ON SPECTRA OF QUADRATIC OPERATOR PENCILS WITH RANK ONE GYROSCOPIC LINEAR PART

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Abstract. The spectrum of a selfadjoint quadratic operator pencil of the form $\lambda^2 M - \lambda G - A$ is investigated where $M \geq 0$, $G \geq 0$ are bounded operators and A is selfadjoint bounded below is investigated. It is shown that in the case of rank one operator G the eigenvalues of such a pencil are of two types. The eigenvalues of one of these types are independent of the operator G. Location of the eigenvalues of both types is described. Examples for the case of the Sturm-Liouville operators A are given.

Keywords: quadratic operator pencil, gyroscopic force, eigenvalues, algebraic multiplicity.

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1. INTRODUCTION

Quadratic operator pencils of the form $L(\lambda) = \lambda^2 M - \lambda G - A$ with a selfadjoint operator A bounded below describing potential energy, a bounded symmetric operator $M \geq 0$ describing inertia of the system and an operator G bounded or subordinate to A occur in different physical problems where, in most of cases, they have spectra consisting of normal eigenvalues (see Definition 2.2). Usually the operator G is symmetric (see, e.g. [7, 20] and Chapter 4 in [13]) or antisymmetric (see [18] and Chapter 2 in [13]). In the first case G describes gyroscopic effect while in the latter case damping forces.

The problems in which gyroscopic forces occur can be found in [1,2,10–12,14,21,23]. The spectra of problems described in these papers may contain complex eigenvalues what leads to instability of the corresponding dynamical system. The gyroscopic stabilization in terms of spectral theory means, roughly speaking, that the spectrum

of an operator pencil $L(\lambda) = \lambda^2 M - \lambda G - A$ with selfadjoint A and symmetric M and G can be real while the spectrum of $L_0(\lambda) = \lambda^2 M - A$ contains pure imaginary eigenvalues. A sufficient condition for the pencil L to have only real eigenvalues is the overdamping condition:

$$(Gy, y)^2 + 4(My, y)(Ay, y) \geq 0 \qquad (1.1)$$

here and in what follows $(\cdot, \cdot)$ stands for the inner product in the corresponding Hilbert space. In [8] pencils which satisfy (1.1) are called hyperbolic.

In the present paper we consider the case of symmetric operator $G \geq 0$, i.e. $(Gy, y) \geq 0$ for all $y \in D(G)$. As far as we know location of complex eigenvalues of the corresponding operator pencil was not considered before. Up to our knowledge, also the case of rank one operator G in the pencil $L(\lambda)$ was not considered.

In Section 2 we obtain some general results on location of eigenvalues of the pencil L.

In Section 3 we consider the case of rank one operator G (problems with such operator G can be found e.g. in [6]). We show that the real eigenvalues are in certain sense interlaced. In case of $A \gg 0$ (i.e. $A \geq \epsilon I$ for some $\epsilon > 0$) this interlacing is equivalent to the "self-interlacing" described in [22] for finite dimensional (matrix) problems.

It should be mentioned that the spectra of problems described by the pencil $\lambda^2 M - i\lambda K - A$ with selfadjoint $K \geq 0$, $M \geq 0$ (i.e. $(Ky, y) \geq 0$, $(My, y) \geq 0$ for all $y \in D(K)$ and $y \in D(M)$, respectively) and selfadjoint A bounded below can be described also as sets of zeros of the shifted generalized Hermite-Biehler functions of the form $\omega(\lambda) = P(\lambda^2) + i\lambda Q(\lambda^2)$ where $P(\lambda^2)$, $\lambda Q(\lambda^2)$ (or $\lambda P(\lambda^2)$, $\lambda^2 Q(\lambda^2)$) are sine-type functions and $\frac{Q(z)}{P(z)}$ is essentially positive Nevanlinna function (see [19] or [13]).

Similarly, the spectra of problems described by the pencil $\lambda^2 M - \lambda G - A$ with selfadjoint $G \geq 0$, $M \geq 0$ and selfadjoint A bounded below can be described also as sets of zeros of the functions of the form $\omega(\lambda) = P(\lambda^2) + \lambda Q(\lambda^2)$ where $\frac{Q(z)}{P(z)}$ is again essentially positive Nevanlinna function (see the function ω in (4.11) in Section 4 as an example).

In Section 4 the results obtained in Section 3 are applied to a Sturm-Liouville boundary value problems with a boundary condition linear in the spectral parameter. Our results give necessary conditions (Theorem 4.1) for a sequence of numbers to be the spectrum of such a problem (problem (4.1)–(4.3)). In our future publication we will show that these conditions are also sufficient and give a method of recovering the potential of the Sturm-Liouville equation. We will also show that it is enough to know one spectrum of such a boundary value problem (problem (4.1)–(4.3) to recover the potential of the Sturm-Liouville equation).

We give also in Section 4 an example of a boundary value problem the spectrum of which consists of two subsequences one of which is symmetric with respect to the real and to the imaginary axes and independent of G.

2. ABSTRACT RESULTS

Let us denote by $B(H)$ the set of bounded closed operators acting on a separable Hilbert space H. We deal with the following quadratic operator pencil

$$L(\lambda) = \lambda^2 M - \lambda G - A,$$

where $M \in B(H)$, $G \in B(H)$ and A is a selfadjoint operator on H. Since $D(M) = D(G) = H$, the domain of the pencil is chosen as usually:

$$D(L(\lambda)) = D(M) \cap D(G) \cap D(A) = D(A).$$

Thus, it is independent of λ.

In what follows we assume the following condition to be satisfied:

Condition I. $M \in B(H)$, $M \geq 0$, $G \in B(H)$ and $G \geq 0$, $A = A^* \geq -\beta I$ (β is a positive number); for some $\beta_1 > \beta$ there exists $(A + \beta_1 I)^{-1} \in S_\infty$, where by S_∞ we denote the set of compact operators on H; $\ker A \cap \ker G \cap \ker M = \{0\}$.

Definition 2.1. The set of values $\lambda \in \mathbb{C}$ such that $L^{-1}(\lambda) := L(\lambda)^{-1}$ exists in $B(H)$ is said to be the resolvent set $\rho(L)$ of L. The spectrum of the pencil L is denoted by $\sigma(L)$, i.e. $\sigma(L) = \mathbb{C} \backslash \rho(L)$.

Definition 2.2 (see, e.g. [13], Definition 1.1.3)**.**

1. A number $\lambda_0 \in \mathbb{C}$ is said to be an eigenvalue of the pencil L if there exists a vector $y_0 \in D(A)$ (called an eigenvector of L, corresponding to λ_0) such that $y_0 \neq 0$ and $L(\lambda_0)y_0 = 0$. Vectors $y_1, \ldots, y_{m-1}$ are called associated to y_0 if

$$\sum_{s=0}^{k} \frac{1}{s!} \left. \frac{d^s L(\lambda)}{d\lambda^s} \right|_{\lambda=\lambda_0} y_{k-s} = 0, \quad (k = 1, 2, \ldots, m-1). \tag{2.1}$$

The number m is said to be the length of the chain composed of the eigen- and associated vectors $y_0, y_1, \ldots, y_{m-1}$.

