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ON THE BOUNDEDNESS OF EQUIVARIANT HOMEOMORPHISM GROUPS

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Abstract. Given a principal G-bundle $\pi : M \to B$, let $\mathcal{H}_G(M)$ be the identity component of the group of G-equivariant homeomorphisms on M. The problem of the uniform perfectness and boundedness of $\mathcal{H}_G(M)$ is studied. It occurs that these properties depend on the structure of $\mathcal{H}(B)$, the identity component of the group of homeomorphisms of B, and of B itself. Most of the obtained results still hold in the C^r category.

Keywords: principal G-manifold, equivariant homeomorphism, uniformly perfect, bounded, C^r equivariant diffeomorphism.

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1. INTRODUCTION

Let M be a topological manifold and let $\mathcal{H}(M)$ denote the group of all homeomorphisms of M that can be joined to the identity by a compactly supported isotopy. Mather proved in [14] that the group $\mathcal{H}(\mathbb{R}^m)$ is acyclic. His result combined with a fragmentation property for homeomorphisms implies that $\mathcal{H}(M)$ is perfect, i.e. equal to its commutator subgroup. Observe that $\mathcal{H}(M)$ is simple as well, provided M is connected and $\partial_M = \emptyset$ (Corollary 2.11).

As a topological group, $\mathcal{H}(M)$ will be endowed with the majorant (or graph, or Whitney) topology, cf. section 2. If M is compact this topology coincides with the compact-open topology.

Assume that G is a Lie group acting freely and properly on M. Then M can be regarded as the total space of a principal G-bundle $\pi : M \to B$. (see, e.g., Theorem 1.11.4 in [6]). An equivariant homeomorphism f of M will be called *transversely compactly supported* if $\pi(\mathrm{supp}(f))$ is compact. In the case of G compact it is the same as being compactly supported. Next, an isotopy is said to be transversely compactly supported if it consists of transversely compactly supported elements. By $\mathcal{H}_G(M)$ we will denote the group of all equivariant homeomorphisms of M which are isotopic to the identity through transversely compactly supported equivariant isotopies. In [16] the third-named author showed the following result.

Theorem 1.1. *Under the above assumption, the group $\mathcal{H}_G(M)$ is perfect.*

Actually, in [16] Theorem 1.1 was formulated for G compact, but its proof is still valid in our case.

Analogous theorems for equivariant C^r-diffeomorphisms ($1 \leqslant r \leqslant \infty$, $r \neq \dim M - \dim G + 1$) was earlier proved by Banyaga ([3, 4]) for G being a torus, and by Abe and Fukui ([1]) for an arbitrary compact Lie group G, provided G acts freely on M. Abe and Fukui in [2] showed also similar results for equivariant Lipschitz homeomorphisms.

Recently Fukui studied in [10] the uniform perfectness of the identity component of the group of compactly supported equivariant C^r-diffeomorphisms under the above assumptions. He showed a relation between the uniform perfectness of this group and the uniform perfectness of the identity component of the group of C^r-diffeomorphisms of B.

In the present paper we will generalize the above results in various directions. Firstly, we would like to extend the main result in [10] to all homeomorphisms, partly by using similar methods. In particular, we observe that the groups studied in [16] are uniformly perfect under some assumptions.

For a topological group G by $\mathcal{P}G$ we denote the isotopy (or path) group of G, that is the totality of $f : I \to G$ with $f(0) = e$, $I = [0, 1]$.

We will need the following notions. By $P : \mathcal{H}_G(M) \to \mathcal{H}(B)$ we denote the homomorphism given by $P(f)(\pi(x)) = \pi(f(x))$. Given a perfect group H, the symbol cld_H will stand for the commutator length diameter of H. If $\mathcal{G} \leq \mathcal{H}(N)$ is a homeomorphism group of a manifold N then $\mathrm{fd}_{\mathcal{G}}$ will denote the diameter of the fragmentation norm on $\mathcal{G}$. See Section 3 for more details.

Theorem 1.2. *Let $\pi : M \to B$ be a principal G-bundle, where G is a Lie group, and $n = \dim B$. Assume that either B is a compact manifold (possibly with boundary, provided $\dim B \geq 2$), or B is an open metrizable manifold. Then:*

1. *If $\mathcal{H}_G(M)$ is uniformly perfect then so is $\mathcal{H}(B)$.*
2. *If the fragmentation norm on the isotopy group $\mathcal{P}\mathcal{H}(B)$ of $\mathcal{H}(B)$ is bounded, then $\mathcal{H}_G(M)$ is uniformly perfect. Moreover, the commutator length diameter $\mathrm{cld}_{\mathcal{H}_G(M)}$ of $\mathcal{H}_G(M)$ satisfies*
$$\mathrm{cld}_{\mathcal{H}_G(M)} \leq \mathrm{fd}_{\mathcal{P}\mathcal{H}(B)} + 2(n+1).$$
3. *If $\mathcal{H}(B)$ is uniformly perfect and $\ker(P)$ possesses a finite number of components, then $\mathcal{H}_G(M)$ is uniformly perfect and*
$$\mathrm{cld}_{\mathcal{H}_G(M)} \leq \mathrm{cld}_{\mathcal{H}(B)} + 2(n+1) + l,$$
where l is the number of components of $\ker(P)$.

Secondly, we consider, more generally, the boundedness of the groups in question.

