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GRAPHONS AND RENORMALIZATION OF LARGE FEYNMAN DIAGRAMS

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Abstract. The article builds a new enrichment of the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams in the language of graph functions.

Keywords: graph functions, Dyson–Schwinger equations, Connes–Kreimer renormalization Hopf algebra.

Mathematics Subject Classification: 05C05, 05C63, 81T16, 81T18.

1. INTRODUCTION

Feynman diagrams and their corresponding finite or infinite formal sums are the central objects in Lagrangian approach to Quantum Field Theory (QFT). This research article aims to explain a new combinatorial formalism in dealing with these physical type of diagrams and expansions on the basis of graph functions. The main strategy is to discuss the concept of convergence for an arbitrary sequence of Feynman diagrams with respect to cut-distance topology. The original objective is to build a new description for the algebraic renormalization machinery on infinite formal expansions of Feynman diagrams (originated from fixed point equations of Green's functions in QFT) in the language of the theory of graphons.

Perturbative QFT, as the result of path integral machinery, generates Green's functions to encode fundamental information of physical theories on the basis of a class of elementary decorated graphs and their formal expansions. These elementary graphs, which are known as one particle irreducible Feynman graphs, work as the building blocks of the physical theory to analyze complicated Feynman diagrams together with nested or overlapping sub-divergencies. In a general configuration, divergencies originated from Green's functions could be classified and studied under two different groups. The first group concerns sub-divergencies of each single Feynman diagram which contribute to Green's functions. These sub-divergencies have been studied under perturbative renormalization where thanks to the advanced mathematical treatments and tools, we already have various theoretical methods and practical techniques to generate finite values from ill-defined Feynman integrals. However there remain many complicated problems in this group which require new advanced mathematical tools to handle sub-divergencies. The second group concerns divergencies originated from QFT-model physical theories with strong coupling constants. In this situation we should deal with infinite formal expansions of Feynman diagrams where the self-similar nature of Green's functions has been concerned to formulate fixed point equations. These equations, which are known as Dyson–Schwinger equations, address complex situations beyond perturbation theory. Dealing with these equations is the main challenge in High Energy Theoretical Physics [6, 13, 31, 32].

There are standard numerical and analytic methods for the computation of some non-perturbative parameters in physical theories but in a general setting, these investigations have not improved rigorously our knowledge about non-perturbative phenomena. Our best chance in this situation is to make stronger the mathematical foundations of non-perturbative QFT. In this direction, search for some new advanced mathematical structures originated from Dyson–Schwinger equations could be helpful for the better understanding of the phenomenology of non-perturbative parameters [33, 35, 36, 40, 41].

Advanced mathematical aspects of Feynman diagrams, modern QFTs and related topics have confirmed the extraordinary applications of combinatorial techniques for the study of quantum systems with infinite degrees of freedom. We can refer the reader to the following selected works in this direction [7, 9–12, 18–21, 29, 30, 33–37, 39–41]. In a big picture, the original achievement of this research work is to address another application of the theory of combinatorics to QFT. We provide a new class of combinatorial methods for the study of sequences of Feynman diagrams such as formal expansions generated by fixed point equations of Green's functions. We aim to search for a new distance on the set of Feynman graphs, making it a compact complete space, which allows us to understand the concept of a large Feynman graph as the limit of a sequence of finite expansions of Feynman graphs with respect to this distance. Our framework is to embed Feynman graphs of a physical theory in the set of graphons via the Connes–Kreimer rooted tree construction represented by pixel pictures. Then we explain that how graph functions are capable of being useful to create a new methodology in dealing with Dyson–Schwinger equations where as the consequence, we will formulate a generalization of the Connes–Kreimer BPHZ renormalization for the level of large Feynman graphs.

Graphons play a central role in the theory of graph limits which has been introduced and developed by Lovász, Szegedy, Borgs, Chayes, Sós and Vesztergombi [2, 17, 27, 28]. These infinite combinatorial objects are actually symmetric measurable functions of the form $f : \Omega \times \Omega \longrightarrow [0, 1]$ for an arbitrary probability space Ω and they could be interpreted as the limits of enough large sequences of finite graphs such as weighted graphs, directed graphs, multigraphs, posets, etc. If we choose the closed unit interval as the probability space, then it is possible to show that every graph limit can be represented by a graphon but such representations of graph limits are not unique. In a short period of time, graphons have been applied in several different fields of research such as theory of large networks in Computer Science, Theory of Probability and Statistics, Combinatorics, Measure Theory and Functional Analysis [3–5, 16, 22].

Problem about convergence and equivalency of graphons have led people to work on cut norm and cut metric where converging to graph limits have been described under different but equivalent settings. The original idea in this direction is to assign limits to sequences of unlabeled graphs when their number of vertices tends to infinity. In other words, a sequence of graphs is convergent, if its corresponding sequence of random graphs converges when the number of vertices of graphs tends to infinity. This perspective enables us to concern the notion of convergence in the context of homomorphism densities [2, 28].

Search for any possible applications of graphons to Quantum Physics has already been started in [36] where a new differential calculus machinery for the study of Dyson–Schwinger equations of Green's functions is built. This article plans to search for new interconnections between the theory of graphons and infinite formal expansions of Feynman diagrams which contribute to Green's functions of a given QFT-model physical theory with strong coupling. We find a machinery to interpret limits of sequences of Feynman diagrams in the language of graph functions where as the fundamental result, at first, we will obtain an enriched version of the Connes–Kreimer renormalization Hopf algebra on graphons and at second, we will obtain a generalization of the BPHZ perturbative renormalization which works at the level of graph limits and cut metric. The outputs of this work is capable to provide some new combinatorial tools in dealing with non-perturbative parameters originated from Green's functions.

We aim to provide a new interpretation of infinite formal sums of Feynman diagrams (originated from solutions of Dyson–Schwinger equations) in the language of graphons. For this purpose, we formulate a graph function interpretation of infinite trees as the limits of sequences of trees (Corollary 2.2). Then we apply the rooted tree representation of Feynman diagrams to obtain a new graph function interpretation of these physical diagrams (Proposition 3.1). This new perspective leads us to achieve the concept of convergence for the infinite sequences of Feynman graphs (Corollaries 4.1 and 4.3). As the immediate applications of these investigations, we formulate a Hopf algebraic structure on graphons (Proposition 3.2) originated from the Kreimer's renormalization coproduct. Then we obtain a new enrichment of the Connes–Kreimer Hopf algebra of Feynman diagrams (Corollary 4.4) which is completed with respect to cut-distance topology. Furthermore, we provide a new approach to deal with Dyson–Schwinger equations in the language of graph functions and cut-distance topology (Proposition 4.6). We show that solutions of these non-perturbative type of equations could be encoded by classes of unlabeled graphons. Actually, these solutions belong to a completed version of the Connes–Kreimer renormalization Hopf algebra with respect to the n-adic topology. This topology is the result of the graduation parameter originated from number of internal edges or vertices of Feynman graphs. When we deal with large graphs in physical theories with strong couplings, the number of vertices or internal edges tends to infinity which makes the n-adic distance equal to zero which is useless. Search for a non-trivial distance on infinite graphs could be helpful for the computation of non-perturbative parameters generated by Dyson–Schwinger equations. In this direction, we show that the topology generated by cut-distance metric provides some new opportunities where we apply the graph function representation of (large) Feynman graphs and the enriched renormalization Hopf algebra to obtain a new modification of the BPHZ renormalization machinery for solutions of Dyson–Schwinger equations (Corollary 4.7).

1.1. OUTSTANDING RESULTS

There are two fundamental achievements in this work which could be useful in dealing with non-perturbative parameters. Thanks to the graph function representation of Feynman diagrams on the basis of rooted trees (Sections 2 and 3), Corollary 4.3 and Proposition 4.6 lead us to obtain a new description of infinite formal expansions of Feynman diagrams in the language of random graphs. In addition, thanks to the existence of a new distance on the set of Feynman graphs with respect to the cut-distance topology, Corollary 4.7 describes mathematically the structure of an algebraic renormalization machinery which works on infinite Feynman diagrams originated from solutions of Dyson–Schwinger equations.

2. FUNDAMENTAL STRATEGY: A NEW APPROACH TO INFINITE TREES

In this section, we plan to apply graph functions for the study of infinite sequences of decorated rooted trees to bring a new understanding of convergent or divergent sequences. We will present an infinite tree as a graph function generated as the limit of a sequence of pixel pictures. In other words, if we generate pixel pictures in terms of information which come from some simple graphs or matrices such as rooted trees, then we will enable to release the notion of convergence for infinite sequences of rooted trees with respect to the cut-distance topology.

An overview on the basic structure of graphons has been provided in Appendix A and here we directly deal with decorated rooted trees.

Definition 2.1. A rooted tree is a finite simply connected graph t which contains a set $V(t)$ of vertices, a set $E(t)$ of edges and a distinguished vertex $r_t \in V(t)$. It is called planar, if there exists an extra information which embeds this tree in the plane. Otherwise, it is called non-planar.

- (i) A rooted tree t is called vertex-decorated by a set S if there exists a bijection map $\alpha_S : V(t) \longrightarrow S$ which determines the label of each vertex of t.
- (ii) A rooted tree t is called edge-decorated by a set T if there exists a bijection map $\alpha_T : E(t) \longrightarrow T$ which determines the label of each edge of t.
- (iii) A rooted tree t is called (vertex,edge)-decorated by a pair (S, T) if there exists a couple (α_S, α_T) of bijective maps which obey the conditions (i) and (ii).

