Opuscula Math. 38, no. 4 (2018), 501–535
https://doi.org/10.7494/OpMath.2018.38.4.501 Opuscula Mathematica

BANACH $*$-ALGEBRAS GENERATED BY SEMICIRCULAR ELEMENTS INDUCED BY CERTAIN ORTHOGONAL PROJECTIONS

Ilwoo Cho and Palle E.T. Jorgensen

Communicated by P.A. Cojuhari

Abstract. The main purpose of this paper is to study structure theorems of Banach $*$-algebras generated by semicircular elements. In particular, we are interested in the cases where given semicircular elements are induced by orthogonal projections in a C^*-probability space.

Keywords: free probability, orthogonal projections, weighted-semicircular elements, semicircular elements.

Mathematics Subject Classification: 46L10, 46L54, 47L15, 47L30, 47L55.

1. INTRODUCTION

The main purposes of this paper are (i) to introduce a way to construct *weighted-semicircular*, and *semicircular elements* generated by *orthogonal projections* in a certain *Banach $*$-probability space*, (ii) to study *operator-algebraic properties* of *Banach $*$-algebras* generated by our weighted-semicircular or semicircular elements, and (iii) to apply the results of (ii) to *operator-theoretic* (or *spectral*) data of operators in such Banach $*$-algebras via *free probability theory*.

There are different approaches to establish semicircular elements (e.g., [1,11,13] and [17]) in topological $*$-probability spaces (e.g., *C^*-probability spaces*, or *W^*-probability spaces*, or *Banach $*$-probability spaces*, etc). Our construction of semicircular elements is different from those already known. It is highly motivated by *weighted-semicircularity* and *semicircularity* in the sense of [3,4] and [5].

1.1. MOTIVATION

Semicircularity is extremely important not only in mathematical (classical, or operator-valued) *probability theory* and *statistics*, but also in *mathematical physics* (quantum physics, quantum chaos theory, etc, e.g., see [8] and [10]). In particular, in *free probability theory*, studying semicircularity of *free random variables* is one of the main branches (e.g., [1, 9, 11–14, 16] and [17]).

Independently, *p-adic* and *Adelic analysis* provide a fundamental tools, and play important roles not only in *number theory* and *geometry* at "very small" distance (e.g., [7] and [15]), but also in related mathematical or scientific fields (e.g., [2–4] and [5]). So, we cannot help emphasizing their importances and valuable applications.

In [5], the we studied certain semicircular-like elements, called *weighted-semicircular elements*, and corresponding *semicircular elements* induced from *p-adic number fields* $\mathbb{Q}_p$, motivated by the earlier studies of free-probabilistic structures over $\mathbb{Q}_p$ (e.g., see [2] and cited papers therein). In [4], the author extended the "p-adic local" weighted-semicircular laws and the semicircular law of [5] to "Adelic universal" weighted-semicircular laws and the semicircular laws under *free product* of Banach $*$-probability spaces in the sense of [5], over *primes p*. As application of the main results in [4], we studied *free stochastic integration* for *free stochastic processes* determined by the weighted-semicircular laws and the semicircular law of [4] in [3]. Also, see [6] and [12].

In this paper, we apply the construction of weighted-semicircular elements, and that of semicircular element of [3, 4] and [5] to construct such free random variables in certain Banach $*$-probability spaces induced by $|\mathbb{Z}|$-many orthogonal projections in a C^*-algebra.

1.2. OVERVIEW

Our study starts from the assumption that there exists integer-many orthogonal projections $\{q_j\}_{j\in\mathbb{Z}}$ in an arbitrarily given C^*-algebra A. One can have such cases artificially (e.g., see Section 3 below), or naturally (e.g., [3,4] and [5]). Define a C^*-subalgebra Q of A generated by the family. And then construct a suitable *radial operator* l *acting on* Q. Define a cyclic Banach $*$-algebra $\mathfrak{L} = \overline{\mathbb{C}[\{l\}]}$ generated by $\{l\}$. Then, each element of $\mathfrak{L}$ is a Banach-space operator acting on Q. Then construct the tensor product algebra

$$\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q.$$

Generating operators $u_j = l \otimes q_j$ of $\mathfrak{L}_Q$ form a weighted-semicircular elements in the free product Banach $*$-probability space,

$$\underset{j\in\mathbb{Z}}{\star}(\mathfrak{L}_Q,\ \tau_j),$$

for suitable linear functionals τ_j on $\mathfrak{L}_Q$.

And hence, there exist $s_j \in \mathbb{R}$ in $\mathbb{C}$, such that $s_j u_j$ become semicircular elements, for all $j \in \mathbb{Z}$.

The construction of our weighted-semicircular elements is one of the main results of this very paper. As we discussed above, the construction is very similar to those in [4]. From our (weighted-)semicircular elements, we construct several different types of Banach $*$-algebras, and study operator-algebraic properties of such $*$-algebras, and consider operator-theoretic properties of certain operators of these Banach $*$-algebras under *spectral data*, expressed by *free distributions* of operators.

1.3. SKETCH OF MAIN RESULTS

In Sections 3 and 4, we established self-adjoint operators $\{u_j\}_{j\in\mathbb{Z}}$ in a certain Banach $*$-algebra $\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q$ induced by the C^*-algebra Q generated by a family $\mathbf{Q} = \{q_j\}_{j\in\mathbb{Z}}$ of mutually orthogonal projections q_j's. By defining a family $\{\tau_j\}_{j\in\mathbb{Z}}$ of suitable linear functionals τ_j's on $\mathfrak{L}_Q$, we show our self-adjoint Banach-space operators u_j are weighted-semicircular in the Banach $*$-probability spaces $(\mathfrak{L}_Q, \tau_j)$, for all $j \in \mathbb{Z}$, i.e., the algebra $\mathfrak{L}_Q$ is sectionized by $\{\tau_j\}_{j\in\mathbb{Z}}$, and in each sectionized Banach $*$-probability space of $\mathfrak{L}_Q$, our operator u_j is weighted-semicircular. The construction of our weighted-semicircular elements, itself, is one of the main results of this paper.

In Section 5, we construct semicircular elements $\{U_j\}_{j\in\mathbb{Z}}$ in $\mathfrak{L}_Q$ generated by our weighted-semicircular elements $\{u_j\}_{j\in\mathbb{Z}}$ of Section 4. It shows that whenever one can take a family of $|\mathbb{Z}|$-many mutually orthogonal projections, the corresponding semicircular elements U_j exist.

In Section 6, based on the constructions of weighted-semicircularity and semicircularity in Sections 4 and 5, we study weighted-semicircularity induced by different types of Banach $*$-algebras generated by "finite" copies of $\mathfrak{L}_Q$'s. In particular, we are interested in the weighted-semicircularity on Banach $*$-algebras generated by finite-copies of $\mathfrak{L}_Q$'s under *direct product*, *tensor product*, and *free product*. Check the weighted-semicircularity (6.12) on direct products of $\mathfrak{L}_Q$'s, and the weighted-semicircularity (6.32), and the semicircularity (6.34) on tensor products of $\mathfrak{L}_Q$'s, and the semicircularity (6.49) on free products of $\mathfrak{L}_Q$'s.

In Section 7, we consider an example from our main results of Sections 4, 5 and 6. Especially, we study weighted-semicircular elements and semicircular elements determined by the mutually orthogonal *rank-1 projections* on the Hilbert space $l^2(\mathbb{Z})$.

2. PRELIMINARIES

Readers can check fundamental analytic-and-combinatorial free probability theory from [14] and [16] (and the cited papers therein). *Free probability* is understood as the noncommutative operator-algebraic version of classical probability theory and statistics. The classical *independence* is replaced by the *freeness*. It has various applications not only in pure mathematics (e.g., [13]), but also in related applied topics (e.g., [2–4] and [5]). In particular, we will use combinatorial free probabilistic approach of *Speicher* (e.g., [14]). *Free moments* and *free cumulants* of operators will be computed without introducing detailed concepts.

Also, readers can check the weighted-semicircularity and semicircularity induced from p-adic number fields $\mathbb{Q}_p$ in [4] and [5]. The main results, including the constructions, of [4] and [5] motivate our study.

3. FUNDAMENTAL SETTINGS

In this section, we establish backgrounds of our study. Let (A, ψ) be a *C^*-probability space*, where A is a *unital C^*-algebra* in the *operator algebra $B(H)$* consisting of all (bounded linear) operators on a *Hilbert space H*, and ψ is a *bounded linear functional on A*, satisfying $\psi(1_A) = 1$, where 1_A is the *identity operator of A*.

An operator a of A is said to be a *free random variable* whenever it is regarded as an element of (A, ψ). As usual, we say a is *self-adjoint*, if $a^* = a$ in A, where a^* is the *adjoint of* a.

We say a self-adjoint free random variable $a \in (A, \psi)$ is an *even element*, if it satisfies

$$\psi(a^{2n-1}) = 0, \text{ for all } n \in \mathbb{N}.$$

Definition 3.1. A self-adjoint free random variable a is said to be weighted--semicircular in (A, ψ) with its weight $t_0 \in \mathbb{C}$, (or in short, t_0-semicircular), if a satisfies the free cumulant computation,

$$k_n(a, \ldots, a) = \begin{cases} k_2(a,\ a) = t_0 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $k_n(\ldots)$ means the free cumulant on A in terms of ψ (in the sense of [14]).

If $t_0 = 1$ in (3.1), the 1-semicircular element a is simply said to be semicircular in (A, ψ), i.e., a is semicircular in (A, ψ), if a satisfies

$$k_n(a, \ldots, a) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

for all $n \in \mathbb{N}$.

By the Möbius inversion of [14], one can characterize the weighted-semicircularity (3.1) as follows: a self-adjoint operator a is t_0-semicircular in (A, ψ), if and only if

$$\psi(a^{2n-1}) = 0, \text{ i.e., it is even in } (A, \psi),$$

and

$$\psi(a^{2n}) = t_0^n c_n$$

for all $n \in \mathbb{N}$, where

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

is the *n-th Catalan number*.

In short, a is t_0-semicircular in (A, ψ), if and only if

$$\psi(a^n) = \omega_n \left(t_0^{\frac{n}{2}} c_{\frac{n}{2}} \right), \tag{3.3}$$

where

$$\omega_n \overset{def}{=} \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$.

Similarly, a free random variable a is semicircular in (A, ψ), if and only if

$$\psi(a^n) = \omega_n c_{\frac{n}{2}}, \tag{3.4}$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (3.3).

So, we will use the t_0-semicircularity (3.1) and its characterization (3.3) alternatively; and similarly, one can use the semicircularity (3.2) and its characterization (3.4), alternatively.

Note that, if a is a self-adjoint free random variable in (A, ψ), then the free moments $\{\psi(a^n)\}_{n=1}^{\infty}$, and the free cumulants $\{k_n(a, \ldots, a)\}_{n=1}^{\infty}$ provide equivalent *free distribution of* a in (A, ψ), which are also equivalent to the *spectral data of* a (e.g., [14] and [16]). Indeed, the *Möbius inversion* of [14] satisfies

$$\psi(a^n) = \sum_{\pi \in NC(n)} \left(\underset{V \in \pi}{\Pi} k_{|V|}(a, \ldots, a) \right),$$

and

$$k_n(a, \ldots, a) = \sum_{\pi \in NC(n)} \left(\underset{V \in \theta}{\Pi} \psi(a^{|V|}) \right) \mu(\pi, 1_n),$$

where $NC(n)$ is the *lattice* consisting of all *noncrossing partitions* over $\{1, \ldots, n\}$, and "$V \in \pi$" means "V is a *block* of π," and where

$$\mu\left(0_k, 1_k\right) = (-1)^{k+1} c_{k+1}, \text{ and } \sum_{\pi \in NC(n)} \mu(\pi, 1_n) = 0,$$

where c_m are m-th Catalan numbers for all $m \in \mathbb{N}$.

Therefore, the free-moment formula (3.3) (resp., (3.4)) is equivalent to the free-cumulant definition (3.1) (resp., (3.2)).

Now, let (A, ψ) be a fixed C^*-probability space, and let $q_j \in A$ be a *projection* in the sense that

$$q_j^* = q_j = q_j^2 \text{ in } A, \tag{3.5}$$

for all $j \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all *integers*. Moreover, assume that the projections $\{q_j\}_{j \in \mathbb{Z}}$ are *mutually orthogonal* in A, in the sense that:

$$q_i q_j = \delta_{i,j} q_j \text{ in } A, \text{ for all } i, j \in \mathbb{Z}, \tag{3.6}$$

where δ is the *Kronecker delta*. Also, suppose, for convenience, that

$$\psi(q_j) \neq 0 \text{ in } \mathbb{C}, \text{ for all } j \in \mathbb{Z}. \tag{3.7}$$

Now, we fix the family $\{q_j\}_{j \in \mathbb{Z}}$ of mutually orthogonal projections of A, and we denote it by $\mathbf{Q}$,

$$\mathbf{Q} = \{q_j : j \in \mathbb{Z}, q_j\text{'s satisfy (3.5), (3.6) and (3.7)}\}, \tag{3.8}$$

in A.

Remark 3.2. Assume that one can find mutually orthogonal projections $\{q_1, \ldots, q_N\}$ in a C^*-algebra A_0, for some $N \in \mathbb{N} \cup \{\infty\}$. Assume that $N < \infty$, for instance, A_0 is a matricial algebra $M_N(\mathbb{C})$, and

$$q_j = [t_{ij}]_{N \times N}, \text{ with } t_{jj} = 1, \text{ and } t_{kl} = 0,$$

for all $(k, l) \neq (j, j)$ in $\{1, \ldots, N\}^2$, etc.