2. The geometric multiplicity of an eigenvalue is defined to be the number of the corresponding linearly independent eigenvectors. The algebraic multiplicity of an eigenvalue is defined to be the greatest value of the sum of the lengths of chains corresponding to linearly independent eigenvectors. An eigenvalue is called semisimple if its algebraic multiplicity equals its geometric multiplicity. An eigenvalue is called simple if its algebraic multiplicity is 1.

3. An eigenvalue λ_0 is said to be isolated if it has some deleted neighborhood contained in the resolvent set $\rho(L)$. An isolated eigenvalue λ_0 of finite algebraic multiplicity is said to be normal.

In the case of linear monic operator pencils $\lambda I - A$ with bounded operator A this definition of a normal eigenvalue coincides with the corresponding definition in [4] (Chapter I, Paragraph 2) for operators.

Lemma 2.3.

1. *The operator pencil L is Fredholm valued with index 0.*
2. $L(\lambda)^* = L(\bar{\lambda})$ *for all* $\lambda \in C$. *In particular, the spectrum of L is symmetric with respect to the real axis.*

The proof of this lemma is the same as the proof of Lemma 1.1.11 in [13].

Lemma 2.4.

1. *The spectrum of the pencil L consists of normal eigenvalues.*
2. *All (if any) the nonreal eigenvalues of L are located in the closed right half-plane.*
3. *If* $G > 0$, *i.e.* $(Gy, y) > 0$ *for all* $y \in D(G)$, $y \neq 0$, *then all (if any) nonreal eigenvalues of L are located in the open right half-plane.*
4. *If* $A \geq 0$, *then the spectrum* $\sigma(L)$ *of L (if not empty) is located on the real axis.*
5. *If* $A >> 0$, *i.e.* $A \geq \epsilon I$, $\epsilon > 0$, *then* $\sigma(L) \subset \mathbb{R} \backslash \{0\}$.

Proof. Since the pencil L is an analytic Fredholm operator valued function, its spectrum consists of eigenvalues of finite algebraic multiplicity and either $\sigma(L) = \mathbb{C}$ or all the eigenvalues are normal, see e.g., [3], Chapter XI, Corollary 8.4.

Let y_0 be an eigenvector corresponding to the eigenvalue λ_0. Then

$$(L(\lambda_0)y_0, y_0) = 0, \quad (y_0 \neq 0),$$

where $(\cdot, \cdot)$ is the inner product in H and consequently,

$$((\text{Re}\lambda_0)^2 - (\text{Im}\lambda_0)^2)(My_0, y_0) - \text{Re}\lambda_0(Gy_0, y_0) - (Ay_0, y_0) = 0 \qquad (2.2)$$

and

$$\text{Im}\lambda_0(2\text{Re}\lambda_0(My_0, y_0) - (Gy_0, y_0)) = 0. \qquad (2.3)$$

If $\text{Im}\lambda_0 \neq 0$, then in view of Condition I the inequality $\text{Re}\lambda_0 \geq 0$ follows from (2.3). Hence $\sigma(L) = \mathbb{C}$ is impossible and assertions 1 and 2 are proved.

If $G > 0$ and $\text{Im}\lambda_0 \neq 0$, then (2.3) implies $\text{Re}\lambda_0 \neq 0$ and therefore $\text{Re}\lambda_0 > 0$. Assertion 3 is proved.

Now let $A \geq 0$. If $\text{Im}\lambda_0 \neq 0$ then multiplying (2.3) by $\frac{\text{Re}\lambda_0}{\text{Im}\lambda_0}$ we subtract the resulting equation from (2.2) and obtain

$$-((\text{Re}\lambda_0)^2 + (\text{Im}\lambda_0)^2)(My_0, y_0) - (Ay_0, y_0) = 0.$$

This implies $(My_0, y_0) = (Ay_0, y_0) = 0$. Since the operators M and A are selfadjoint we obtain $My_0 = Ay_0 = 0$, and consequently $Gy_0 = 0$. This contradicts Condition I. Thus, statement 4 is proved.

If $A >> 0$ then in addition to the previous arguments we have $0 \notin \sigma(L)$. That means statement 5 is valid. $\qquad \square$

Lemma 2.5.

1. *All the eigenvalues of L located on* $(-\infty, 0)$ *are semisimple, i.e. they do not possess associated vectors.*
2. *If* $G > 0$ *on* $\ker A$, *then all the eigenvalues of L located on* $(-\infty, 0]$ *are semisimple.*

Proof. 1. Let λ_0 be an eigenvalue of L located on $(-\infty, 0)$. Let us denote by y_0 (one of) the corresponding eigen- and by y_1 the first associated vector of the chain. By Definition 2.2

$$\lambda_0^2 My_1 - \lambda_0 Gy_1 - Ay_1 + 2\lambda_0 My_0 - Gy_0 = 0. \tag{2.4}$$

Multiplying (2.4) by y_0 we obtain

$$((\lambda_0^2 - \lambda_0 G - A)y_1, y_0) + ((2\lambda_0 M - G)y_0, y_0) = 0. \tag{2.5}$$

Taking into account that λ_0 is real we obtain from (2.5):

$$(y_1, (\lambda_0^2 M - \lambda_0 G - A)y_0) + ((2\lambda_0 M - G)y_0, y_0) = 0,$$

what means

$$((2\lambda_0 M - G)y_0, y_0) = 0. \tag{2.6}$$

Equation (2.6) is possible for $\lambda_0 < 0$ only if $(My_0, y_0) = (Gy_0, y_0) = 0$. Since M and G are selfadjoint we arrive at $My_0 = Gy_0 = 0$ and, consequently, $L(\lambda_0)y_0 = -Ay_0 = 0$. Thus, $y_0 \in \ker M \cap \ker G \cap \ker A$. Then due to Conditions I we have $y_0 = 0$, a contradiction.

2. Let $G > 0$ on $ker A$ and let $y_0 \neq 0$ be an eigenvector corresponding to the eigenvalue $\lambda_0 = 0$ and therefore $y_0 \in ker A$. Then (2.6) implies $(Gy_0, y_0) = 0$, a contradiction. $\square$

Lemma 2.6. *If $M >> 0$ (i.e. $(My, y) \geq m\|y\|^2$, $m > 0$) then all (if any) the algebraically nonsimple real eigenvalues lie on the interval $[0, \frac{1}{2m}\|G^{\frac{1}{2}}\|^2]$.*

Proof. This result follows immediately from (2.6). $\square$

Lemma 2.7. *If $M + G \geq \epsilon I$ ($\epsilon > 0$), $dim ker A > 0$ and $dim(ker A \cap ker G) = p \geq 0$, then the algebraic multiplicity of $\lambda = 0$ as an eigenvalue of L is equal to $p + dim \, ker A$.*