An isomorphism between two decorated rooted trees t_1, t_2 is a pair of bijections (f_V, f_E) such that $f_V : V(t_1) \longrightarrow V(t_2)$ and $f_E : E(t_1) \longrightarrow E(t_2)$ which preserve roots, decorations and all incidences. Incidence means that for each vertices $v_1, v_2 \in V(t_1)$, $f_V(v_1), f_V(v_2) \in V(t_2)$, $f_E(v_1v_2) = f_E(v_1)f_E(v_2) \in E(t_2)$.

If we have labels of the class vertex-decorated, then the beginning and ending vertices of each edge in the tree have decorations. This information provides natural decorations on edges of the tree. For example, the edge between two vertices v_i and v_j is decorated by $v_i v_j$.

A rooted forest vertex-decorated by S is an ordered set $F = \{t_1, \dots, t_n\}$ of decorated rooted trees. Two rooted forest $F_1 = \{t_1, \dots, t_n\}$ and $F_2 = \{t'_1, \dots, t'_m\}$ are isomorphic if $m = n$ and there exists a permutation $\rho \in S_n$ together with isomorphisms $f_i : t_i \simeq t'_{\sigma(i)}$.

Each isomorphism class of labeled rooted trees contains different possible decorations which could be defined on edges of each tree. It is possible to produce a multi-edge complicated graph with respect to each class where each edge in this new graph is the symbol of a particular decoration. Work on the sequences of these multi-edges graphs could be interesting when we want to discuss about the notion of homomorphism density. At this level, since we plan to concern the graph limit of an infinite sequence of finite graphs, we use basic format of trees for the presentation of classes of trees to simplify the original idea of our machinery. Let us discuss about the existence of a limit for an infinite sequence of classes of rooted trees in terms of some examples.

As the first example, we proceed by finding a limit for the infinite sequence $(l_n)_{n\geq 1}$ of classes of ladder trees given as in Figure 1.

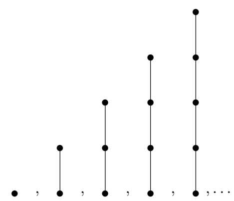

Fig. 1. A sequence of ladder trees

We can associate a pixel picture to each ladder tree l_n where if there exists a direct edge between two vertices, then the corresponding box in the pixel graph would be black and otherwise, the box would be white. In terms of this rule, the sequence in Figure 1 could be replaced by the following sequence of finite pixel pictures in Figure 2.

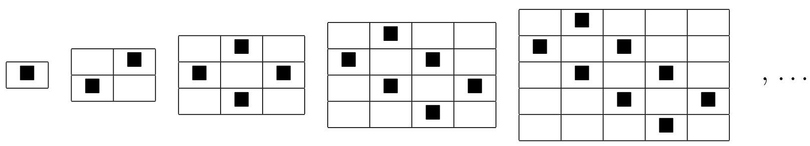

Fig. 2. A pixel picture representation of ladder trees

Since the adjacency matrix is a symmetric matrix, we connect the black points in the above of the main diagonal by segments and separately, connect the black points in the below of the main diagonal by segments. When n tends to infinity, this sequence of pixel pictures together with added segments almost surely converges to the following presentation in Figure 3, which could be considered as the domain of a symmetric Lebesgue measurable function from $[0,1] \times [0,1] \subset \mathbb{R}^2$ to $[0,1] \subset \mathbb{R}$.

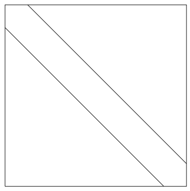

Fig. 3. The domain of a labeled graphon for the infinite ladder tree

The labeled graphons associated with the diagram in Figure 3 can be determined by the class of graph functions (with respect to the relation (6.1) given in Appendix A) which contains functions with the general form f^{ϵ} (for each $\epsilon > 0$) such that it has the value 1 on the set

$$\{(x,y) \in [0,1] \times [0,1] : y = (1-\epsilon) - x \quad \text{or} \quad y = (1+\epsilon) - x\}$$

and 0 on other points.

Therefore the unlabeled graphon originated from Figure 3 leads us to the existence of a convergence for the initial sequence (in Figure 1) of ladder trees.

As the second example, we proceed by finding a limit for the following infinite sequence of classes of rooted trees given in Figure 4.

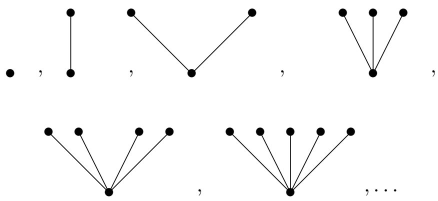

Fig. 4. A sequence of non-planar rooted trees with increasing number of leaves

We can replace the sequence in Figure 4 by the following infinite sequence of pixel pictures (labeled graphons) given in Figure 5.

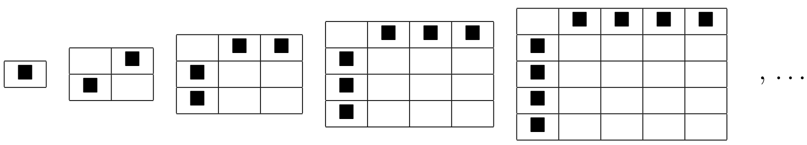

Fig. 5. A pixel picture representation

If we connect the black points in the horizontal direction by adding segments and separately, connect the black points in the vertical direction by adding segments, then whenever n tends to infinity, this sequence of pixel pictures together with added segments almost surely converges to the following presentation in Figure 6.

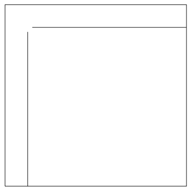

Fig. 6. The domain of a labeled graphon for the tree with infinite number of leaves

This presentation could be considered as the domain of a symmetric Lebesgue measurable function from $[0,1] \times [0,1] \subset \mathbb{R}^2$ to $[0,1] \subset \mathbb{R}$. The labeled graphons associated with the diagram in Figure 6 can be determined by the class of graph functions (with respect to the relation (6.1) given in Appendix A) with the general form g^{ϵ} (for each $\epsilon > 0$) such that it has the value 1 on the set

$$\{(x,y) \in [0,1] \times [0,1] : x = \epsilon \text{ and } 0 \leq y < 1-\epsilon\}$$
$$\cup \{(x,y) \in [0,1] \times [0,1] : y = 1-\epsilon \text{ and } \epsilon < x \leq 1\}$$

and the value 0 on its complement.

Therefore the unlabeled graphon originated from Figure 6 leads us to the existence of a convergence for the initial sequence (Figure 4) of rooted trees.

As the third example, we proceed by finding a limit for the following infinite sequence of classes of non-planar rooted trees given in Figure 7.

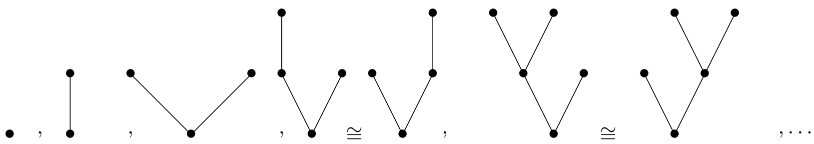

Fig. 7. A sequence of non-planar binary rooted trees

We can replace the sequence in Figure 7 by the following sequence of pixel pictures (as labeled graphons) given in Figure 8.

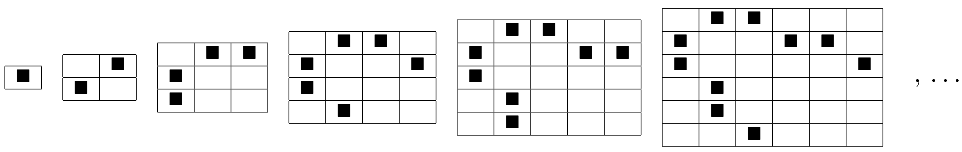

Fig. 8. A pixel picture representation of binary trees

If we connect the black points in the above of the main diagonal by adding horizontal and vertical segments and separately, connect the black points in the below of the main diagonal by adding vertical and horizontal segments, then whenever n tends to infinity, this sequence of pixel pictures together with added segments almost surely converges to the following presentation in Figure 9.

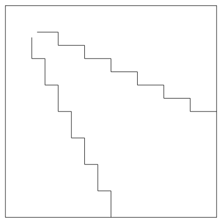

Fig. 9. The domain of a labeled graphon for the infinite binary tree

This presentation could be considered as the domain of a symmetric Lebesgue measurable function from $[0,1] \times [0,1] \subset \mathbb{R}^2$ to $[0,1] \subset \mathbb{R}$. The separate steps in this graphon, which live in the above and below of the main diagonal, come from steps of length two in the matrices which goes away when n tends to infinity. But the distance between steps (vertical segments in the above part and horizontal segments in the below part) could be decreased into arbitrary small $\epsilon > 0$. The labeled graphons associated with the diagram (in Figure 9) can be determined by the class of graph functions such as h^{ϵ} (with respect to the relation (6.1) given in Appendix A). It contains the characteristic function for the set of step by step sub-intervals together with added small segments presented in the above diagram.

Therefore the unlabeled graphon originated from Figure 9 leads us to the existence of a convergence for the initial sequence (in Figure 7) of rooted trees.

