+1} *belongs to* $\mathcal{S}_r[a, b]$ *for some* $n + 1 \leq r \leq m + n + 1$.
(b) *Assume* (2.9) *with* $d = b$ *holds for some even integer* $n \geq 0$, (2.8) *with* $c = a$ *and* (2.13) *hold for some integer* $m \geq 0$, *and* (2.11) *holds. Then in addition to the conclusion in Theorem 2.4, SLP* (2.7), (1.3) *also has a positive eigenvalue* λ_{m+n+1} *which satisfies that* $\lambda_{m+n+1} < \lambda_{m+n+1}^D$. *Moreover, the eigenfunction* y_{m+n+1} *associated with* λ_{m+n+1} *belongs to* $\mathcal{S}_r[a, b]$ *for some* $m + 1 \leq r \leq m + n + 1$.
(c) *Assume that* (2.8) *with* $c = a$, (2.9) *with* $d = b$, (2.12), (2.13) *hold for some even integers* $m, n \geq 0$, *and* (2.10) *and* (2.11) *hold. Then in addition to the conclusions to Parts (a) and (b) above, SLP* (2.7), (1.3) *also has a positive eigenvalue* λ_{m+n} *which satisfies that* $\lambda_{m+n} < \lambda_{m+n}^D$. *Moreover, the eigenfunction* y_{m+n} *associated with* λ_{m+n} *belongs to* $\mathcal{S}_r[a, b]$ *for some* $0 \leq r \leq m + n$.

The following corollary follows directly from Theorem 2.8.

Corollary 2.9. *Assume* (2.8) *with* $c = a$, (2.9) *with* $d = b$, (2.12), (2.13) *hold for* $m = n = 0$, *and* (2.10) *and* (2.11) *hold. Then SLP* (2.7), (1.3) *has a sequence of positive eigenvalues* $\{\lambda_r\}_{r=0}^{\infty}$ *which satisfy the interlacing relation with* $\{\lambda_r^D\}_{r=0}^{\infty}$

$$\lambda_{r-2}^D < \lambda_r < \lambda_r^D \quad \text{for } r \geq 2$$

and

$$\lambda_0 < \lambda_0^D \quad \text{and} \quad \lambda_1 < \lambda_1^D,$$

and the eigenfunction y_r *associated with* λ_r *belongs to* $\mathcal{S}_r[a, b]$ *for* $r \geq 0$.

3. PROOFS OF MAIN RESULTS

Proof of Theorem 2.2. Let $y(t, \lambda)$ be the solution of Eq. (1.1) on $[c, b]$ such that

$$y(c) = 1 \quad \text{and} \quad (py')(c) = 0. \tag{3.1}$$

Let $\theta(t,\lambda)$ be the Prüfer angle of $y(t,\lambda)$, i.e., $\theta(t,\lambda)$ is a continuous function on $[c,b]$ such that

$$\tan\theta(t,\lambda) = y(t,\lambda)/(py')(t,\lambda) \text{ and } \theta(c,\lambda) = \pi/2.$$

It is well known, see [24, Theorem 4.5.3], that for $t \in (c,b]$, $\theta(t,\lambda)$ is strictly increasing in λ and

$$\lim_{\lambda\to-\infty} \theta(t,\lambda) = 0 \text{ and } \theta(t,\lambda) = \infty.$$

For ease of notation, let $\mu_i^{[1]} := \mu_i^{[1]}(c)$, the i-th eigenvalue of SLP (1.1) (1.2). Thus for $\mu_i^{[1]} < \lambda < \mu_{i+1}^{[1]}$, with $i \ge m$, we have

$$(i+1)\pi = \theta(b,\mu_i^{[1]}) < \theta(b,\lambda) < \theta(b,\mu_{i+1}^{[1]}) = (i+2)\pi. \tag{3.2}$$