Proof. Let $0 \neq y_0 \in \ker A$ and let y_1 be the first associated vector of the chain. Then

$$\left.\frac{dL(\lambda)}{d\lambda}\right|_{\lambda=0} y_0 + L(0)y_1 = -Gy_0 - Ay_1 = 0. \tag{2.7}$$

If $y_0 \in \ker G$ then y_1 can be chosen equal to 0. If $y_0 \notin \ker G$, then (2.7) implies

$$-(Gy_0, y_0) - (Ay_1, y_0) = -(Gy_0, y_0) - (y_1, Ay_0) = -(Gy_0, y_0) = 0.$$

Combining the last equation with the condition $G \geq 0$ we obtain $Gy_0 = 0$, a contradiction. It remains to prove that in case of $y_0 \in ker G$, $y_1 = 0$ the third vector of the chain (see Definition 2.2) does not exist. Suppose it does exist and denote it by y_2. Then, see (2.1),

$$-Ay_2 - Gy_1 + My_0 = -Ay_2 + My_0 = 0.$$

Consequently,

$$0 = -(Ay_2, y_0) + (My_0, y_0) = (My_0, y_0).$$

Then $My_0 = 0$ and $My_0 = Gy_0 = Ay_0 = 0$ what contradicts Condition I. $\square$

Let us introduce the following parameter-dependent operator pencil:

$$L(\lambda, \eta) = \lambda^2 M - \lambda \eta G - A.$$

It is clear that $L(\lambda, 1) = L(\lambda)$ and $L(\lambda, 0) = \lambda^2 M - A$.

Theorem 2.8 ([13, Theorem 1.2.7])**.** *Let $\eta_0 \in \mathbb{C}$ and let the open domain $\Omega \subset \mathbb{C}$ be a domain which contains exactly one eigenvalue λ_0 of the pencil $L(\lambda, \eta_0)$. Denote by m the algebraic multiplicity of λ_0. Then there exist numbers $\epsilon > 0$ and $m_1 \in \mathbb{N}$ $m_1 \leq m$, such that the following assertions are true in the deleted neighborhood $0 < |\eta - \eta_0| < \epsilon$.*

1. *The pencil $L(\lambda, \eta)$ possesses exactly m_1 distinct eigenvalues inside the domain Ω. Those eigenvalues can be arranged in groups $\lambda_{sj}(\eta)$, $(s = \overline{1, l}; j = \overline{1, p_s}; \sum_{s=1}^{l} p_s = m_1)$ in such a way that the functions of the group, i.e. $\lambda_{s1}, \lambda_{s2}, \ldots, \lambda_{sp_s}$ compose a complete set of a p_s-valued function. In this case those eigenvalues can be presented in the form of the following Puiseux series expansion*

$$\lambda_{sj}(\eta) = \lambda_0 + \sum_{k=1}^{\infty} a_{sk} (((\eta - \eta_0)^{\frac{1}{p_s}})_j)^k, \quad (j = 1, 2, \ldots, p_s),$$

where, for $j = 1, 2, \ldots, p_s$

$$((\eta - \eta_0)^{\frac{1}{p_s}})_j = |\eta - \eta_0|^{\frac{1}{p_s}} \exp\left(\frac{2\pi(j-1) + i \arg(\eta - \eta_0)}{p_s}\right).$$

2. *A basis of the eigen-space corresponding to $\lambda_{sj}(\eta)$ can be presented in the following form*

$$y_{sj}^{(q)}(\eta) = y_{s0}^{(q)} + \sum_{k=1}^{\infty} y_{sk}^{(q)} (((\eta - \eta_0)^{\frac{1}{p_s}})_j)^k, \quad (j = 1, 2, \ldots, p_s, \quad q = 1, 2, \ldots, \alpha_s),$$

where α_s is the geometric multiplicity of $\lambda_{sj}(\eta)$, $y_{s0}^{(q)}$ belong to the eigen-subspace of $L(\lambda, \eta_0)$ corresponding to the eigenvalue λ_0.

It should be mentioned that Theorem 2.8 is a generalization of the Weierstrass theorem on function analytic in two variables (see, e.g. [9, p. 476]). It was proved in [5] for bounded operators.

Remark 2.9. If $\lambda_0 \neq 0$ is a real or pure imaginary eigenvalue of $L(\cdot, \eta_0)$ for some real η_0 and if eigenvalues of $L(\cdot, \eta)$ near λ_0 for real η near η_0 are also real or pure imaginary, then $p_s = 1$ for all s in Theorem 2.8.

Remark 2.10. If both η and η_0 are real so that $\arg(\eta - \eta_0)$ is an integer multiple of π, then we can put

$$((\eta - \eta_0)^{\frac{1}{p_s}})_j = |\eta - \eta_0|^{\frac{1}{p_s}} exp\left(\frac{2\pi(j-1) + i\delta_i \arg(\eta - \eta_0)}{p_s}\right), \qquad (2.8)$$

where δ_i is an odd integer. This would change the numeration of the roots for $\eta - \eta_0 < 0$ if $\frac{\delta_i - 1}{p_s}$ is not even integer but allows us when taking $\delta_s = p_s$ if p_k is odd to have $((\eta - \eta_0)^{\frac{1}{p_s}})_1$ real for all η in a real neighborhood of η_0.

If we additionally assume that G is bounded and boundedly invertible then we can write, assuming $\lambda \neq 0$,

$$L(\lambda, \eta) = \lambda G^{\frac{1}{2}}(-\eta I + \lambda G^{-\frac{1}{2}} M G^{-\frac{1}{2}} - \lambda^{-1} G^{-\frac{1}{2}} A G^{-\frac{1}{2}}) G^{\frac{1}{2}}.$$

Hence

$$L(\lambda, \eta) = \lambda G^{\frac{1}{2}} Q(\lambda, \eta) G^{\frac{1}{2}}$$

where

$$Q(\lambda, \eta) = -\eta I + \lambda G^{-\frac{1}{2}} M G^{-\frac{1}{2}} - \lambda^{-1} G^{-\frac{1}{2}} A G^{-\frac{1}{2}}.$$

For $\lambda \neq 0$ we note that if $ker Q(\lambda, \eta) \neq \{0\}$ then its dimension is the geometric multiplicity of the eigenvalue λ of the pencil $Q(\cdot, \eta)$ as well as the geometric multiplicity of the eigenvalue η of the pencil $Q(\lambda, \cdot)$. The algebraic multiplicities can be different, but for $\lambda \in \mathbb{R}$, we have a standard problem for a self-adjoint operator with the spectral parameter η, and hence all eigenvalues of $Q(\lambda, \cdot)$ for real λ are real and semisimple. Thus, we have

Lemma 2.11 ([13, Lemma 1.2.10]). *Assume that $G >> 0$, let $\lambda_0 \in \mathbb{R} \backslash \{0\}$ and let η_0 be an eigenvalue of the pencil $Q(\lambda_0, \cdot)$ with (geometric) multiplicity 1. Then there are $\epsilon > 0$ and l real analytic functions*

$$\eta_s(\lambda) = \eta_0 + \sum_{k=p_s}^{\infty} b_{s,k}(\lambda - \lambda_0)^k, \quad s = 1, 2, \dots, l, \quad |\lambda - \lambda_0| < \epsilon, \tag{2.9}$$

where $p_s \in \mathbb{N}$, $b_{sp_s} \in \mathbb{R} \backslash \{0\}$, $b_{sk} \in \mathbb{R}$ for $k > p_s$, such that $(\eta_s(\lambda))_{s=1}^{l}$ represents the eigenvalues near η_0 of the pencil $Q(\lambda, \cdot)$, counted with multiplicity, for each $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \epsilon$.

