Abstract. A graph $G$ is called hypohamiltonian if $G$ is not hamiltonian, but $G - x$ is hamiltonian for each vertex $x$ of $G$. We present a list of 331 forbidden configurations which do not appear in hypohamiltonian graphs.

Keywords: hypohamiltonian graph, forbidden configuration, long cycle.

Mathematics Subject Classification: 05C38, 05C45.

1. INTRODUCTION

Throughout this paper, we consider connected graphs without loops or multiple edges. The used graph terminology is taken from the book [14]. A cycle in a graph $G$ which contains all vertices of $G$ is called hamiltonian cycle, and a graph containing such a cycle is called hamiltonian. A vertex of degree $k$ is referred to as a $k$-vertex. A configuration in a graph $G = (V, E)$ is a pair $(H, f)$ where $H \subseteq G$ is a connected subgraph of $G$ and $f : V(H) \rightarrow \mathbb{Z}^+$ is a mapping such that, for each $x \in V(H)$, $f(x) \geq \deg_H(x)$ (less formal, the configuration is a subgraph with specified degrees of its vertices in the supergraph).

One of classical topics in research of hamiltonian graphs is the study of nonhamiltonian graphs whose all vertex-deleted subgraphs are hamiltonian; these graphs are called hypohamiltonian graphs. The smallest such graph is the Petersen graph (see [8]) and, by results from [2–4,11], and [1], $n$-vertex hypohamiltonian graphs exist for all $n \geq 10, n \notin \{11, 12, 14, 17\}$ (even more, the number of nonisomorphic such graphs grows exponentially with $n$, see [10]). A lot of work was also done in study of planar hypohamiltonian graphs, concerning mainly their constructions and looking for smallest examples, see [12, 7, 17, 15, 9].
In this paper, we are interested in forbidden configurations for hypohamiltonian graphs. It is easy to see that a hypohamiltonian graph cannot contain, as a subgraph, a 3-cycle with 3-valent vertex. Surprisingly, to our knowledge, no systematic study of other configurations which cannot appear in hypohamiltonian graphs was done (very recently in [6], it was shown that a hypohamiltonian graph cannot contain an edge common to two 3-cycles and incident with a 4-valent vertex). In order to fill this gap, we present a list of 329 new forbidden configurations for hypohamlicity. A part of this list is used in our recent related paper [5] on local structure of planar hypohamiltonian graphs from the point of view of existence of unavoidable configurations (which are still only little explored; along the classical result of Thomassen [13] that each planar hypohamiltonian graph contains a 3-valent vertex, C. Zamfirescu [16] very recently showed that it contains at least four such vertices). Note that our list is not complete as we considered mainly the configurations of small diameter whose central vertex has small degree (the complete list is very likely infinite). The absence of these configurations yields that planar hypohamiltonian graphs are, in certain sense, locally sparse and the upper bound for the number of their edges might be much lower than the general upper bound $3n - 6$ for planar graphs ($n$ being the number of vertices). In particular, we believe that the multiplicative coefficient 3 could be decreased; anyway, this cannot be achieved using the presented configuration list as it contains also configurations where vertices of degrees 5 or more are surrounded with triangles only.

2. RESULTS

We start with auxiliary lemma whose instances will be used later in the analysis of long cycle structure in hypohamiltonian graphs:

**Lemma 2.1.** Let $G$ be a graph, $v_0 \in V(G)$, and let $C$ be a hamiltonian cycle of $G - v_0$. If $C$ contains

(a) an edge $v_1v_2$, where $v_0v_1, v_0v_2 \in E(G)$, or

(b) a subpath $P = [v_1, v_2, v_3, v_4]$, where $v_0v_1, v_0v_3, v_2v_4 \in E(G)$, or

(c) two disjoint subpaths $P_1 = [v_1, v_2]$ and $P_2 = [v_3, v_4, v_5]$, where $v_0v_1, v_0v_4, v_2v_4,
v_3v_5 \in E(G)$, or

(d) two disjoint subpaths $P_1 = [v_1, v_2, v_3]$ and $P_2 = [v_4, v_5]$, where $v_0v_1, v_0v_3, v_2v_4,
v_2v_5 \in E(G)$, or

(e) two disjoint subpaths $P_1 = [v_1, v_2, v_3, v_4]$ and $P_2 = [v_5, v_6, v_7]$, where $v_0v_1, v_0v_3, v_2v_6, v_4v_6, v_5v_7 \in E(G)$, or