Theorem 1.3. *Under the above assumption, the following statements hold:*

1. *If $\mathcal{H}_G(M)$ is bounded then $\mathcal{H}(B)$ is bounded as well.*
2. *If the fragmentation norm of the isotopy group $\mathcal{PH}(B)$ of $\mathcal{H}(B)$ is bounded, then $\mathcal{H}_G(M)$ is bounded too. Furthermore, for any conjugation-invariant norm ν on $\mathcal{H}_G(M)$ and for all $f \in \mathcal{H}_G(M)$ one has*

$$\nu(f) \leq 14(\mathrm{fd}_{\mathcal{PH}(B)} + n + 1)\nu(\varphi),$$

where φ is a moving map *in a ball, cf. Section 3.*

Observe that the group $\mathcal{PH}(B)$ is fragmentable due to Corollaries 2.7 and 2.10 below.

The proofs will be given in Section 4. The last section is devoted to some remarks on the smooth case. Namely we formulate analogs of Theorems 1.2 and 1.3 in this case.

Throughout we will assume that all manifolds are metrizable and so paracompact.

2. FRAGMENTATIONS OF HOMEOMORPHISMS AND ISOTOPIES

Let N be a topological manifold. The following type of fragmentations is important when studying groups of homeomorphisms.

Definition 2.1. A subgroup $\mathcal{G} \leq \mathcal{H}(N)$ is called *locally continuously fragmentable* with respect to a finite open cover $\{U_i\}_{i=1}^d$ and a neighborhood $\mathcal{U}$ of $\mathrm{id} \in \mathcal{G}$ if there exist continuous mappings $\sigma_i : \mathcal{U} \to \mathcal{G}$, $i = 1, \ldots, d$, such that for all $f \in \mathcal{U}$ one has

$$f = \sigma_1(f) \ldots \sigma_d(f), \quad \mathrm{supp}(\sigma_i(f)) \subset U_i, \text{ for all } i.$$

Moreover, we assume that $\mathrm{supp}(\sigma_i(f))$ is compact. If $\mathcal{U} = \mathcal{G}$ then $\mathcal{G}$ is called *globally continuously fragmentable.*

For a topological space X and a topological group $\mathcal{G} \leq \mathcal{H}(N)$, let $\mathcal{C}(X, \mathcal{G})$ stand for the group of continuous maps $X \to \mathcal{G}$ with the pointwise multiplication and the compact-open topology. For $f \in \mathcal{C}(X, \mathcal{G})$ we define $\mathrm{supp}(f) = \bigcup_{x \in X} \mathrm{supp}(f^x)$, where $f^x : p \in N \mapsto f(x)(p) \in N$. Then Definition 2.1 extends obviously for $\mathcal{C}(X, \mathcal{G})$.

Lemma 2.2. *If $\mathcal{G}$ is a topological group then $\mathcal{C}(X, \mathcal{G})$ is a topological group too.*

The proof is trivial.

Proposition 2.3. *If $\mathcal{G}$ is locally continuously fragmentable then so is $\mathcal{C}(X, \mathcal{G})$.*

Proof. Let $\sigma_i : \mathcal{U} \to \mathcal{G}$, $i = 1, \ldots, d$, be as in Definition 2.1. If $Y \subset X$ is compact, set $\mathcal{V} = \{f \in \mathcal{C}(X, \mathcal{G}) : f(Y) \subset \mathcal{U}\}$ and define continuous maps $\sigma_i^{\mathcal{C}} : \mathcal{V} \to \mathcal{C}(X, \mathcal{G})$ by the formulae $\sigma_i^{\mathcal{C}}(f)(x) = \sigma_i(f^x)$, where $f \in \mathcal{C}(X, \mathcal{G})$, $x \in X$. It follows that $\mathrm{supp}(\sigma_i^{\mathcal{C}}(f)^x) \subset U_i$ for all i and x. Consequently we have $\mathrm{supp}(\sigma_i^{\mathcal{C}}(f)) \subset U_i$. $\square$

For a topological group $\mathcal{G}$ we denote by $\mathcal{PG} = \{f \in \mathcal{C}(I, \mathcal{G}) : f(0) = e\}$ the isotopy (or path) group. If $\mathcal{G} = \mathcal{H}(N)$, in view of [13, Lemma 41.7], for all $f \in \mathcal{PG}$ there is a compact subset $K \subset N$ such that $\operatorname{supp}(f) = \bigcup_{t\in I} \operatorname{supp}(f^t) \subset K$.

Let G be a Lie group. Given a principal G-bundle $\pi : M \to B$ the *gauge group* $\operatorname{Gau}(M)$ is the subgroup of $\mathcal{H}_G(M)$ of all elements that project to id_B. That is, $\operatorname{Gau}(M)$ is the space of G-equivariant transversely compactly supported mappings $\mathcal{C}(M, (G, \mathrm{conj}))^G$. It follows that $\operatorname{Gau}(M)$ identifies with $\mathcal{C}(B \leftarrow M[G, \mathrm{conj}])$, the space of compactly supported sections of the associated bundle $M[G, \mathrm{conj}]$. Consequently, any $f \in \operatorname{Gau}(M)$ in a trivialization of π over $U_i \subset B$ identifies with a mapping $f^{(i)} : U_i \to G$ such that $f(p) = p.f^{(i)}(\pi(p))$.

Denote by $P : \mathcal{H}_G(M) \to \mathcal{H}(B)$ the mapping defined by $P(f)(\pi(x)) = \pi(f(x))$. Then $\ker P \subset \operatorname{Gau}(M)$.

Let $\dim B = n$. By a *ball* U in B we will mean relatively compact, open ball imbedded with its closure in B. Similarly we define a half-ball if B has boundary. For B compact, let $d = d_B$ is the smallest integer such that $B = \bigcup_{i=1}^d U_i$ where U_i is a ball or half-ball such that $\mathrm{cl}(U_i) \neq B$ for each i. Clearly $d \leq n + 1$ for B compact. Next let B be an open manifold. Then B admits an open cover $\{V_j\}_{j=1}^{n+1}$ (a so-called *Palais cover*) such that each V_j is the union of a countable, locally finite family of balls with pairwise disjoint closures.