Then one can construct a C^*-algebra

$$A = A_0^{\oplus|\mathbb{Z}|} = C^* \left(\ldots \sqcup \ A_0 \sqcup A_0 \sqcup \ldots \right),$$

under suitable product topology. Then one can obtain the mutually orthogonal projections,

$$\{\ldots, q_1^{(-1)}, \ldots, q_N^{(-1)}, q_1^{(0)}, \ldots, q_N^{(0)}, q_1^{(1)}, \ldots, q_N^{(1)}, \ldots, \}$$

in A, with identity:

$$q_j^{(k)} = q_j \text{ in the direct summand } A_0 \text{ of } A, \text{ for all } k \in \mathbb{Z},$$

for all $j = 1, \ldots, N$.

Similarly, if one can find mutually orthogonal projections $\{q_1, \ q_2, \ q_3, \ \ldots\}$ in a C^*-algebra A_0 (i.e., $N = \infty$), then one can construct a C^*-algebra

$$A = A_0 \oplus A_0,$$

and the corresponding orthogonal projections,

$$\{\ldots, q_3^{(1)}, q_2^{(1)}, q_1^{(1)}, q_1^{(2)}, q_2^{(2)}, q_3^{(2)}, \ldots\}$$

in A, with

$$q_j^{(k)} = q_j \text{ in the direct summand } A_0 \text{ of } A, \text{ for } k = 1, 2,$$

for all $j \in \mathbb{N}$.

Thus, we can assume the existence of mutually orthogonal $|\mathbb{Z}|$-many projections induced by a fixed C^*-algebra A.

And let Q be the C^*-subalgebra of A generated by $\mathbf{Q}$ of (3.8),

$$Q \overset{def}{=} C^* (\mathbf{Q}) \subseteq A, \tag{3.9}$$

where $\mathbf{Q}$ is in the sense of (3.8).

Proposition 3.3. *Let Q be a C^*-subalgebra* (3.9) *of a given C^*-algebra A generated by $\mathbf{Q}$ of* (3.8). *Then*

$$Q \overset{*\text{-}iso}{=} \underset{j \in \mathbb{Z}}{\oplus} (\mathbb{C} \cdot q_j) \overset{*\text{-}iso}{=} \mathbb{C}^{\oplus|\mathbb{Z}|}, \text{ in } A. \tag{3.10}$$

Proof. The proof of the isomorphism theorem (3.10) is straightforward by the mutually-orthogonality (3.5) of the generator set $\mathbf{Q}$ of Q in A. □

Define now linear functionals ψ_j on Q by

$$\psi_j (q_i) = \delta_{ij} \psi(q_j), \text{ for all } i \in \mathbb{Z}, \tag{3.11}$$

for all $j \in \mathbb{Z}$, where ψ is a fixed linear functional in the fixed C^*-probability space (A, ψ). The linear functionals $\{\psi_j\}_{j \in \mathbb{Z}}$ of (3.11) are well-defined on Q by the structure theorem (3.10) of the C^*-subalgebra Q of (3.9) in A.

So, if $q \in Q$, then

$$q = \sum_{j \in \mathbb{Z}} t_j q_j \qquad (\text{with } t_j \in \mathbb{C}),$$

by (3.10), and hence,

$$\psi_j(q) = \psi_j \left(\sum_{j \in \mathbb{Z}} t_j q_j \right) = t_j \psi(q_j),$$

by (3.11), for all $j \in \mathbb{Z}$.

Definition 3.4. The C^*-probability spaces $(Q,\ \psi_j)$ are called the j-th C^*-probability spaces of Q in a fixed C^*-probability space (A, ψ), where Q is in the sense of (3.9), and ψ_j are in the sense of (3.11), for all $j \in \mathbb{Z}$.

Now, let us define a *Banach-space operator* c and a "acting on the C^*-algebra Q" by a bounded linear transformations satisfying that

$$c\,(q_j) = q_{j+1}, \text{ and } a\,(q_j) = q_{j-1}, \tag{3.12}$$

for all $j \in \mathbb{Z}$. Then c and a are indeed well-defined bounded linear operators "on Q".

Definition 3.5. We call these Banach-space operators c and a of (3.12), the *creation*, respectively, the *annihilation* on Q. Define now a new Banach-space operator l acting on Q, by

$$l = c + a. \tag{3.13}$$

This operator l of (3.13) is said to be the *radial operator on* Q.

By the definition (3.13), one has

$$l \left(\sum_{j \in \mathbb{Z}} t_j\ q_j \right) = \sum_{j \in \mathbb{Z}} t_j \left(q_{j+1} + q_{j-1} \right), \text{ on } Q.$$

Now, define a cyclic *Banach algebra*

$$\mathfrak{L} \overset{def}{=} \overline{\mathbb{C}[\{l\}]}^{\|\cdot\|}, \tag{3.14}$$

generated the radial operator l of (3.13), equipped with the *operator norm*,

$$\|T\| = \sup\{\|Tq\|_Q : \|q\|_Q = 1\},$$

where $\|\cdot\|_Q$ is the C^*-*norm on* Q, where $\overline{X}^{\|\cdot\|}$ mean the operator-norm closures of subsets X of the *operator space* $B(Q)$, consisting of all bounded linear transformations on Q.

On the Banach algebra $\mathfrak{L}$ of (3.14), define a unary operation $(*)$ by

$$\left(\sum_{n=0}^{\infty} t_n l^n \right)^* = \sum_{n=0}^{\infty} \overline{t_n} l^n \text{ in } \mathfrak{L}, \tag{3.15}$$

where $\overline{z}$ mean the *conjugates of* z, for all $z \in \mathbb{C}$.

Then it is not difficult to check that the operation (3.15) forms an *adjoint on* $\mathfrak{L}$, i.e., the Banach algebra $\mathfrak{L}$ of (3.14) forms a *Banach* $*$*-algebra* with its adjoint (3.15).

Definition 3.6. We call the Banach $*$-algebra $\mathfrak{L}$ of (3.14), the *radial* (*Banach* $*$-)*-algebra on* Q.

Now, let $\mathfrak{L}$ be the radial algebra on Q. Construct now the *tensor product Banach* $*$*-algebra*,

$$\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q. \tag{3.16}$$

Since the radial algebra $\mathfrak{L}$ of Q is a Banach $*$-algebra, and since Q is a C^*-algebra, the topological tensor product $\mathfrak{L}_Q$ of them is a well-determined Banach $*$-algebra under the product topology.

Definition 3.7. We call the tensor product Banach $*$-algebra $\mathfrak{L}_Q$ of (3.16), the radial projection (Banach $*$-)algebra on Q.

4. WEIGHTED-SEMICIRCULAR ELEMENTS INDUCED BY **Q**

Throughout this section, we use the same settings, notations and concepts of Section 3. We here construct weighted-semicircular elements induced by the family $\mathbf{Q}$ of mutually orthogonal projections in the sense of (3.8). Let (Q, ψ_j) be j-th C^*-probability space of Q in (A, ψ), where ψ_j are in the sense of (3.11), for all $j \in \mathbb{Z}$, and let $\mathfrak{L}_Q$ be the radial projection algebra (3.16) on Q.

Remark that, if

$$u_j = l \otimes q_j \in \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.1}$$

then

$$u_j^n = (l \otimes q_j)^n = l^n \otimes q_j, \text{ for all } n \in \mathbb{N},$$

since $q_j^n = q_j$, for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$.

One can construct a linear functional φ_j on the radial projection algebra $\mathfrak{L}_Q$ by a linear morphism satisfying that

$$\varphi_j \left((l \otimes q_i)^n \right) = \varphi_j \left(l^n \otimes q_i \right) \overset{def}{=} \psi_j \left(l^n(q_i) \right), \tag{4.2}$$

for all $n \in \mathbb{N}$, for all $i, j \in \mathbb{Z}$.

We call the Banach $*$-probability spaces

$$\left(\mathfrak{L}_Q, \ \varphi_j \right), \text{ for all } j \in \mathbb{Z}, \tag{4.3}$$

the j*-th* (*Banach-*$*$*-*)*probability spaces on* Q.

Now, observe the elements $l^n\left(q_i\right)$ in Q, for all $n \in \mathbb{N}, i \in \mathbb{Z}$. If c and a are the creation, respectively, the annihilation on Q in the sense of (3.12), then

$$ca = ac = 1_Q, \text{ the identity operator on } Q, \tag{4.4}$$

where

$$1_Q(q_j) = q_j, \text{ for all } j \in \mathbb{Z},$$

and hence,

$$1_Q\,(x) = x, \text{ for all } x \in Q.$$

Indeed, for any $q_j \in \mathbf{Q}$ in Q,

$$ca\,(q_j) = c\,(a\,(q_j)) = c\,(q_{j-1}) = q_{j-1+1} = q_j,$$

and

$$ac\,(q_j) = a\,(c\,(q_j)) = a\,(q_{j+1}) = q_{j+1-1} = q_j,$$

for all $j \in \mathbb{Z}$. More generally, one has

$$c^n a^n = 1_Q = a^n c^n, \text{ for all } n \in \mathbb{N}, \tag{4.5}$$

by induction on (4.4).

By (4.4) and (4.5), we also obtain that

$$c^{n_1} a^{n_2} = a^{n_2} c^{n_1}, \text{ for all } n_1, n_2 \in \mathbb{N}, \tag{4.6}$$

too.

Thus, one obtains that

$$l^n = (c+a)^n = \sum_{k=0}^{n} \binom{n}{k} c^k a^{n-k}, \tag{4.7}$$

by (4.4), (4.5), and (4.4)$''$, for all $n \in \mathbb{N}$.

Note that, for any $n \in \mathbb{N}$,

$$l^{2n-1} = \sum_{k=0}^{2n-1} \binom{2n-1}{k} c^k a^{n-k}, \tag{4.8}$$

by (4.7). So, the formula (4.8) does not contain 1_Q-terms, by (4.4) and (4.5).

Note also that, for any $n \in \mathbb{N}$, one has

$$l^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} c^k a^{n-k} = \binom{2n}{n} c^n a^n + [\text{Rest terms}], \tag{4.9}$$

by (4.4), (4.5), and (4.7).

Proposition 4.1. *Let l be the radial operator* (3.13) *generating the radial algebra $\mathfrak{L}$ on Q. Then*

$$l^{2n-1} \textit{ does not contain } 1_Q\textit{-terms in } \mathfrak{L}, \tag{4.10}$$

$$l^{2n} \textit{ contains } \binom{2n}{n} \cdot 1_Q \textit{ in } \mathfrak{L}. \tag{4.11}$$

Proof. The statement (4.10) is proven by (4.8), with help of (4.4), (4.5) and (4.7). And, the statement (4.11) is proven by (4.9) and (4.5). □

Since

$$u_j^n = (l \otimes q_j)^n = l^n \otimes q_j,$$

one can get that

$$\varphi_j\left(u_j^{2n-1}\right) = \psi_j\left(l^{2n-1}(q_j)\right) = 0, \tag{4.12}$$

for all $n \in \mathbb{N}$, and $j \in \mathbb{Z}$, by (3.11) and (4.10).

Similarly, we have

$$\varphi_j\left(u_j^{2n}\right) = \psi_j\left(l^{2n}(q_j)\right) = \psi_j\left(\binom{2n}{n} q_j + [\text{Rest terms}]q_j\right)$$

by (4.9)

$$= \binom{2n}{n}\psi_j(q_j) = \binom{2n}{n}\psi(q_j), \tag{4.13}$$

by (3.11), for all $n \in \mathbb{N}$, i.e., by (4.10) and (4.11), we obtain the following free-distributional data on the j-th probability space $(\mathfrak{L}_Q,\ \varphi_j)$, for $j \in \mathbb{Z}$.

Theorem 4.2. *Fix $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be the corresponding generating operator* (4.1) *of the j-th probability space $(\mathfrak{L}_Q, \varphi_j)$. Then*

$$\varphi_j\left(u_j^n\right) = \omega_n\left(\left(\frac{n}{2}+1\right)\psi(q_j)\right)c_{\frac{n}{2}}, \tag{4.14}$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (3.3), *and c_k are the k-th Catalan numbers, for all $k \in \mathbb{N}$.*

Proof. By (4.12) and (4.13), one can get that:

$$\varphi_j\left(u_j^{2n-1}\right) = 0,$$

and

$$\begin{aligned}\varphi_j\left(u_j^{2n}\right) &= \binom{2n}{n}\psi(q_j) = \left(\frac{n+1}{n+1}\right)\binom{2n}{n}\psi(q_j)\\ &= ((n+1)\psi(q_j))\left(\frac{1}{n+1}\binom{2n}{n}\right)\\ &= ((n+1)\psi(q_j))\,c_n,\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, the formula (4.14) holds. □

Motivated by the free-distributional data (4.14) of the generating operator u_j of $\mathfrak{L}_Q$, we define the following morphism

$$E_{j,Q} : \mathfrak{L}_Q \to \mathfrak{L}_Q$$

by a surjective linear transformation satisfying

$$E_{j,Q}\left(u_i^n\right) = \delta_{j,i}\left(\frac{\psi(q_j)^{n-1}}{\left(\left[\frac{n}{2}\right]+1\right)}u_j^n\right), \tag{4.15}$$

for all $n \in \mathbb{N}, i, j \in \mathbb{Z}$, where δ is the Kronecker delta, and $[\frac{n}{2}]$ mean the *minimal integers* greater than or equal to $\frac{n}{2}$, for example,

$$\left[\frac{3}{2}\right] = 2 = \left[\frac{4}{2}\right].$$

The linear transformations $E_{j,Q}$ of (4.15) are well-defined linear transformations on $\mathfrak{L}_Q$, because of the construction (3.16) of $\mathfrak{L}_Q$, and by the structure theorem (3.10) of Q, i.e., all elements T of $\mathfrak{L}_Q$ has its form,

$$\sum_{k=1}^{\infty} \sum_{j \in \mathbb{Z}} t_{k,j} u_j^{n_k} \quad (\text{with } n_k \in \mathbb{N}, t_{k,j} \in \mathbb{C}).$$

So, under linearity, the morphisms $E_{j,Q}$ of (4.15) are indeed well-defined on the radial projection algebra $\mathfrak{L}_Q$, for all $j \in \mathbb{Z}$.