Proof. For real λ, the eigenvalues of the self-adjoint operator function $Q(\lambda, \cdot)$ are real. Hence the lemma immediately follows from Theorem 2.8 and Remark 2.9. □

Lemma 2.12 ([13, Lemma 1.4.1]). *Let the conditions of Lemma 2.11 be satisfied. Let $\lambda_0 > 0$ and $\eta_0 > 0$, then in some neighborhood of (λ_0, η_0), i.e. in*

$$\{(\lambda, \eta) : |\lambda - \lambda_0| < \epsilon, |\eta - \eta_0| < \delta, \epsilon > 0, \delta > 0\}$$

all the eigenvalues are given by the following formula

$$\lambda_j(\eta) = \lambda_0 + \sum_{k=1}^{\infty} \beta_k((\eta - \eta_0)_j^{\frac{1}{r}})^k, \quad (j = 1, 2, \dots, r), \tag{2.10}$$

where $\beta_1 \neq 0$ is real if r is odd and real or pure imaginary if r is even, $(\eta - \eta_0)_j^{\frac{1}{r}}$, $(j = 1, 2, \dots, r)$ means the complete set of branches of the root.

Proof. We obtain this result immediately after inverting (2.9). □

Theorem 2.13. *Let, in addition to Condition I, $M >> 0$ and $G >> 0$ then for negative eigenvalues λ_{-j} of L there exists a subsequence of positive eigenvalues (denote them λ_j ($\lambda_{j+1} \geq \lambda_j$)) such that*

$$\lambda_j + \lambda_{-j} \geq 0. \tag{2.11}$$

Proof. We index the eigenvalues in such a way that if $\lambda_{-j}(0) < 0$ then $\lambda_j(0) = -\lambda_{-j}(0)$ and $\lambda_j^2(0) \leq \lambda_{j+1}^2(0)$.

The eigenvalues of $L(\cdot, \eta)$ are piecewise analytic functions of η. The eigenvalues may loose analyticity only when they collide. This follows from the results above.

The eigenvalues located on $(-\infty, 0)$ are analytic functions of $\eta > 0$ because there are no nonreal eigenvalues in the open left half-plane (see Lemma 2.4, Statement 2) and, consequently, for all $\eta > 0$ and $\lambda \in (-\infty, 0)$ the collision happens according to (2.10) with $r = 1$ and $\beta_1 \in \mathbb{R}\backslash\{0\}$.

Differentiating $L(\lambda_j(\eta), \eta)y_j(\eta) = 0$ with respect to η and taking the inner product with $y_j(\eta)$ leads to

$$\lambda_j'(\eta)((2\lambda_j(\eta)M - G)y_j(\eta), y_j(\eta)) - \lambda_j(\eta)(Gy_j(\eta), y_j(\eta)) = 0$$

and therefore

$$\lambda_j'(\eta) = \frac{\lambda_j(\eta)\,(Gy_j(\eta), y_j(\eta))}{2\lambda_j(\eta)\,(My_j(\eta), y_j(\eta)) - \eta\,(Gy_j(\eta), y_j(\eta))}. \tag{2.12}$$

Here $(My_j(\eta), y_j(\eta))$ and $(My_j(\eta), y_j(\eta))$ are positive and depend continuously on η. It follows from (2.12) that if $\lambda_j(\eta) < 0$ then $\lambda_j'(\eta) > 0$. These negative eigenvalues do not cross the origin due to Lemma 2.7.

The denominator of (2.12) is positive for sufficiently small $\eta > 0$ and $\lambda_j(\eta) > 0$ while the numerator is nonnegative. We therefore have shown that $\lambda_j(\eta)$ and $\lambda_{-j}(\eta)$ are nondecreasing, which gives

$$\lambda_j(\eta) + \lambda_{-j}(\eta) \geq 0$$

for $\eta \geq 0$ small enough.

While $\eta > 0$ increases, $\lambda_j'(\eta)$ can change its sign only when the denominator in the right-hand side of (2.12) vanishes, i.e. when eigenvalues collide. If such a collision takes place on the interval $(0, \infty)$, then the eigenvalues involved behave according to formula (2.10). Such a coalescence on the half-axis $(0, \infty)$ is of one of the following three kinds.

The first one is the case of r odd in (2.10) and $\beta_1 > 0$ real. In this case we identify the eigenvalue moving to the right along the real axis after the collision with the one which moved to the right along the real axis before the collision.

By a collision of the second kind we mean one which has r even and $\beta_1 \in \mathbb{R}\backslash\{0\}$ in (2.10). After such a collision two new real eigenvalues appear which are moving in opposite directions along the real axis, and such a collision cannot violate Theorem 2.13.

The third kind of collision has even r and pure imaginary $\beta_1 \neq 0$. Let $\lambda_j(\eta)$ take part in such a collision at $\eta = \eta_0 \in (0, 1]$. Then a collision of the second kind indeed

occurred at some $\eta \in (0, \eta_0,)$ in some point $\lambda_\times \in (\lambda_j(\eta_0), \infty)$ on the real axis. In this case the eigenvalue that has arisen after this collision and is moving to the right is identified as $\lambda_j(\eta)$. □

Remark 2.14. The eigenvalues for which the denominator

$$2\lambda_j(\eta)(My_j(\eta), y_j(\eta)) - \eta(Gy_j(\eta), y_j(\eta))$$

is positive (negative) are called eigenvalues of type I (type II) in [7]. However, we use terms "type I" and "type II" in Section 3 differently.

Lemma 2.15. *Let $M \geq mI$, $m > 0$, then for all $\eta \in [0,1]$ the nonreal eigenvalues (if any) lie in a bounded domain $\{\lambda : 0 \leq \mathrm{Re} \leq \frac{\eta}{2m}\|G^{\frac{1}{2}}\|^2\}$, $|\mathrm{Im}\lambda| \leq m^{-\frac{1}{2}}\beta^{\frac{1}{2}}$.*

Proof. Let $\lambda \notin \mathbb{R}$ be an eigenvalue of $L(\cdot, \eta)$ and y be the corresponding eigenvector. Then

$$\lambda^2(My, y) - \eta\lambda(Gy, y) - (Ay, y) = \lambda^2(My, y) - \eta\lambda\|G^{\frac{1}{2}}y\|^2 - (Ay, y) = 0$$

and, since λ is not real,

$$\mathrm{Re}\lambda = \frac{1}{2}\eta\|G^{\frac{1}{2}}y\|^2(My, y)^{-1}, \quad \mathrm{Im}\lambda = \frac{1}{2}(My, y)^{-1}\sqrt{|\eta^2\|G^{\frac{1}{2}}y\|^4 + 4(My, y)(Ay, y)|}$$

what implies the statement of the lemma. □

In what follows, writing *number of eigenvalues in a domain* we mean the number with account of their algebraic multiplicities.