(f) two disjoint subpaths $P_1 = [v_1, v_2, v_3, v_4]$ and $P_2 = [v_6, v_7]$, where $v_0v_1, v_0v_3, v_2v_5, v_4v_6, v_4v_7 \in E(G)$, or
(g) two disjoint subpaths $P_1 = [v_1, v_2]$ and $P_2 = [v_3, v_4, v_5, v_6]$, where $v_0v_1, v_0v_4, v_2v_5, v_3v_6 \in E(G)$, or

(h) a subpath $P = [v_1, v_2, v_3, v_4, v_5, v_6, v_7]$, where $v_0v_1, v_0v_4, v_2v_7, v_3v_6 \in E(G)$, or

(i) a subpath $P = [v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_4v_6]$, where $v_0v_1, v_0v_5, v_2v_7, v_3v_6 \in E(G)$, or

(j) two disjoint subpaths $P_1 = [v_1, v_2, v_3, v_4, v_5]$ and $P_2 = [v_6, v_7]$, where $v_0v_1, v_0v_5, v_2v_7, v_4v_6 \in E(G)$, or

then $G$ is hamiltonian.

Proof. Let $C$ be a hamiltonian cycle of $G - v_0$. For each of the cases described above, we find a hamiltonian cycle of $G$:

(a) take $C - [v_1, v_2] + [v_1, v_0, v_2]$ (Figure 1a),

(b) take $C - P + [v_1, v_0, v_3, v_2, v_4]$ (Figure 1b),

(c) take $C - P_1 + [v_1, v_0, v_4, v_2] - P_2 + [v_3, v_5]$ (Figure 1c),

(d) take $C - P_1 + [v_1, v_0, v_3] - P_2 + [v_4, v_2, v_5]$ (Figure 1d),

(e) take $C - P_1 + [v_1, v_0, v_3, v_2, v_6, v_4] - P_2 + [v_5, v_7]$ (Figure 1e),

(f) take $C - P_1 + [v_1, v_0, v_3, v_2, v_5] - P_2 + [v_6, v_4, v_7]$ (Figure 1f),

(g) take $C - P_1 + [v_1, v_0, v_4, v_5, v_2] - P_2 + [v_3, v_6]$ (Figure 1g),

(h) take $C - P + [v_1, v_0, v_4, v_5, v_6, v_3, v_2, v_7]$ (Figure 1h),

(i) take $C - P + [v_1, v_0, v_5, v_2, v_3, v_4, v_7]$ (Figure 1i),

(j) take $C - P_1 + [v_1, v_0, v_5] - P_2 + [v_6, v_4, v_3, v_2, v_7]$ (Figure 1j).
This implies that the above described routings of a cycle which omits exactly one vertex in a hypohamiltonian graph $G$ are not possible.
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Fig. 3. Configurations with central vertex of degree 3, 5, or 6
Fig. 4. Configurations with central vertex of degree 3, 5, or 6
Fig. 5. Configurations with central vertex of degree 3, 5, 6 or 4
Fig. 6. Configurations with central vertex of degree 4 or 7
**Theorem 2.2.** No hypohamiltonian graph contains any of the forbidden configurations represented on Figures 2–6.

**Proof.** Let $G$ be a hypohamiltonian graph.

**Configurations $F_3$ and $F_3'$**

**Case 1.** Suppose that $G$ contains $F_3$ (i.e. a triangle $[v_0, v_1, v_2]$ with 3-vertex $v_2$). Let $C$ be a hamiltonian cycle of $G - v_0$. Since $v_2$ is a 2-vertex of $G - v_0$, the edge $v_1v_2$ belongs to $C$. Thus by Lemma 2.1(a), $G$ is hamiltonian, a contradiction.

**Case 2.** Suppose that $G$ contains $F_3'4$ (Figure 7). Let $C$ be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), $C$ does not contain the edge $v_2v_3$, thus $P = [v_1, v_4, v_2, v_5]$ is a subpath of $C$. Hence by Lemma 2.1(b), $G$ is hamiltonian (where $C - P + [v_1, v_0, v_2, v_4, v_5]$ is a hamiltonian cycle of $G$), a contradiction.