Define now a new linear functional τ_j on $\mathfrak{L}_Q$ by

$$\tau_j \overset{def}{=} \varphi_j \circ E_{j,Q} \text{ on } \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.16}$$

where φ_j is the linear functional (4.2) on $\mathfrak{L}_Q$. By the linearity of φ_j and $E_{j,Q}$, the above morphisms τ_j are indeed well-defined linear functionals on $\mathfrak{L}_Q$, for all $j \in \mathbb{Z}$.

Definition 4.3. The well-defined Banach ∗-probability spaces

$$\mathfrak{L}_Q(j) \overset{denote}{=} (\mathfrak{L}_Q, \tau_j) \tag{4.17}$$

are called the j-th filtered (Banach-∗-)probability spaces of $\mathfrak{L}_Q$, for all $j \in \mathbb{Z}$.

On the j-th filtered probability space $\mathfrak{L}_Q(j)$ of (4.17), one can obtain that

$$\begin{aligned} \tau_j\left(u_j^n\right) &= \varphi_j\left(E_{j,Q}\left(u_j^n\right)\right) \\ &= \varphi_j\left(\frac{\psi\left(q_j\right)^{n-1}}{\left([\frac{n}{2}]+1\right)}\left(u_j^n\right)\right) = \frac{\psi\left(q_j\right)^{n-1}}{\left([\frac{n}{2}]+1\right)}\varphi_j\left(u_j^n\right) \\ &= \frac{\psi\left(q_j\right)^{n-1}}{\left([\frac{n}{2}]+1\right)}\omega_n\left(\left(\frac{n}{2}+1\right)\psi\left(q_j\right)\right)c_{\frac{n}{2}} \end{aligned}$$

by (4.14), i.e., we can get that

$$\tau_j\left(u_j^n\right) = \begin{cases} \psi\left(q_j\right)^n c_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{4.18}$$

for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$.

Theorem 4.4. *Let $\mathfrak{L}_Q(j) = (\mathfrak{L}_Q, \tau_j)$ be the j-th filtered probability space of $\mathfrak{L}_Q$, for $j \in \mathbb{Z}$. Then*

$$\tau_j\left(u_k^n\right) = \delta_{j,k}\left(\omega_n \psi\left(q_j\right)^n c_{\frac{n}{2}}\right) \tag{4.19}$$

for all $n \in \mathbb{N}$, for all $k \in \mathbb{Z}$, where ω_n are in the sense of (3.3).

Proof. If $k = j$ in $\mathbb{Z}$, then the free-distributional data (4.19) holds true by (4.18), for all $n \in \mathbb{N}$.

If $k \neq j$ in $\mathbb{Z}$, then, by the very definition (4.15) of the j-th filterization $E_{j,Q}$, and also by the definition (4.2) of φ_j,

$$\tau_j\left(u_k^n\right) = 0.$$

Therefore, the above formula (4.19) holds true, for all $k \in \mathbb{Z}$. □

The following theorem is the direct consequence of the above free distributional data (4.19).

Theorem 4.5. *Let $\mathfrak{L}_Q(j)$ be the j-th filtered probability space $(\mathfrak{L}_Q,\ \tau_j)$ of $\mathfrak{L}_Q$, for $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be the j-th generating operator of $\mathfrak{L}_Q$ in $\mathfrak{L}_Q(j)$. Then u_j is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$.*

Proof. First of all, the generating operators u_k are self-adjoint in $\mathfrak{L}_Q$. Indeed,

$$u_k^* = (l \otimes q_k)^* = l^* \otimes q_k^* = l \otimes q_k = u_k,$$

by (3.15) and by the self-adjointness of projections q_k, for all $k \in \mathbb{Z}$.

Let us fix $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be a generating free random variable of the j-th filtered probability space $\mathfrak{L}_Q(j)$. Then it is self-adjoint in $\mathfrak{L}_Q(j)$.

Also, by (4.19), we have that

$$\tau_j\left(u_j^n\right) = \omega_n\left(\psi\left(q_j\right)\right)^n c_{\frac{n}{2}} = \omega_n\left(\psi(q_j)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}},$$

where ω_n are in the sense of (3.3), for all $n \in \mathbb{N}$.

Therefore, by the free-momental characterization (3.3) of weighted-semicircularity, the element u_j is $\psi(q_j)^2$-semicircular in the j-th filtered probability space $\mathfrak{L}_Q(j)$ of $\mathfrak{L}_Q$. □

The above theorem shows that, for any $k \in \mathbb{Z}$, the k-th generating operator $u_k = l \otimes q_k$ of the k-th filtered probability space $\mathfrak{L}_Q(k)$ of Q is $\psi(q_k)^2$-semicircular.

Observe also that if $k_n^j(\ldots)$ is the free cumulant in terms of the linear functional τ_j on $\mathfrak{L}_Q$ in the sense of [14], then

$$k_n^j(\underbrace{u_j, u_j, \ldots, u_j}_{n\text{-times}}) = \begin{cases} k_2^j\left(u_j,\ u_j\right) = \psi(q_j)^2 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{4.20}$$

for all $n \in \mathbb{N}$, by (4.19), and by the Möbius inversion of [14]. So, by (3.1), the j-th generating operators u_j are $\psi(q_j)^2$-semicircular in the j-th filtered probability spaces $\mathfrak{L}_Q(j)$, for all $j \in \mathbb{Z}$.

5. SEMICIRCULAR ELEMENTS INDUCED BY **Q**

As in Section 4, we will keep working on the j-th filtered probability space $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$. The main results of Section 4 show that, for any fixed $j \in \mathbb{Z}$, the j-th generating operator $u_j = l \otimes q_j$ of $\mathfrak{L}_Q$ is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$, satisfying that

$$\tau_j\left(u_j^n\right) = \omega_n \psi(q_j)^n c_{\frac{n}{2}},$$

equivalently,

$$k_n^j\left(u_j,\ \ldots,\ u_j\right) = \begin{cases} \psi(q_j)^2 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

for all $n \in \mathbb{N}$, where $k_n^j(\ldots)$ means the free cumulant on $\mathfrak{L}_Q$ in terms of the linear functional τ_j of (4.16).

Corollary 5.1. *Let $u_j = l \otimes q_j$ be the j-th generating operator of the radial projection algebra $\mathfrak{L}_Q$, for $j \in \mathbb{Z}$. If the projection q_j of Q satisfies $\psi(q_j) = 1$, then u_j is semicircular in the j-th filtered probability space $\mathfrak{L}_Q(j)$.*

However, generally $\psi(q_j)$ may not be identical to 1, for $j \in \mathbb{Z}$. So, we consider the following semicircularity induced by our weighted-semicircularity in $\mathfrak{L}_Q(j)$.

Define operators U_j of $\mathfrak{L}_Q(j)$ by

$$U_j = \frac{1}{\psi(q_j)} u_j \in \mathfrak{L}_Q(j), \tag{5.2}$$

for all $j \in \mathbb{Z}$, where u_j are the $\psi(q_j)^2$-semicircular elements in $\mathfrak{L}_Q(j)$. Then these operators U_j can be semicircular in $\mathfrak{L}_Q(j)$ under a certain additional condition.

Theorem 5.2. *Let $U_j = \frac{1}{\psi(q_j)}\, u_j$ be a free random variable* (5.2) *of the j-th filtered probability space $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$. If $\psi(q_j) \in \mathbb{R}$ in $\mathbb{C}$, then U_j is semicircular in $\mathfrak{L}_Q(j)$.*

Proof. Fix $j \in \mathbb{Z}$, and assume that $\psi(q_j) \in \mathbb{R}$ in $\mathbb{C}$. Then

$$U_j^* = \left(\frac{1}{\psi(q_j)} u_j\right)^* = U_j,$$

by the self-adjointness of u_j in $\mathfrak{L}_Q$. So, the operator U_j is self-adjoint in $\mathfrak{L}_Q$.

Consider now that: if $k_n^j(\ldots)$ is the free cumulant on $\mathfrak{L}_Q$ in terms of τ_j, then

$$\begin{aligned} k_n^j(\underbrace{U_j, U_j, \ldots, U_j}_{n\text{-times}}) &= k_n^j\left(\frac{1}{\psi(q_j)}\ u_j, \ldots, \frac{1}{\psi(q_j)} u_j\right) \\ &= \left(\frac{1}{\psi(q_j)}\right)^n k_n^j(\underbrace{u_j, u_j, \ldots, u_j}_{n\text{-times}}) \end{aligned}$$

by the bimodule-map property of free cumulants (e.g., [14])

$$= \begin{cases} \left(\frac{1}{\psi(q_j)}\right)^2 k_2^j\left(u_j,\ u_j\right) & \text{if } n = 2, \\ 0 & \text{otherwise} \end{cases}$$

by the $\psi(q_j)^2$-semicircularity of u_j in $\mathfrak{L}_Q(j)$

$$= \begin{cases} \left(\frac{1}{\psi(q_j)^2}\right)(\psi(q_j))^2 = 1 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{5.3}$$

for all $n \in \mathbb{N}$. Therefore, if $\psi(q_j) \in \mathbb{R}$ in $\mathbb{C}$, then the self-adjoint operator U_j satisfies the free-distributional data (5.3). Thus, by (5.3) and (3.2), this operator U_j is semicircular in $\mathfrak{L}_Q(j)$. □

The above theorem shows that, from the $\psi(q_j)^2$-semicircular elements $u_j = l \otimes q_j$ in $\mathfrak{L}_Q(j)$, one can construct semicircular elements $U_j = \frac{1}{\psi(q_j)} u_j$ in $\mathfrak{L}_Q(j)$, whenever $\psi(q_j) \in \mathbb{R}$, for all $j \in \mathbb{Z}$.

6. WEIGHTED-SEMICIRCULARITY ON BANACH $*$-ALGEBRAS GENERATED BY COPIES OF $\mathfrak{L}_Q$

Let (A, ψ) be a fixed C^*-probability space, and Q, the C^*-subalgebra of A generated by the family $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ of $|\mathbb{Z}|$-many projections satisfying the conditions (3.4), (3.6) and (3.7). And let $\mathfrak{L}_Q$ be the radial projection algebra (3.16), inducing its j-th filtered probability spaces $\mathfrak{L}_Q(j) = (\mathfrak{L}_Q, \tau_j)$ in the sense of (4.17), for all $j \in \mathbb{Z}$. Then the free random variables

$$u_j = l \otimes q_j \in \mathfrak{L}_Q \tag{6.1}$$

are $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$, for all $j \in \mathbb{Z}$.

If $\psi(q_j) \in \mathbb{R}$ in $\mathbb{C}$, for $j \in \mathbb{Z}$, then the free random variables

$$U_j = \frac{1}{\psi(q_j)} u_j \in \mathfrak{L}_Q \tag{6.2}$$

are semicircular in $\mathfrak{L}_Q(j)$.

In the rest of this section, we assume $\psi(q_j) \in \mathbb{R}$ in $\mathbb{C}$, for all $j \in \mathbb{Z}$, for convenience.

6.1. ON THE BANACH $*$-PROBABILITY SPACES $\mathfrak{L}_Q^{\oplus N}(j_1, \ldots, j_N)$

Let $W = (j_1, \ldots, j_N)$ be an N-tuple of integers $j_1, \ldots, j_N$ in $\mathbb{Z}$, for $N \in \mathbb{N}$. Remark that the chosen integer-entries $j_1, \ldots, j_N$ of W are not necessarily distinct in $\mathbb{Z}$, and define the *direct product Banach $*$-algebra* $\mathfrak{L}_Q^{\oplus N}$ by

$$\mathfrak{L}_Q^{\oplus N} \overset{def}{=} \underbrace{\mathfrak{L}_Q \oplus \ldots \oplus \mathfrak{L}_Q}_{N\text{-times}}, \tag{6.3}$$

where $\oplus$ means the *direct sum of Banach $*$-algebras* under product topology.

Then on a new Banach $*$-algebra $\mathfrak{L}_Q^{\oplus N}$ of (6.3), one can define a conditional expectation

$$E_W : \mathfrak{L}_Q^{\oplus N} \to \mathbb{C}^{\oplus N}$$

by the linear transformation satisfying that

$$E_W\left((T_1,\ \ldots,\ T_N)\right) = \left(\tau_{j_1}(T_1),\ \ldots,\ \tau_{j_N}(T_N)\right), \tag{6.4}$$

for all

$$(T_1, \ldots, T_N) \overset{denote}{=} \underset{l=1}{\overset{N}{\oplus}} T_l \in \mathfrak{L}_Q^{\oplus N},$$

where $\mathbb{C}^{\oplus N}$ is the N-*dimensional algebra* (which is regarded as the *diagonal subalgebra* in the *matricial algebra* $M_N(\mathbb{C})$), and τ_{j_k} are the linear functionals (4.16) on $\mathfrak{L}_Q$, inducing the corresponding j_k-th filtered probability space $\mathfrak{L}_Q(j_k) = (\mathfrak{L}_Q, \tau_{j_k})$, for $k = 1, \ldots, N$, where $j_1, \ldots, j_N$ are the entries of the fixed N-tuple W.

Then the morphism E_W of (6.4) is indeed a well-defined *conditional expectation from* $\mathfrak{L}_Q^{\oplus N}$ *onto* $\mathbb{C}^{\oplus N}$ (e.g., [14]).

Proposition 6.1. *The linear morphism* $E_W : \mathfrak{L}_Q^{\oplus N} \to \mathbb{C}^{\oplus N}$ *of* (6.4)*, for a fixed* N*-tuple* $W = (j_1, \ldots, j_N)$ *of integers* $j_1, \ldots, j_N$*, is a conditional expectation.*

Proof. By the very definition (6.4) of the linear morphism E_W, it is a well-determined bounded linear transformation from $\mathfrak{L}_Q^{\oplus N}$ "onto" $\mathbb{C}^{\oplus N}$, because τ_{j_l} are bounded, for all $l = 1, \ldots, N$.