Theorem 2.16. *Let the conditions of Theorem 2.13 be satisfied. If the negative eigenvalues λ_{-j} are arranged in pairs with positive eigenvalues λ_j such that (2.11) is true then the number of positive eigenvalues not included in the pairs is*

$$2\kappa_A - \kappa_c,$$

where κ_A is the number of the negative eigenvalues of the operator A and κ_c is the number nonreal eigenvalues.

Proof. According to Theorem 2.8 the total multiplicity is preserved locally. Due to Lemma 2.15 the nonreal eigenvalues do not disappear at infinity while η grows from 0 to 1. For $\eta = 0$ we have $2\kappa_A - \kappa_c = 0$ and there are no positive eigenvalue not included in a pair with a negative eigenvalue.

Thus, the number of the eigenvalues in the open upper half-plane may reduce only at collisions on the real axis. Let s eigenvalues reach a point on the real axis at $\eta = \eta_0$ moving from the open upper half-plane. Due to the symmetry of the spectrum with respect to the real axis the same number of eigenvalues reaches the same point moving from the open lower half-plane. This results in disappearing of s eigenvalues from each of open upper and lower half-planes and appearing $2s$ real eigenvalues. □

3. PENCILS WITH RANK ONE LINEAR PART

Let us consider the quadratic operator pencil $L(\lambda, \eta)$ with operators M, G, A acting in the Hilbert space $H \oplus \mathbb{C}$ and satisfying Condition I. Moreover, in this section let the following condition be valid:

Condition II.

$$G = b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad b > 0.$$

Lemma 3.1.

1. *Let $\lambda > 0$ and $-\lambda$ be eigenvalues of the operator pencil $L(\cdot, \eta_0)$ for some $\eta_0 \in (0, 1]$. Then λ and $-\lambda$ are eigenvalues of $L(\cdot, \eta)$ for all $\eta \in [0, 1]$.*
2. *Let $\lambda = i\tau$ with $\tau \in \mathbb{R} \backslash \{0\}$ be an eigenvalue of the operator pencil $L(\cdot, \eta_0)$, where $\eta_0 \in (0, 1]$. Then $i\tau$ and $-i\tau$ are eigenvalues of $L(\cdot, \eta)$ for all $\eta \in [0, 1]$.*

Proof. 1. Let $y_1 = \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix}$ be an eigenvector of $L(\cdot, \eta_0)$ corresponding to the eigenvalue λ and let $y_2 = \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix}$ be an eigenvector of $L(\cdot, \eta_0)$ corresponding to $-\lambda$. Then

$$(\lambda^2 M - \lambda\eta_0 G - A)y_1 = 0, \tag{3.1}$$

$$(\lambda^2 M + \lambda\eta_0 G - A)y_2 = 0, \tag{3.2}$$

and consequently

$$\lambda^2(y_2, My_1) - \lambda\eta_0(y_2, Gy_1) - (y_2, Ay_1) = 0,$$
$$\lambda^2(My_2, y_1) + \lambda\eta_0(Gy_2, y_1) - (Ay_2, y_1) = 0.$$

Taking into account that M, G and A are selfadjoint, the difference of the above equations gives

$$0 = (Gy_2, y_1) = by_{22}\overline{y}_{12}.$$

Then one of the factors must be zero, say $y_{12} = 0$, which gives $Gy_1 = 0$. Hence (3.1) and (3.2) lead to $L(\pm\lambda, \eta)y_1 = 0$, which completes the proof of part 1.

2. Due to the symmetry of the spectrum, see Lemma 2.3 part 2, it follows that if $i\tau$ ($\tau \in \mathbb{R} \backslash \{0\}$) is an eigenvalue of $L(\cdot, \eta_0)$, then also $-i\tau$ is an eigenvalue $L(\cdot, \eta_0)$. Let y be an eigenvector of $L(\cdot, \eta_0)$ corresponding to the eigenvalue $i\tau$. Then

$$-\tau^2(My, y) - i\tau\eta_0(Gy, y) - (Ay, y) = 0.$$

Since M, G and A are selfadjoint, all three inner products are real. Therefore $\lambda \neq 0$ and $\eta_0 > 0$ give $(Gy, y) = 0$ and thus $Gy = 0$ because $G \geq 0$. It follows $L(\pm\lambda, \eta)y = L(\lambda, \eta_0)y = 0$ for all $\eta \in [0, 1]$. $\qquad \square$

Definition 3.2 (see Definition 1.5.2 in [13])**.** Let $\eta_0 \in (0,1]$ and $m_I(\lambda) := \min_{\eta \in (0,1]} m(\lambda, \eta)$, where $m(\lambda, \eta)$ denotes the multiplicity of an eigenvalue λ of the pencil $L(\cdot, \eta)$.

1. An eigenvalue λ of $L(\cdot, \eta_0)$ is said to be an eigenvalue of type I if λ is an eigenvalue of the pencil $L(\cdot, \eta)$ for each $\eta \in (0,1]$.
2. For $\lambda \in \mathbb{C}$ let $m_0(\lambda) = dim(ker L(\lambda) \cap ker G)$. If $m_0(\lambda) > 0$, then each nonzero vector in $ker L(\lambda) \cap ker G$ is called an eigenvector of type I for L at λ.
3. An eigenvalue λ of the pencil $L(\cdot, \eta_0)$ is said to be an eigenvalue of type II if $m(\lambda, \eta_0) \neq m_I(\lambda)$.

Remark 3.3.

1. An eigenvalue can be both of type I and type II. If this is the case for some η, then we say that λ is an eigenvalue of the pencil $L(\cdot, \eta)$ of type I multiplicity $m_I(\lambda)$ and of type II multiplicity $m(\lambda, \eta) - m_I(\lambda)$.
2. If 0 is an eigenvalue of the pencil L, then it follows from Lemma 2.7 that 0 is an eigenvalue of $L(\cdot, \eta)$ for all $\eta \in (0,1]$, and, if $dim ker A = n$, the algebraic multiplicity $m(0, \eta)$ is $2n$ if $Gy = 0$ for any $y \in ker A$ and $2n - 1$ if there exists $0 \neq y \in ker A$ such that $Gy \neq 0$.
3. If $ker M \cap ker A = \{0\}$, then the pencil $L(\cdot, 0)$ satisfies Condition I. Since eigenvalues λ of type I are eigenvalues of the pencil $L(\cdot, \eta)$ for all $\eta \in (0,1]$, it follows from (2.10) that $m_I(\lambda)$ branches of the eigenvalue λ are constant near $\eta = 0$, so that $m_I(\lambda) \leq m(\lambda, 0)$, whereas the remaining $m(\lambda, 0) - m_I(\lambda)$ branches are not constant.