Corollary 2.2 (first fundamental definition)**.** *A sequence $\{t_n\}_{n\geq 1}$ of classes of labeled rooted trees is convergent when n goes to infinity, if the corresponding sequence $\{[f^{t_n}]\}_{n\geq 1}$ of unlabeled graphons converges to a unique unlabeled graphon with respect to the cut-distance topology.*

Proof. Unlabeled graphons and cut-distance topology have been introduced in Appendix A. The above examples lead us to formulate the limit of a sequence of rooted trees in terms of the behavior of its corresponding sequence of pixel pictures (as labeled graphons). But we have mentioned that by changing label or the style of pixel picture, we will get another result which makes problem for a well-defined concept. If we apply the notion of "unlabeled graphon", which eliminates the dependency of graphons on labels by working on the equivalence class (6.1) defined in Appendix A, we can achieve a well-defined concept. In a general configuration, a convergent sequence $\{[f^{t_n}]\}_{n\geq 1}$ of unlabeled graphons under the cut-distance topology could determine the domain of a symmetric Lebesgue measurable function such as U from $[0,1] \times [0,1] \subset \mathbb{R}^2$ to $[0,1] \subset \mathbb{R}$ with the corresponding unlabeled graphon class $[U]$. The uniqueness of this limit is discussed in Appendix B. Now if we consider the pixel picture presentation of the class $[U]$ as one of its labeled graphons, then we can associate an infinite graph t

to U with the corresponding graphon class f^t. Thanks to the uniqueness of the limit and the fact that $f^t \in [U]$, we have $[U] = [f^t]$.

The infinite graph t with respect to the class $[f^t]$, which we name it a large tree, is the unique limit of the sequence $\{t_n\}_{n\geq 1}$ with respect to the cut-distance topology. □

The first fundamental definition means that for a given sequence $\{t_n\}_{n\geq 1}$ of classes of decorated rooted trees, each sequence $\{\mathrm{lg}_n\}_{n\geq 1}$ of labeled graphons where each term $\mathrm{lg}_n \in [f^{t_n}]$ is convergent to a labeled graphon $\mathrm{lg} \in [f^t]$ with respect to the metric (6.4) given in Appendix A.

Now we have the concept of convergence at the level of trees but there still remains one question about the identification of the infinite graph t which is addressed by the graphon $[f^t]$. Since we are dealing with infinite graphs, we can approximate them by using random graphs. Lemma 6.6 in Appendix A confirms that each sequence of random graphs with respect to the graphon $[f^t]$ converges to that graphon. We can consider random graphs as labeled graphons, therefore for each $n \geq 1$, let $[R(n, f^t)]$ be the random graph class (as an unlabeled graphon) of order n with respect to the unlabeled graphon $[f^t]$ such that n is the number of selected nodes in the closed interval $[0, 1]$. Now for enough large n, its corresponding random graph class presents approximately the large tree t.

In Appendix B, we have provided a proof about the uniqueness of the limit of a convergent sequence of decorated rooted trees with respect to the cut-distance topology.

As the last result of this section, we provide a new interpretation of large trees in the language of random graphs.

Corollary 2.3. *For a given large tree t, which is the result of the convergence of an infinite sequence $\{t_n\}_{n\geq 1}$ of decorated rooted trees, there exists a sequence $(R_n)_{n\geq 1}$ of random graphs associated to trees t_ns which converges to the unique unlabeled graphon $[f^t]$ with respect to the cut distance topology.*

Proof. Let there exists an infinite sequence $\{t_n\}_{n\geq 1}$ of decorated rooted trees which converges to the large tree t under the cut-distance topology. Following the first fundamental definition (Corollary 2.2), we know that the corresponding sequence $\{[f^{t_n}]\}_{n\geq 1}$ of unlabeled graphons converges to the unlabeled graphon $[f^t]$.

Usually we can see rooted trees as partially ordered sets. Consider the standard orientation on rooted trees which begins from the root and ends in leaves of a tree. This orientation inherits a partial order relation $\leq$ on vertices of a rooted tree which allows us to interpret a rooted tree as a poset where the root is the minimal object and leaves are the maximal elements [34].

For each $n \geq 1$, choose a finite subset $V(t_n, \rho_n)$ in the closed interval $[0, 1]$ which contains $|V(t_n)|$ nodes in $[0, 1]$ such that these nodes are selected by projections of vertices of the tree t_n under a fixed injective embedding map ρ_n of partial orders from t_n in $[0, 1]$. This means that

$$\forall v_i, v_j \in t_n : \quad v_i \leq v_j \Longleftrightarrow \rho_n(v_i) \leq \rho_n(v_j).$$

In the next step, for each $n \geq 1$, build a random graph R_n (as a labeled graphon which belongs to $[f^{t_n}]$) in terms of the points in the subset $V(t_n, \rho_n)$ where for each pair $v_i, v_j \in V(t_n, \rho_n)$, the corresponding edge $v_i v_j$ is included with probability $f^{t_n}(v_i, v_j)$.

Now thanks to the formula (6.4), Lemma 6.6 in Appendix A and the first fundamental definition (Corollary 2.2) with probability one, the sequence $\{R_n\}_{n\geq 1}$ converges to the graphon f^t under the cut-distance topology when n goes to infinity. □

3. FEYNMAN DIAGRAMS UNDER A NEW COMBINATORIAL SETTING

Under a mathematical setting, a Feynman diagram Γ is a finite decorated graph which contains a set of vertices as the symbol of interactions, a set of internal edges as the symbol of virtual elementary particles and a set of external edges as the symbol of elementary particles. Decorations provide essential physical information such as types of elementary particles and their momenta which obey the conservation law. Sub-divergencies which live in iterated Feynman integrals have been encoded via nested or overlapping loops. Removing these infinities has been interpreted via a Hopf algebraic formalism where the renormalization coproduct is the mathematical reformulation of the Zimmermann's forest formula. This mathematical treatment, which generates an infinite dimensional complex Lie group, has led us to a spectrum of advanced mathematical tools for the production finite values from ill-defined Feynman integrals [8, 14, 15, 24, 29, 34].

In this part we plan to explain a new mathematical interpretation of Feynman diagrams on the basis of graph functions. We apply rooted tree representation of Feynman diagrams to determine a class of infinite graphs which contribute to limits of sequences of finite Feynman diagrams under a topological setting. This study will have two fundamental achievements where we will obtain the structure of a Hopf algebra on a class of graphons and then we will find a new description of solutions of Dyson–Schwinger equations in the language of graphons and cut-distance topology.

The Connes–Kreimer Hopf algebra of rooted trees enjoys a universal property which enables us to study perturbative renormalization machinery in a renormalizable QFT via combinatorial tools. The basic strategy in this universal setting is to use decorations on vertices and edges such that each vertex in a tree is the symbol for a simple primitive loop and each edge between two vertices in a tree presents the positions of the related loops (sub-divergencies) with respect to each other and the whole Feynman diagram. If there is no direct edge or any sequence of edges which connects two vertices in the tree, then it means that the corresponding two loops are independent from each other in the original graph. This model works nice for Feynman diagrams together with nested sub-divergencies. For overlapping sub-divergencies, there are some challenges but the corresponding tree representation could be a linear combination of decorated rooted trees. We refer the reader to [1, 12, 23, 24] for further details in this issue.

Generally speaking, there exists an injective Hopf algebraic homomorphism Ξ from the Connes–Kreimer renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$ to the renormalization Hopf algebra of non-planar rooted trees decorated by primitive 1PI Feynman diagrams in Φ. For each Feynman graph Γ with the general form

$$\Gamma = \prod_{i=1}^{k_j} \Gamma_j \star_{j,i} \gamma_{j,i},$$

where $r, k_1, \ldots, k_r$ are some positive integers and Γ_js are 1PI primitive sub-graphs,

$$\Xi : \Gamma \longmapsto \sum_{j=1}^{r} B^{+}_{\Gamma_j, G_{j,i}} \Big[\prod_{i=1}^{k_j} \Xi(\gamma_{j,i}) \Big] \tag{3.1}$$

such that $G_{j,i}$s are the gluing information with respect to the insertion operator $\star_{j,i}$ [24]. This morphism is the key tool for us to interpret Feynman diagrams in terms of decorated non-planar rooted trees. Now according to our explained strategy, which relates rooted trees with the theory of graphons under pixel pictures, we are going to interpret Feynman diagrams in terms of pixel pictures.

As an example, consider the following infinite sequence of Feynman diagrams (see Figure 10) which are diagrams in a simplified toy model theory.

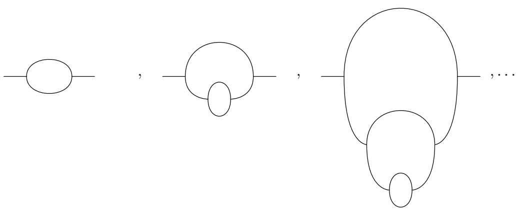

Fig. 10. A sequence of Feynman diagrams with increasing number of nested loops

This sequence shows Feynman diagrams with many many number of nested loops for large n which reports the appearance of iterated integrals together with sub-divergencies inside of the main integral. The main question is that where does this sequence go and what does it mean? Work on its decorated tree version could be helpful. So choose a type of decoration which restores each simple loop such as in Figure 11 in a vertex of the tree such that edges in the tree make clear positions of all loops in the original Feynman diagram with respect to each other and the original diagram.

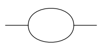

Fig. 11. A primitive 1PI Feynman diagram

Therefore the sequence in Figure 10 generates the infinite sequence in Figure 1 of decorated non-planar rooted trees which tends to the graphon in Figure 3 when n

goes to infinity. This observation, as an intension, encourages us to interpret the convergence of the original sequence in Figure 10 on the basis of the infinite type graphon in Figure 3.

As other example, consider another infinite sequence of Feynman diagrams in a simplified toy model theory which is given in Figure 12.