Clearly, if one takes $T = (t_1 \cdot I, \ldots, t_N \cdot I) \in \mathfrak{L}_Q^{\oplus N}$, for

$$I \overset{def}{=} 1_{\mathfrak{L}_Q} = 1_{\mathfrak{L}} \otimes 1_Q \text{ in } \mathfrak{L}_Q,$$

for $t_1, \ldots, t_N \in \mathbb{C}$, where $1_{\mathfrak{L}}$ is the identity element l^0 of the radial algebra $\mathfrak{L}$, satisfying $1_{\mathfrak{L}}(l^n) = l^n$, for all $n \in \mathbb{N}$, then

$$E_W(T) = (t_1, \ldots, t_N) = T,$$

by understanding $T \in \underset{k=1}{\overset{N}{\oplus}} (\mathbb{C} \cdot I) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus N}$ in $\mathfrak{L}_Q^{\oplus N}$. Therefore, under linearity, we have

$$E_W(T) = T, \text{ whenever } T \in \mathbb{C}^{\oplus N} \text{ in } \mathfrak{L}_Q^{\oplus N}. \tag{6.5}$$

Now, let $S_1 = (t_1, \ldots, t_N), S_2 = (t'_1, \ldots, t'_N) \in \mathbb{C}^{\oplus N}$ in $\mathfrak{L}_Q^{\oplus N}$, with $t_k, t'_k \in \mathbb{C}$, for all $k = 1, \ldots, N$, and let $T = (T_1, \ldots, T_N) \in \mathfrak{L}_Q^{\oplus N}$. Then

$$\begin{aligned}
E_W(S_1 T S_2) &= E_W\left((t_1 T_1 t'_1, \ldots, t_1 T_N t'_N)\right) \\
&= \left(\tau_{j_1}\left(t_1 T_1 t'_1\right), \ldots, \tau_{j_N}\left(t_N T_N t'_N\right)\right) \\
&= \left(t_1\ \tau_{j_1}(T_1)\ t'_1,\ \ldots,\ t_N\ \tau_{j_N}(T_N)\ t'_N\right) \\
&= (t_1, \ldots, t_N)\left(\tau_{j_1}(T_1), \ldots, \tau_{j_N}(T_N)\right)(t'_1, \ldots, t'_N)
\end{aligned}$$

in the algebra $\mathbb{C}^{\oplus N}$

$$= S_1 E_W(T) S_2.$$

So, for any $S_1, S_2 \in \mathbb{C}^{\oplus N}$, and $T \in \mathfrak{L}_Q^{\oplus N}$, one obtains that

$$E_W(S_1 T S_2) = S_1 E_W(T) S_2 \text{ in } \mathbb{C}^{\oplus N}. \tag{6.6}$$

Finally, observe that, if $T = (T_1, \ldots, T_N) \in \mathfrak{L}_Q^{\oplus N}$, then $T^* = (T_1^*, \ldots, T_N^*)$ in $\mathfrak{L}_Q^{\oplus N}$. So,

$$E_W(T^*) = (\tau_{j_1}(T_1^*), \ldots, \tau_{j_N}(T_N^*)) = \left(\overline{\tau_{j_1}(T_1)}, \ldots, \overline{\tau_{j_N}(T_N)}\right),$$

where $\overline{z}$ are the conjugates of z, for all $z \in \mathbb{C}$

$$= (\tau_{j_1}(T_1), \ldots, \tau_{j_N}(T_N))^*$$

in the algebra $\mathbb{C}^{\oplus N}$

$$= (E_W(T))^*.$$

Thus, for any $T \in \mathfrak{L}_Q^{\oplus N}$, one has that

$$E_W(T^*) = (E_W(T))^* \text{ in } \mathbb{C}^{\oplus N}. \tag{6.7}$$

Therefore, by (6.5), (6.6) and (6.7), the bounded surjective linear transformation is a conditional expectation from $\mathfrak{L}_Q^{\oplus N}$ onto $\mathbb{C}^{\oplus N}$. □

Let tr_N be the usual *normalized trace* on the matricial algebra $M_N(\mathbb{C})$, i.e.,

$$tr_N([t_{ij}]_{N\times N}) = \frac{1}{N}\sum_{j=1}^{N} t_{jj},$$

for all $(N \times N)$-matrices with their (i,j)-entries t_{ij} in $\mathbb{C}$.

Since the N-dimensional algebra $\mathbb{C}^{\oplus N}$ is the diagonal subalgebra of $M_N(\mathbb{C})$ consisting of all $(N \times N)$-diagonal matrices, one can naturally restrict the normalized trace tr_N on $M_N(\mathbb{C})$ to the linear functional $tr_N \mid_{\mathbb{C}^{\oplus N}}$, also denoted simply by tr_N, on $\mathbb{C}^{\oplus N}$, i.e.,

$$tr_N((t_1, \ldots, t_N)) \overset{def}{=} \frac{1}{N}\left(\sum_{j=1}^{N} t_j\right), \tag{6.8}$$

for all $(t_1, \ldots, t_N) \in \mathbb{C}^{\oplus N}$.

Now, define a linear functional τ_W on the direct product Banach $*$-algebra $\mathfrak{L}_Q^{\oplus N}$ by

$$\tau_W \overset{def}{=} tr_N \circ E_W \text{ on } \mathfrak{L}_Q^{\oplus N}, \tag{6.9}$$

where tr_N is the restricted trace (6.8) on $\mathbb{C}^{\oplus N}$, and E_W is the conditional expectation (6.4) from $\mathfrak{L}_Q^{\oplus N}$ onto $\mathbb{C}^{\oplus N}$.

Then the linear morphism η_W of (6.9) is indeed a well-defined linear functional on $\mathfrak{L}_Q^{\oplus N}$, moreover, it is a *trace* in the sense that:

$$\tau_W\left(T_1T_2\right) = \tau_W(T_2T_1), \text{ for all } T_1, T_2 \in \mathfrak{L}_Q^{\oplus N}.$$

So, the pair $\left(\mathfrak{L}_Q^{\oplus N},\ \tau_W\right)$ forms a well-defined tracial Banach ∗-probability space.

Definition 6.2. We denote the Banach ∗-probability space $\left(\mathfrak{L}_Q^{\oplus N}, \tau_W\right)$ simply by $\mathfrak{L}_Q^{\oplus N}(W)$, where η_W is the linear functional (6.9) on $\mathfrak{L}_Q^{\oplus N}$, i.e.,

$$\mathfrak{L}_Q^{\oplus N}(W) \overset{def}{=} \left(\mathfrak{L}_Q^{\oplus N},\ \tau_W\right), \tag{6.10}$$

for all N-tuples W in $\mathbb{Z}$.

Let us take an operator u_W in $\mathfrak{L}_Q^{\oplus N}$, as a free random variable of the Banach ∗-probability space $\mathfrak{L}_Q^{\oplus N}(W)$ in the sense of (6.10), where

$$u_W \overset{def}{=} \frac{1}{N}\left(U_{j_1}, U_{j_2}, \ldots, U_{j_N}\right) \in \mathfrak{L}_Q^{\oplus N}, \tag{6.11}$$

and

$$U_{j_k} \overset{def}{=} \frac{1}{\psi(q_{j_k})} u_{j_k} = \frac{1}{\psi(q_{j_k})}\left(l \otimes q_{j_k}\right) \in \mathfrak{L}_Q(j_k)$$

are our semicircular elements in the j_k-th filtered probability space $\mathfrak{L}_Q(j_k) = \left(\mathfrak{L}_Q,\ \tau_{j_k}\right)$, for all $k = 1, \ldots, N$. (Recall and remark that in the rest of this paper, we automatically assumed $\psi(q_j) \in \mathbb{R}$ in $\mathbb{C}$, for all $j \in \mathbb{Z}$. So, the direct summands U_{j_k} of the operator u_W of (6.11) in $\mathfrak{L}_Q^{\oplus}(W)$ are well-determined semicircular elements in $\mathfrak{L}_Q(j_k)$, for $k = 1, \ldots, N = |W|$.)

Theorem 6.3. *Let u_W be a free random variable* (6.11) *in* $\mathfrak{L}_Q^{\oplus N}(W)$ *of* (6.10)*, for a fixed N-tuple $W = (j_1, \ldots, j_N)$ of integers. Then u_W is $\frac{1}{N^2}$-semicircular in $\mathfrak{L}_Q^{\oplus N}(W)$, i.e., for any $W \in \mathbb{Z}^N$, there exists $u_W \in \mathfrak{L}_Q^{\oplus |W|}$, such that*

$$u_W \text{ is } \frac{1}{|W|^2}\text{-semicircular in } \mathfrak{L}_Q^{\oplus |W|}(W). \tag{6.12}$$

Proof. Let u_W be in the sense of (6.11) in the Banach ∗-probability space $\mathfrak{L}_Q^{\oplus N}(W) = \left(\mathfrak{L}_Q^{\oplus N},\ \tau_W\right)$ of (6.10), for a fixed $W = (j_1,\ \ldots,\ j_N) \in \mathbb{Z}^N$. Clearly, the operator u_W is self-adjoint in $\mathfrak{L}_Q^{\oplus N}$, because every direct summand U_{j_k} is self-adjoint by its semicircularity, for all $k = 1, \ldots, N$.

By the definition (6.9) of τ_W, one can get that

$$\begin{aligned}\tau_W\left(u_W^n\right) &= \tau_W\left(\frac{1}{N^n}(U_{j_1}, U_{j_2}, \ldots, U_{j_N})^n\right)\\ &= \frac{1}{N^n}\tau_W\left((U_{j_1}^n, U_{j_2}^n, \ldots, U_{j_N}^n)\right)\\ &= \frac{1}{N^n}tr_N\left(E_W\left((U_{j_1}^n,\ U_{j_2}^n, \ldots, U_{j_N}^n)\right)\right)\\ &= \frac{1}{N^n}tr_N\left(\left(\tau_{j_1}\left(U_{j_1}^n\right), \ldots, \tau_{j_N}\left(U_{j_N}^n\right)\right)\right)\\ &= \frac{1}{N^n}tr_N\left(\left(\omega_n c_{\frac{n}{2}}, \ldots, \omega_n c_{\frac{n}{2}}\right)\right)\end{aligned}$$

by the semicircularity of U_{j_k} in $(\mathfrak{L}_Q, \tau_{j_k})$, for $k = 1, \ldots, N$

$$\begin{aligned}&= \frac{\omega_n}{N^n}\ tr_N\left(\left(c_{\frac{n}{2}}, \ldots, c_{\frac{n}{2}}\right)\right)\\ &= \frac{\omega_n}{N^n}\left(\frac{1}{N}\sum_{k=1}^{N} c_{\frac{n}{2}}\right)\end{aligned}$$

by the definition (6.8) of the normalized trace tr_N on $\mathbb{C}^{\oplus N}$

$$= \frac{\omega_n}{N^n}\left(\frac{N}{N}c_{\frac{n}{2}}\right) = \frac{\omega_n}{N^n}c_{\frac{n}{2}} = \omega_n\left(\frac{1}{N}\right)^n c_{\frac{n}{2}}, \tag{6.13}$$

where ω_n are in the sense of (3.3), for all $n \in \mathbb{N}$.

Therefore, by (6.13) and (3.3), one can conclude that the free random variable T_W is $\frac{1}{N^2}$-semicircular in $\mathfrak{L}_Q^{\oplus N}(W)$. Just for sure, we consider the following "extra" parts of the proof.

By the free-moment computation (6.13), we obtain that: if $k_n^W(\ldots)$ is the free cumulant on $\mathfrak{L}_Q^{\oplus N}(W)$ with respect to the linear functional τ_W, then

$$k_2^W\left(u_W,\ u_W\right) = \tau_W\left(u_W^2\right) - \left(\tau_W(u_W)\right)^2 = \tau_W\left(u_W^2\right) = \frac{1}{N^2},$$

under the Möbius inversion of [14], i.e., one obtains that

$$k_2^W\left(u_W,\ u_W\right) = \frac{1}{N^2}. \tag{6.14}$$

Now, for any $n \in \mathbb{N}$,

$$k_{2n-1}^W\left(u_W, \ldots, u_W\right) = \sum_{\pi \in NC(2n-1)}\left(\prod_{V \in \pi}\tau_W\left(u_W^{|V|}\right)\right)\mu\left(\pi,\ 1_{2n-1}\right)$$

by the Möbius inversion of [14]

$$= 0,$$

because every noncrossing partition π over $\{1, \ldots, 2n-1\}$ contains at least one "odd" block V_0, satisfying the block-depending free moment $\tau_W(u_W^{|V_0|}) = 0$, by the evenness from the formula (6.13), i.e.,

$$k_n^W(u_W, \ldots, u_W) = 0, \text{whenever } n \text{ is odd in } \mathbb{N}. \tag{6.15}$$

Observe that, for any $n \in \mathbb{N} \setminus \{1\}$,

$$k_{2n}^W(u_W, \ldots, u_W) = \sum_{\theta \in NC_e(2n)} \left(\prod_{V \in \theta} \eta_W \left(u_W^{|V|} \right) \right) \mu(\theta, 1_{2n})$$

where

$$NC_e(2n) = \{\theta \in NC(2n) : V \in \theta \Longleftrightarrow |V| \text{ is even}\},$$

and hence

$$= \sum_{\theta \in NC_e(2n)} \left(\prod_{V \in \theta} \frac{1}{N^{|V|}} c_{\frac{|V|}{2}} \right) \mu(\theta, 1_{2n})$$

by (6.13)

$$\begin{aligned}
&= \sum_{\theta \in NC_e(2n)} \left(\prod_{V \in \theta} \frac{1}{N^{|V|}} \right) \left(\prod_{V \in \theta} c_{\frac{|V|}{2}} \right) \mu(\theta, 1_{2n}) \\
&= \sum_{\theta \in NC_e(2n)} \left(\frac{1}{N} \right)^{\underset{V \in \theta}{\Sigma} |V|} \left(\prod_{V \in \theta} c_{\frac{|V|}{2}} \right) \mu(\theta, 1_{2n}) \\
&= \sum_{\theta \in NC_e(2n)} \left(\frac{1}{N} \right)^n \left(\prod_{V \in \theta} c_{\frac{|V|}{2}} \right) \mu(\theta, 1_{2n}) \\
&= \left(\left(\frac{1}{N} \right)^n c_{n+1} \right) \sum_{\theta \in NC_e(2n)} \mu(\theta, 1_{2n}) \\
&= \left(\left(\frac{1}{N} \right)^n c_{n+1} \right) \left(\sum_{\pi \in NC(n)} \mu(\pi,\ 1_n) \right) = 0,
\end{aligned}$$

by [14], i.e.,

$$k_{2n}^W(u_W, \ldots, u_W) = 0, \text{ whenever } n > 1 \text{ in } \mathbb{N}. \tag{6.16}$$

Thus, by (6.14), (6.15) and (6.16), indeed, we obtain that

$$k_n^W(u_W, \ldots, u_W) = \begin{cases} \frac{1}{N^2} & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{6.17}$$

for all $n \in \mathbb{N}$.