Proposition 2.4. *Let G be a Lie group and let $\pi : M \to B$ be a principal G-bundle. Assume that either B is a compact manifold* (*possibly with boundary*) *or B is an open manifold. Then $\mathcal{P}\operatorname{Gau}(M)$ is globally continuously fragmentable* (*in the obvious sense*) *with respect to $\{U_i\}_{i=1}^d$ in the former case, and with respect to $\{V_j\}_{j=1}^{n+1}$ in the latter.*

Proof. (See also [10].) We show the proof in the first case, the second is the same. Assume that $\{\lambda_i\}_{i=1}^d$ is a partition of unity subordinate to $\{U_i\}_{i=1}^d$. Let $g = \{g^t\} \in \mathcal{P}\operatorname{Gau}(M)$. Then clearly $g^t \in \ker P$ for all t. In general, $\mathcal{P}\operatorname{Gau}(M) = \mathcal{P}\ker P$.

Put

$$h_1^t(p) = g^{\lambda_1(\pi(p))t}, \quad h_2^t(p) = (h_1^1)^{-1} g^{\lambda_1(\pi(p))+\lambda_2(\pi(p))t},$$

and for $i = 3, \ldots, d$, we define

$$h_i^t(p) = (h_1^1 \cdots h_{i-1}^1)^{-1} g^{\lambda_1(\pi(p))+\cdots+\lambda_{i-1}(\pi(p))+\lambda_i(\pi(p))t}.$$

Then $\operatorname{supp}(P(h_i^t)) \subset U_i$ for all i and t, and $g = h_1^1 \ldots h_d^1$. Moreover, the maps $\sigma_i : g \mapsto h_i = \{h_i^t\}$ are continuous. □

We will use the deformation properties for the spaces of imbeddings obtained by Edwards and Kirby in [7]. See also Siebenmann [18]. First let us recall some notions and the main theorem of [7]. From now on N is a metrizable topological manifold and $I = [0, 1]$. If U is a subset of N, a *proper imbedding* of U into N is an imbedding $h : U \to N$ such that $h^{-1}(\partial N) = U \cap \partial N$. An *isotopy* of U into N is a family of imbeddings $h_t : U \to N$, $t \in I$, such that the map $h : U \times I \to N$ defined by $h(x, t) = h_t(x)$ is continuous. An isotopy is *proper* if each imbedding in it is proper. Now let C and U be subsets of N with $C \subseteq U$. By $I(U, C; N)$ we denote the space of proper imbeddings of U into N which equal the identity on C, endowed with the compact-open topology.

Suppose X is a space with subsets A and B. A *deformation of A into B* is a continuous mapping $\varphi : A \times I \to X$ such that $\varphi|_{A\times 0} = \mathrm{id}_A$ and $\varphi(A \times 1) \subseteq B$. If $\mathcal{U}$ is a subset of $I(U;N)$ and $\varphi : \mathcal{U} \times I \to I(U;N)$ is a deformation of $\mathcal{U}$, we may equivalently view φ as a map $\varphi : \mathcal{U} \times I \times U \to N$ such that for each $h \in \mathcal{U}$ and $t \in I$, the map $\varphi(h,t) : U \to N$ is a proper imbedding.

If $W \subseteq U$, a deformation $\varphi : \mathcal{U} \times I \to I(U;N)$ is *modulo W* if $\varphi(h,t)|_W = h|_W$ for all $h \in \mathcal{U}$ and $t \in I$.

Suppose $\varphi : \mathcal{P} \times I \to I(U;N)$ and $\psi : \mathcal{V} \times I \to I(U;N)$ are deformations of subsets of $I(U;N)$ and suppose that $\varphi(\mathcal{U} \times 1) \subseteq \mathcal{V}$. Then the *composition* of ψ with φ, denoted by $\psi \star \varphi$, is the deformation $\psi \star \varphi : \mathcal{U} \times I \to I(U;N)$ defined by

$$\psi \star \varphi(h,t) = \begin{cases} \varphi(h,2t) & \text{for } t \in [0,1/2], \\ \psi(\varphi(h,1), 2t-1) & \text{for } t \in [1/2,1]. \end{cases} \tag{2.1}$$

The main result in [7] is the following theorem.

Theorem 2.5. *Let N be a topological manifold and let U be a neighborhood in N of a compact subset C. For any neighborhood $\mathcal{V}$ of the inclusion $i : U \subset N$ in $I(U;N)$ there are a neighborhood $\mathcal{U}$ of $i \in I(U;N)$ and a deformation $\varphi : \mathcal{U} \times I \to \mathcal{V}$ into $I(U,C;N)$ which is modulo of the complement of a compact neighborhood of C in U and such that $\varphi(i,t) = i$ for all t. Moreover, if $D_i \subset V_i$, $i = 1,\dots,q$, is a finite family of closed subsets D_i with their neighborhoods V_i, then φ can be chosen so that the restriction of φ to $(\mathcal{U} \cap I(U, U \cap V_i; N)) \times I$ assumes its values in $I(U, U \cap D_i; N)$ for each i.*

Now we wish to show that $\mathcal{H}(N)$ is locally continuously fragmentable (Definition 2.1) provided N is compact.