![Fig. 7. Particular configurations with central 3-vertex (Cases 2–4)](image)

**Case 3.** Suppose that $G$ contains $F_333$, $F_334$ or $F_344$ (Figure 7). Let $C$ be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), $C$ does not contain the edges $v_3v_7$ (of $F_3334$ and $F_344$) and $v_1v_8$ (of $F_344$), thus $C = [v_1, v_2, v_3, v_4, v_5, v_6]$, a contradiction (note that $F_3333$ itself is not hypohamiltonian).

**Case 4.** Suppose that $G$ contains $F_335$ (Figure 7). Let $C$ be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), $C$ does not contain the edges $v_1v_4$ and $v_1v_8$, thus $[v_3, v_4, v_5]$ is a subpath of $C$. If $C$ contains the edge $v_2v_3$, then (with respect to 3-vertices $v_2, v_3, v_8$) $[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ is a subpath of $C$, thus $G$ is hamiltonian by Lemma 2.1(h). Otherwise, $C$ contains the edge $v_0v_2$, thus $[v_1, v_2, v_7, v_8, v_3, v_4, v_5]$ is a subpath of $C$. Hence $G$ is hamiltonian by Lemma 2.1(i), with contradiction in both cases.

In the sequel, we will analyze 83 configurations $F_3'ijk$ of Figures 3–5. They share the following common features: each of them contains a cycle $[v_0, v_1, v_2, v_3, v_4, v_5]$ and a 3-vertex $v_3$, which is a common neighbour of $v_0, v_2, v_4$, whereas $v_1, v_3, v_5$ have degrees $i, j, k$ ($3 \leq i, j, k \leq 6$), respectively, and $v_1$ ($v_5, v_6$) is incident with $i - 3$ ($j - 3, k - 3$) triangles (without common edge).

**Case 5.** Suppose that $G$ contains a configuration $F_3'ijk$; assume that the indexing of vertices of $F_3'ijk$ (Figure 8) is chosen in the way that $v_0$ is the special vertex (marked by x-cross in Figures 3–5). Let $C$ be a hamiltonian cycle of $G - v_0$. Clearly, $[v_2, v_3, v_4]$ is a subpath of $C$, since $v_3$ is a 2-vertex of $G - v_0$. 

Claim 5a. C does not contain the edges $v_2v_6$ and $v_4v_6$.

Proof of Claim 5a. Suppose that $C$ contains $v_4v_6$ (the case when $C$ contains $v_2v_6$ is treated in similar way). Note that in this case $C$ does not contain $v_4v_5$.

i. If $v_5$ is a 3-vertex (Figure 8a), then $C$ contains at most one edge incident with $v_5$, a contradiction.

ii. If $v_5$ is a 4-vertex, then by Lemma 2.1(a), $C$ does not contain the edge $v_5v_7$ (Figure 8b), or by Lemma 2.1(c), $C$ does not contain the path $[v_9, v_5, v_{10}]$ (Figure 8c), or else by Lemma 2.1(g), $C$ does not contain the edge $v_5v_{11}$ (Figure 8d). We obtain that $C$ contains at most one edge incident with $v_5$, a contradiction.

iii. If $v_5$ is a 5-vertex, then by Lemma 2.1(a) and (c), $C$ contains neither the edge $v_5v_7$ nor the path $[v_9, v_5, v_{10}]$ (Figure 8e), or by Lemma 2.1(a) and (g), $C$ does not contain the edges $v_5v_7$ and $v_5v_{11}$ (Figure 8f), or else by Lemma 2.1(c) and (g), $C$ contains neither the path $[v_9, v_5, v_{10}]$ nor the edge $v_5v_{11}$ (Figure 8g); thus $C$ contains at most one edge incident with $v_5$, a contradiction.

iv. If $v_5$ is a 6-vertex, then by Lemma 2.1(a), (c), and (g), $C$ contains neither the edge $v_5v_7$ nor the path $[v_9, v_5, v_{10}]$ nor the edge $v_5v_{11}$ (Figure 8h). Thus $C$ contains at most one edge incident with $v_5$, a contradiction as well. □

If $v_6$ is a 3-vertex, then by Claim 5a, $C$ contains at most one edge incident with $v_6$, a contradiction. In the rest of the Case 5, let $v_6$ have degree at least 4.
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If the edge $v_1v_2$ ($v_4v_5$) does not belong to any triangle of $F_3^{ijk}$, then $C$ contains $v_1v_2$ ($v_4v_5$).