Therefore, by (6.17) and (3.1), the free random variable u_W is indeed $\frac{1}{N^2}$-semicircular in $\mathfrak{L}_Q^{\oplus N}(W)$. □

The above theorem shows that, for any finite sequences W of integers, if we define an operator

$$u_W = \frac{1}{|W|}\left(\underset{j\in W}{\oplus} U_j\right) \text{ in } \mathfrak{L}_Q^{\oplus|W|},$$

then it is $\frac{1}{|W|^2}$-semicircular in the Banach $*$-probability space $\mathfrak{L}_Q^{\oplus|W|}(W)$, where $U_j = \frac{1}{\psi(q_j)}(l \otimes q_j)$ are the semicircular elements of $(\mathfrak{L}_Q, \tau_j)$, for all $j \in W$ in $\mathbb{Z}$.

6.2. ON THE BANACH $*$-PROBABILITY SPACES $\mathfrak{L}_Q^{\otimes N}(j_1, \ldots, j_N)$

Like in Section 6.1, let $W = (j_1, \ldots, j_N) \in \mathbb{Z}^N$ be an arbitrarily fixed N-tuple of integers $j_1, \ldots, j_N$, which are not necessarily distinct from each other in $\mathbb{Z}$, and let $\mathfrak{L}_Q$ be the radial projection algebra of the C^*-algebra Q of the fixed C^*-probability space (A, ψ). Let τ_j be the linear functionals (4.16) on $\mathfrak{L}_Q$, inducing the corresponding j-th filtered probability space $\mathfrak{L}_Q(j) = (\mathfrak{L}_Q,\ \tau_j)$ of Q. Also, let

$$U_j = \frac{1}{\psi(q_j)} u_j = \frac{1}{\psi(q_j)}(l \otimes q_j)$$

be the semicircular element in $\mathfrak{L}_Q(j)$, for all $j \in \mathbb{Z}$.

Now, define the tensor product Banach $*$-algebra $\mathfrak{L}_Q^{\otimes N}$ by

$$\mathfrak{L}_Q^{\otimes N} \overset{def}{=} \underbrace{\mathfrak{L}_Q \otimes_{\mathbb{C}} \mathfrak{L}_Q \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \mathfrak{L}_Q}_{N\text{-times}} \tag{6.18}$$

under product topology.

Define a linear transformation F_W from the tensor product Banach $*$-algebra $\mathfrak{L}_Q^{\otimes N}$ of (6.18) onto the finite-dimensional algebra $\mathbb{C}^{\otimes N}$ by the linear morphism,

$$F_W \overset{denote}{=} \underset{k=1}{\overset{N}{\otimes}} \tau_{j_k} \text{ on } \mathfrak{L}_Q^{\otimes N}, \tag{6.19}$$

satisfying that

$$F_W\left(\underset{k=1}{\overset{N}{\otimes}} T_k\right) = \underset{k=1}{\overset{N}{\otimes}} \left(\tau_{j_k}(T_k) \cdot I\right),$$

for all $\underset{k=1}{\overset{N}{\otimes}} T_k \in \mathfrak{L}_Q^{\otimes N}$, where I is the identity element of $\mathfrak{L}_Q$ introduced in Section 6.1.

Proposition 6.4. *Let $\mathfrak{L}_Q^{\otimes N}$ be in the sense of* (6.18) *for a fixed N-tuple W with $N = |W|$. Then the linear morphism F_W of* (6.19) *is a conditional expectation from $\mathfrak{L}_Q^{\otimes N}$ onto $\mathbb{C}^{\otimes N}$.*

Proof. By the boundedness of the linear functionals τ_{j_k} of (4.16) on $\mathfrak{L}_Q$, for $k = 1, \ldots, N$, the linear transformation F_W is a bounded linear transformation from $\mathfrak{L}_Q^{\otimes N}$ to $\mathbb{C}^{\otimes N}$. Also, by the very definition (6.19), it is surjective.

Now, take $T = \underset{k=1}{\overset{N}{\otimes}} t_k I$ in $\mathfrak{L}_Q^{\otimes N}$, for $t_1, \ldots, t_N \in \mathbb{C}$, i.e., $T \in \mathbb{C}^{\otimes N} \subset \mathfrak{L}_Q^{\otimes N}$. Then

$$F_W\,(T) = \underset{k=1}{\overset{N}{\otimes}} \tau_{j_k}(t_k I) = \underset{k=1}{\overset{N}{\otimes}} t_k \text{ in } \mathbb{C}^{\otimes N}.$$

So, under linearity, one has

$$F_W(S) = S \text{ in } \mathbb{C}^{\otimes N}, \text{ for all } S \in \mathfrak{L}_Q^{\otimes N}. \tag{6.20}$$

Now, let $T_1 = \underset{k=1}{\overset{N}{\otimes}} t_k I$ (with $t_k \in \mathbb{C}$), and $T_2 = \underset{k=1}{\overset{N}{\otimes}} t'_k I$ (with $t'_k \in \mathbb{C}$) be in $\mathbb{C}^{\otimes N}$, and $S = \underset{k=1}{\overset{N}{\otimes}} S_k \in \mathfrak{L}_Q^{\otimes N}$ (with $S_k \in \mathfrak{L}_Q$). Then

$$F_W\,(T_1 S T_2) = F_W \left(\underset{k=1}{\overset{N}{\otimes}} (t_k S_k t'_k) \right) = \underset{k=1}{\overset{N}{\otimes}} (t_k \tau_{j_k}(S_k) t'_k)$$

by (6.19)

$$\begin{aligned} &= \left(\underset{k=1}{\overset{N}{\otimes}} t_k I \right) \left(\underset{k=1}{\overset{N}{\otimes}} \tau(S_k) \right) \left(\underset{k=1}{\overset{N}{\otimes}} t'_k I \right) \\ &= T_1 F_W(S) T_2. \end{aligned}$$

Thus, under linearity, for all $T_1, T_2 \in \mathbb{C}^{\otimes N}$, and $S \in \mathfrak{L}_Q^{\otimes N}$, one obtains that

$$F_W\,(T_1 S T_2) = T_1 F_W(S) T_2 \text{ in } \mathbb{C}^{\otimes N}. \tag{6.21}$$

Take $T = \underset{k=1}{\overset{N}{\otimes}} T_k \in \mathfrak{L}_Q^{\otimes N}$. Then $T^* = \underset{k=1}{\overset{N}{\otimes}} T_k^*$ in $\mathfrak{L}_Q^{\otimes N}$. So, by (6.19),

$$F_W(T^*) = F_W(T)^*. \tag{6.22}$$

Therefore, the surjective bounded linear transformation F_W is a conditional expectation from $\mathfrak{L}_Q^{\otimes N}$ onto $\mathbb{C}^{\otimes N}$, by (6.20), (6.21) and (6.22). □

On the N-dimensional tensor product algebra $\mathbb{C}^{\otimes N}$, define a linear functional φ_N by a linear morphism satisfying that:

$$\varphi_N \left(\underset{k=1}{\overset{N}{\otimes}} t_k \right) \overset{def}{=} \prod_{k=1}^{N} t_k, \text{ for all } \underset{k=1}{\overset{N}{\otimes}} t_k \in \mathbb{C}^{\otimes N}. \tag{6.23}$$

Define now a linear functional φ_W^o on $\mathfrak{L}_Q^{\otimes N}$ by

$$\varphi_W^o \overset{def}{=} \varphi_N \circ F_W \text{ on } \mathfrak{L}_Q^{\otimes N}, \tag{6.24}$$

where φ_N is in the sense of (6.23) and F_W is the conditional expectation (6.19).

By the definition (6.24) of φ_W^o, it is a well-determined linear functional on $\mathfrak{L}_Q^{\otimes N}$ satisfying

$$\varphi_W^o\left(\underset{k=1}{\overset{N}{\otimes}} T_k\right) = \prod_{k=1}^{N} \tau_{j_k}\left(T_k\right). \tag{6.25}$$

Define an element I_{W,j_k} of $\mathfrak{L}_Q^{\otimes N}$ by

$$I_{W,j_k} \overset{def}{=} \left(1_{\mathfrak{L}} \otimes \hat{q_{j_k}}\right), \tag{6.26}$$

with

$$\hat{q_{j_k}} = 1_Q \otimes \ldots \otimes 1_Q \otimes \underset{k\text{-th}}{q_{j_k}} \otimes 1_Q \otimes \ldots \otimes 1_Q,$$

for all $j_k \in W$ in $\mathbb{Z}$, for $k = 1, \ldots, N$.

From the operators I_{W,j_k} of (6.26), define an operator I_W of $\mathfrak{L}_Q^{\otimes N}$ by

$$I_W = \sum_{k=1}^{N} I_{W,j_k} \tag{6.27}$$

Note that, if the chosen projections $q_{j_1}, \ldots, q_{j_N}$ are "mutually orthogonal" on Q, equivalently, if $W = (j_1, \ldots, j_N)$ is an N-tuple of "mutually distinct" integers in $\mathbb{Z}^N$, then one can get that

$$\begin{aligned} I_W^n &= \left(\sum_{k=1}^{N}\left(1_{\mathfrak{L}} \otimes \hat{q_{j_k}}\right)\right)^n = \sum_{k=1}^{N}\left(1_{\mathfrak{L}} \otimes \hat{q_{j_k}}\right)^n \\ &= \sum_{k=1}^{N}\left(1_{\mathfrak{L}}^n \otimes \left(\hat{q_{j_k}}\right)^n\right) = \sum_{k=1}^{N}\left(1_{\mathfrak{L}} \otimes \hat{q_{j_k}}\right) = I_W, \end{aligned} \tag{6.28}$$

for all $n \in \mathbb{N}$, by (6.26) and (6.27).

So, if I_W is in the sense of (6.27) in $\mathfrak{L}_Q^{\otimes N}$, then one can get that

$$\varphi_W^o\left(I_W^n\right) = \varphi_W^o\left(I_W\right) = \prod_{k=1}^{N} \tau_{j_k}\left(1_{\mathfrak{L}} \otimes q_{j_k}\right) = \prod_{k=1}^{N} \psi\left(q_{j_k}\right),$$

and

$$\varphi_W^o\left(I_{W,j_k}^n\right) = \varphi_W^o\left(I_{W,j_k}\right) = \prod_{k=1}^{N} \tau_{j_k}\left(1_{\mathfrak{L}} \otimes q_{j_k}\right) = \psi\left(q_{j_k}\right), \tag{6.29}$$

in $\mathbb{C}$, by (6.25) and (6.28), for all $n \in \mathbb{N}$, where φ_W^o is in the sense of (6.24).

For convenience, we denote the $\mathbb{C}$-quantity $\varphi_W^o\left(I_W\right)$ of (6.29), simply by ψ_W, below. Note that, since $N < \infty$, the quantity ψ_W converges in $\mathbb{C}$.

Now, define new linear functionals $\varphi_{W,k}$'s on $\mathfrak{L}_Q^{\otimes N}$ by

$$\varphi_{W,k} \overset{def}{=} \frac{\psi(q_{j_k})}{\psi_W}\varphi_W^o \text{ on } \mathfrak{L}_Q^{\otimes N}, \tag{6.30}$$

for all $k = 1, \ldots, N$, where φ_W^o is the linear functional (6.24) on $\mathfrak{L}_Q^{\otimes N}$ satisfying (6.25), and where ψ_W is the $\mathbb{C}$-quantity (6.29).

Then the linear functionals $\varphi_{W,k}$ of (6.30) are well-defined on $\mathfrak{L}_Q^{\otimes N}$, and hence, we obtain the tensor product Banach ∗-probability spaces $\mathfrak{L}_Q^{\otimes N}(k, W)$,

$$\mathfrak{L}_Q^{\otimes N}(k, W) \overset{def}{=} \left(\mathfrak{L}_Q^{\otimes N},\ \varphi_{W,k}\right), \tag{6.31}$$

where $\varphi_{W,k}$ are in the sense of (6.30), for all $k = 1, \ldots, N$.

On our tensor product Banach ∗-probability space $\mathfrak{L}_Q^{\otimes N}(k, W)$ of (6.31), one can obtain the following semicircular elements in $\mathfrak{L}_Q^{\otimes N}$.