Proposition 2.6. *Let N be a compact topological manifold and let $\{U_i\}_{i=1}^d$ be an open relatively compact cover of N. Then there exist $\mathcal{U}$, a neighborhood of the identity in $\mathcal{H}(N)$, and continuous mappings $\sigma_i : \mathcal{U} \to \mathcal{H}(N)$, $i = 1,\dots,d$, such that $h = \sigma_1(h)\dots\sigma_d(h)$ and $\mathrm{supp}(\sigma_i(h)) \subset U_i$ for all i and all $h \in \mathcal{U}$. That is, $\mathcal{H}(N)$ satisfies Definition 2.1.*

Proof. (See also [7].) First we have to shrink the cover $\{U_i\}_{i=1}^d$ d times, that is we choose an open $U_{i,j}$ for every $i = 1,\dots,d$ and $j = 0,\dots,d$ with $U_{i,0} = U_i$ such that $\bigcup_{i=1}^d U_{i,j} = N$ for all j and such that $\mathrm{cl}(U_{i,j+1}) \subset U_{i,j}$ for all i,j. We make use of Theorem 2.5 d times with $q = 1$. Namely, for $i = 1,\dots,d$ we have a neighborhood $\mathcal{U}_i$ of the identity in $I(N, \bigcup_{\alpha=1}^{i-1} U_{\alpha,i-1}; N)$ and a deformation $\phi_i : \mathcal{U}_i \times I \to \mathcal{H}(N)$ which is modulo $N \setminus U_{i,0}$ and which takes its values in $I(N, \bigcup_{\alpha=1}^{i} \mathrm{cl}(U_{\alpha,i}); N)$ and such that $\phi_i(\mathrm{id}, t) = \mathrm{id}$ for all t. Here we apply Theorem 2.5 with $C = \mathrm{cl}(U_{i,i})$, $U = U_{i,0}$, $D_1 = \bigcup_{\alpha=1}^{i-1} \mathrm{cl}(U_{\alpha,i})$ and $V_1 = \bigcup_{\alpha=1}^{i-1} U_{\alpha,i-1}$. Taking a neighborhood $\mathcal{U}$ of id small enough, we have that $\phi_d \star \dots \star \phi_1$ restricted to $\mathcal{U} \times I$ is well defined. For every $h \in \mathcal{U}$ we set $h_0 = h$ and $h_i = \phi_i \star \dots \star \phi_1(h,1)$, $i = 1,\dots,d$. It follows that $h_d = \mathrm{id}$ and $h = \prod_{i=1}^d h_i h_{i-1}^{-1}$. It suffices to define $\sigma_i : \mathcal{U} \to \mathcal{H}(N)$ by $\sigma_i(h) = h_i h_{i-1}^{-1}$ for all i. $\square$

It follows from Propositions 2.6 and 2.3 the following result.

Corollary 2.7. *Assume that N is compact and $\mathcal{H}(N)$ is locally continuously fragmentable with respect to a cover $\{U_i\}_{i=1}^d$ of N and a neighborhood $\mathcal{U}$ of* id *(Definition 2.1). Then the isotopy group $\mathcal{PH}(N)$ is locally continuously fragmentable with respect to $\{U_i\}_{i=1}^d$ and $\hat{\mathcal{U}} = \{f \in \mathcal{C}(I, N) : f(I) \subset \mathcal{U}\}$.*

Recall the notion of the majorant or graph topology (see, e.g., [7] or [13]). Let X and Y be Hausdorff spaces and let $\mathcal{C}(X, Y)$ be the space of all continuous mappings $X \to Y$. For $f \in \mathcal{C}(X, Y)$ by $\mathrm{graph}_f : X \to X \times Y$ we denote the graph mapping. The *majorant topology* (or the *graph topology*, or *Whitney topology*) on $\mathcal{C}(X, Y)$ is given by the basis of all sets of the form $\{f \in \mathcal{C}(X, Y) : \mathrm{graph}_f(X) \subset U\}$, where U runs over all open sets in $X \times Y$. The majorant topology is Hausdorff since it is finer than the compact-open topology. If X is paracompact and (Y, d) is a metric space then for $f \in \mathcal{C}(X, Y)$ one has a basis of neighborhoods of the form $\{g \in \mathcal{C}(X, Y) : d(f(x), g(x)) < \varepsilon(x),\ \forall x \in X\}$, where ε runs over all positive continuous functions on X. This topology is independent of the choice of metric.

Remark 2.8 (See [7], Remark on p. 79.)**.** Theorem 2.5 can be generalized to the consideration of proper imbeddings of a neighborhood U of a *closed* subset C of a manifold N. In this case $I(U, C; N)$ is endowed with the majorant topology and the deformations considered are majorant deformations, defined in a way analogous to the above definition.

Proposition 2.9. *Let N be an open manifold and $\{V_j\}_{j=1}^{n+1}$ a Palais cover of N. Then $\mathcal{H}(N)$ is locally continuously fragmentable with respect to $\{V_j\}_{j=1}^{n+1}$.*

Proof. We apply the version of Theorem 2.5 mentioned in Remark 2.8 and repeat the proof of 2.6 with $\{V_j\}_{j=1}^{n+1}$ instead of $\{U_i\}_{i=1}^d$. There is also the problem whether $\mathrm{supp}(h_i^t)$ is compact (the last requirement in Definition 2.1), since V_j need not be relatively compact. But this is clearly fulfilled because $\mathrm{supp}(h)$ is compact and the deformations involved are modulo the complement of a neighborhood of C. □

A homeomorphism group $\mathcal{G} \leq \mathcal{H}(N)$ is called *fragmentable* with respect to some open cover $\mathcal{O}$ of N if each element of $\mathcal{G}$ can be written as a product of homeomorphisms supported in elements of $\mathcal{O}$. Clearly, if $\mathcal{G}$ is locally continuously fragmentable with respect to $\mathcal{O}$ then it is fragmentable with respect to $\mathcal{O}$.