Proof of Claim 5b. Assume that $v_1v_2$ does not belong to any triangle (similarly for $v_4v_5$).

i. If $v_1$ is a 3-vertex, then $C$ contains the edge $v_1v_2$ (Figure 9a).

ii. If $v_1$ is a 4-vertex, then by Lemma 2.1(a), $C$ does not contain the edge $v_1v_{12}$ (Figure 9b), or by Lemma 2.1(c), $C$ does not contain the path $[v_{14}, v_1, v_{15}]$ (Figure 9c). Thus $C$ contains the edge $v_1v_2$.

iii. If $v_1$ is a 5-vertex, then by Lemma 2.1(a), $C$ does not contain the edge $v_1v_{12}$ and by Lemma 2.1(c), $C$ does not contain the path $[v_{14}, v_1, v_{15}]$ (Figure 9d). Thus $C$ contains the edge $v_1v_2$ as well.

It is easy to check (on Figures 3–5) that (in the case when $\deg(v_6) \geq 4$) at least one of the edges $v_1v_2$ and $v_4v_5$ is incident with no triangle of $F_3^{ijk}$. Suppose $v_1v_2$ has this property. Then by Claim 5b, $[v_1, v_2, v_3, v_4]$ is a subpath of $C$.

Claim 5c. $C$ does not contain the edge $v_4v_5$ and the edge $v_4v_6$ belongs to a triangle of $F_3^{ijk}$.

Proof of Claim 5c. Recall that $v_6$ has degree at least 4.

i. If $v_6$ is a 4-vertex and the edge $v_4v_6$ does not belong to a triangle, then by Lemma 2.1(d), $C$ does not contain the edge $v_6v_{20}$ (Figure 10a), or by Lemma 2.1(e), $C$ does not contain the path $[v_{18}, v_6, v_{19}]$ (Figure 10b). Thus $C$ contains at most one edge incident with $v_6$, a contradiction.

ii. If $v_6$ is a 4-vertex and the edge $v_4v_6$ belongs to a triangle (Figure 10c), then $C$ contains the edge $v_6v_{16}$; thus by Lemma 2.1(d), $C$ does not contain the edge $v_4v_5$.

iii. If $v_6$ is a 5-vertex and the edge $v_4v_6$ does not belong to a triangle (Figure 10d), then by Lemma 2.1(d), $C$ does not contain the edge $v_6v_{20}$ and by Lemma 2.1(e), $C$ does not contain the path $[v_{18}, v_6, v_{19}]$. Thus $C$ contains at most one edge incident with $v_6$, a contradiction.

iv. If $v_6$ is a 5-vertex and the edge $v_4v_6$ belongs to a triangle, then by Lemma 2.1(d), $C$ does not contain the edge $v_6v_{20}$ (Figure 10e), or by Lemma 2.1(e), $C$ does not contain the path $[v_{18}, v_6, v_{19}]$ (Figure 10f). Hence $C$ contains the edge $v_6v_{16}$ and therefore by Lemma 2.1(d), $C$ does not contain the edge $v_4v_5$. 

*Fig. 9.* Possible neighbourhoods of $v_1$ in configurations $F_3^{ijk}$ (Case 5)
v. If \( v_6 \) is a 6-vertex, then by Lemma 2.1(d), \( C \) does not contain the edges \( v_6v_{20} \) and by Lemma 2.1(e), \( C \) does not contain the path \([v_{18}, v_6, v_{19}]\) (Figure 10g). Hence \( C \) contains the edge \( v_6v_{16} \) and therefore by Lemma 2.1(d), \( C \) does not contain the edge \( v_4v_5 \).

\[ \square \]

\( \begin{array}{c}
\begin{array}{ccc}
\text{(a)} & \text{(b)} & \text{(c)} \\
\text{(d)} & \text{(e)} & \text{(f)} \\
\text{(g)}
\end{array}
\end{array} \]

Fig. 10. Possible neighbourhoods of \( v_6 \) in configurations \( F'_3ijk \) (Case 5)

Now, \( C \) contains the path \([v_1, v_2, v_3, v_4]\), \( C \) does not contain the edge \( v_4v_5 \), and the edge \( v_4v_6 \) belongs to a triangle. It is easy to check (on Figures 3–5) that (for \( v_4v_6 \) belonging to a triangle and \( v_1v_2 \) not belonging to any triangle) the edge \( v_4v_5 \) does not belong to any triangle, thus by Claim 5b, \( C \) contains the edge \( v_4v_5 \), a contradiction.