Lemma 3.4. *Assume that $ker M \cap ker A = \{0\}$. Then the eigenvalues of type I of $L(\cdot, \eta)$, which are independent of $\eta \in (0,1]$, are located on imaginary and real axes symmetrically with respect to 0. If additionally $M + G \geq \epsilon I$ ($\epsilon > 0$), at most finitely many of the eigenvalues of type I are on the imaginary axis.*

    The proofs of this and the next lemmas are the same as the proofs of Lemmas 1.5.4 and 1.5.5 in [13], respectively, with the only change K for G.

Lemma 3.5.

1. *For all $\lambda, \eta \in \mathbb{C}$, $ker L(\lambda, \eta) \cap ker G = ker L(\lambda) \cap ker G$, that is $ker L(\lambda, \eta) \cap ker G$ is independent of η. In particular, $m_0(\lambda) \leq m_I(\lambda)$.*
2. *Let $\lambda \neq 0$, $\eta \in (0,1]$ and assume that $ker L(\lambda) \cap ker G \neq \{0\}$. Then no eigenvector $y_0 \in ker L(\lambda \cap ker G = ker L(\lambda) \cap ker G$ of $L(\cdot, \eta)$ at λ has an associated vector.*
3. *If $ker M \cap ker A = \{0\}$, then $m_I(\lambda) = m_0(\lambda)$ for all $\lambda \in \mathbb{C} \backslash \{0\}$.*

Theorem 3.6. *Assume that $ker M \cap ker A = \{0\}$.*

1. *$\lambda \neq 0$ is an eigenvalue of type I of the operator pencil L if and only if λ is an eigenvalue of the pencil $L(\cdot, 0)$ having an eigenvector of the form $(y_0, 0)^T$, and $m_I(\lambda)$ is the dimension of the space of eigenvectors of this form.*
2. *If $\lambda \neq 0$ is an eigenvalue of type I of the pencil L, but not an eigenvalue of type II, then λ is semisimple.*

3. *If $\lambda \neq 0$ is an eigenvalue of the pencil L of type II, then $\ker L(\lambda)$ has a basis consisting of $m_I(\lambda)$ eigenvectors of type I and one eigenvector y_0 with $Gy_0 \neq 0$ with maximal chain length $m(\lambda) - m_I(\lambda)$, that is, there is a chain $(y_j)_{j=0}^{m(\lambda)-m_I(\lambda)-1}$ of the eigenvector y_0 and, if $m(\lambda) - m_I(\lambda) > 1$, associated vectors of L at λ.*
4. *If $\lambda \neq 0$ is an eigenvalue of type II of the pencil L, then $-\lambda$ is not an eigenvalue of type II of the pencil L.*

The proof of Theorem 3.6 is the same as the proof of Theorem 1.5.6 in [13] with the change of K for G.

Let us describe location of the eigenvalues $\{\lambda_j^{II}(\eta)\}$ of type II. Let us enumerate them as follows:

$$(\lambda_1^{II}(0))^2 < (\lambda_2^{II}(0))^2 < \ldots < (\lambda_{\kappa_{II}}^{II}(0))^2 < 0 \leq (\lambda_{\kappa_{II}+1}^{II}(0))^2 < (\lambda_{\kappa_{II}+2}^{II}(0))^2 < \ldots,$$

$$\lambda_{-j}^{II}(0) = -\lambda_j^{II}(0).$$

Here $2\kappa_{II}$ is the number of pure imaginary eigenvalues of $L(\cdot, 0)$ of type II.

Proposition 3.7. *The negative eigenvalues of type II preserve their order:*

$$\ldots < \lambda_{-j-1}^{II}(\eta) < \lambda_{-j}^{II}(\eta) < \ldots < \lambda_{-\kappa_{II}-1}^{II}(\eta) \leq 0.$$

Proof. We know from Lemma 2.5 that all the negative eigenvalues are semisimple. From Statement 3 of Theorem 3.6 we conclude that the eigenvalues of type II are geometrically simple (however, an eigenvalue of type II can for some value of η coincide with an eigenvalue of type I). Since $\lambda_j^{II}(\eta)$ are continuous functions of η we arrive at the statement of the proposition. $\square$

Theorem 3.8. *Assume that $\ker M \cap \ker A = \{0\}$ and that $M + G >> 0$. Then the eigenvalues of type II of the operator pencil L possess the following properties.*

1. *$L(\cdot, \eta)$ has nonreal eigenvalues of type II located symmetrically with respect to the real axis. Denote their number by $2\kappa_{II}(\eta) \geq 0$.*
2. *For all $\eta \in (0, 1]$ these nonreal eigenvalues of type II lie in the open right half-plane.*
3. *$|\lambda_{-j}^{II}(\eta)| \notin \{\lambda_j^{II}(\eta)\}$ for all $j \geq \kappa_{II}(0) + 1$ if $\lambda_{\kappa_{II}(0)+1} > 0$ and for all $j \geq \kappa_{II}(0) + 2$ if $\lambda_{\kappa_{II}(0)+1}(0) = 0$.*
4. *The number $n_j(\eta)$ of eigenvalues of type II in each interval $(|\lambda_{-j}^{II}(\eta)|, |\lambda_{-j-1}^{II}(\eta)|)$ $(j \geq \kappa_{II}(0) + 1$ if $\lambda_{\kappa_{II}(0)+1}^{II}(0) > 0$ and $j \geq \kappa_{II}(0) + 2$ if $\lambda_{\kappa_{II}(0)+1}^{II}(0) = 0)$ is odd.*
5. *If $\lambda_{\kappa_{II}+1}^{II}(0) > 0$ the number $n_0(\eta)$ of eigenvalues of type II in the interval $(0, |\lambda_{-\kappa_{II}-1}^{II}(\eta)|)$ is even.*
6. *Denote by*

$$\tilde{\kappa}_{II}(\eta) = \frac{1}{2} \begin{cases} n_0(\eta) - 1 + \sum_{j=1}^{\infty} (n_j(\eta) - 1) & \text{if } \lambda_{-\kappa_{II}}^{II}(0) = 0, \\ n_0(\eta) + \sum_{j=1}^{\infty} (n_j(\eta) - 1) & \text{if } \lambda_{-\kappa_{II}}^{II}(0) \neq 0. \end{cases}$$

Then $\tilde{\kappa}_{II}(\eta) + \kappa_{II}(\eta) + \kappa_I = \kappa_A$ where κ_A is the number of negative eigenvalues of the operator pencil $\lambda M - A$ and $2\kappa_I$ is the number (independent of η) of pure imaginary eigenvalues of $L(\cdot, \eta)$ of type I.

Proof. Statement 1 follows from Lemma 2.3.

By Lemma 2.15 we know that the nonreal eigenvalues lie in the closed right half-plane. But if such an eigenvalue is pure imaginary, then it is of type I. This means that nonreal eigenvalues of type II lie in the open right half-plane. Statement 2 is proved.