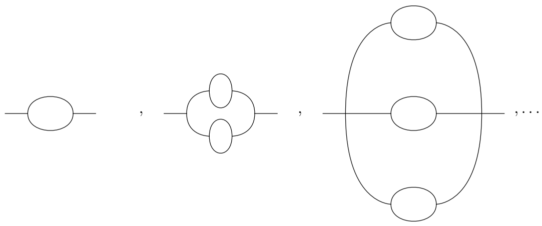

Fig. 12. A sequence of Feynman diagrams with increasing number of independent loops

This sequence shows Feynman diagrams with many many independent nested loops for large n. If we want to understand where this sequence goes, then we need to apply its decorated tree interpretation. Using the type of decorations applied in the previous example leads us to the infinite sequence in Figure 4 of decorated rooted trees corresponding to the sequence in Figure 12 which tends to the graphon in Figure 6 when n goes to infinity. This observation, as an intension, encourages us to interpret the convergence of the original sequence in Figure 12 on the basis of the infinite type graphon in Figure 6.

Now it is time to concern the possibility of rebuilding a Feynman graph in terms of graphons. For this purpose we require to modify the renormalization coproduct at the level of pixel pictures to obtain a procedure which reconstructs uniquely a Feynman diagram (in a toy model) in terms of information given by an unlabeled graphon or an equivalence class of labeled graphons. By working on trees (as simple graphs), we can see what is happening to the pixel picture related to a Feynman diagram during the application of this specific coproduct. As an example, consider the Feynman diagram Γ in a simplified toy model theory (see Figure 13).

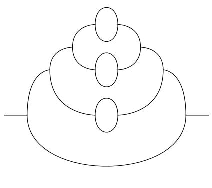

Fig. 13. A Feynman diagram with nested and independent loops

Its rooted tree representation t_Γ is given in Figure 14 by the decorated non-planar rooted tree.

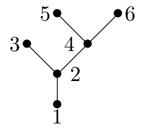

Fig. 14. The tree representation of a Feynman diagram

The pixel picture p_{t_Γ} associated to the tree t_Γ can be determined by this rule that for any direct edge between two vertices, its corresponding box is black and otherwise the box is white.

It is important to remark that the tree t_Γ is decorated such that the vertex with number 1 (as the root) is the symbol of the main divergent subdiagram in Γ and other numbers in vertices tell us about nested loops in the original graph Γ and their positions with respect to each other. The proper sub-trees generated by the renormalization coproduct determine the corresponding pixel pictures $p_{t_1}, \ldots, p_{t_{11}}$ such that the initial pixel picture p_{t_Γ} is the result of the union of them. But since there are some intersections among these pixel pictures, we can not uniquely correspond a sub-tree to each pixel picture to rebuild uniquely the original graph. In general, each of these pixel pictures does not deliver us a unique rooted tree because for each two vertices v_i and v_j, which are connected to each other, we have three possibilities to identify edges. The one candidate is the edge from v_i to v_j, other is the edge from v_j to v_i and the third one is both edges at the same time (which is not a tree). Sometimes this issue does not allow us to identify correctly the root. For example, we can have a pixel picture which could give us two classes of sub-trees with different vertices 1 and 2 as roots. There is also another issue about the used labeling for pixel pictures which should be paid attention. Generally speaking, if we change the labeling on pixel pictures, then we can get completely different rooted trees.

This class of challenges could be solved by modifying the decorations of trees. In this case, for any given Feynman diagram, we concern orientations as a new class of decorations on edges of the corresponding tree. These orientations on edges allow us to uniquely identify the situation between two nested loops. Under this new labeling, we can relate a class of pixel pictures to these oriented labeled rooted trees where if there exists an edge from v_i to v_j, then the corresponding box in the pixel picture has a black color.

Proposition 3.1. *For each Feynman diagram Γ in a physical theory, there exists a unique graphon class $[f^{t_\Gamma}]$ as the unlabeled graphon with respect to Γ such that this graphon class has enough information to rebuild uniquely the original diagram Γ.*

Proof. In general, we have two classes of sub-divergencies in Feynman diagrams and therefore we divide the proof into two different situations.

Case (i). For a given Feynman graph Γ with no overlapping sub-divergencies, there exists a unique decorated oriented non-planar rooted tree t_Γ. Since rooted trees are simple type graphs, Lemma 6.5 in Appendix A identifies uniquely the graphon class $[f^{t_\Gamma}]$ (as the unlabeled graphon) with respect to the diagram Γ. At first we build a labeled graphon f^{t_Γ} which belongs to the class $[f^{t_\Gamma}]$. Let $v_1, \ldots, v_n$ be decorations of vertices of the tree t_Γ such that each v_i is the symbol of a nested loop (a primitive 1PI Feynman sub-graph of Γ) and each edge between v_i and v_j reports the position of nested loops with respect to each other. If $x \in (0,1]$, we define $i = \lceil nx \rceil$ and set $u_x = v_i$ to be the ith vertex of t_Γ. For $x = 0$ we set $u_x = v_1$ and define

$$f^{t_\Gamma}(x,y) := \begin{cases} 1 & \text{if } u_x \text{ and } u_y \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the pixel picture p_{t_Γ} (as a labeled graphon) generated by the tree t_Γ such that $p_{t_\Gamma} \in [f^{t_\Gamma}]$. It produces an adjacency matrix which is a symmetric matrix. Restrict this matrix into its corresponding upper triangular matrix. The class of decorated oriented non-planar rooted trees with respect to this upper triangular type matrix provides enough materials to rebuild the original graph Γ.

Case (*ii*). Suppose Γ has overlapping sub-divergencies. There exists a linear combination $u_\Gamma := t_1 + \ldots + t_n$ of finite number of decorated oriented non-planar rooted trees which corresponds to the original diagram Γ. Now thanks to Lemma 6.5 in Appendix A, we can determine a unique graphon class $[f^{u_\Gamma}]$ (as the unlabeled graphon) with respect to the diagram Γ. The labeled graphons f^{u_Γ} corresponding to u_Γ can be determined in terms of the normalization of the combination of labeled graphons $f^{t_1}, \ldots, f^{t_n}$ which means that

$$f^{u_\Gamma}(x,y) := \frac{f^{t_1}(x,y) + \ldots + f^{t_n}(x,y)}{|f^{t_1}(x,y) + \ldots + f^{t_n}(x,y)|}.$$

Now similar to the Case (i), the pixel picture p_{u_Γ} guides us to rebuild the original graph Γ. □

Proposition 3.2. *Let $\mathcal{G}^{\text{simple}}_{[0,1]}$ be the set of all isomorphism classes which are unlabeled graphons with respect to finite simple graphs. Consider the commutative polynomial algebra generated by symbols $[f^t]$, which represent unlabeled graphons in $\mathcal{G}^{\text{simple}}_{[0,1]}$, over the field $\mathbb{K}$ with characteristic zero. Then there exists a renormalization type of Hopf algebra structure on $\mathcal{G}^{\text{simple}}_{[0,1]}$.*

Proof. Proposition 3.1 enables us to modify the renormalization coproduct on Feynman diagrams (and rooted trees) for graphons in $\mathcal{G}^{\text{simple}}_{[0,1]}$.

At the first stage, this algebra is unital associative such that the graphon class with respect to the empty graph is the unit. In addition, the number of vertices of simple graphs determines a grading structure on this algebra.

At the second stage, we extend this polynomial type algebra to a bialgebra H^{simple} such that its counit $\varepsilon : H^{\text{simple}} \longrightarrow \mathbb{K}$ is given by

$$\varepsilon([f^{t_1}] \ldots [f^{t_n}]) = \begin{cases} 0 & \text{if } [f^{t_1}] \ldots [f^{t_n}] \neq 1, \\ 1 & \text{otherwise.} \end{cases}$$

On the other hand, define a coproduct on the generators of H^{simple} and then extend it in a linear way which produces an algebra morphism. Consider an unlabeled graphon $[f^t]$ with the corresponding decorated oriented non-planar rooted tree t. Applying the decoration type explained by Figure 11 allows us to identify a unique Feynman graph Γ corresponding to the tree t in a simplified toy model physical theory. Now thanks to the renormalization coproduct on Feynman diagrams, define

$$\Delta_{\text{simple}}([f^t]) = \sum [f^{t_\gamma}] \otimes [f^{t_{\Gamma/\gamma}}] \tag{3.2}$$

such that the sum is taken over all unlabeled graphons $[f^{t_\gamma}]$ generated by the rooted tree representation t_γ of γ and $[f^{t_{\Gamma/\gamma}}]$ is the unlabeled graphon corresponding to the complement graph Γ/γ.

At the third stage, thanks to the addressed grading structure on $\mathcal{G}^{\text{simple}}_{[0,1]}$ and the coproduct (3.2), the required antipode on H^{simple} could be defined inductively. □

4. A NEW PERSPECTIVE ON INFINITE EXPANSIONS OF FEYNMAN DIAGRAMS ON THE BASIS OF GRAPHONS

The reinterpretation of the Bogoliubov–Zimmermann's forest formula in the language of the theory of words ([6, 8, 41]) has opened a new approach in dealing with Feynman diagrams on the basis of new combinatorial tools such as decorated trees and theory of Hall sets. Now thanks to the first fundamental definition (Corollary 2.2), it is possible to initiate a new understanding of infinite sequences of Feynman graphs in the context of graphons. This new perspective will lead us to deal with infinite formal series of Feynman diagrams originated from Dyson–Schwinger equations under a topological setting.

We have enough materials to inherit a notion of distance among combinatorial Feynman diagrams which leads us to define convergence or divergency of infinite sequences of Feynman diagrams with respect to the cut-distance topology.