**Configurations \( F_4 \) and \( F'_4 \)**

**Case 6.** Suppose that \( G \) contains \( F_4 \) (Figure 11). Let \( C \) be a hamiltonian cycle of \( G - v_0 \). By Lemma 2.1(a), \( C \) does not contain edges \( v_1v_2 \) and \( v_2v_3 \), thus \( C \) contains at most one edge incident with \( v_2 \), a contradiction.

**Case 7.** Suppose that \( G \) contains \( F'_4a, F'_4b, \) or \( F'_45 \) (Figure 11). Let \( C \) be a hamiltonian cycle of \( G - v_0 \). By Lemma 2.1(a), \( C \) contains neither the edge \( v_1v_3 \) nor the edge \( v_5v_6 \) (of \( F'_4a \) and \( F'_45 \)) and by Lemma 2.1(c), \( C \) does not contain the path \([v_7, v_5, v_8]\) (of \( F'_4b \) and \( F'_45 \)). Thus \([v_2, v_3, v_4, v_5]\) is a subpath of \( C \), hence by Lemma 2.1(b), \( G \) is hamiltonian, a contradiction.
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In the sequel, we will analyze 40 configurations \( F'_{4ijk} \) of Figures 5–6. They share the following common features: each of them contains a cycle \([v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7]\) and a 4-vertex \( v_4 \), which is a common neighbour of \( v_0, v_1, v_2, v_3, v_5 \), whereas \( v_2, v_6, v_7 \) have degrees \( i, j, k \) (\( 3 \leq i, j, k \leq 6 \)), respectively, and \( v_2 (v_6, v_7) \) is incident with \( i - 3 \) \((j - 3, k - 3)\) triangles (without common edge).

Case 8. Suppose that \( G \) contains a configuration \( F'_{4ijk} \) (except of \( F'_{4335c}, F'_{4336}, F'_{4344c}, F'_{4344j}, F'_{4345e}, F'_{4444c}, F'_{4444f}, \) and \( F'_{4445c} \)); assume that the indexing of vertices of \( F'_{4ijk} \) (Figure 12) is chosen in the way that \( v_0 \) is the special vertex (marked by x-cross in Figures 5–6). Note that in each of 32 considered configurations \( F'_{4ijk} \), the edges \( v_1v_2, v_3v_7, \) and \( v_5v_6 \) do not belong to any triangle of \( F'_{4ijk} \). Let \( C \) be a hamiltonian cycle of \( G - v_0 \). By Lemma 2.1(a), \( C \) does not contain the edge \( v_1v_4 \). Thus \([v_3, v_4, v_5]\) is a subpath of \( C \) (Figure 12).

Fig. 11. Particular configurations with central 4-vertex (Cases 6 and 7)

Fig. 12. Possible neighbourhoods of \( v_6 \) in configurations \( F'_{4ijk} \) (Case 8)
Claim 8a. \( C \) contains the edge \( v_5v_6 \).

Proof of Claim 8a.

i. If \( v_6 \) is a 3-vertex, then it is a 2-vertex of \( G - v_0 \) (Figure 12a), thus \( C \) contains the edge \( v_5v_6 \).

ii. If \( v_6 \) is a 4-vertex, then by Lemma 2.1(a), \( C \) does not contain the edge \( v_6v_8 \) (Figure 12b), or by Lemma 2.1(c), \( C \) does not contain the path \([v_9, v_6, v_{10}]\) (Figure 12c). Thus \( C \) contains the edge \( v_5v_6 \).

iii. If \( v_6 \) is a 5-vertex, then by Lemma 2.1(a), \( C \) does not contain the edge \( v_6v_8 \) and by Lemma 2.1(c), \( C \) does not contain the path \([v_9, v_6, v_{10}]\) (Figure 12d). Thus \( C \) contains the edge \( v_5v_6 \) as well. 

\[ \square \]

Fig. 13. Possible neighbourhoods of \( v_7 \) in configurations \( F_{ijk}^4 \) (Case 8)

Now, \( P_1 = [v_3, v_4, v_5, v_6] \) is a subpath of \( C \).

Claim 8b. \( C \) contains the edge \( v_3v_7 \).