If $|\lambda_{-j}^{II}(\eta)| \in \{\lambda_j^{II}(\eta)\}$ for some $j \geq \kappa_{II}(0)+1$ if $\lambda_{\kappa_{II}(0)+1} > 0$ or some $j \geq \kappa_{II}(0)+2$ if $\lambda_{\kappa_{II}(0)+1} = 0$ then by Lemma 3.1 $\lambda_j^{II}(\eta)$ is independent of η and therefore is of type II. This contradiction proves Statement 3.

Since the right-hand side of (2.12) is nonnegative for small $\eta > 0$ and $\lambda_j(\eta) \in \mathbb{R}$ the real eigenvalues for such η move to the right along the real axis. Therefore, the number of eigenvalues of type II in each interval $(|\lambda_{-j}(\eta)|, |\lambda_{-j-1}(\eta)|)$ $(j \geq \kappa_{II}(0)+1$ if $\lambda_{\kappa_{II}(0)+1}^{II} > 0$ and $j \geq \kappa_{II}(0)+2$ if $\lambda_{\kappa_{II}(0)+1}^{II} = 0$) is 1 for $\eta > 0$ small enough. This number can increase only when a collision of the second kind of eigenvalues of type II happens on the real axis. Such a collision increases the number of the eigenvalues in such an interval by 2. Then this number can increase by 2 as the result of a collision of the second kind and decrease by 2 as the result of a collisions of the third type. Thus $n_j(\eta)$ is odd for all $\eta > 0$ what proves Statement 4.

Let $\lambda_{\kappa_{II}+1}^{II}(0) > 0$. Then for $\eta > 0$ small enough there are no eigenvalues of type II on the interval $(0, |\lambda_{-\kappa_{II}(0)-1}|)$. This number can increase by 2 as a result of collision of the second kind and decrease by 2 as a result of collision of the third kind. Statement 5 follows.

It is clear that $\tilde{\kappa}_{II}(0) = 0$ and $\kappa_{II}(0) + \kappa_I = \kappa_A$. The number $2\kappa_{II}(\eta)$ of nonreal eigenvalues of type II increases (decreases) by 2 at collisoons of the second (third) kind. But simultaneously $\tilde{\kappa}_{II}$ decreases (increases) by 2. Thus, the sum $\kappa_{II}(\eta) + \tilde{\kappa}_{II}(\eta)$ remains unchanged at collisions of any of the three kinds. This proves Statement 6. $\quad\square$

4. APPLICATIONS

4.1. We consider a Sturm-Liouville problem with the gyroscopic condition at the right end

$$-y'' + q(x)y = \lambda^2 y, \tag{4.1}$$

$$y(0) = 0, \tag{4.2}$$

$$y'(a) + \lambda \alpha y(a) = 0. \tag{4.3}$$

Here λ is the spectral parameter, the parameter $\alpha > 0$ and the potential q is real-valued and belongs to $L_2(0, a)$.

Let us introduce the operators A, G and M acting in the Hilbert space $H = L_2(0, a) \oplus \mathbb{C}$ according to the formulae:

$$A \begin{pmatrix} v(x) \\ c \end{pmatrix} = \begin{pmatrix} -v''(x) + q(x)v(x) \\ v'(a) \end{pmatrix},$$

$$D\left(A\right) = \left\{ \begin{pmatrix} v(x) \\ c \end{pmatrix} : \quad v(x) \in W_2^2\left(0, a\right), \quad v\left(0\right) = 0, \quad c = v(a) \right\},$$

$$M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}.$$

The operator A is selfadjoint and bounded below $(A \geq \beta I)$ and there exists $-\beta_1 < -\beta \leq \|y\|^{-2}(Ay, y)$ such that $(A + \beta_1 I)^{-1}$ is a compact operator (see [13, Proposition 2.1.1]).

We associate the quadratic operator pencil

$$L\left(\lambda\right) = \lambda^2 M - \lambda G - A \tag{4.4}$$

with problem (4.1)–(4.3).

We identify the spectrum of problem (4.1)–(4.3) with the spectrum of the pencil $L\left(\lambda\right)$. It is clear that $M \geq 0$, $G \geq 0$ and $M + G \gg 0$. The spectrum of the pencil consists of normal eigenvalues (see Section 2 of this paper).

Let us prove that all of these eigenvalues are of type II. Suppose a pure imaginary $i\tau \neq 0$ is an eigenvalue. Then by Lemma 3.1 $-i\tau$ is also an eigenvalue and according to the proof of Lemma 3.1 $c = v(a) = 0$ in (4.3). Therefore, the second component of the equation $L(\lambda_0)Y = 0$ gives $v'(a) = 0$ what contradicts $v(a) = 0$. In the same way, one can prove that there are no symmetrically located real eigenvalues and that the possible eigenvalue at the origin is simple.

Thus, the conditions of Theorem 3.8 are satisfied and all the eigenvalues are of type II and therefore statements 1)–6) of Theorem 3.8 are valid. Thus we obtain

Theorem 4.1.

1. *Problem (4.1)–(4.3) may have nonreal eigenvalues located symmetrically with respect to the real axis. Denote their number by $2\kappa \geq 0$.*
2. *All the nonreal eigenvalues lie in the open right half-plane.*
3. *If we denote by $\{\lambda_j\}_{-\infty}^{\kappa_0}$ the negative eigenvalues of problem (4.1)–(4.3) then $|\lambda_k| \notin \{\lambda_j\}_{-\infty, j\neq 0}^{\infty}$ for all $-\infty < k \leq \kappa_0$.*
4. *The number n_j of eigenvalues in each interval $(|\lambda_j|, |\lambda_{j-1}|)$ $(-\infty < j \leq \kappa_0)$ is odd.*
5. *If $0 \notin \{\lambda_j\}_{-\infty, j\neq 0}^{\infty}$ then the number n_0 of eigenvalues in the interval $(0, |\lambda_{\kappa_0}|)$ is even. If $0 \in \{\lambda_j\}_{-\infty, j\neq 0}^{\infty}$ then the number of eigenvalues in the interval $(0, |\lambda_{\kappa_0}|)$ is odd and 0 is a simple eigenvalue.*
6. *Denote by*

$$\tilde{\kappa} = \frac{1}{2} \begin{cases} n_0 + \sum\limits_{j=1}^{\infty}(n_j - 1) & \text{if } 0 \notin \{\lambda_j\}_{-\infty, j\neq 0}^{\infty}, \\ n_0 - 1 + \sum\limits_{j=1}^{\infty}(n_j - 1) & \text{if } 0 \in \{\lambda_j\}_{-\infty, j\neq 0}^{\infty}. \end{cases}$$