Corollary 4.1 (second fundamental definition)**.** *A sequence $\{\Gamma_n\}_{n\geq 1}$ of Feynman diagrams in a given physical theory is convergent when n goes to infinity, if its corresponding sequence $\{[f^{t_{\Gamma_n}}]\}_{n\geq 1}$ of unlabeled graphons converges to the class $[f^{t_\Gamma}]$ with respect to the cut-distance topology.*

Proof. We have almost explained the machinery to interpret a given finite Feynman diagram Γ_n via its corresponding unique unlabeled graphon class $[f^{t_{\Gamma_n}}]$. In short, we consider the injective Hopf algebraic homomorphism Ξ (3.1) to present Γ_n in terms of a decorated oriented non-planar rooted tree t_{Γ_n}. Lemma 6.5 in Appendix A enables us to formulate the unlabeled graphon class $[f^{t_{\Gamma_n}}]$ with respect to the simple graph t_{Γ_n}.

Now thanks to Proposition 3.1 and the first fundamental definition (Corollary 2.2), we can formulate the concept of convergence for the sequence $\{\Gamma_n\}_{n\geq 1}$ with respect to the behavior of the corresponding sequence $\{[f^{t_{\Gamma_n}}]\}_{n\geq 1}$ of unlabeled graphons. In other words, the convergent sequence $\{[f^{t_{\Gamma_n}}]\}_{n\geq 1}$ of unlabeled graphons under the cut-distance topology could determine the domain of a symmetric Lebesgue measurable function such as W from $[0,1]\times[0,1]$ to $[0,1]$ with the corresponding unlabeled graphon class $[W]$. The uniqueness of this limit is discussed in Appendix B. Now if we consider the pixel picture presentation of the class $[f^{t_\Gamma}]$ as one of its labeled graphons, then we can associate a large tree t_Γ to W with the corresponding graphon class $[f^{t_\Gamma}]$. Thanks to the uniqueness of the limit and the fact that $f^{t_\Gamma} \in [W]$, we have $[f^{t_\Gamma}] = [W]$. On the other hand, thanks to the injective Hopf algebraic homomorphism Ξ (3.1), we can determine uniquely a physical graph Γ with respect to t_Γ.

This infinite type of graph Γ, which we name it a large Feynman diagram, is the unique limit of the sequence $\{\Gamma_n\}_{n\geq 1}$ with respect to the cut-distance topology. □

The second fundamental definition tells us that each sequence $\{\mathrm{lg}_n\}_{n\geq 1}$ of labeled graphons where each term $\mathrm{lg}_n \in [f^{t_{\Gamma_n}}]$ is convergent to an infinite labeled graphon $\mathrm{lg} \in [f^{t_\Gamma}]$ with respect to the cut-distance topology generated by (6.2) in Appendix A. The uniqueness of this limit has been discussed in Appendix B.

The distance between two (large) Feynman diagrams Γ_1, Γ_2 can be defined by the cut-distance metric between their corresponding unlabeled graphons $[f^{t_{\Gamma_1}}]$ and $[f^{t_{\Gamma_2}}]$.

Thanks to this new graphon approach to Feynman diagrams, now we enable to apply random graphs to interpret the structure of the large graph Γ with more details.

Corollary 4.2. *An unlabeled graphon $[f^{t_\Gamma}]$ could be approximated by a sequence of random graphs.*

Proof. According to Lemma 6.6 in Appendix A, each sequence of random graphs, which is generated by the unlabeled graphon $[f^{t_\Gamma}]$, is convergent to that graphon. Consider random graphs as labeled graphons such that for each $n \geq 1$, let $[R(n, f^{t_\Gamma})]$ is the random graph class (unlabeled graphon) of order n with respect to the graphon $[f^{t_\Gamma}]$ and n selected nodes in the closed interval $[0, 1]$. So now for enough large n, the corresponding random graph class could be an estimation of the large Feynman graph Γ. □

Corollary 4.3. *For a given large Feynman graph Γ, which is the result of the convergence of an infinite sequence $\{\Gamma_n\}_{n\geq 1}$ of finite Feynman diagrams, there exists a sequence $(R_n)_{n\geq 1}$ of random graphs associated to graphs Γ_ns which converges to the unique unlabeled graphon $[f^{t_\Gamma}]$ with respect to the cut distance topology.*

Proof. Let there exists an infinite sequence $\{\Gamma_n\}_{n\geq 1}$ of finite Feynman diagrams which converges to the large graph Γ under the cut-distance topology. In terms of the second fundamental definition (Corollary 4.1), there exists a sequence of unlabeled graphons such as $\{[f^{t_{\Gamma_n}}]\}_{n\geq 1}$ which converges to $[f^{t_\Gamma}]$.

For each $n \geq 1$, make a finite subset $V(\Gamma_n, \rho_n)$ in the closed interval $[0, 1]$ with respect to the Feynman diagram Γ_n. This set contains $|V(t_{\Gamma_n})|$ nodes in $[0, 1]$ which is the number of vertices in the rooted tree t_{Γ_n} corresponding to the graph Γ_n. These nodes are selected by projections of the vertices in the graph t_{Γ_n} under a fixed injective poset type embedding map ρ_n from the tree t_{Γ_n} (as a poset) in the interval $[0, 1]$. Now define a sequence $\{R_n\}_{n\geq 1}$ of random graphs such that for each $n \geq 1$, the graph R_n (as a labeled graphon which belongs to $[f^{t_{\Gamma_n}}]$) could be built in terms of the points in the subset $V(\Gamma_n, \rho_n)$ where for each pair $v_i, v_j \in V(\Gamma_n, \rho_n)$, the corresponding edge $v_i v_j$ is included with probability $f^{t_{\Gamma_n}}(v_i, v_j)$.

Thanks to the formula (6.4) and Lemma 6.6 in Appendix A, Corollary 2.3, Proposition 3.1, the second fundamental definition (Corollary 4.1) and Lemma 7.2 in

Appendix B, with probability one, the sequence $\{R_n\}_{n\geq 1}$ converges to the graphon f^{t_Γ} under the cut-distance topology when n tends to infinity. □

Another consequence of the second fundamental definition is to enrich the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams with respect to the cut-distance topology.

Corollary 4.4. *There exists a complete and compact metric structure on the space of Feynman diagrams of a given physical theory Φ which is comaptible with the renormalization Hopf algebra.*

Proof. We plan to show that the renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$ on Feynman diagrams in Φ can be equipped by the cut-distance metric $d_\diamond$. It leads us to obtain a complete and compact metric structure on the space of Feynman diagrams. In addition, we show that the renormalization coproduct respects convergence under the cut-distance topology.

Let Γ be a large graph as the limit of a sequence of finite Feynman graphs such as $\{\Gamma_n\}_{n\geq 0}$. Thanks to Corollary 4.1, the corresponding sequence $\{[f^{t_{\Gamma_n}}]\}_{n\geq 0}$ of unlabeled graphons associated to finite (simple) graphs t_{Γ_n} converges to $[f^{t_\Gamma}]$ when n tends to infinity with respect to the cut distance topology. Thanks to Proposition 3.2, by induction, it can be seen that the sequence $\{\Delta_{\mathrm{simple}}([f^{t_{\Gamma_n}}])\}_{n\geq 0}$ converges to $\Delta_{\mathrm{simple}}([f^{t_\Gamma}])$ with respect to the cut-distance topology when n tends to infinity. It shows the compatibility of the enriched coproduct with the cut-distance topology. Thanks to the standard graduation parameter on the Hopf algebra of graphons, the antipode could be formulated inductively in terms of the coproduct which means that we can have the compatibility of the antipode with respect to the cut-distance topology.

Now it remains to modify this compatibility for the Connes–Kreimer renormalization coproduct and for this purpose consider the injective morphism which sends each finite Feynman graph Γ to its unique graphon class $[f^\Gamma]$ which is determined by Proposition 3.1. Then apply Proposition 3.2, Corollary 4.1 and Lemma 7.2 in Appendix B to extend this morphism naturally to achieve a Hopf algebraic injective from the Connes–Kreimer renormalization Hopf algebra $H_{\mathrm{FG}}(\Phi)$ to the Hopf algebra H^{simple} of graphons.

The resulting Hopf algebra is called the enriched renormalization Hopf algebra and denoted by $H^{d_\diamond}_{\mathrm{FG}}(\Phi)$. The distance $d_\diamond$ is obtained in terms of the embedding in the space of graphons. A completion, which concerns infinite graphs, should be taken in order to obtain a compact metric space. It means that $H^{d_\diamond}_{\mathrm{FG}}(\Phi)$ is an enrichment of the renormalization Hopf algebra. □

This completion will help us improve our previous efforts about the construction of a renormalization program on Dyson–Schwinger equations under an algebro-geometric setting [33, 35]. Infinite graphs, which live in the boundary of this complete space, are actually capable to encode solutions of Dyson–Schwinger equations. This means that the enriched Connes–Kreimer renormalization Hopf algebra is capable to encode the coproduct of a given large Feynman diagram. It can be done in terms of the limit of a sequence of co-products of finite Feynman graphs which tends to the original large graph with respect to the cut-distance topology. We plan to concern this observation with more details which leads us to build the modified version of the BPHZ perturbative renormalization for Dyson–Schwinger equations in the language of graph functions. For this purpose, at first, we show that solutions of Dyson–Schwinger equations could be encoded by the enriched Hopf algebra $H^{d_\diamond}_{\mathrm{FG}}(\Phi)$ which leads us to describe the convergence of this class of non-perturbative type of equations in the context of graphons and cut-distance topology.