For all $t \in [c,b]$ and $\lambda \ge \mu_m^{[1]} \ge \mu_{m_0}^{[1]} > 0$, define an energy function for $y(t,\lambda)$ by

$$E(t,\lambda) = \frac{1}{2p(t)}[p(t)y'(t,\lambda)]^2 + \frac{1}{2}\left(q^* - q(t) + \lambda w(t)\right)[y(t,\lambda)]^2. \tag{3.3}$$

By (H1), $E(t,\lambda) > 0$ for all $t \in [a,b]$. For ease of notation in the following, let $p := p(t)$, $q := q(t)$, $w := w(t)$, and $y := y(t,\lambda)$. It follows from Eq. (1.1) and (H1) that

$$\begin{aligned}
E'(t,\lambda) &= -\frac{p'}{2p^2}[py']^2 - \frac{1}{2}q'y^2 + q^*yy' + \frac{1}{2}\lambda w'y^2 \\
&\ge -\frac{p'}{2p^2}[py']^2 - \frac{1}{2}q'y^2 - \frac{1}{2}q^*[y^2 + (y')^2] + \frac{1}{2}\lambda w'y^2 \\
&= -\frac{(p'+q^*)}{2p^2}[py']^2 - \frac{1}{2}[q'+q^*]y^2 + \frac{1}{2}\lambda w'y^2 \\
&\ge -\frac{l^-}{2p}[py']^2 - \frac{l^-}{2}[q^*-q]y^2 - \frac{l^-}{2}\lambda wy^2 \\
&= -l^-E(t,\lambda).
\end{aligned}$$

From this, we have $E'(t,\lambda) + l^-E(t,\lambda) \ge 0$ for all $t \in [a,b]$. It follows that for $s \in [a,b]$,

$$\ln\frac{E(b,\lambda)}{E(s,\lambda)} = \int_s^b \frac{E'(t,\lambda)}{E(t,\lambda)}\,dt \ge -\int_a^b l^-\,dt = -l^-(b-a).$$

Thus

$$E(s,\lambda) \le e^{l^-(b-a)}E(b,\lambda), \quad s \in [a,b]. \tag{3.4}$$

For $\lambda = \mu_i^{[1]}$ and $\lambda = \mu_{i+1}^{[1]}$ with $i \ge m$, we have for $s \in [a,b]$

$$E(s,\lambda) \ge \frac{1}{2}[q^* - q(s) + \lambda w(s)][y(s,\lambda)]^2 \tag{3.5}$$

and

$$E(s,\lambda) \ge \frac{1}{2p(s)}[p(s)y'(s,\lambda)]^2$$

along with

$$E(b,\lambda) = \frac{1}{2p(b)}[p(b)y'(b,\lambda)]^2;$$

and therefore for $s \in [a,b]$ we have

$$|y(s,\lambda)| \le \sqrt{\frac{2E(s,\lambda)}{q^* - q(s) + \lambda w(s)}} \quad \text{and} \quad |y(b,\lambda)| = 0, \tag{3.6}$$

along with

$$|p(s)y'(s,\lambda)| \le \sqrt{2p(s)E(s,\lambda)} \quad \text{and} \quad |p(b)y'(b,\lambda)| = \sqrt{2p(b)E(b,\lambda)}. \tag{3.7}$$

Define

$$\Gamma(\lambda) = \delta_{21} y(b,\lambda) + \delta_{22}(py')(b,\lambda) - \int_a^b [y(s,\lambda)\, d\xi_1(s) + (py')(s,\lambda)\, d\xi_2(s)].$$

Let $i \ge m$ such that $i = 2k$ for some $k \in \mathbb{N}_0$. Note that $y(b,\mu_{2k}^{[1]}) = y(b,\mu_{2k+1}^{[1]}) = 0$, $y'(b,\mu_{2k}^{[1]}) < 0$, and $y'(b,\mu_{2k+1}^{[1]}) > 0$. Also, (2.4) implies that $\delta_{22} > 0$. Then by (3.6), (3.7), (3.4), and (2.4), we have