Proof of Claim 8b.

i. If \( v_7 \) is a 3-vertex, then \( C \) does not contain the edge \( v_7v_5 \) (Figure 13a). Thus \( C \) contains the edge \( v_3v_7 \).

ii. If \( v_7 \) is a 4-vertex, then by Lemma 2.1(d), \( C \) does not contain the edge \( v_7v_{12} \) (Figure 13b), or by Lemma 2.1(e), \( C \) does not contain the path \([v_{13}, v_7, v_{14}]\) (Figure 13c). Thus \( C \) contains the edge \( v_3v_7 \).

iii. If \( v_7 \) is a 5-vertex, then by Lemma 2.1(d), \( C \) does not contain the edge \( v_7v_{12} \) and by Lemma 2.1(e), \( C \) does not contain the path \([v_{13}, v_7, v_{14}]\) (Figure 13d). Thus \( C \) contains the edge \( v_3v_7 \) as well. 

\[ \square \]

Fig. 14. Possible neighbourhoods of \( v_2 \) in configurations \( F_{ijk}^4 \) (Case 8)

Now, \( P_2 = [v_7, v_3, v_4, v_5, v_6] \) is a subpath of \( C \).
Claim 8c. $C$ contains the edge $v_1v_2$.

Proof of Claim 8c.

i. If $v_2$ is a 3-vertex, then it is a 2-vertex of $G - P_2$ (Figure 14a), thus the edge $v_1v_2$ belongs to $C$.

ii. If $v_2$ is a 4-vertex (Figure 14b), then by Lemma 2.1(f) $C$ does not contain the edge $v_2v_17$, thus the edge $v_1v_2$ belongs to $C$.

Now, $C$ contains the path $P_2 = [v_7, v_3, v_4, v_5, v_6]$ and the edge $v_1v_2$, thus by Lemma 2.1(g), $G$ is hamiltonian, a contradiction.

Case 9. Suppose that $G$ contains $F'_335c$ or $F'_3336$ (Figure 15). Let $C$ be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), $C$ does not contain the edges $v_1v_4$ and $v_6v_8$. Thus $[v_3, v_4, v_5]$ is a subpath of $C$. Since $v_7$ is a 3-vertex, $C$ contains either $v_3v_7$ or $v_5v_7$.

If $C$ contains $v_3v_7$, then $C$ does not contain $v_2v_3$, thus the edge $v_1v_2$ belongs to $C$, and by Lemma 2.1(g), $G$ is hamiltonian, a contradiction. If $C$ contains $v_5v_7$, then by Lemma 2.1(e), $C$ does not contain the path $P = [v_9, v_6, v_10]$ (for $F'_3336$), thus $C$ contains the edge $v_6v_11$, and by Lemma 2.1(g), $G$ is hamiltonian, a contradiction as well.

![Fig. 15. Particular configurations $F'_{ij}k$ (Case 9)](image)

Case 10. Suppose that $G$ contains one of $F'_3344c$, $F'_3344j$, $F'_3345e$, $F'_3444c$, $F'_3444f$, and $F'_3445c$ (Figure 16). Let $C$ be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), $C$ does not contain the edge $v_1v_9$. Thus $[v_2, v_1, v_8]$ is a subpath of $C$. By Lemma 2.1(b), $C$ does not contain the path $[v_2, v_4, v_3]$, thus $C$ contains the edge $v_4v_5$.

Claim 10a. $C$ contains the edge $v_5v_6$.

Proof of Claim 10a.

i. If $v_6$ is a 3-vertex, then it is a 2-vertex of $G - v_0$ ($F'_4344c$), thus $C$ contains the edge $v_5v_6$.

ii. If $v_6$ is a 4-vertex, then by Lemma 2.1(a), $C$ does not contain the edge $v_6v_{10}$ ($F'_3444j$, $F'_3444f$), or by Lemma 2.1(e), $C$ does not contain the path $[v_{11}, v_6, v_{12}]$ ($F'_3444e$). Thus $C$ contains the edge $v_5v_6$.

iii. If $v_1$ is a 5-vertex, then by Lemma 2.1(a), $C$ does not contain the edge $v_6v_{10}$ and by Lemma 2.1(c), $C$ does not contain the path $[v_{11}, v_6, v_{12}]$ ($F'_3345e$, $F'_3445c$). Thus $C$ contains the edge $v_5v_6$ as well. □
Fig. 16. Particular configurations $F'_{ijk}$ (Case 10)

Now, $P_1 = [v_2, v_1, v_8]$ and $P_2 = [v_4, v_5, v_6]$ are subpaths of $C$.