Consider Φ as a renormalizable physical theory with the corresponding Hopf algebra $H = H_{\mathrm{FG}}(\Phi)$ of Feynman diagrams. Define a chain complex $C = \{C_n, \mathbf{b}\}_{n\geq 0}$ such that for each n, C_n is the set of all linear maps T from H to $H^{\otimes n}$ and C_0 is the field. Thanks to the renormalization coproduct, consider the operator

$$\mathbf{b}T := (id \otimes T)\Delta + \sum_{i=1}^{n} (-1)^i \Delta_i T + (-1)^{n+1} T \otimes \mathbb{I} \tag{4.1}$$

as the coboundary operator. The corresponding cohomology group H^1 generates Hochschild one cocycle. For each primitive 1PI Feynman diagram γ, if the insertion of a diagram into another graph does not break the compatibility between B^+_γ and the renormalization coproduct, then the operator B^+_γ is a 1-cocycle.

Definition 4.5. Given a family $\{\gamma_n\}_{n\geq 1}$ of primitive 1PI Feynman diagrams with the corresponding Hochschild one cocycles $\{B^+_{\gamma_n}\}_{n\geq 1}$, a class of combinatorial DSEs is defined by

$$X = \mathbb{I} + \sum_{n\geq 1} \alpha^n \omega_n B^+_{\gamma_n}(X^{n+1}). \tag{4.2}$$

Each equation DSE determines an infinite formal expansion of Feynman diagrams and the huge challenge would be to find a meaningful solution for this class of equations. The unique solution $X = \sum_{n\geq 1} \alpha^n X_n$ of DSE can be given in terms of the recursive relations of the form

$$X_n = \sum_{j=1}^{n} \omega_j B^+_{\gamma_j}\Big(\sum_{k_1+\ldots+k_{j+1}=n-j,\ k_i\geq 0} X_{k_1} \ldots X_{k_{j+1}} \Big) \tag{4.3}$$

such that X_0 is the empty tree. Each term X_n is produced on the basis of the terms with the lower degrees and furthermore, terms X_ns play the role of generators for a Hopf sub-algebra associated to the equation DSE [1, 18, 19, 25, 26, 40, 41]. This presentation of the solution X, which belongs to the completion of $H[[\alpha]]$ with respect to the n-adic topology, has been applied as a starting point to build some new mathematical structures which are capable to encode non-perturbative parameters [33, 35, 36].

Proposition 4.6. *Consider an equation DSE of the form* (4.2) *with the unique solution* $X = \sum_{n\geq 0} X_n$. *Make a new sequence* $\{Y_n\}_{n\geq 1}$ *of partial sums of* $\{X_n\}_{n\geq 0}$ *such that each element of this sequence includes a finite number of* X_n*s as the following way*

$$Y_n := X_1 + \ldots + X_n. \tag{4.4}$$

The sequence $\{Y_n\}_{n\geq 1}$ *converges to* X *with respect to the cut-distance topology.*

Proof. We will show that the construction of the $n-$adic topology originated from the graduation parameter on Feynman graphs leads us to determine a sequence of random graphs corresponding to each Dyson–Schwinger equation. In this situation, thanks to Lemma 6.6 in Appendix A, we will show that this sequence is convergent with respect to the cut-distance topology. This new relation will be built on the basis of the graph function representation of infinite rooted trees and rooted tree representation of Feynman diagrams which have led us to a new graphon type interpretation from large Feynman graphs explained in the previous parts.

We previously have seen that a pixel picture is a graphic model which is produced in terms of the adjacency matrix. They correspond to particular decorations of the original graph which means that an unlabeled graph could have several representations.

We need to introduce a sequence of scaling pixel pictures (or graphons) such as $(R_n)_{n\geq 0}$ which converges to the unique class $[f^{t_X}]$ as the unlabeled graphon associated to the large graph X when n tends to infinity.

Our plan is to make each R_n as a random graph with a particular edge probability. On the one side, the Connes–Kreimer Hopf algebra $H = H_{\mathrm{FG}}(\Phi)$ is connected graded finite type with respect to the number of internal edges. So we have $H = \bigoplus_{n\geq 0} H_n$ such that for each n, H_n is the homogeneous component of degree n. This grading gives us an increasing filtration $H = \bigoplus_{n\geq 0} H^n$ such that for each n, $H^n = \bigoplus_{k=0}^{n} H_k$. Therefore for each Feynman diagram $\Gamma \in H$, there exists some components of H which contains Γ. Define a new parameter

$$\mathrm{val}(\Gamma) := \max \Big\{ n \in \mathbb{N} : \Gamma \in \bigoplus_{k\geq n} H_k \Big\}. \tag{4.5}$$

It leads us to generate a concept of distance with respect to the filtration on elements of H which is given by

$$d(\Gamma_1, \Gamma_2) := 2^{-\mathrm{val}(\Gamma_1 - \Gamma_2)}. \tag{4.6}$$

The induced topology corresponding to (4.6) is known as the n-adic topology. The completion of the Hopf algebra H with respect to this topology is the extended Hopf algebra $\overline{H} = \prod_{n\geq 0} H_n$ which has elements of the form $\sum_{n\geq 0} \Gamma_n$ such that $\Gamma_n \in H_n$. It is shown that solutions of combinatorial DSEs belong to this completed Hopf algebra [1, 19, 25, 26, 40, 41].

On the other side, for each n, X_n is a finite Feynman diagram which guarantees the finiteness of the terms Y_ns. Each Y_n could be constructed from Y_{n-1} by a growing (not generally uniform) attachment graph sequence.

Thanks to these both sides, we plan to make a sequence of random graphs which converges to the unlabeled graphon $[f^{t_X}]$ when n goes to infinity.

For each n, let $V(Y_n, \rho_n)$ be a finite subset of the closed interval which contains

$$|V(t_{Y_n})| := |V(t_{X_1})| + |V(t_{X_2})| + \ldots + |V(t_{X_n})| \tag{4.7}$$

nodes in $[0, 1]$ which are selected by projections of the vertices of t_{Y_n} under a fixed poset type injective embedding map ρ_n from t_{Y_n} in $[0, 1]$.

On the other hand, since $X = \sum_{n\geq 0} X_n \in \overline{H}$ exists uniquely, it is possible to associate a labeled graphon f_d with respect to the n-adic metric which belongs to the class $[f^{t_X}]$.

Now for each $n \geq 1$, make a new random type graph R_n by using the points in $V(Y_n, \rho_n)$ and the n-adic metric. In other words, for each pair $v_i, v_j \in V(Y_n, \rho_n)$, let $\rho_n^{-1}(v_i) \in \Gamma_{k_i} \subset X_{k_i}$ and $\rho_n^{-1}(v_j) \in \Gamma_{k_j} \subset X_{k_j}$. The corresponding edge $v_i v_j$ in the random graph is included by the probability

$$f_d(v_i, v_j) := d(\Gamma_{k_i}, \Gamma_{k_j}) = 2^{-\mathrm{val}(\Gamma_{k_i} - \Gamma_{k_j})}. \tag{4.8}$$

As the consequence, thanks to Proposition 3.1 and Corollary 4.3, the sequence $(R_n)_{n\geq 1}$ is convergent to the labeled graphon $[f^{tx}]$ when n tends to infinity. □

In terms of the standard graduation factor on Feynman diagrams, the renormalization Hopf algebra is of finite type which means that the number of generators in each order is finite. Therefore this Hopf algebra is capable to encode a formal expansion of Feynman diagrams which all terms except a finite number in the expansion consider to be zero. This lack has already been covered by working on the completion of the renormalization Hopf algebra with respect to the n-adic topology which enables us to deal with infinite series and polynomials. Proposition 3.2 addresses the new enriched model $H_{\mathrm{FG}}^{d_\diamond}(\Phi)$ of the renormalization Hopf algebra which contains large Feynman graphs. Moreover, Proposition 4.6 shows us the capability of this new enriched Hopf algebra for the description of solutions of combinatorial DSEs. The immediate consequence of this investigation is to obtain a new extended version of the BPHZ renormalization formalism at the level of large Feynman graphs. In the standard version we deal with a single graph, decompose it into its primitive components and then eliminate sub-divergencies from each simple component at the same time [24, 26, 38, 39, 41]. But at this developed version, we enable to work on a large graph which is the convergent limit of an infinite sequence of Feynman diagrams with respect to the cut-distance topology.

Corollary 4.7. *Let $X = \sum_{n\geq 0} X_n$ be a large Feynman graph generated by a given equation DSE of the form (4.2) in a QFT Φ. Renormalized value and counterterm associated to X could be computed in terms of its generators under the BPHZ formalism.*

Proof. Let ϕ be the regularized Feynman rules character associated to Φ. Consider the undeformed character $\phi \circ S$ such that S is the antipode of $H_{\mathrm{FG}}(\Phi)$ and then deform it by the minimal subtraction map. Now for each finite Feynman diagram Γ, the BPHZ renormalization can be summarized by the equations

$$S^{\phi}_{R_{ms}}(\Gamma) = -R_{ms}(\phi(\Gamma)) - R_{ms}(\sum_{\gamma\subset\Gamma} S^{\phi}_{R_{ms}}(\gamma)\phi(\Gamma/\gamma)), \tag{4.9}$$

$$\Gamma \longmapsto S^{\phi}_{R_{ms}} * \phi(\Gamma). \tag{4.10}$$

It is easy to see that

$$S^{\phi}_{R_{ms}} * \phi(\Gamma) = \overline{R}(\Gamma) + S^{\phi}_{R_{ms}}(\Gamma) \tag{4.11}$$

such that the Bogoliubov operation $\overline{R}$ is given by

$$\overline{R}(\Gamma) = U^z_\mu(\Gamma) + \sum_{\gamma\subset\Gamma} c(\gamma) U^z_\mu(\Gamma/\gamma) = \phi(\Gamma) + \sum_{\gamma\subset\Gamma} S^{\phi}_{R_{ms}}(\gamma)\phi(\Gamma/\gamma). \tag{4.12}$$

The formulas $S^{\phi}_{R_{ms}}(\Gamma)$ and $S^{\phi}_{R_{ms}} * \phi(\Gamma)$ give us counterterm and renormalized value related to the Feynman integral $U(\Gamma)$ [6, 14, 15, 41].