$$\begin{aligned}
\Gamma(\mu_{2k}^{[1]}) &= \delta_{22}(py')(b,\mu_{2k}^{[1]}) - \int_a^b y(s,\mu_{2k}^{[1]})\, d\xi_1(s) - \int_a^b (py')(s,\mu_{2k}^{[1]})\, d\xi_2(s) \\
&\le -\delta_{22}|p(b)y'(b,\mu_{2k}^{[1]})| + \int_a^b |y(s,\mu_{2k}^{[1]})|\, d\xi_1^+(s) + \int_a^b |p(s)y'(s,\mu_{2k}^{[1]})|\, d\xi_2^+(s) \\
&\le -\delta_{22}\sqrt{2p(b)E(b,\mu_{2k}^{[1]})} + \int_a^b \sqrt{\frac{2E(s,\mu_{2k}^{[1]})}{q^* - q(s) + \mu_{2k}^{[1]} w(s)}}\, d\xi_1^+(s) \\
&\quad + \int_a^b \sqrt{2p(s)E(s,\mu_{2k}^{[1]})}\, d\xi_2^+(s) \\
&\le \sqrt{2p(b)E(b,\mu_{2k}^{[1]})}\Bigg(- \delta_{22} + \int_a^b \frac{e^{(b-a)l^-/2} d\xi_1^+(s)}{\sqrt{p(b)(q^* - q(s) + \mu_{2k}^{[1]} w(s))}} \\
&\qquad + \int_a^b e^{(b-a)l^-/2} \sqrt{\frac{p(s)}{p(b)}}\, d\xi_2^+(s)\Bigg) < 0.
\end{aligned}$$

In the same way we can show that $\Gamma(\mu_{2k+1}) > 0$. By the continuity of $\Gamma(\lambda)$, there exists $\lambda_{2k+1} \in (\mu_{2k}^{[1]}, \mu_{2k+1}^{[1]})$ such that $\Gamma(\lambda_{2k+1}) = 0$. Similarly, if $i \ge m$ such that $i = 2k+1$

for some $k \in \mathbb{N}_0$, there exists $\lambda_{2k+2} \in (\mu^{[1]}_{2k+1}, \mu^{[1]}_{2k+2})$ such that $\Gamma(\lambda_{2k+2}) = 0$. For both cases, λ_{i+1} is an eigenvalue of SLP (1.1), (1.2) and $y(t, \lambda_{i+1})$ is a corresponding eigenfunction. Moreover, from (3.2),

$$(i+1)\pi < \theta(b, \lambda_{i+1}) < (i+2)\pi.$$

Then (2.6) follows from the monotone property of $\theta(t, \lambda)$ with respect to λ. We observe that

$$\theta'(t, \lambda) = \frac{1}{p(t)} \cos^2 \theta(t, \lambda) + (\lambda w(t) - q(t)) \sin^2 \theta(t, \lambda).$$

Hence $\theta(t, \lambda)$ is strictly increasing at points where $\theta(t, \lambda) = 0 \pmod{\pi}$. Note that $y(t) = 0$ if an only if $\theta(t, \lambda) = 0 \pmod{\pi}$. Hence, y has exactly $i + 1$ zeros on (c, b), and so $y(t, \lambda_{i+1}) \in \mathcal{S}_{i+1}[c, b]$ for $i \geq m$. □

To prove Theorem 2.4, we state a counterpart to Theorem 2.2 for the SLP consisting of Eq. (1.1) and the Riemann–Stieltjes BC

$$\begin{cases} \delta_{11} y(a) + \delta_{12}(py')(a) - \int\limits_a^b [y(s)\, d\eta_1(s) + (py')(s)\, d\eta_2(s)] = 0, \\ (py')(d) = 0. \end{cases} \tag{3.8}$$

Lemma 3.1. *Assume for some integer $n \geq n_0$,* (2.5) *holds. Then SLP* (1.1), (3.8) *has an infinite number of positive eigenvalues $\{\lambda_j(d)\}_{j=n+1}^{\infty}$ which satisfy the interlacing relation with $\{\mu^{[2]}_j(d)\}_{j=n}^{\infty}$*

$$\mu^{[2]}_n(d) < \lambda_{n+1}(d) < \mu^{[2]}_{n+1}(d) < \lambda_{n+2}(d) < \cdots < \mu^{[2]}_j(d) < \lambda_{j+1}(d) < \cdots .$$

Moreover, the eigenfunction y_j associated with $\lambda_j(d)$ belongs to $\mathcal{S}_j[a, d]$ for $j \geq n + 1$.