Claim 10b. $C$ contains the edge $v_3v_7$.

Proof of Claim 10b.

i. If $v_7$ is a 3-vertex, then it is a 2-vertex of $G - P_2$, thus the edge $v_3v_7$ belongs to $C$ (concerning the configurations $F'_{344c}, F'_{344j}, F'_{444c}$).

ii. If $v_7$ is a 4-vertex, then by Lemma 2.1(d), $C$ does not contain the edge $v_7v_{14}$ (concerning the configurations $F'_{344c}, F'_{444c}, F'_{444f}, F'_{445c}$). Thus $C$ contains the edge $v_3v_7$. \hfill $\square$

Now, if $C$ contains the edge $v_3v_4$, then by Lemma 2.1(g), $G$ is hamiltonian. Otherwise $C$ contains the edge $v_2v_4$ and by Lemma 2.1(j), $G$ is hamiltonian as well, a contradiction in both cases.

Fig. 17. Configuration $F_{4}3333$ (Case 11)
Case 11. Suppose that $G$ contains $F_4^{3333}$ (Figure 17). Let $C$ be a hamiltonian cycle of $G - v_0$. Then the cycle $[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ is a subgraph of $C$, a contradiction (as the smallest hypohamiltonian graph has 10 vertices).

Configurations $F_6$

In the sequel, we will analyze 42 configurations $F_6^{ijk}$, each of them results from corresponding $F_3^{\ell mn}$ (with $\ell = i - 1$, $m = j - 1$, $n = k - 1$) by adding all three dashed edges (Figures 3–5). They share the following common features: each of them contains 6-wheel with the central 6-vertex $v_3$ and a rim cycle $[v_0, v_1, v_2, v_6, v_4, v_5]$, whereas $v_1, v_5, v_6$ have degrees $i, j, k$ ($5 \leq i, j, k \leq 7$), respectively, and $v_1$ ($v_5, v_6$) is incident with $i - 2 (j - 2, k - 2)$ triangles.

Case 12. Suppose that $G$ contains a configuration $F_6^{ijk}$; assume that the indexing of vertices of $F_6^{ijk}$ (Figure 18) is chosen in the way that $v_0$ is the special vertex (marked by x-cross in Figures 3–5). Let $C$ be a hamiltonian cycle of $G - v_0$. Then by Lemma 2.1(a), $C$ does not contain the edges $v_1v_3$ and $v_3v_5$ (that is, two of three edges in which $F_6^{ijk}$ differs from $F_3^{\ell mn}$). If $C$ does not contain the edge $v_3v_6$ (the third edge of $F_6^{ijk}$ not occurring in $F_3^{\ell mn}$), then the proof is the same as for $F_3^{\ell mn}$.

Suppose that $C$ contains $v_3v_6$. Assume first that $v_5$ is a 5-vertex and $v_0v_1$ belongs to two triangles of $F_6^{ijk}$ (Figure 18a; the case when $v_1$ is a 5-vertex and $v_0v_1$ belongs to two triangles of $F_6^{ijk}$ is symmetric). Then, by Lemma 2.1(a), $C$ does not contain the edge $v_5v_7$, thus $C$ contains the edge $v_4v_5$. Moreover, if $C$ contains the edge $v_3v_4$, then by Lemma 2.1(b), $G$ is hamiltonian, otherwise $C$ contains the edge $v_2v_3$ and by Lemma 2.1(c), $G$ is hamiltonian as well, a contradiction in both cases.

In the remaining configurations $F_6^{ijk}$ — that is, when $v_5$ is not a 5-vertex or $v_0v_1$ belongs to exactly one triangle of $F_6^{ijk}$ (and, symmetrically, $v_1$ is not a 5-vertex or $v_0v_1$ belongs to exactly one triangle of $F_6^{ijk}$) — it is easy to check (on Figures 3–5) that both edges $v_1v_2$ and $v_4v_5$ belong to exactly one triangle of $F_6^{ijk}$ (i.e. to $[v_1, v_2, v_3]$ or $[v_3, v_4, v_5]$, respectively).

Fig. 18. Possible neighbourhoods of $v_5$ in configurations $F_6^{ijk}$ (Case 12)
Claim 12a. C does not contain the edges $v_2v_3$ and $v_3v_4$.