On the other hand, thanks to Proposition 4.6, we know that

$$X = \lim_{n \longrightarrow \infty} Y_n \tag{4.13}$$

with respect to the cut-distance topology. Now the linear property of the character ϕ and the antipode and also, the compatibility of the renormalization coproduct with the enriched metric structure on the Hopf algebra $H_{\rm FG}(\Phi)$ (Proposition 3.2 and Corollary 4.4) allow us to compute counterterm and related renormalized value with respect to the equation DSE in terms of generators of the unique solution $X = \sum_{n \geq 0} X_n$ via the following process.

For the counterterm, we have

$$\begin{aligned} S^{\phi}_{R_{ms}}(X) &= \lim_{n \longrightarrow \infty} S^{\phi}_{R_{ms}}(Y_n) = \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} S^{\phi}_{R_{ms}}(X_i) \\ &= \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} \Big(- R_{ms}(\phi(X_i)) - R_{ms}\Big(\sum_{\gamma \subset X_i} S^{\phi}_{R_{ms}}(\gamma) \phi(X_i/\gamma) \Big) \Big) \end{aligned} \tag{4.14}$$

and for the renormalized value, we have

$$\begin{aligned} S^{\phi}_{R_{ms}} * \phi(X) &= \lim_{n \longrightarrow \infty} S^{\phi}_{R_{ms}} * \phi(Y_n) \\ &= \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} S^{\phi}_{R_{ms}} * \phi(X_i). \end{aligned} \tag{4.15}$$

All the above sequences could be reinterpreted in terms of graphons in terms of our explained framework. Since the unique solution X for DSE exists, those limits are convergent with respect to the cut-distance topology. It means that the limits given by the equations (4.14) and (4.15) are well-defined. □

5. FUTURE DIRECTIONS

This research work will open two new general research directions in Mathematics and Physics. The first direction is to search for possible connections between renormalization Hopf algebra of graphons and other combinatorial Hopf algebras. The second direction is to search for some new interpretations of non-perturbative parameters in QFTs with strong coupling constants in the language of random graphs. Our study is capable to open some new practical techniques in dealing with non-perturbative parameters in QFT. For instance, there are many sub-divergencies in the structure of an infinite formal expansion of Feynman diagrams which contribute to the solution of a given Dyson–Schwinger equation where we can classify them in terms of some primitive 1PI Feynman diagrams. On the other hand, theory of graphons provides the notion of homomorphism density which considers the density of sub-graphs in a big complex graph. According to these investigations and thanks to the built methodology in this article, we will have a new possibility to study the quantity of a particular class of sub-divergencies in a large Feynman graph. This capability could be useful whenever we want to achieve some approximations in the computational processes of non-perturbative parameters in theories with strong coupling constants. It means that we can ignore those sub-divergencies with the lowest quantities in a large Feynman graph to obtain some optimal estimations of non-perturbative parameters. In addition, thanks to Corollary 2.3, we can propose another unexplored project about any possible representation of Dyson–Schwinger equations of gauge field theories on the basis of random graphs.

6. APPENDIX A: A PRELIMINARY DISCUSSION ABOUT GRAPHONS

In the first Appendix, we review the basic structure of graph functions under combinatorial and topological settings.

A finite graph G can be described as a pair $(V(G), E(G))$ of finite sets such that $V(G)$ identifies all vertices in the graph and $E(G)$ determines all edges among vertices of $V(G)$. It is weighted graph, if it accepts a weight value in $[0,1]$ for each of its edges. For arbitrary vertices v_i and v_j in G, label the corresponding edge v_iv_j with probability equal to its weight. This gives us a simple random graph R_G with respect to G. For given finite graphs G_1, G_2, a homomorphism from G_1 to G_2 is defined as a map ϕ with $V(G_1)$ as its domain and $V(G_2)$ as its image such that it preserves the edge adjacency. It means that for each edge $(v,w) \in E(G_1)$, the pair $(\phi(v), \phi(w))$ is an edge in $E(G_2)$.

Consider the family of symmetric, Lebesgue measurable functions from $[0,1]^2 \subset \mathbb{R}^2$ to $[0,1] \subset \mathbb{R}$ which is equipped by the almost everywhere equal relation as an equivalent relation. The resulting class of functions can be seen as some weighted graphs on the vertex set $[0,1]$ which are called graph functions or graphons. In a topological setting, a graphon appears as the limit of a sequence of finite graphs with respect to the cut-distance metric.

Definition 6.1. For a given finite graph G, the adjacency matrix with respect to G is a $|V(G)| \times |V(G)|$ matrix $M_G = (m_{ij})$ such that each array m_{ij} shows the number of edges which exists directly between v_i and v_j in G.

If the graph has no self-loop, then its adjacency matrix is a symmetric matrix such that the arrays on its diameter are all zero. There exists a class of finite graphs which has no self-loop such that between each two vertices of the given graph there exists at most one edge. They are called simple graphs.

Definition 6.2. A pixel picture with respect to a given labeled finite simple graph G is a graded square with the area $|V(G)|^2$ such that each small unit square is black if its corresponding array in the adjacency matrix is 1 and is white if its corresponding array in the adjacency matrix is 0.

Definition 6.3. A labeled graphon is a symmetric Lebesgue measurable function from $[0,1] \times [0,1] \subset \mathbb{R}^2$ to $[0,1] \subset \mathbb{R}$. A relabeling on a graphon is described as the result of applying an invertible transformation to the closed interval $[0,1]$ which preserves the chosen measure.

An unlabeled graphon is a graphon up to relabeling. Therefore according to Definition 6.3, each unlabeled graphon could be described by an equivalence class of labeled graphons. This class is given by

$$[\mathrm{lg}] := \{\mathrm{lg}^{\varphi} : (x,y) \longmapsto \mathrm{lg}(\varphi(x), \varphi(y)), \quad \varphi\} \tag{6.1}$$

such that φ is an invertible and measure preserving transformation of the closed unit interval in $\mathbb{R}$. We can work on such equivalence classes to neutralize the appearance of different labels.

Remark 6.4. Regarding the class (6.1), if we change the vertex-decorations of a given finite graph G, then we can get a different pixel picture which means that we can associate various labeled graphons to G. All of these graphons can be encoded by the unlabeled graphon class $[f^G]$ associated to the graph G.

Lemma 6.5. *For each given finite simple labeled graph G, the adjacency matrix determines a unique class of labeled graphons.*

Proof. The adjacency matrix of G gives the function

$$A^G(u,v) := \begin{cases} 1 & u \text{ and } v \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Let the vertices of G are decorated by the set $\{1, 2, \dots, n\}$ and consider the probability space $\Omega = (0,1]$ which is divided by the partition $I_{in} := (\frac{i-1}{n}, \frac{i}{n}]$ of sub-intervals. Now thanks to the adjacency matrix of G define a new function

$$f^G(x,y) := A^G(i,j), \quad x \in I_{in},\ y \in I_{jn}.$$

which is a graph function on $[0,1] \times [0,1]$ such that

$$f^G(x,y) = \begin{cases} 1 & (\lceil nx \rceil, \lceil ny \rceil) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

where $E(G)$ is the set of edges of G, $\lceil nx \rceil$, $\lceil ny \rceil$ are the least integer greater than or equal to the real numbers nx, ny, respectively.

f^G is a bounded symmetric measurable function on $[0,1] \times [0,1]$ which provides measure preserving transformation of the closed unit interval. As the result, f^G is a labeled graphon for the graph G where thanks to (6.1), the class $[f^G]$ can be identified as the unique class of graph functions which encodes the graph G in terms of its corresponding adjacency matrix. It is called the unlabeled graphon with respect to the finite decorated graph G. □

This lemma suggests the existence of a pattern to rebuild a labeled finite graph uniquely in terms of information derived by its corresponding unlabeled graphon.

On the one hand, pixel pictures, as elementary examples of labeled graphons, could play the role of a bridge between graph functions and Feynman diagrams. On the other hand, thanks to Lemma 6.5, we have a natural link between graphs and graphons which is central in the graph limit theory. These investigations enable us to encode the combinatorial information of Feynman diagrams via unlabeled graphons.

Set $\mathcal{G}_{[0,1]}$ as the space of labeled graphons which contains all bounded symmetric Lebesgue measurable functions of the form $f : [0,1]^2 \longrightarrow [0,1]$. Up to the almost everywhere equal relation on measurable functions, there exists a natural equivalence relation $\sim$ on $\mathcal{G}_{[0,1]}$. We say that two graphons are weakly equivalent iff their corresponding unlabeled measurable functions g_1, g_2 are the same almost everywhere. This leads us to work on class functions which belong to the quotient space $\mathcal{G}_{[0,1]}/\sim$.