Proof. This follows immediately from Theorem 2.2 after applying the linear transformation $t = a + b - s$ with $d = a + b - c$. □

Proof of Theorem 2.4. For any $r \geq m_0 + n_0 + 2$, choose $i \geq m_0$ and $j \geq n_0$ such that $r = i+j+2$. For any $c \in [a, b)$ and $d \in (a, b]$, denote by $\mu^{[1]}_i(c)$ the i-th eigenvalue of SLP (1.1), (2.1) and $\mu^{[2]}_j(d)$ the j-th eigenvalue of SLP (1.1), (2.2). Following [8, Remark 3.3], we can show that $\{\lambda^{[1]}_i(c) : c \in [a, b)\}$ and $\{\lambda^{[2]}_j(d) : d \in (a, b]\}$ form continuous eigenvalue branches of SLP (1.1), (3.8). Then $\mu^{[1]}_i(a) > 0$ and $\mu^{[2]}_j(b) > 0$. It is known, see [12, Theorem 4.1] and [14, Theorems 2.2 and 2.3], that $\mu^{[1]}_i(c)$ is strictly increasing and $\lim_{c \to b^-} \mu^{[1]}_i(c) = \infty$; and $\mu^{[2]}_j(d)$ is strictly decreasing and $\lim_{d \to a^+} \mu^{[2]}_j(d) = \infty$. Thus, $\mu^{[1]}_i(c), \mu^{[2]}_j(d) > 0$ for $c, d \in (a, b)$.

By Theorem 2.2 and Lemma 3.1, we have

$$\lambda^{[1]}_{i+1}(c) > \mu^{[1]}_i(c) \text{ and } \lambda^{[2]}_{j+1}(d) > \mu^{[2]}_j(d) \text{ for all } c, d \in (a, b).$$

It follows that $\lambda_{i+1}^{[1]}(c) > 0$ and $\lambda_{j+1}^{[2]}(d) > 0$ for all $c, d \in (a,b)$, and $\lim_{c\to b^-} \lambda_{i+1}^{[1]}(c) = \infty$ and $\lim_{d\to a^+} \lambda_{j+1}^{[2]}(d) = \infty$. By the continuity of $\lambda_{i+1}^{[1]}(c)$ and $\lambda_{j+1}^{[2]}(d)$, there exists $c^* = d^* \in (a,b)$ such that $\lambda_{i+1}^{[1]}(c^*) = \lambda_{j+1}^{[2]}(d^*)$. Let

$$\lambda_r = \lambda_{i+1}^{[1]}(c^*) = \lambda_{j+1}^{[2]}(d^*). \tag{3.9}$$

Thus, any eigenfunction of SLP (1.1), (1.2) associated with $\lambda_{i+1}^{[1]}(c^*)$ is an eigenfunction of SLP (1.1), (3.8) associated with $\lambda_{j+1}^{[2]}(d^*)$, and vice versa. We denote the common eigenfunction of the two problems on $[a,b]$ as y_r. By Theorem 2.2 and Lemma 3.1, we have $y_r \in \mathcal{S}_{i+1}[c^*, b] \cap \mathcal{S}_{j+1}[a, d^*]$, i.e., $y_r \in \mathcal{S}_r[a,b]$ since $r = i + j + 2$.

By an argument similar to above, we see that there exists $c^{**} = d^{**} \in (a,b)$ such that $\mu_i^{[1]}(c^{**}) = \mu_j^{[2]}(d^{**})$. This, along with the fact that $c^* = d^*$, the monotone properties of $\mu_i^{[1]}(c)$ and $\mu_j^{[2]}(d)$, and Theorem 2.2 and Lemma 3.1, shows that

(i) when $c^{**} \le c^*$, $\mu_i^{[1]}(c^{**}) \le \mu_i^{[1]}(c^*) < \lambda_{i+1}^{[1]}(c^*)$;
(ii) when $d^{**} \ge d^*$, $\mu_j^{[2]}(d^{**}) \le \mu_j^{[2]}(d^*) < \lambda_{j+1}^{[2]}(d^*)$.