Corollary 2.10. *Let N be an open manifold. Then $\mathcal{PH}(N)$ is locally continuously fragmentable with respect to $\{V_j\}_{j=1}^{n+1}$. Consequently, $\mathcal{PH}(N)$ is fragmentable with respect to the family of balls.*

Indeed, this follows from Lemma 2.2 and Propositions 2.3 and 2.9. The second assertion is trivial.

Corollary 2.11. *The group $\mathcal{H}(N)$ is perfect and, provided N is connected and $\partial_N = \emptyset$, $\mathcal{H}(N)$ is also simple. The universal covering group $\mathcal{H}(N)^\sim$ is perfect as well.*

Proof. According to Mather [14] the group $\mathcal{H}(\mathbb{R}^n)$ is perfect. Next Proposition 2.6 or 2.9 applied to $\mathcal{H}(N)$ implies that $\mathcal{H}(N)$ is perfect. The simplicity then follows from [15] (see also [8]).

To show the second assertion we employ a reasoning from Mather [14] concerning the perfectness for $\mathcal{H}(U)^{\sim}$ instead of $\mathcal{H}(U)$, U being a ball or half-ball (see also the proof of Lemma 3.5(1) below) with some modifications as in [15]. Next it suffices to apply Corollary 2.7 or 2.10. □

Observe that the group $\mathrm{Homeo}_c(N)$ of all compactly supported homeomorphisms of N need not be path-connected. In fact, this is so for N as follows. Delete from the cylinder $\mathbb{S}^1 \times \mathbb{R}$ a countable number of small discs centered at $(1, n)$, where $n \in \mathbb{Z}$. Then N is an open manifold obtained by sewing up a copy of $\mathbb{T}^2 \setminus B^2$ to each boundary component of the above space.

However $\mathrm{Homeo}_c(\mathbb{R}^m)$is path-connected due to the Alexander trick: if $\mathrm{supp}(g)$ is compact, we define an isotopy $g^t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in I$, from the identity to g, by

$$g^t(x) = \begin{cases} tg\left(\frac{1}{t}x\right) & \text{for } t > 0, \\ x & \text{for } t = 0. \end{cases}$$

Remark 2.12. When we consider the identity component of the compactly supported C^r diffeomorphism group on a smooth manifold N, it is well-known that its universal covering is perfect, provided $r \neq \dim N + 1$ ([4]). However, we do not know whether its isotopy group is perfect.

3. BOUNDEDNESS OF THE GROUPS OF EQUIVARIANT HOMEOMORPHISMS

A group H is called *bounded* if it is bounded with respect to any bi-invariant metric on it. This notion is strictly related to the notion of a conjugation-invariant norm.

A *conjugation-invariant norm* on H is a function $\nu : H \to [0, \infty)$ which satisfies the following conditions. For any $g, h \in H$,

1. $\nu(h) > 0$ if and only if $h \neq e$,
2. $\nu(h^{-1}) = \nu(h)$,
3. $\nu(hg) \leq \nu(h) + \nu(g)$,
4. $\nu(ghg^{-1}) = \nu(h)$.

It is easily seen that H is bounded if and only if any conjugation-invariant norm on H is bounded.

The following general fact will be in use.

Lemma 3.1 ([5])**.** *If a group H admits a homomorphism onto an unbounded group, then H itself is unbounded.*

Observe that the commutator length cl_H is a conjugation-invariant norm on $[H, H]$. In particular, if H is a perfect group then cl_H is a conjugation-invariant norm on H. For any perfect group H denote by cld_H the commutator length diameter of H, $\mathrm{cld}_H := \sup_{h \in H} \mathrm{cl}_H(h)$. Then H is called *uniformly perfect* if $\mathrm{cld}_H < \infty$.

Now let $\mathcal{G}$ is a subgroup of $\mathcal{H}(N)$ and assume that $\mathcal{G}$ is fragmentable (with respect to balls and half-balls). For $h \in \mathcal{G}$, $h \neq \mathrm{id}$, we define the *fragmentation norm* $\mathrm{frag}_{\mathcal{G}}(h)$ to be the least integer $r > 0$ such that $h = h_1 \dots h_r$ with $\mathrm{supp}(h_i) \subset U_i$ for some ball (or half-ball) U_i. By definition $\mathrm{frag}_{\mathcal{G}}(\mathrm{id}) = 0$. Denote by $\mathrm{fd}_{\mathcal{G}}$ the diameter of fragmentation norm, namely $\mathrm{fd}_{\mathcal{G}} = \sup_{h \in \mathcal{G}} \mathrm{frag}_{\mathcal{G}}(h)$.

We also have the following concept of conjugation-invariant norm.

Definition 3.2. Let H be a connected topological group and let U be a neighborhood of $e \in H$. By $\tilde{U}$ we denote the "saturation" of U with respect to conj_h for $h \in H$ and the inversion i, that is $\tilde{U} = \bigcup_{h \in H} hUh^{-1} \cup hU^{-1}h^{-1}$. Then for $h \in H$, $h \neq e$, by $\mu_U(h)$ we denote the smallest $k \in \mathbb{N}$ such that $h = h_1 \dots h_k$ with $h_i \in \tilde{U}$ for all i. It is easily seen that μ_U is a conjugation-invariant norm.

Clearly if $U \subset V$ then $\mu_V \leq \mu_U$.