Theorem 6.5. *Let $W = (j_1, \ldots, j_N)$ be an N-tuple of "mutually distinct" integers $j_1, \ldots, j_N$ in $\mathbb{Z}$, for $N \in \mathbb{N}$, and fix $k_0 \in \{1, \ldots, N\}$. For a fixed k_0, let $\mathfrak{L}_Q^{\otimes N}(k_0, W)$ be the corresponding Banach ∗-probability space* (6.31) *equipped with its linear functional $\varphi_{k_0,W}$ of* (6.30)*. Define the free random variables U_{W,k_0}, for some $k_0 \in \{1, \ldots, N\}$, by*

$$U_{W,k_0} \overset{def}{=} \left(\underset{k=1}{\overset{N}{\otimes}} S_{j_k}\right) \in \mathfrak{L}_Q^{\otimes N}(k_0, W) = (\mathfrak{L}_Q^{\otimes N}, \varphi_{k_0,W}), \tag{6.32}$$

where $\psi_W \in \mathbb{C}$ is in the sense of (6.29)*, and*

$$S_{j_k} = \begin{cases} u_{j_{k_0}} & \text{if } k = k_0, \\ I_{W,j_k} & \text{otherwise,} \end{cases}$$

for all $k = 1, \ldots, N$, where $u_{j_{k_0}} = \left(l \otimes q_{j_{k_0}}\right)$ is our $\psi(q_{j_{k_0}})^2$-semicircular element of the j_{k_0}-th filtered probability space $\mathfrak{L}_Q(j_{k_0})$ of Q, and I_{W,j_k} are in the sense of (6.26)*, for $k \neq k_0$ in $\{1, \ldots, N\}$. Then U_{W,k_0} of* (6.32) *is a $\psi(q_{j_{k_0}})^2$-semicircular element in $\mathfrak{L}_Q^{\otimes N}(W)$.*

Proof. By the very definition (6.32) of U_{W,k_0}, it is self-adjoint in $\mathfrak{L}_Q^{\otimes N}$. Indeed, the tensor factors I_{W,j_k} and $u_{j_{k_0}}$ are self-adjoint in $\mathfrak{L}_Q$, and hence, the operator U_{W,k_0} is self-adjoint in $\mathfrak{L}_Q^{\otimes N}$.

Observe now that

$$\varphi_{k_0,W}\left(U^n_{W,k_0}\right) = \varphi_{k_0,W}\left(\left(I^n_{W,j_1} \otimes \ldots \otimes I^n_{W,j_{k_0-1}} \otimes \underset{k_0\text{-th}}{u^n_{j_{k_0}}} \otimes I^n_{W,j_{k_0+1}} \otimes \ldots \otimes I^n_{W,j_N}\right)\right)$$

$$= \varphi_{k_0,W}\left(\left(I_{W,j_1} \otimes \ldots \otimes I_{W,j_{k_0-1}} \otimes \underset{k_0\text{-th}}{u^n_{j_{k_0}}} \otimes I_{W,j_{k_0+1}} \otimes \ldots \otimes I_{W,j_N}\right)\right)$$

by (6.27)

$$= \frac{\psi(q_{j_{k_0}})}{\psi_W}\left(\tau_{j_1}\left(I_{W,j_1}\right)\right)\ldots\left(\tau_{j_{k_0}}\left(u^n_{j_{k_0}}\right)\right)\ldots\left(\tau_{j_N}\left(I_{W,j_N}\right)\right)$$

by (6.25) and (6.30)

$$= \frac{\psi(q_{j_{k_0}})}{\psi_W}\left(\psi(q_{j_1})\right)\ldots\left(\omega_n\psi(q_{j_{k_0}})^n c_{\frac{n}{2}}\right)\ldots\left(\psi(q_{j_N})\right)$$

$$= \frac{\psi(q_{j_{k_0}})\left(\prod_{k=1}^{k_0-1}\psi(q_{j_k})\right)\left(\prod_{k=k_0+1}^{N}\psi(q_{j_k})\right)}{\psi_W}\left(\omega_n\ \psi(q_{j_{k_0}})^n\ c_{\frac{n}{2}}\right)$$

by (6.29)

$$= \omega_n\left(\psi(q_{j_{k_0}})\right)^n c_{\frac{n}{2}}, \tag{6.33}$$

for all $n \in \mathbb{N}$.

Therefore, by (6.33) and (3.3), the free random variable U_{W,k_0} is $\psi(q_{j_{k_0}})$-semicircular in $\mathfrak{L}_Q^{\otimes N}(k_0, W)$. □

The above theorem shows that, whenever W is a finite sequence of mutually distinct $|W|$-many integers of $\mathbb{Z}$, then the operators $U_{W,k}$ of (6.32) in $\mathfrak{L}_Q^{\otimes|W|}$ are $\psi(q_{j_k})^2$-semicircular in $\mathfrak{L}_Q^{\otimes|W|}(k,W)$, for all $k = 1, \ldots, |W|$.

By the weighted-semicircularity (6.33) of the operator $U_{W,k}$ of (6.17), one can obtain the following corollary.

Corollary 6.6. *Let $W = (j_1, \ldots, j_N)$ be the N-tuple of mutually distinct integers $j_1, \ldots, j_N$ in $\mathbb{Z}$, and let $\mathfrak{L}_Q^{\otimes N}(k,W)$ be the Banach $*$-probability space in the sense of* (6.31)*, for $k = 1, \ldots, N$. Define a free random variable $T_{W,k}$ by*

$$T_{W,k} \overset{def}{=} I_{W,j_1} \otimes \ldots \otimes I_{W,j_{k-1}} \otimes \underset{k\text{-}th}{U_{j_k}} \otimes I_{W,j_{k+1}} \otimes \ldots \otimes I_{W,j_N}, \tag{6.34}$$

where I_{W,j_k} are in the sense of (6.26)*, and*

$$U_{j_k} = \frac{1}{\psi(q_{j_k})}\left(l \otimes q_{j_k}\right)$$

is our semicircular element in $\mathfrak{L}_Q(j_k)$, for all $k = 1, \ldots, N$. Then the operator $T_{W,k}$ is semicircular in $\mathfrak{L}_Q^{\otimes N}(k,W)$, for $k = 1, \ldots, N$.

Proof. Similar to (6.33), one can get that

$$\varphi_{k,W}\left(T_{W,k}^{n}\right)=\omega_{n}c_{\frac{n}{2}},\ \text{ for all } n\in\mathbb{N}. \tag{6.35}$$

Thus, by (6.35) and (3.4), the free random variables $T_{W,k}$ are semicircular in $\mathfrak{L}_{Q}^{\otimes N}(k,W)$, for all $k=1,\ldots,N$. □

6.3. ON THE BANACH $*$-PROBABILITY SPACES $\mathfrak{L}_{Q}^{N}(j_{1},\ldots,j_{N})$

Let Q be the C^{*}-subalgebra of a fixed C^{*}-probability space $(A,\,\psi)$ generated by the family $\mathbf{Q}=\{q_{j}\}_{j\in\mathbb{Z}}$ of mutually orthogonal projections satisfying

$$\psi(q_{j})\in\mathbb{R}\text{ in }\mathbb{C},\ \text{ for all } j\in\mathbb{Z},$$

and let $\mathfrak{L}_{Q}$ be the radial projection algebra of Q, having the corresponding j-th filtered probability spaces $\mathfrak{L}_{Q}(j)=(\mathfrak{L}_{Q},\tau_{j})$, for all $j\in\mathbb{Z}$.

As in Sections 6.1 and 6.2, let $W=(j_{1},\ldots,j_{N})$ be the N-tuple of integers $j_{1},\ldots,j_{N}$ in $\mathbb{Z}$, for $N\in\mathbb{N}$. As in Section 6.1 (and different from the main results of Section 6.2), the integers $j_{1},\ldots,j_{N}$ are not necessarily distinct from each other in $\mathbb{Z}$.

For a fixed N-tuple W, define the *free product Banach $*$-probability space* $\mathfrak{L}_{Q}^{N}(W)$ by the Banach $*$-probability space,

$$\begin{aligned}\mathfrak{L}_{Q}^{N}(W)&\overset{denote}{=}\left(\mathfrak{L}_{Q}^{N},\ \tau^{W}\right)\overset{def}{=}\underset{k=1}{\overset{N}{\star}}\mathfrak{L}_{Q}(j_{k})\\&=\underset{k=1}{\overset{N}{\star}}(\mathfrak{L}_{Q},\ \tau_{j_{k}})=\left(\underset{k=1}{\overset{N}{\star}}\mathfrak{L}_{Q},\ \ \underset{k=1}{\overset{N}{\star}}\tau_{j_{k}}\right)\end{aligned} \tag{6.36}$$

in the sense of [14] and [16], where

$$\tau^{W}=\underset{k=1}{\overset{N}{\star}}\tau_{j_{k}}$$

is the *free product linear functional* on

$$\mathfrak{L}_{Q}^{N}=\underset{k=1}{\overset{N}{\star}}\mathfrak{L}_{Q}=\underbrace{\mathfrak{L}_{Q}\star\mathfrak{L}_{Q}\star\ldots\star\mathfrak{L}_{Q}}_{N\text{-times}}.$$

Note that the above free product Banach $*$-probability space $\mathfrak{L}_{Q}^{N}(W)$ of (6.36) is highly dictated by the choice of linear functionals $\{\tau_{j_{k}}\}_{k=1}^{N}$ (e.g., see [14] and [16]).

Now, define a set $\mathcal{N}$ by the family of all finite sequences in $\{1,\ldots,N\}$, i.e.,

$$\mathcal{N}\overset{def}{=}\underset{l=1}{\overset{\infty}{\sqcup}}\mathcal{N}(l), \tag{6.37}$$

with its partition blocks

$$\mathcal{N}(l)\overset{def}{=}\{(i_{1},\ldots,i_{l}):i_{j}\in\{1,\ldots,N\}\text{ for all }j=1,\ldots,l\},$$

for all $l\in\mathbb{N}$.

For each $l \in \mathbb{N}$, define the subsets $\mathcal{AN}(l)$ of $\mathcal{N}(l)$ by

$$\mathcal{AN}(l) \overset{def}{=} \left\{ w \in \mathcal{N}(l) \middle| \begin{array}{c} w \text{ is alternating in the sense that:} \\ \text{if } w = (i_1, \ldots, i_l), \text{ then} \\ i_1 \neq i_2, \ i_2 \neq i_3, \ldots, i_{l-1} \neq i_l \end{array} \right\}. \tag{6.38}$$

Suppose $N = 3$, and let

$$w_1 = (1, 2, 2, 3, 1, 1) \in \mathcal{N}(6) \text{ in } \mathcal{N},$$

and

$$w_2 = (1, 3, 1, 2, 1, 2) \in \mathcal{N}(6) \text{ in } \mathcal{N},$$

where $\mathcal{N}$ and $\mathcal{N}(6)$ are in the sense of (6.37). Then

$$w_1 \notin \mathcal{AN}(6), \text{ but } w_2 \in \mathcal{AN}(6),$$

in $\mathcal{N}$, where $\mathcal{AN}(6)$ is in the sense of (6.38).

Define now the subset $\mathcal{AN}$ of $\mathcal{N}$ by

$$\mathcal{AN} \overset{def}{=} \underset{l=1}{\overset{\infty}{\sqcup}} \mathcal{AN}(l), \text{ in } \mathcal{N}. \tag{6.39}$$

Now, let us take $w \in \mathcal{AN}(n)$ in $\mathcal{AN}$ of (6.39), and choose an operator T_w,

$$T_w = \underset{l \in w}{\Pi} T_{j_l} \in \mathfrak{L}_Q^N(W), \tag{6.40}$$

as a free "reduced" word in $\{T_{j_1}, \ldots, T_{j_N}\}$, where T_{j_k} are in the free blocks $\mathfrak{L}_Q(j_k)$ in $\mathfrak{L}_Q^N(W)$, for all $k = 1, \ldots, N$.

Remark 6.7. By [14] and [16], if $\mathfrak{L}_Q^N(W)$ is identified with $\left(\mathfrak{L}_Q^N, \tau^W\right)$ in the sense of (6.36), the free product Banach $*$-algebra $\mathfrak{L}_Q^N$ of $\mathfrak{L}_Q(j_1), \ldots, \mathfrak{L}_Q(j_N)$ has its Banach-space expression,

$$\mathfrak{L}_Q^N \overset{\text{Banach-Sp}}{=} \mathbb{C} \oplus \left(\underset{l=1}{\overset{\infty}{\oplus}} \underset{w \in \mathcal{AN}(l)}{\oplus} \mathfrak{L}^o(w) \right), \tag{6.41}$$

where

$$\mathfrak{L}^o(w) \overset{\text{Banach-Sp}}{=} \mathfrak{L}_Q^o(j_{i_1}) \otimes \mathfrak{L}_Q^o(j_{i_1}) \otimes \ldots \otimes \mathfrak{L}_Q^o(j_{i_l}),$$

whenever $w = (i_1, \ldots, i_l) \in \mathcal{AN}(l)$ in $\mathcal{AN}$, where

$$\mathfrak{L}_Q^o(j_{i_k}) \overset{\text{Banach-Sp}}{=} \mathfrak{L}_Q \ominus \mathbb{C}, \text{ for all } k = 1, \ldots, l.$$

Here, "$\overset{\text{Banach-Sp}}{=}$" means "being Banach-space isomorphic" and $\oplus$ is the direct product, $\otimes$ is the tensor product, and $\ominus$ is the complement of Banach spaces under topology, i.e., if an operator T is contained in $\mathfrak{L}^o(w)$, and if $w \in \mathcal{AN}$, then T is understood as the free reduced words in $\mathfrak{L}_Q^N$.

By (6.36), (6.38) and (6.39), one obtains the following result.

Proposition 6.8. *If T_w is in the sense of* (6.40) *in $\mathfrak{L}_Q^N(W)$, then*

$$\tau^W(T_w) = \prod_{l \in w} \tau_{j_l}(T_{j_l}), \tag{6.42}$$

whenever $w \in \mathcal{AN}$.

Proof. The free-moment formula (6.42) is proven by (6.36), (6.38) and (6.39) (See [14] and [16]). □

By the free-moment formula (6.42), one can get that, if

$$w = (i_1, \ldots, i_n) \in \mathcal{AN}(n) \text{ in } \mathcal{AN},$$

and if

$$M_w^{r_1,\ldots,r_n} \overset{def}{=} \underset{l \in w}{\Pi} U_{j_l}^{r_l} = \prod_{l=1}^{n} U_{j_{i_l}}^{r_l} \in \mathfrak{L}_Q^N(W), \tag{6.43}$$

for all $r_1, \ldots, r_n \in \mathbb{N}$, where $U_j = \frac{1}{\psi(q_j)}(l \otimes q_j)$ are our semicircular elements of $\mathfrak{L}_Q(j)$, for all $j \in \mathbb{Z}$, then one has

$$\tau^W(M_w^{r_1,\ldots,r_n}) = \prod_{l \in w} \tau_{j_l}(U_{j_l}^{r_l}) = \prod_{l=1}^{n} \tau_{j_{i_l}}\left(U_{j_{i_l}}^{r_l}\right),$$

by (6.42)

$$= \prod_{l=1}^{n} \omega_{r_l} c_{\frac{r_l}{2}} \tag{6.44}$$

by the semicircularity of U_{j_l}, for all $l = 1, \ldots, n$, where c_k are the k-th Catalan numbers for all $k \in \mathbb{N}$.