Recall that λ_r^D is the r-th eigenvalue of the SLP consisting of Eq. (1.1) and the Dirichlet BC $y(a) = y(b) = 0$. Using the fact that $r - 2 = i + j$ and counting the zeros of the eigenfunctions on the intervals (c^{**}, b), (a, d^{**}), and (a,b), we see that $\lambda_{r-2}^D = \mu_i^{[1]}(c^{**}) = \mu_j^{[2]}(d^{**})$ with the same eigenfunction. This together with (3.9) shows that $\lambda_{r-2}^D < \lambda_r$.

With the same reasoning, we can find $c^{***} = d^{***} \in (a,b)$ such that $\mu_{i+1}^{[1]}(c^{***}) = \mu_{j+1}^{[2]}(d^{***})$. Similar to above,

(i) when $c^* \le c^{***}$, $\lambda_{i+1}^{[1]}(c^*) < \mu_{i+1}^{[1]}(c^*) \le \mu_{i+1}^{[1]}(c^{***})$;
(ii) when $d^* \ge d^{***}$, $\lambda_{j+1}^{[2]}(d^*) < \mu_{j+1}^{[2]}(d^*) \le \mu_{j+1}^{[2]}(d^{***})$.

Since $r = i + j + 2$, we have $\lambda_r^D = \mu_{i+1}^{[1]}(c^{***}) = \mu_{j+1}^{[2]}(d^{***})$. By (3.9), we see that $\lambda_r < \lambda_r^D$. □

Proof of Theorem 2.6. We need only show that under the assumptions, SLP (2.7), (1.2) has an additional eigenvalue $\lambda_m(c)$ associated with eigenfunction $y_m \in \mathcal{S}_i[c,b]$, for some $0 \le i \le m$, which satisfies $0 < \lambda_m(c) < \mu_m^{[1]}(c)$.

From (3.1), we have that $y(t,\lambda)$ is a solution of Eq. (2.7) satisfying $y(c,\lambda) = 1$ and $y'(c,\lambda) = 0$. Since $\mu_m^{[1]}(c) > 0$, then for $0 < \lambda < \mu_m^{[1]}(c)$ we have

$$0 < \theta(b,\lambda) < \theta(b, \mu_m^{[1]}(c)) = (m+1)\pi. \tag{3.10}$$

For $p(t) \equiv 1$ and $q(t) \equiv 0$, define an energy function for $y(t,\lambda)$ as in (3.3). Then (3.4), (3.6), and (3.7) hold for $\lambda = \mu_m^{[1]}(c)$. Since m is even implies that $y'(b, \mu_m^{[1]}(c)) < 0$, then by a similar process as in the proof of Theorem 2.2, we arrive at $\Gamma(\mu_m^{[1]}(c)) < 0$.

For $\lambda = 0$, we have $y(t,0) \equiv 1$ and $y'(t,0) \equiv 0$ for all $t \in [a,b]$. By the continuous dependence of solutions on parameters, it follows that $y'(t,\lambda) = o(1)$ uniformly for all $t \in [a,b]$ as $\lambda \to 0$. Since

$$y(t,\lambda) = y(c,\lambda) + \int_c^t y'(s,\lambda)\,ds,$$

then for sufficiently small λ, we have

$$y(t,\lambda) = 1 + o(1) > 1/2 \quad \text{for all } t \in [a,b].$$

Then from (2.11), we have for λ sufficiently small

$$\begin{aligned}
\Gamma(\lambda) &= y(b,\lambda)\left(\delta_{21} + \delta_{22}\frac{y'(b,\lambda)}{y(b,\lambda)} - \int_a^b \left[\frac{y(s,\lambda)}{y(b,\lambda)}\,d\xi_1(s) + \frac{y'(s,\lambda)}{y(b,\lambda)}\,d\xi_2(s)\right]\right)\\
&\geq y(b,\lambda)\left(\delta_{21} + o(1) - \int_a^b \left[(1+o(1))\,d\xi_1^+(s) + o(1)\,d\xi_2^+(s)\right]\right)\\
&= y(b,\lambda)\left(\delta_{21} - \int_a^b d\xi_1^+(s)\right) + o(1) > 0.
\end{aligned}$$