Then $\tilde{\kappa} + \kappa = \kappa_A$ where κ_A is the number of negative eigenvalues of the operator pencil $\lambda M - A$ or what is the same of problem

$$-y'' + q(x)y = \lambda y, \tag{4.5}$$

$$y(0) = y'(a) = 0. \tag{4.6}$$

4.2. In the second example we have eigenvalues of both types. Let $q = const > 0$. Consider the following spectral problem

$$-y_j'' + qy_j = \lambda^2 y_j, \quad j = 1, 2, \tag{4.7}$$

$$y_1(0) = y_2(0) = 0, \tag{4.8}$$

$$y_1(a) = y_2(a), \tag{4.9}$$

$$y_1'(a) = y_2'(a) + \lambda\alpha y_2(a). \tag{4.10}$$

Let us introduce the operators A, G and M acting in the Hilbert space $H = L_2(0, a) \oplus L_2(0, a) \oplus \mathbb{C}$ according to the formulae:

$$A \begin{pmatrix} v_1(x) \\ v_2(x) \\ c \end{pmatrix} = \begin{pmatrix} -v_1''(x) + qv_1(x) \\ -v_2''(x) + qv_2(x) \\ v_2'(a) \end{pmatrix},$$

$$D(A) = \left\{ \begin{pmatrix} v_1(x) \\ v_2(x) \\ c \end{pmatrix} : v_1(x) \in W_2^2(0, a), \, v_2(x) \in W_2^2(0, a), \, v_1(0) = v_2(0) = 0, \, c = v_2(a) \right\},$$

$$M = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G = \alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The corresponding operator pencil $L(\lambda)$ given by (4.4) satisfies conditions of Theorem 3.8. Moreover, in case of $q = const$ we can find the characteristic function of this problem, i.e. the function the set of zeros of which coincides with the spectrum of problem (4.7)–(4.9):

$$\omega(\lambda) = \frac{\sin\sqrt{\lambda^2 + q}\,a}{\sqrt{\lambda^2 + q}} \left(2\cos\sqrt{\lambda^2 + q}\,a + \alpha\lambda\frac{\sin\sqrt{\lambda^2 + q}\,a}{\sqrt{\lambda^2 + p}} \right). \tag{4.11}$$

The spectrum of problem (4.7)–(4.10), i.e. the set of zeros of ω consists of two subsequences $\{\lambda_j^I\}_{-\infty, j\neq 0}^\infty$ and $\{\lambda_j^{II}\}_{-\infty, j\neq 0}^\infty$ where

$$\lambda_j^I = \pm\sqrt{\left(\frac{\pi j}{a}\right)^2 - q}, \quad j = \pm 1, \pm 2, \dots \tag{4.12}$$

and $\{\lambda_j^{II}\}_{-\infty, j\neq 0}^\infty$ is the set of zeros of the function

$$\left(2\cos\sqrt{\lambda^2 + q}\,a + \alpha\lambda\frac{\sin\sqrt{\lambda^2 + q}\,a}{\sqrt{\lambda^2 + p}} \right).$$

According to Definition 3 2 (with $\eta = \alpha$) we obtain the following theorem.

Theorem 4.2. *The zeros* $\{\lambda_j^I\}_{-\infty, j\neq 0}^\infty$ *are eigenvalues of the type I of problem* (4.7)–(4.10) *while the zeros* $\{\lambda_j^{II}\}_{-\infty, j\neq 0}^\infty$ *are the eigenvalues of type II.*

Analyzing the behavior of the function

$$\left(2\cos\sqrt{\lambda^2+q}\,a + \alpha\lambda\frac{\sin\sqrt{\lambda^2+q}\,a}{\sqrt{\lambda^2+p}}\right)$$

we conclude (for $q = const$) what follows:

(i) $n_j = 1$ for all j and $n_0 = 0$ if $0 \in \{\lambda_j\}_{-\infty,j\neq 0}^{\infty}$ and $n_0 = 1$ if $0 \notin \{\lambda_j\}_{-\infty,j\neq 0}^{\infty}$.

(ii) An eigenvalue of type I cannot coincide with an eigenvalue of type II for any value of $\alpha > 0$ because if

$$\frac{\sin\sqrt{\lambda^2+q}\,a}{\sqrt{\lambda^2+q}} = 0 \tag{4.13}$$

and

$$\left(2\cos\sqrt{\lambda^2+q}\,a + \alpha\lambda\frac{\sin\sqrt{\lambda^2+q}\,a}{\sqrt{\lambda^2+p}}\right) = 0$$

then

$$\cos\sqrt{\lambda^2+q}\,a = 0$$

what contradicts (4.13).

(iii) In the same way we conclude that $\{-\lambda_j^{II}\}_{-\infty,j\neq 0}^{\infty} \cap \{\lambda_j^{I}\}_{-\infty,j\neq 0}^{\infty} = \emptyset$.

To avoid considering many cases we give explicit result for one particular case. Here we describe the eigenvalues $\{\lambda_j\}_{-\infty,j\neq 0}^{\infty} = \{\lambda_j^{I}\}_{-\infty,j\neq 0}^{\infty} \cup \{\lambda_j^{II}\}_{-\infty,j\neq 0}^{\infty}$ identifying $\lambda_{2j-1} = \lambda_j^{II}$ and $\lambda_{2j} = \lambda_j^{I}$ ($j \in \mathbb{N}$).

Using (i)–(iii) and Theorem 3.8 we obtain the following result.

Corollary 4.3. *Let $q > 0$ and $q^{\frac{1}{2}}a\pi^{-1} \in \mathbb{N}$. Then*

a) *$\lambda = 0$ is an eigenvalue of geometric multiplicity 1 and algebraic multiplicity 2;*

b) *there are $2(q^{\frac{1}{2}}a\pi^{-1} - 1)$ simple pure imaginary (nonzero) eigenvalues located symmetrically with respect to the origin which can be found by (4.12) with $j = \pm 1, \pm 2, \ldots, \pm(q^1 2a\pi^{-1} - 1)$;*

c) *there are $2q^{\frac{1}{2}}a\pi^{-1}$ complex (neither real nor pure imaginary) eigenvalues located symmetrically with respect to the real axis in the open right half plane;*

d) *real eigenvalues can be indexed such that $\lambda_j < 0$ for $-\infty < j \le -2q^{\frac{1}{2}}a\pi^{-1} - 1 := \kappa_0$ and the interval $(0, |\lambda_{\kappa_0}|)$ is free of eigenvalues*

e) *if we denote by $\{\lambda_j\}_{-\infty}^{\kappa_0}$ the negative eigenvalues of problem (4.7)–(4.10) then $|\lambda_{2k-1}| \notin \{\lambda_j\}_{-\infty,j\neq 0}^{\infty}$ for all $-\infty < 2k - 1 \le \kappa_0$;*

f) *the number of eigenvalues in each interval $(|\lambda_{2j-1}|, |\lambda_{2j-3}|)$ $(-\infty < 2j - 1 \le \kappa_0)$ is 2 (one of type I and one of type II);*

g) *the number of eigenvalues in each the intervals*

$$\left(-((\pi(j+1)a^{-1})^2 - q)^{\frac{1}{2}}, -((\pi j a^{-1})^2 - q)^{\frac{1}{2}}\right)$$

and in each the intervals

$$\left(((\pi j a^{-1})^2 - q)^{\frac{1}{2}}, ((\pi(j+1)a^{-1})^2 - q)^{\frac{1}{2}}\right)$$

for $j = q^{\frac{1}{2}}a\pi^{-1}, q^{\frac{1}{2}}a\pi^{-1} + 1, \ldots$ is 1.

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