Theorem 6.9. *Let $w = (j_{i_1}, \ldots, j_{i_n}) \in \mathcal{AN}(n)$ in $\mathcal{AN}$, for some $n \in \mathbb{N}$, and let $M_w^{r_1,\ldots,r_n}$ be a free random variable* (6.43) *in $\mathfrak{L}_Q^N(W)$ of* (6.36)*. Then*

$$\tau^W(M_w^{r_1,\ldots,r_n}) = \omega_{r_1,\ldots,r_n} \prod_{l=1}^{n} c_{\frac{r_l}{2}}, \tag{6.45}$$

with

$$\omega_{r_1,\ldots,r_n} = \begin{cases} 1 & \text{if all } r_1, \ldots, r_n \text{ are even,} \\ 0 & \text{otherwise,} \end{cases}$$

for $r_1, \ldots, r_n \in \mathbb{N}$.

Proof. The free-moment formula (6.45) is proven by (6.44). □

Similar to the free-moment formulas (6.45) for a free random variable $M_w^{r_1,\ldots,r_n}$ of (6.43) in $\mathfrak{L}_Q^N(W)$, one can get the following result.

Corollary 6.10. *Let $w = (j_1, \ldots, j_n) \in \mathcal{AN}(n)$ in $\mathcal{AN}$, for some $n \in \mathbb{N}$. For a fixed $w \in \mathcal{AN}(n)$, and fixed quantities $r_1, \ldots, r_n \in \mathbb{N}$, define an operator $m_w^{r_1,\ldots,r_n} \in \mathfrak{L}_Q^N$ by*

$$m_w^{r_1,\ldots,r_n} \overset{def}{=} \prod_{l=1}^{n} u_{j_{i_l}}^{r_k} \in \mathfrak{L}_Q^N(W), \tag{6.46}$$

where $u_j = l \otimes q_j$ are our $\psi(q_j)^2$-semicircular elements of $(\mathfrak{L}_Q, \tau_j)$, for all $j \in \mathbb{Z}$. Then

$$\tau^W\left(m_w^{r_1,\ldots,r_n}\right) = \omega_{r_1,\ldots,r_n} \prod_{l=1}^{n} \left(\psi(q_{j_{i_l}})^{r_l} c_{\frac{r_l}{2}}\right), \tag{6.47}$$

for all $k \in \mathbb{N}$, where $\omega_{r_1,\ldots,r_n}$ is in the sense of (6.45)*.*

Proof. Let $m_w^{r_1,\ldots,r_n} \in \mathfrak{L}_Q^N(W)$ be in the sense of (6.46), for $w \in \mathcal{AN}(n)$, and $n, r_1, \ldots, r_n \in \mathbb{N}$. Observe that

$$\tau^W\left(m_w^{r_1,\ldots,r_n}\right) = \tau^W\left(\prod_{l=1}^{n} u_{j_{i_l}}^{r_l}\right) = \tau^W\left(\prod_{l=1}^{n} u_{j_{i_l}}^{r_l}\right)$$

by understanding $m_w^{r_1,\ldots,r_n}$ as a free reduced word in $\mathfrak{L}^o(w)$ of (6.41)

$$= \prod_{l=1}^{n} \tau_{j_{i_l}}\left(u_{j_{i_l}}^{r_l}\right) = \prod_{l=1}^{n} \omega_{r_l} \psi(q_{j_{i_l}})^{r_l} c_{\frac{r_l}{2}},$$

by the $\psi(q_{i_l})^2$-semicircularity of $q_{j_{i_l}}$. Thus, we obtain the free-distributional data (6.47). □

The above free-moment formulas (6.45) and (6.47) provide ways to compute free moments of certain free random variables of $\mathfrak{L}_Q^N(W)$ generated by our semicircular elements U_j, and weighted-semicircular elements u_j of the j-th filtered probability spaces $(\mathfrak{L}_Q, \tau_j)$, for $j \in \mathbb{Z}$.

Also, by (6.45) and (6.47), we obtain the following trivial, but interesting free-distributional data on $\mathfrak{L}_Q^N(W)$.

Theorem 6.11. *Let $\mathfrak{L}_Q^N(W) = \left(\mathfrak{L}_Q^N, \tau^W\right)$ be the Banach $*$-probability space* (6.36) *for a fixed N-tuple $W = (j_1, \ldots, j_N)$ in $\mathbb{Z}$. Let*

$$u_{j_k} = l \otimes q_{j_k} \in \mathfrak{L}_Q(j_k) \text{ in } \mathfrak{L}_Q^N(W), \text{ for all } k = 1, \ldots, N, \tag{6.48}$$

where $\mathfrak{L}_Q(j_1), \ldots, \mathfrak{L}_Q(j_N)$ are the free blocks of $\mathfrak{L}_Q^N(W)$. Then u_{j_k} are $\psi(q_{j_k})^2$-semicircular in $\mathfrak{L}_Q^N(W)$, for all $k = 1, \ldots, N$. Now, let

$$U_{j_k} = \frac{1}{\psi(q_{j_k})} u_{j_k} \in \mathfrak{L}_Q(j_k) \text{ in } \mathfrak{L}_Q^N(W), \text{ for all } k = 1, \ldots, N. \tag{6.49}$$

Then they are semicircular in $\mathfrak{L}_Q^N(W)$.

Proof. Let u_{j_k} be in the sense of (6.48) in $\mathfrak{L}_Q^N(W)$, for $k \in \{1, \ldots, N\}$. Then it forms a free reduced word with its length-1 in the free block $\mathfrak{L}_Q(j_k)$ of $\mathfrak{L}_Q^N(W)$, inducing free reduced words $u_{j_k}^n$ with their length-1 in $\mathfrak{L}_Q(j_k)$ in $\mathfrak{L}_Q^N(W)$, for all $n \in \mathbb{N}$. So, one obtains that

$$\tau^W\left(u_{j_k}^n\right) = \tau_{j_k}\left(u_{j_k}^n\right) = \omega_n\left(\psi(q_{j_k})^2\right)^{\frac{n}{2}} c_{\frac{n}{2}},$$

by (6.47), for all $n \in \mathbb{N}$, for all $k = 1, \ldots, N$.

Therefore, the operators u_{j_k} of (6.48) are $\psi(q_{j_k})^2$-semicircular in $\mathfrak{L}_Q^N(W)$, for all $k = 1, \ldots, N$.

Similarly, one obtains that, if U_{j_k} are in the sense of (6.49) in $\mathfrak{L}_Q^N(W)$, then

$$\tau^W\left(U_{j_k}^n\right) = \tau_{j_k}\left(U_{j_k}^n\right) = \omega_n c_{\frac{n}{2}},$$

by (6.45), for all $n \in \mathbb{N}$, for all $k = 1, \ldots, N$.

So, the operators U_{j_k} of (6.49) are semicircular in $\mathfrak{L}_Q^N(W)$, for all $k = 1, \ldots, N$. □

The above theorem shows that our weighted-semicircularity, and semicircularity in free blocks $\mathfrak{L}_Q(j)$, the j-th filtered probability spaces, for $j \in W$, are preserved by those in the free product Banach ∗-probability space $\mathfrak{L}_Q^{|W|}(W)$.

7. EXAMPLE: FROM ORTHOGONAL RANK-1 PROJECTIONS ON $l^2(\mathbb{Z})$

Let $H = l^2(\mathbb{Z})$ be the canonical l^2-*Hilbert space* with its *orthonormal basis* $\mathcal{B}_H = \{\xi_j\}_{j \in \mathbb{Z}}$ satisfying

$$\langle \xi_{j_1}, \xi_{j_2} \rangle_2 = \delta_{j_1, j_2}, \text{ for all } j_1, j_2 \in \mathbb{Z}, \tag{7.1}$$

where

$$\xi_j = \Big(\ldots, 0, \ldots, 0, \underset{j\text{-th}}{1}, 0, \ldots, 0, \ldots\Big), \text{ for all } j \in \mathbb{Z},$$

and where $\langle \cdot, \cdot \rangle_2$ is the *usual* l^2-*inner product on* H,

$$\langle (\ldots, t_{-1}, t_0, t_1, \ldots), (\ldots, s_{-1}, s_0, s_1, \ldots) \rangle_2 = \sum_{k=-\infty}^{\infty} t_k \overline{s_k}.$$

Then on the operator algebra $B(H)$, one can naturally define the *rank*-1 *projections* q_j by

$$q_j = \langle \cdot, \xi_j \rangle_2 \xi_j, \text{ for all } j \in \mathbb{Z}. \tag{7.2}$$

i.e., for any $h \in H$, one has

$$q_j(h) = \langle h, \xi_j \rangle_2 \xi_j, \text{ for all } j \in \mathbb{Z},$$

and hence, if $h = \sum_{n \in \mathbb{Z}} t_n \xi_n \in H$, with $t_n \in \mathbb{C}$, then

$$q_j(h) = \left\langle \sum_{n \in \mathbb{Z}} t_n \xi_n,\ \xi_j \right\rangle_2 \xi_j = t_j \xi_j,$$

by (7.2), for all $j \in \mathbb{Z}$.

Thus, one can get the family

$$\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$$

of mutually orthogonal projections q_j's of (7.2), and the corresponding C^*-subalgebra

$$Q = C^*(\mathbf{Q}) \tag{7.3}$$

of the C^*-algebra $B(H)$, which is $*$-isomorphic to $\mathbb{C}^{\oplus |\mathbb{Z}|}$.

Now, fix an arbitrary vector h_0 of H,

$$h_0 = \sum_{n \in \mathbb{Z}} t_n \xi_n \in H, \text{ with } t_n \in \mathbb{C}.$$

For the fixed vector h_0 of H, define now a linear functional ψ on $B(H)$ by

$$\psi(T) = \langle T(h_0), h_0 \rangle_2, \text{ for all } T \in B(H). \tag{7.4}$$

By the sesqui-linearity of the inner product $\langle \cdot, \cdot \rangle_2$, the morphism ψ of (7.4) is indeed a linear functional on $B(H)$, i.e., $(B(H),\ \psi)$ forms a well-defined C^*-probability space.

Observe that the linear functional ψ of (7.4) on $B(H)$ satisfies that

$$\begin{aligned}\psi(q_j) &= \langle q_j(h_0),\ h_0 \rangle_2 = \langle t_j \xi_j,\ h_0 \rangle_2 \\ &= t_j \langle \xi_j,\ h_0 \rangle_2 = t_j^2,\end{aligned}$$

by (7.1) and (7.2), i.e.,

$$\psi(q_j) = t_j^2, \text{ for all } j \in \mathbb{Z}, \tag{7.5}$$

where t_j is the *j-th coefficient* of a fixed vector h_0 of (7.3).

Clearly, one can restrict the linear functional ψ on $B(H)$ to that on the C^*-subalgebra Q of $B(H)$, in the sense of (7.3), also denoted by ψ. Then, for any

$$T = \sum_{j \in \mathbb{Z}} s_j q_j \in Q, \text{ with } s_j \in \mathbb{C},$$

one has

$$\psi(T) = \sum_{j \in \mathbb{Z}} s_j t_j^2 \text{ in } \mathbb{C},$$

by (7.5).

So, one can determine the family $\{\psi_j\}_{j \in \mathbb{Z}}$ of linear functionals providing sectionized free probability on Q, where they are linear functional satisfying that

$$\psi_j \left(\sum_{n \in \mathbb{Z}} s_n q_n \right) = \psi(s_j q_j) = s_j \psi(q_j) = s_j t_j^2, \tag{7.6}$$

by (7.5), for all $j \in \mathbb{Z}$, i.e., we obtain the following sectionized C^*-probability spaces,

$$\{(Q, \psi_j) : j \in \mathbb{Z}\},$$

for the linear functionals ψ_j of (7.6).

Now, construct the radial projection algebra $\mathfrak{L}_Q$,

$$\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q \overset{def}{=} \overline{\mathbb{C}[\{l\}]}^{B(Q)} \otimes_{\mathbb{C}} Q \tag{7.7}$$

of Q, by defining the radial operator l on Q by the linear transformation satisfying

$$l\,(q_j) = c\,(q_j) + a\,(q_j) = q_{j+1} + q_{j-1},$$

for all $j \in \mathbb{Z}$, where c and a are the creation, respectively, the annihilation on Q.

On the Banach ∗-algebra $\mathfrak{L}_Q$ of (7.7), define the sectionized linear functionals φ_j by linear morphisms satisfying that

$$\varphi_j\left((l \otimes q_i)^n\right) = \varphi_j\left(l^n \otimes q_i\right) = \psi_j\left(l^n(q_i)\right), \tag{7.8}$$

for all $i, j \in \mathbb{Z}$, for all $n \in \mathbb{N}$, where ψ_j are in the sense of (7.6).

Note that, by (7.6), one has that

$$\begin{aligned}
\psi_j\left(l^n(q_i)\right) &= \psi_j\left(\sum_{k=0}^{n} \binom{n}{k} c^k a^{n-k}(q_i)\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \psi_j\left(q_{i+k-(n-k)}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \psi_j\left(q_{i+2k-n}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\delta_{j,i+2k-n}\ t_j^2\right),
\end{aligned}$$

with help of (7.5), for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$.