By the continuity of $\Gamma(\lambda)$, there exists $\lambda_m(c) \in (0, \mu_m^{[1]}(c))$ such that $\Gamma(\lambda_m(c)) = 0$. Thus, $\lambda_m(c)$ is an eigenvalue of SLP (1.1), (1.2) and $y(t,\lambda_m(c))$ is the corresponding eigenfunction. Moreover, from (3.10) and the monotone property of $\theta(t,\lambda)$ with respect to λ, we have $y(t,\lambda_m(c)) \in \mathcal{S}_i[c,b]$ for $0 \leq i \leq m$. □

The following lemma is a counterpart to Theorem 2.6.

Lemma 3.2. *Assume* (2.9) *holds for some even integer* $n \geq 0$ *and* (2.11) *holds. Then addition to the conclusion of Lemma 3.1, SLP* (2.7), (3.8) *also has a positive eigenvalue* $\lambda_n(d)$ *which satisfies that* $\lambda_n(d) < \mu_n^{[2]}(d)$. *Moreover, the eigenfunction associated with* $\lambda_n(d)$ *belongs to* $\mathcal{S}_i[a,d]$ *for some* $0 \leq i \leq n$.

Proof. By applying the linear transformation stated in the proof of Lemma 3.1 to Theorem 2.6, this result obtained. It is therefore omitted. □

Proof of Theorem 2.8. (a) Let $\{\lambda_m^{[1]}(c) : c \in [a,b)\}$ be the m-th continuous eigenvalue branch for SLP (2.7), (1.2) given in the proof of Theorem 2.6 above. Unlike the case in Theorem 2.4, we do not expect $\lim_{c\to b^-} \lambda_m^{[1]}(c) = \infty$. Instead, we claim that with the condition (2.13)

$$\limsup_{c\to b^-} \lambda_m^{[1]}(c) \geq \mu_{n+1}^{[2]}(b) > \lambda_{n+1}^{[2]}(b). \tag{3.11}$$

For otherwise, $\limsup_{c\to b^-} \lambda_m^{[1]}(c) < \mu_{n+1}^{[2]}(b)$. Hence, there exists a sequence $\{c_k\}_{k=0}^{\infty} \subset [a,b)$ such that $c_k \to b$ and

$$\lim_{k\to\infty} \lambda_m^{[1]}(c_k) = \bar{\lambda} < \mu_{n+1}^{[2]}(b).$$

Let $y_k^{[1]}(t)$ be the eigenfunction associated with $\lambda_m^{[1]}(c_k)$ such that $y_k^{[1]}(c_k) = 1$. Then $y_k^{[1]}(t)$ satisfies Eq. (2.7) with $\lambda = \lambda_m^{[1]}(c_k)$ and BC (1.2). Obviously, $(y_k^{[1]})'(c_k) = 0$. Let $\bar{y}(t)$ be the solution of Eq. (1.1) with $\lambda = \bar{\lambda}$ satisfying that $\bar{y}(b) = 1$ and $\bar{y}'(b) = 0$. By the continuous dependence of solutions of initial value problems on the initial conditions and parameters, we see that $\lim_{k\to\infty} y_k^{[1]}(t) = \bar{y}(t)$ uniformly on $[a,b]$. This shows that $\bar{y}(t)$ satisfies BC (1.2) and hence is an eigenfunction of SLP (2.7), (1.2) with $\lambda = \bar{\lambda}$. Define an energy function for $\bar{y}(t)$ by (3.3) with $p(t) \equiv 1$ and $q(t) \equiv 0$. Then