Set $\mathcal{G}$ as the vector space of all bounded symmetric measurable functions of the form $f : [0,1]^2 \longrightarrow \mathbb{R}$. This space includes $\mathcal{G}_{[0,1]}$ as a subset. The map $\|\cdot\|$ given by

$$\|f\|_{\mathrm{cut}} := \sup_{A,B\subset[0,1]} \Big| \int_{A\times B} f(x,y)dxdy \Big|$$

defines a seminorm on $\mathcal{G}$ where A, B are Lebesgue measurable subsets of $[0,1]$. Invertible maps which preserve measure produce a group denoted by $\mathcal{S}_{[0,1]}$. This group acts on $\mathcal{G}_{[0,1]}$ by

$$\forall \sigma \in \mathcal{S}_{[0,1]} : f^{\sigma}(x,y) := f(\sigma(x), \sigma(y)).$$

Now by applying this action we can obtain a pseudo-metric structure on $\mathcal{G}_{[0,1]}$ which is formulated by

$$\delta_{\diamond}(g_1, g_2) := \inf_{\sigma\in\mathcal{S}_{[0,1]}} \left\| g_1 - g_2^{\sigma} \right\|_{\mathrm{cut}}. \tag{6.2}$$

The infimum parameter in the definition of $\delta_{\diamond}$ makes it well-defined on the space of unlabeled graphons but it is not yet a metric. The pseudo-metric property of $\delta_{\diamond}$ on the space of graphons means that there exists at least one pair of distinct unlabeled graphons which are not identified one by one by $\delta_{\diamond}$. This lack could be covered by working on the quotient space of unlabeled graphons with respect to the equivalence relation on these graphons under the weakly equivalent relation which says that

$$g_1 \sim g_2 \Longleftrightarrow \delta_{\diamond}(g_1, g_2) = 0. \tag{6.3}$$

The distance between two classes $[g_1], [g_2]$ (as unlabeled graphons) is determined by

$$\inf_{\varphi,\psi} \sup_{A,B} \left| \int_{A\times B} g_1(\varphi(x), \varphi(y)) - g_2(\psi(x), \psi(y))dxdy \right|. \tag{6.4}$$

such that the infimum is taken over all different relabeling φ of g_1 and ψ of g_2, and the supremum is taken over all Lebesgue measurable subsets A, B of the unit closed interval $[0,1]$.

$\delta_\diamond$ is a metric on the space of all unlabeled graphons up to the weak isomorphism. The topology generated by this metric is called cut-distance topology where at the end of the day, we can build a complete compact metric structure on the quotient space $\mathcal{G}_{[0,1]}/\sim$ [16, 17, 22, 27].

A graphon g and a finite subset $S \subset [0,1]$ of n points which are chosen uniformly and independently from $[0,1]$ address a weighted graph $H_{S,g}$. This new graph has $|S| = n$ vertices such that for each arbitrary vertices $s_i, s_j \in S$, the corresponding edge $s_i s_j$ has weight $g(s_1, s_2)$. The random graph with respect to the weighted graph $H_{S,g}$ is denoted by $G_H(n, g)$.

Lemma 6.6. *For a given graphon g and for every $n \geq 1$, let R_n be a random graph $G_H(n,g)$ with respect to the graphon g and selected finite subset S_n of $[0,1]$. Then, with probability one and with respect to the cut-distance topology, the sequence $(R_n)_{n\geq 1}$ converges to g* [16, 17, 27, 28].

7. APPENDIX B: UNIQUENESS

In the second Appendix, we concern the uniqueness property of convergent limits of sequences of decorated rooted tress which leads us to identify this uniqueness for the level of Feynman diagrams.

Lemma 7.1. *Up to unlabeled graphons and weakly equivalent relation, the limit of a convergent sequence of decorated rooted trees is unique.*

Proof. Let $\{t_n\}_{n\geq 1}$ be a sequence of classes of labeled rooted trees which converges to unlabeled graphons $[f^t]$ and $[f^s]$ with respect to the topology induced by the cut-distance metric (6.4). Thanks to Corollary 2.2, we have

$$\forall \epsilon > 0 \ \exists N_t \in \mathbb{N}, \quad \forall n > N_t \Longrightarrow \delta_\diamond([f^{t_n}], [f^t]) < \epsilon/2, \tag{7.1}$$

$$\forall \epsilon > 0 \ \exists N_s \in \mathbb{N}, \quad \forall n > N_s \Longrightarrow \delta_\diamond([f^{t_n}], [f^s]) < \epsilon/2. \tag{7.2}$$

For each $n > \max\{N_t, N_s\}$, we plan to show that

$$\delta_\diamond([f^t], [f^s]) \leq \delta_\diamond([f^t], [f^{t_n}]) + \delta_\diamond([f^{t_n}], [f^s]) \tag{7.3}$$

such that thanks to (7.1) and (7.2), we will obtain $\delta_\diamond([f^t], [f^s]) < \epsilon$. Therefore $\delta_\diamond([f^t], [f^s]) = 0$ where thanks to the the equivalence relation (6.3), we will have $[f^t] = [f^s]$.

Since graph functions are Lebesgue measurable on $[0,1] \times [0,1]$, it is enough to prove (7.3) for the level of step functions and then applying the approximation method enables us to extend the result to general graph functions.

Let f^t, f^s, f^{t_n} are step functions with corresponding partitions

$$\Omega_t := \bigcup_{i=1}^{I} A_i, \quad \Omega_{t_n} := \bigcup_{j=1}^{J} B_j, \quad \Omega_s := \bigcup_{k=1}^{K} C_k,$$

such that A_i, B_j, C_k are non-zero Lebesgue measurable subsets of the closed interval $[0,1]$. Thanks to Lemma 6.3 in [22], we have

$$\|f^{t,\pi_t} - f^{t_n,\pi_{t_n}}\|_{\text{cut}} < \delta_\diamond(f^t, f^{t_n}) + \epsilon,$$

$$\|f^{t_n,\pi_{t_n}} - f^{s,\pi_s}\|_{\text{cut}} < \delta_\diamond(f^{t_n}, f^s) + \epsilon$$

such that π_l is the projection onto Ω_l. Thanks to the Lebesgue measure on Ω_l, define a new product measure μ on $\Omega_t \times \Omega_{t_n} \times \Omega_s$ such that for each subset E, it is given by

$$\mu(E) := \sum_{i,j,k} \frac{m_{t,t_n}(A_i \times B_j) m_{t_n,s}(B_j \times C_k)}{m_{t_n}(B_j)} \times \frac{m_t \times m_{t_n} \times m_s(E \cap (A_i \times B_j \times C_k))}{m_t(A_i) m_{t_n}(B_j) m_s(C_k)},$$

where m_t, m_{t_n}, m_s are Lebesgue measure on $[0,1]$ and $m_{t,t_n}, m_{t_n,s}$ are Lebesgue measure on $[0,1] \times [0,1]$. It can be shown that the projections

$$\pi_l : (\Omega_t \times \Omega_{t_n} \times \Omega_s, \mu) \longrightarrow (\Omega_l, m_l)$$

are measure preserving. In addition, the projections $\pi_{tt_n}, \pi_{t_n s}$ map the probability measure μ to new measures ρ_{tt_n} on $\Omega_t \times \Omega_{t_n}$ and $\rho_{t_n s}$ on $\Omega_{t_n} \times \Omega_s$ such that thanks to Lemma 6.3 in [22] we will have

$$\|f^{t,\pi_t} - f^{t_n,\pi_{t_n}}\|_{\text{cut},\Omega_t \times \Omega_{t_n},\rho_{tt_n}} = \|f^{t,\pi_t} - f^{t_n,\pi_{t_n}}\|_{\text{cut},\Omega_t \times \Omega_{t_n}, m_{t,t_n}},$$

$$\|f^{t,\pi_t} - f^{t_n,\pi_{t_n}}\|_{\text{cut},\Omega_t \times \Omega_{t_n} \times \Omega_s,\mu} = \|f^{t,\pi_t} - f^{t_n,\pi_{t_n}}\|_{\text{cut},\Omega_t \times \Omega_{t_n},\rho_{tt_n}}.$$

As the result, we will have

$$\|f^{t,\pi_t} - f^{t_n,\pi_{t_n}}\|_{\text{cut},\mu} < \delta_\diamond(f^t, f^{t_n}) + \epsilon,$$

$$\|f^{t_n,\pi_{t_n}} - f^{s,\pi_s}\|_{\text{cut},\mu} < \delta_\diamond(f^{t_n}, f^s) + \epsilon.$$

Now apply the triangle inequality for $\|\cdot\|_{\text{cut},\mu}$ to obtain

$$\begin{aligned}
\delta_\diamond(f^t, f^s) &\leq \|f^{t,\pi_t} - f^{s,\pi_s}\|_{\text{cut},\mu} \\
&\leq \|f^{t,\pi_t} - f^{t_n,\pi_{t_n}}\|_{\text{cut},\mu} + \|f^{t_n,\pi_{t_n}} - f^{s,\pi_s}\|_{\text{cut},\mu} \\
&< \delta_\diamond(f^t, f^{t_n}) + \delta_\diamond(f^{t_n}, f^s) + 2\epsilon,
\end{aligned}$$

where $\epsilon > 0$ is arbitrary which means that we have obtained (7.3) for the level of step functions. □

Lemma 7.2. *Up to unlabeled graphons and weakly equivalent relation, the limit of a convergent sequence of Feynman diagrams is unique.*

Proof. It is a direct result of the second fundamental definition (Corollary 4.1) and Lemma 7.1. □

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