Define now a new linear functional τ_j on the radial projection algebra $\mathfrak{L}_Q$ of (7.7) by a linear morphism satisfying

$$\tau_j\left((l \otimes q_j)^n\right) = \varphi_j\left(\delta_{j,i}\left(\frac{\psi\,(q_j)^{n-1}}{\left([\frac{n}{2}]+1\right)}\,(l \otimes q_j)^n\right)\right), \tag{7.9}$$

for all $n \in \mathbb{N}$, for all $j \in \mathbb{Z}$, where φ_j are in the sense of (7.8).

Then, by (4.18), if

$$u_j = l \otimes q_j \in \left(\mathfrak{L}_Q, \tau_j\right),$$

then we obtain that

$$\tau_j\left(u_j^n\right) = \omega_n \psi(q_j)^n c_{\frac{n}{2}} = \omega_n t_j^{2n} c_{\frac{n}{2}}, \tag{7.10}$$

by (7.5) and (7.6), for all $n \in \mathbb{N}$, where $\omega_n = 1$, if n is even, and $\omega_n = 0$, if n is odd in $\mathbb{N}$, for all $j \in \mathbb{Z}$. Recall again that t_j are the j-th entry of the fixed vector h_0 of (7.3) in $H = l^2(\mathbb{Z})$.

Observation 7.1. *Let $u_j = l \otimes q_j$ be a free random variable of the Banach $*$-probability space $(\mathfrak{L}_Q, \tau_j)$, where τ_j are in the sense of* (7.9)*, for all $j \in \mathbb{Z}$. If the j-th entry t_j of the vector h_0 of* (7.3) *is nonzero, then u_j is t_j^4-semicircular in $(\mathfrak{L}_Q, \tau_j)$, for all $j \in \mathbb{Z}$, by* (7.10) *and* (3.3)*. By the Möbius inversion of* [14]*, one obtains the equivalent free-distributional data of* (7.10) *as follows: if $t_j \neq 0$ in h_0, then*

$$k_n^{(j)}(u_j, \ldots, u_j) = \begin{cases} t_j^4 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7.11}$$

for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$.

Therefore, by Observation 7.1, we immediately get the following result.

Observation 7.2. *Let $u_j = l \otimes q_j$ in $\mathfrak{L}_Q$, and let $U_j = t_j^{-2}\, u_j \in \mathfrak{L}_Q$, for some $j \in \mathbb{Z}$, where the j-th coefficients t_j are the "nonzero" in the fixed vector h_0 of* (7.3) *in H. Then U_j is semicircular in $(\mathfrak{L}_Q,\ \tau_j)$, for such $j \in \mathbb{Z}$, by Section 5.*

The above Observations 7.1 and 7.2, we obtain the weighted-semicircularity and corresponding semicircularity on $\mathfrak{L}_Q$, for suitable $j \in \mathbb{Z}$.

From below, let us assume $t_j \neq 0$, as the j-th entry of a fixed Hilbert-space vector h_0 of (7.3), and let

$$u_j = l \otimes q_j, \text{ and } U_j = \frac{1}{\psi(q_j)} u_j = \frac{1}{t_j^2} u_j \text{ in } \mathfrak{L}_Q,$$

for such $j \in \mathbb{Z}$.

For the fixed vector $h_0 \in H = l^2(\mathbb{Z})$ of (7.3), define a subset $Supp(h_0)$ of $\mathbb{Z}$ by

$$Supp(h_0) \overset{def}{=} \{j \in \mathbb{Z} : t_j \neq 0 \text{ in } h_0\}. \tag{7.12}$$

We call this set $Supp(h_0)$ of (7.12), the *support of* h_0.

Observation 7.3. *Let $h_0 = \sum_{n \in \mathbb{Z}} t_n \xi_n \in H$ be in the sense of* (7.3)*, and let $Supp(h_0)$ be the support* (7.12) *of h_0 in $\mathbb{Z}$. If $j \in Supp(h_0)$ in $\mathbb{Z}$, then one has a t_j^4-semicircular element u_j, and a semicircular element U_j in the j-th filtered probability space $\mathfrak{L}_Q(j)$.*

Now, take a finite sequence $W = (j_1, \ldots, j_N)$ of integers $j_1, \ldots, j_N$ in $Supp(h_0)$, for some $N \in \mathbb{N}$. Then, by Observation 7.3, one can take corresponding weighted-semicircular elements $u_{j_1}, \ldots, u_{j_N}$, and semicircular elements $U_{j_1}, \ldots, U_{j_N}$ (which are not necessarily distinct from each other) in $\mathfrak{L}_Q$. Construct the direct product Banach $*$-algebra $\mathfrak{L}_Q^{\oplus N}$ generated by ($|W| = N$)-copies of $\mathfrak{L}_Q$.

On $\mathfrak{L}_Q^{\oplus N}$, define a linear functional τ_W by a linear morphism satisfying that

$$\tau_W \left(\underset{k=1}{\overset{N}{\oplus}} T_k \right) = \frac{1}{N} \sum_{k=1}^{N} \tau_{j_k}(T_k), \tag{7.13}$$

for all $\underset{k=1}{\overset{N}{\oplus}} T_k \in \mathfrak{L}_Q^{\oplus N}$, as in (6.9).

Then we obtain a well-defined Banach ∗-probability space,

$$\mathfrak{L}_Q^{\oplus N}(W) = \left(\mathfrak{L}_Q^{\oplus N}, \tau_W\right), \tag{7.14}$$

where τ_W is in the sense of (7.13).

Observation 7.4. *For a fixed finite sequence W in Supp(h_0), define a free random variable T_W of the Banach ∗-probability space $\mathfrak{L}_Q^{\oplus|W|}(W)$ of (7.14) by*

$$T_W \overset{def}{=} \frac{1}{|W|}\left(\underset{j\in W}{\oplus} U_j\right) \in \mathfrak{L}_Q^{\oplus|W|}(W), \tag{7.15}$$

where U_j are our semicircular elements of $\mathfrak{L}_Q(j)$, for $j \in W$. Then the free random variable T_W of (7.15) is $\frac{1}{|W|^2}$-semicircular in $\mathfrak{L}_Q^{\oplus|W|}(W)$, by (6.12).

Now, for a fixed finite sequence $W = (j_1, \ldots, j_N)$ of "mutually distinct" integers $j_1, \ldots, j_N$ in *Supp*(h_0), define the tensor product Banach ∗-algebra $\mathfrak{L}_Q^{\otimes N}$ generated by N-copies of $\mathfrak{L}_Q$. On this Banach ∗-algebra $\mathfrak{L}_Q^{\otimes N}$, define linear functionals $\{\varphi_{W,k}\}_{k=1}^N$ by the linear morphisms satisfying

$$\varphi_{W,k}\left(\overset{N}{\underset{k=1}{\otimes}} T_k\right) = t_{j_k}^2\left(\prod_{l=1}^N \frac{1}{t_{j_l}^2}\tau_{j_l}(T_l)\right), \tag{7.16}$$

for all $k = 1, \ldots, N$, as in (6.31).

Then one can get the family

$$\left\{\mathfrak{L}_Q^{\otimes N}(W,k) \overset{denote}{=} \left(\mathfrak{L}_Q^{\otimes N},\ \varphi_{W,k}\right)\right\}_{k=1}^N \tag{7.17}$$

of Banach ∗-probability spaces, where $\varphi_{W,k}$ are in the sense of (7.16).

Observation 7.5. *For a fixed finite sequence $W = (j_1, \ldots, j_N)$ of mutually distinct integers in Supp(h_0), one can construct the family (7.17) of Banach ∗-probability spaces $\mathfrak{L}_Q^{\otimes|W|}(W,\ k)$, for $k = 1,\ \ldots,\ N$. For a fixed $k \in \{1, \ldots, N\}$, define a free random variable $T_{W,k}$ of $\mathfrak{L}_Q^{\otimes|W|}(W,k)$ by*

$$T_{W,k} = I_{W,j_1} \otimes \ldots \otimes I_{W,j_{k=1}} \otimes \underset{k\text{-}th}{u_{j_k}} \otimes I_{W,j_{k+1}} \otimes \ldots \otimes I_{W,j_N}, \tag{7.18}$$

where u_{j_k} is our $t_{j_2}^2$-semicircular element of $(\mathfrak{L}_Q,\ \tau_{j_k})$, and

$$I_{W,j_k} = 1_{\mathfrak{L}} \otimes \left(\hat{q_{j_k}}\right) \in \mathfrak{L}_Q,\ \textit{for}\ k = 1, \ldots, N.$$

Then the operator $T_{W,k}$ of (7.18) is $t_{j_k}^4$-semicircular in $\mathfrak{L}_Q^{\otimes|W|}(W,k)$ by (6.33), for all $k = 1, \ldots, N$.

Observation 7.6. *Under the same conditions of Observation 7.5, if we take a free random variable*

$$S_{W,k} = I_{W,j_1} \otimes \ldots \otimes I_{W,j_{k-1}} \otimes \underset{k\text{-}th}{U_{j_k}} \otimes I_{W,j_{k+1}} \otimes \ldots \otimes I_{W,j_N},$$

in $\mathfrak{L}_Q^{\otimes N}(W,k)$, *for* $k \in \{1, \ldots, N\}$, *where* U_{j_k} *is our semicircular element of* $(\mathfrak{L}_Q, \tau_{j_k})$, *then* $S_{W,k}$ *is semicircular in* $\mathfrak{L}_Q^{\otimes N}(W,k)$, *for* $k = 1, \ldots, N$.

Now, let $W = (j_1, \ldots, j_N)$ be an N-tuple of integers in $Supp(h_0)$, and let

$$\mathfrak{L}_Q(j_k) = (\mathfrak{L}_Q,\ \tau_{j_k}),\ \text{for } k = 1, \ldots, N,$$

be the corresponding j_k-filtered probability spaces of Q. Construct the free product Banach $*$-probability space,

$$\mathfrak{L}_Q^N(W) = \left(\mathfrak{L}_Q^N, \tau^W\right) = \left(\underset{k=1}{\overset{N}{\star}} \mathfrak{L}_Q, \underset{k=1}{\overset{N}{\star}} \tau_{j_k}\right). \tag{7.19}$$

Observation 7.7. *Let* W *be a finite sequence of integers in* $Supp(h_0)$, *and let* $\mathfrak{L}_Q^{|W|}(W)$ *be in the sense of* (7.19). *Then the free reduced words* u_k *and* U_k *with their length-1 in the free blocks* $\mathfrak{L}_Q(j_k)$ *are* t_k^4*-semicircular, respectively, semicircular in* $\mathfrak{L}_Q^{|W|}(W)$, *for all* $k = 1, \ldots, |W|$, *by* (6.48) *and* (6.49).

REFERENCES

[1] M. Brannan, K. Kirkpatrick, *Quantum groups and generalized circular elements*, Pacific J. Math. **282** (2016) 1, 35–61.

[2] I. Cho, *On dynamical systems induced by p-adic number fields*, Opuscula Math. **35** (2015) 4, 445–484.

[3] I. Cho, *p-adic free stochastic integrals for p-adic weighted-semicircular motions determined by primes p*, Libertas Math. (New Series) **36** (2016) 2, 65–110.

[4] I. Cho, *Free semicircular families in free product Banach ∗-algebras induced by p-adic number fields*, Complex Anal. Oper. Theory **11** (2017) 3, 507–565.

[5] I. Cho, P.E.T. Jorgensen, *Semicircular elements induced by p-adic number fields*, Opuscula Math. **37** (2017) 5, 665–703.

[6] A. Deya, R. Schott, *On the rough-paths approach to noncommutative stochastic calculus*, J. Funct. Anal. **265** (2013) 4, 594–628.

[7] T. Gillespie, *Prime number theorems for Rankin-Selberg L-functions over number fields*, Sci. China Math. **54** (2011) 1, 35–46.

[8] I. Gomez, M. Caetagnino, *A quantum version of spectral decomposition theorem of dynamical systems*, Quantum Chaos Hierarchy: Ergodic, Mixing and Exact, Chaos Solitons Fractals **70** (2015), 99–116.

[9] N.A. Ivanov, T. Omland, *C^*-simplicity of free products with amalgamation and radical classes of groups*, J. Funct. Anal. **272** (2017) 9, 3712–3741.

[10] Y. Liu, Y. Yu, *On an extension of Stevic-Sharma operator from the general space to weighted-type spaces on the unit ball*, Complex Anal. Oper. Theory **11** (2017) 2, 261–288.

[11] M.A. Navarro, S. Villegas, *Semi-stable radial solutions of p-Laplace equations in* $\mathbb{R}^N$, Nonlinear Anal. **149** (2017), 111–116.

[12] A. Nica, *Free probability aspect of irreducible meandric systems, and some related observations about meanders*, Infinite Dimes. Anal. Quantum Probab. Relat. Top. **19** (2016) 2, 1650011, 22 pp.

[13] F. Radulescu, *Random matrices, amalgamated free products and subfactors of the C^*-algebra of a free group of nonsingular index*, Invent. Math. **115** (1994), 347–389.

[14] R. Speicher, *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, Amer. Math. Soc. Mem. **132** (1998) 627.

[15] V.S. Vladimirov, I.V. Volovich, E.I. Zelenov, *p-adic analysis and mathematical physics*, Ser. Soviet & East European Math., vol. 1, World Scientific, 1994.

[16] D. Voiculescu, K. Dykemma, A. Nica, *Free random variables*, CRM Monograph Series, vol. 1, Amer. Math. Soc., Providence, 1992.

[17] Y. Yin, Z. Bai, J. Hu, *On the semicircular law of large-dimensional random quaternion matrices*, J. Theor. Probab. **29** (2016) 3, 1100–1120.

Ilwoo Cho
choilwoo@sau.edu

Saint Ambrose University
Department of Mathematics and Statistics
421 Ambrose Hall, 518 W. Locust St.
Davenport, Iowa, 52803, USA

Palle E.T. Jorgensen
palle-jorgensen@uiowa.edu

The University of Iowa
Department of Mathematics
Iowa City, IA 52242-1419, USA

Received: June 12, 2017.
Accepted: August 21, 2017.