$$E(b,\bar{\lambda}) = \frac{\bar{\lambda}}{2} w(b)[\bar{y}(b,\bar{\lambda})]^2$$

and (3.4) and (3.5) hold for y and λ replaced with $\bar{y}$ and $\bar{\lambda}$. As a result, we have

$$E(b,\bar{\lambda}) = \frac{1}{2}\bar{\lambda} w(b),$$

$$|\bar{y}(s,\bar{\lambda})| \le \sqrt{\frac{2E(s,\bar{\lambda})}{\bar{\lambda} w(s)}} \quad \text{and} \quad |\bar{y}(b,\bar{\lambda})| = 1, \quad s \in [a,b]$$

and

$$|\bar{y}'(s,\bar{\lambda})| \le \sqrt{2E(s,\bar{\lambda})} \quad \text{and} \quad |\bar{y}'(b,\bar{\lambda})| = 0, \quad s \in [a,b].$$

Thus

$$\begin{aligned}
\Gamma(\bar{\lambda}) &\ge \delta_{21} - \int_a^b \left(|\bar{y}(s,\bar{\lambda})|\, d\xi_1^+(s) + |\bar{y}'(s,\bar{\lambda})|\, d\xi_2^+(s) \right) \\
&\ge \delta_{21} - \int_a^b \left(\sqrt{\frac{2E(s,\bar{\lambda})}{\bar{\lambda} w(s)}}\, d\xi_1^+(s) + \sqrt{2E(s,\bar{\lambda})}\, d\xi_2^+(s) \right) \\
&\ge \delta_{21} - \sqrt{\frac{2E(b,\bar{\lambda})}{\bar{\lambda}}} \int_a^b \left(\frac{e^{(b-a)l^-/2}}{\sqrt{w(s)}}\, d\xi_1^+(s) + e^{(b-a)l^-/2}\sqrt{\bar{\lambda}}\, d\xi_2^+(s) \right) \\
&\ge \sqrt{\frac{2E(b,\bar{\lambda})}{\bar{\lambda}}} \left(\frac{\delta_{21}}{\sqrt{w(b)}} - e^{(b-a)l^-/2} \left[\int_a^b \frac{d\xi_1^+(s)}{\sqrt{w(s)}} + \sqrt{\mu_{n+1}^{[2]}(b)} \int_a^b d\xi_2^+(s) \right] \right) > 0.
\end{aligned}$$

This contradicts the fact that $\bar{\lambda}$ is an eigenvalue of SLP (2.7), (1.2) with associated eigenfunction $\bar{y}$ and hence proves the relationship in (3.11) holds.

From (3.11) and the fact that $\lim_{d\to b^+} \lambda^{[2]}_{n+1}(d) = \infty$, by the continuity of $\lambda^{[1]}_m(c)$ and $\lambda^{[2]}_{n+1}(d)$, there exist $c^* = d^* \in (a, b)$ such that $\lambda^{[1]}_m(c^*) = \lambda^{[2]}_{n+1}(d^*)$. By Lemma 3.1 and Theorem 2.6, the eigenfunctions associated with $\lambda^{[1]}_m(c^*)$ and $\lambda^{[2]}_{n+1}(d^*)$ belong to $\mathcal{S}_i[c^*, b] \cap \mathcal{S}_{n+1}[a, d^*]$ for some $0 \le i \le m$. Then the rest of the proof is essentially the same as the proof of Theorem 2.4 and is omitted.

(b) The proof is essentially the same as that in part (a) above but uses Theorem 2.4 and Lemma 3.2 instead of Theorem 2.4 and Theorem 2.6. It is therefore omitted.

(c) This can be proved similarly as part (a) above with $\lambda^{[2]}_{n+1}(d)$ replaced by $\lambda^{[2]}_n(d)$ and $\lambda^{[1]}_m(c)$ replaced by $\lambda^{[1]}_{m+1}(c)$. We omit the detail. □

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Received: July 20, 2017.
Accepted: August 8, 2017.