Proposition 3.3. *If $\mathcal{G} \leq \mathcal{H}(N)$ is bounded and locally continuously fragmentable with respect to an open cover $\{U_i\}_{i=1}^d$, there is $p \in \mathbb{N}$ such that any $g \in \mathcal{G}$ admits a decomposition $g = g_1 \dots g_p$ with each g_j supported in some $U_{i(j)}$.*

In fact, let $\mathcal{U}$, σ_i, $i = 1, \dots, d$, be as in Definition 2.1. Moreover, assume $\mathcal{U}^{-1} = \mathcal{U}$. In view of the assumption $k_{\mathcal{U}} = \mu_{\mathcal{U}}(\mathcal{G}) < \infty$, where $\mu_{\mathcal{U}}$ is as in Definition 3.2. Take $p = dk_{\mathcal{U}}$.

When considering the boundedness or the uniform perfectness of homeomorphism groups, the idea of displacement is in use. A subgroup K of H is called strongly m-displaceable if there is $h \in H$ such that the subgroups $K, hKh^{-1}, \dots, h^mKh^{-m}$pairwise commute. Then we say that h m-displaces K. Then Theorem 2.2 and Corollary 2.3 in [5] can be formulated as follows.

Theorem 3.4. *Let ν be a conjugation-invariant norm on H and assume that h m-displaces K for every $m \geqslant 1$. For every $f \in [K, K]$ we have:*

1. $\mathrm{cl}_H(f) \leqslant 2$,
2. $\nu(f) \leqslant 14\nu(h)$.

In the case of homeomorphism groups we have the following partial amelioration of Theorem 3.4.

Lemma 3.5 (Basic Lemma)**.** *Let U be a ball or a half-ball of a topological manifold N. If U is a half-ball we assume* $\dim N \geq 2$.

1. *There is $\varphi \in \mathcal{H}(U)$ such that any $f \in \mathcal{H}(U)$ can be written as a commutator of the form $f = [h, \bar{\varphi}]$, where $\bar{\varphi}$ is a conjugate of φ in $\mathcal{H}(U)$ and $h \in \mathcal{H}(U)$.*
2. *If ν is a conjugation-invariant norm on $\mathcal{H}(U)$ then ν is bounded on $\mathcal{H}(U)$ by $2\nu(\varphi)$.*
3. *Let $\pi : M \to B$ be a principal G-bundle. Then for U, a ball or a half-ball in B, any $f \in \mathcal{H}_G(\pi^{-1}(U))$ can be written as a product of two commutators. Moreover, f can be expressed in the form*
$$f = \prod_{i=1}^{7} [h_i, \bar{\varphi}_i],$$
where each $\bar{\varphi}_i$ is a conjugate of the obvious lift to $\mathcal{H}_G(\pi^{-1}(U))$ of φ as in (1) *and each $h_i \in \mathcal{H}_G(\pi^{-1}(U))$.*

4. *Let U be a ball or a half-ball. A conjugation-invariant norm ν on $\mathcal{H}_G(M)$ restricted to the subgroup $\mathcal{H}_G(\pi^{-1}(U))$ is bounded by $14\nu(\varphi)$.*

The homeomorphism φ specified in (1) will be called a *moving map* in a ball U. Abusing the notation φ stands also for the lift of φ to $\mathcal{H}_G(\pi^{-1}(U))$. It is defined uniquely up to conjugation. Note that when we consider the C^r category, φ may be chosen of class C^r. However, (1) and (2) are no longer true in the C^r category, while (3) and (4) still hold.

Proof. (1) Fix $f \in \mathcal{H}(U)$. Then $f \in \mathcal{H}(V)$, where V is a ball with $\mathrm{cl}(V) \subset U$. Next, fix $p \in U \setminus \mathrm{cl}(V)$ and set $V_0 = V$. There exists a sequence of balls $\{V_k\}_{k=1}^{\infty}$ such that $\mathrm{cl}(V_k) \subset U$ for all k, the family $\{\mathrm{cl}(V_k)\}_{k=0}^{\infty}$ is pairwise disjoint, and $V_k \to p$ when $k \to \infty$. Choose a homeomorphism $\varphi \in \mathcal{H}(U)$ such that $\varphi(V_{k-1}) = V_k$ for $k = 1, 2, \ldots$ Observe that V and φ depend on the choice of f; however all such φ can be chosen in such a way that they are pairwise conjugate. Here we use the fact that $\mathcal{H}(U)$ acts transitively on the family of balls in U.

Now we define a continuous homomorphism $S : \mathcal{H}(V) \to \mathcal{H}(U)$ by the formula

$$S(h) = \varphi^k h \varphi^{-k} \quad \text{on } V_k, \ k = 0, 1, \ldots$$

and $S(h) = \mathrm{id}$ outside $\bigcup_{k=0}^{\infty} V_k$. It is clear that $h = [S(h), \varphi]$, as required. Observe that for an arbitrary $f \in \mathcal{H}(U)$ we can repeat this argument with a new φ conjugate to the old one. If U is a half-ball then this procedure is easily repeated with φ moving on the boundary.

(2) In fact, $\nu(f) = \nu([h, \bar{\varphi}]) \leq \nu(h\bar{\varphi}h^{-1}) + \nu(\bar{\varphi}^{-1}) = 2\nu(\varphi)$.

(3) The first assertion follows from Theorems 1.1 and 3.4(1). To show the second we will use the reasoning from section 2.2 in [5]. Given $\psi \in \mathcal{H}(U)$ we say that f is a *ψ-commutator* if $f = \mathrm{conj}_g[\psi, h]$ for some $g, h \in \mathcal{H}(U)$. According to the reasoning from [5, Section 2.2], any element of $[\mathcal{H}(V), \mathcal{H}(V)]$ can be written as a product of seven φ-commutators, provided V and φ are as in (1). Now since $\pi^{-1}(U) = U \times G$, by arguing componentwise, any homeomorphism belonging to the commutator subgroup of $\mathcal{H}_G(\pi^{-1}(V))$ can be expressed as a product of seven $\bar{\varphi}$-commutators, where $\bar{\varphi}(x, g) = (\varphi(x), g)$. V being chosen arbitrarily, the same holds for any element of the commutator subgroup of $\mathcal{H}_G(\pi^{-1}(U))$. Finally we apply Theorem 1.1.

(4) is an immediate consequence of (3). $\square$

The following fact reveals a significance of the fragmentation norm for homeomorphism groups. Analogous theorem holds for diffeomorphism groups, cf. [5].

Theorem 3.6. *The following statements are equivalent:*

1. *The group $\mathcal{H}(N)$ is bounded.*
2. *The fragmentation norm on $\mathcal{H}(N)$ is bounded.*

If the above holds and ν is any conjugation-invariant norm on $\mathcal{H}(N)$ then

$$\nu(h) \leq 2\mathrm{frag}(h)\nu(\varphi),$$

where φ is a moving map in a ball.

Proof. The part (1)⇒(2) is trivial. To show (2)⇒(1), let ν be a conjugation-invariant norm on $\mathcal{H}(N)$. It suffices to show that ν is bounded on $\mathcal{H}(U)$, where U is a ball or a half-ball. In the case of homeomorphisms this can be proved in a simpler manner than in [5]. Namely it is an immediate consequence of Lemma 3.5(2) and the estimates on ν that we obtain are better than in the smooth case, cf. Section 5. □

Some other examples of bounded or unbounded homeomorphism groups, especially diffeomorphism groups, symplectomorphism groups or contactomorphism groups, have been specified in [5, 9, 11, 12, 17].

4. PROOFS OF THEOREMS 1.2 AND 1.3

We will need the following proposition.

Proposition 4.1. *Let $\pi : M \to B$ be a principal G-bundle. Then the homomorphism $P : \mathcal{H}_G(M) \to \mathcal{H}(B)$ is surjective. Furthermore, the induced map for isotopies $\bar{P} : \mathcal{PH}_G(M) \to \mathcal{PH}(B)$ is also surjective.*

Proof. Obviously, the second assertion implies the first. To show the second, let $h \in \mathcal{PH}(B)$. From Corollary 2.7 or 2.10 it follows that $h = h_1 \dots h_p$ with each $h_i \in \mathcal{PH}(B)$ supported in a ball or in a half-ball. Consequently, each h_i can be lifted by means of a trivialization of π to an isotopy $\tilde{h}_i \in \mathcal{PH}_G(M)$ such that $\bar{P}(\tilde{h}_i) = h_i$. Thus $\bar{P}(\tilde{h}) = h$, where $\tilde{h} = \tilde{h}_1 \dots \tilde{h}_p$. □

Proof of Theorem 1.2. (1) In view of the obvious argument it follows from Proposition 4.1.

(2) Let $f \in \mathcal{H}_G(M)$ and let $\bar{f} = \{\bar{f}^t\} \in \mathcal{PH}_G(M)$ joining f to the identity. Then, in view of the assumption,

$$\bar{P}(\bar{f}) = f_1 \dots f_r,$$

where $f_i \in \mathcal{PH}(B)$ is supported in a ball or half-ball U_i and r is bounded by $\mathrm{fd}_{\mathcal{PH}(B)}$. According to Lemma 3.5(1) any f_i can be written as $f_i = [h_i, \varphi_i]$, where h_i is supported in U_i and φ_i is a moving map in U_i. Every $h_i \in \mathcal{PH}(U_i)$ and $\varphi_i \in \mathcal{H}(U_i)$ can be lifted to $\bar{h}_i \in \mathcal{PH}_G(M)$ and $\bar{\varphi}_i \in \mathcal{H}_G(M)$ due to Proposition 4.1. Set

$$\bar{h} = [\bar{h}_1, \bar{\varphi}_1] \dots [\bar{h}_r, \bar{\varphi}_r], \quad \bar{g} = \bar{f}\bar{h}^{-1} \quad \text{and} \quad g = \bar{g}^1.$$

Since $\bar{P}(\bar{h}) = \bar{P}(\bar{f})$, it follows that $\bar{g}$ is an isotopy in $\ker(P)$ joining g to the identity.

Now in view of Proposition 2.4 g can be written as the product of at most $n + 1$ factors, each of them supported in $\pi^{-1}(U)$, where U is a ball or a half-ball provided B is compact. If B is open then U is a finite union of balls with disjoint closures. In view of Lemma 3.5(3) in each case g can be expressed as a product of at most $2(n + 1)$ commutators or $7(n + 1)$ φ-commutators.

Consequently, f can be expressed as a product of $r + 2(n + 1)$ commutators. Therefore $\mathcal{H}_G(M)$ is uniformly perfect and

$$\mathrm{cld}_{\mathcal{H}_G(M)} \leq \mathrm{fd}_{\mathcal{PH}(B)} + 2(n + 1),$$

as required.

(3) The proof is analogous to that of Theorem 3.1. in [10]. □

Proof of Theorem 1.3. (1) It is an immediate consequence of Lemma 3.1 and Proposition 4.1.

(2) For $f \in \mathcal{PH}_G(M)$ one has $\bar{P}(f) \in \mathcal{PH}(B)$ and we may apply Corollary 2.7 or 2.10 to get a fragmentation of isotopies

$$\bar{P}(f) = h_1 \dots h_p,$$

where each h_i is supported in a ball or a half-ball, say U_i. Here p is bounded according to the assumption. Due to Proposition 4.1 we define $\tilde{h}_i \in \mathcal{PH}_G(\pi^{-1}(U_i))$, the lifts of h_i, and we put $\tilde{h} = \tilde{h}_1 \dots \tilde{h}_p$ and $g = f\tilde{h}^{-1} \in \mathcal{P}\ker(P)$ as in the proof of 1.2. Due to Lemma 3.5(4), $\nu(\tilde{h}_i) \leq 14\nu(\varphi)$ for all i, where φ is a moving map in a ball. Now we apply Proposition 2.4 to g and we obtain a fragmentation $g = g_1 \dots g_d$, where $d \leq n+1$ for dimension reasons, and $\mathrm{supp}(g_i) \subset \pi^{-1}(V_i)$, where V_i is either a ball or a half-ball or a finite union of balls with pairwise disjoint closures. According to Lemma 3.5(4), $\nu(g_i) \leq 14\nu(\varphi)$ for all i. Thus

$$\nu(f) \leq 14(\mathrm{fd}_{\mathcal{PH}(B)} + n + 1)\nu(\varphi).$$

This completes the proof. $\square$

5. REMARKS ON THE SMOOTH CASE

Let $\pi : M \to B$ be a smooth principal G-bundle, where G is a compact Lie group. We define $\mathcal{D}^r_G(M)$, $1 \leq r \leq \infty$, to be the group of all compactly supported G-equivariant diffeomorphisms of class C^r that can be joined to the identity by a compactly supported G-equivariant isotopy of class C^r. The main theorem in [1] is the following.

Theorem 5.1 ([1])**.** *The group $\mathcal{D}^r_G(M)$ is perfect if $1 \leq r \leq \infty$ and $r \neq \dim B + 1$.*

Since in the C^r case, $1 \leq r \leq \infty$, fragmentations of diffeomorphisms are constructed by means of isotopies and the corresponding smooth families of vector fields, the following fact has a standard proof (see, e.g., [4]), easier than that in the topological case (cf. Section 2).

Proposition 5.2. *Let N be a C^r manifold. The group $\mathcal{D}^r(N)$ is fragmentable with respect to the family of balls and half-balls. The same is true for the group $\mathcal{P}^r\mathcal{D}^r(N)$ of all C^r-smooth isotopies of $\mathcal{D}^r(N)$.*

In the proof of the following analogue of Theorem 3.6. Here we apply Theorem 3.4(2) instead of Lemma 3.5(2).

Theorem 5.3. *Under the above notation, the following statements are equivalent:*

1. *The group $\mathcal{D}^r(N)$ is bounded.*
2. *The fragmentation norm on $\mathcal{D}^r(N)$ is bounded.*

Moreover, if ν is any conjugation-invariant norm on $\mathcal{D}^r(N)$ then

$$\nu(h) \leq 14\mathrm{frag}(h)\nu(\varphi),$$

where φ is a moving map in a ball.

Now we can extend slightly the main theorem in the Fukui's paper [10] as follows.

Theorem 5.4. *Let $\pi : M \to B$ be a C^r principal G-bundle, where G is a compact Lie group and $1 \leq r \leq \infty$ and $r \neq n+1$, $n = \dim B$. Assume that either B is a compact manifold (possibly with boundary if $\dim B \geq 2$), or B is an open manifold. Then:*

1. *If $\mathcal{D}^r_G(M)$ is uniformly perfect then $\mathcal{D}^r(B)$ is uniformly perfect.*
2. *If the fragmentation norm of the isotopy group $\mathcal{P}^r\mathcal{D}^r(B)$ of $\mathcal{D}^r(B)$ is bounded, then $\mathcal{D}^r_G(M)$ is uniformly perfect, and the commutator length diameter $\mathrm{cld}_{\mathcal{D}^r_G(M)}$ of $\mathcal{D}^r_G(M)$ satisfies*
$$\mathrm{cld}_{\mathcal{D}^r_G(M)} \leq 2(\mathrm{fd}_{\mathcal{P}^r\mathcal{D}^r(B)} + n + 1).$$
3. *If $\mathcal{D}^r(B)$ is uniformly perfect and $\ker(P)$ possesses a finite number of components, then $\mathcal{D}^r_G(M)$ is uniformly perfect and*
$$\mathrm{cld}_{\mathcal{D}^r_G(M)} \leq \mathrm{cld}_{\mathcal{D}^r(B)} + 2(n+1) + l,$$
where l is the number of components of $\ker(P)$.

The proof is completely analogous to that of Theorem 1.2 with the only three exceptions. First we have to use Proposition 5.2 for $\mathcal{P}^r\mathcal{D}^r(B)$. Second, Lemma 3.5 is no longer true in the smooth case and has to be replaced by Theorem 3.4(1) (or by direct reasonings from Tsuboi [19] and Fukui [10]). Third, we have to apply the fact that $\mathcal{D}^r_G(M)$ is perfect (Theorem 5.1).

For the same reasons we have the following smooth counterpart of Theorem 1.3.

Theorem 5.5. *Under the above assumption, we have:*

1. *If $\mathcal{D}^r_G(M)$ is bounded then $\mathcal{D}^r(B)$ is bounded too.*
2. *If the fragmentation norm of the isotopy group $\mathcal{P}^r\mathcal{D}^r(B)$ of $\mathcal{D}^r(B)$ is bounded, then $\mathcal{D}^r_G(M)$ is bounded as well.*

For any conjugation-invariant norm ν on $\mathcal{D}^r_G(M)$ and for all $f \in \mathcal{D}^r_G(M)$ one has
$$\nu(f) \leq 14(\mathrm{fd}_{\mathcal{P}^r\mathcal{D}^r(B)} + n + 1)\nu(\varphi),$$
where φ is a moving map in a ball.

In the proof we make similar modifications as in Theorem 5.4.

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