Proof of Claim 5a. Suppose that C contains $v_3v_4$ (if C contains $v_2v_3$, we argue similarly).

i. If $v_5$ is a 5-vertex, then by Lemma 2.1(c), C does not contain the path $[v_9, v_5, v_{10}]$ (Figure 18b), thus C contains the edge $v_4v_5$ and consequently by Lemma 2.1(b), $G$ is hamiltonian, a contradiction.

ii. If $v_5$ is a 6-vertex, then by Lemma 2.1(a) and (c), C contains neither the edge $v_5v_7$ nor the path $[v_9, v_5, v_{10}]$ (Figure 18c), thus C contains the edge $v_4v_5$ and consequently by Lemma 2.1(b), $G$ is hamiltonian, a contradiction as well. \hfill \Box

Finally, C contains at most one edge incident with $v_3$, a contradiction.

Configurations $F_5$

Case 13. Suppose that G contains $F_55a$, $F_55b$, or $F_56$ (Figure 19). Let C be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), C contains neither the edge $v_1v_3$ nor the edge $v_3v_5$ nor else the edge $v_5v_6$ (of $F_55a$ and $F_56$) and by Lemma 2.1(c), C does not contain the path $[v_7, v_5, v_8]$ (of $F_55b$ and $F_56$). Thus $[v_2, v_3, v_4, v_5]$ is a subpath of C, hence by Lemma 2.1(b), $G$ is hamiltonian, a contradiction.

Fig. 19. Particular configurations with central 5-vertex (Cases 13 and 14)

Case 14. Suppose that G contains $F_533333$ (Figure 19). Let C be a hamiltonian cycle of $G - v_0$. Then the cycle $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}]$ is a subgraph of C, a contradiction (there is no hypohamiltonian graph on 11 vertices).

In the sequel, we will analyze 147 configurations $F_{5ijk}$, each of them results from corresponding $F_5\ell mn$ (with $\ell = i - 1$, $m = j - 1$, $n = k$, or $\ell = i - 1$, $m = j$, $n = k - 1$ or else $\ell = i$, $m = j - 1$, $n = k - 1$) by adding two dashed edges (Figures 3–5). They share the following common features: each of them contains a cycle $R = [v_0, v_1, v_2, v_6, v_4, v_5]$ and a 5-vertex $v_3$, which is a common neighbour of five of the six vertices of $R$, whereas $v_1, v_5, v_6$ have degrees $i, j, k$ ($3 \leq i, j, k \leq 7$), respectively, and $v_1$ ($v_5, v_6$) is incident with $i - 2$ ($j - 2, k - 2$) triangles or with $i - 3$ ($j - 3, k - 3$) triangles, if $v_1$ ($v_5, v_6$) is not adjacent to $v_3$, respectively.
Case 15. Suppose that $G$ contains a configuration $F_{5ijk}$; assume that the indexing of vertices of $F_{5ijk}$ is chosen in the way that $v_0$ is the special vertex (marked by x-cross in Figures 3–5). Let $C$ be a hamiltonian cycle of $G - v_0$. In every configuration $F_{5ijk}$, either $F_{5ijk}$ is a subgraph of corresponding $F_{5pqr}$ (with $p + q + r = i + j + k + 1$, $i \leq p \leq i + 1$, $j \leq q \leq j + 1$, $k \leq r \leq k + 1$) and the proof follows from the proof for $F_{5pqr}$, or $F_{5ijk}$ results from corresponding $F_{5tmm}$ by adding two edges $v_1v_3$ and $v_3v_5$ between neighbours of $v_0$. Then by Lemma 2.1(a), $C$ does not contain the edges $v_1v_3$ and $v_3v_5$ and the proof is the same as for $F_{5tmm}$.

Configurations $F_7$

Case 16. Suppose that $G$ contains $F_755$ (Figure 20). Let $C$ be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), $C$ does not contain the edges $v_1v_9$, $v_1v_2$, $v_2v_3$, and $v_3v_{10}$. Thus $[v_7, v_1, v_8]$ and $[v_4, v_3, v_{12}]$ are subpaths of $C$. Subsequently, by Lemma 2.1(b), $C$ does not contain the edges $e_2v_7$ and $e_2v_4$ and by Lemma 2.1(c), $C$ does not contain the path $[v_5, v_2, v_6]$. Hence $C$ contains at most one edge incident with $v_2$, a contradiction.

![Fig. 20. Configurations with central 7-vertex (Cases 16 and 17)](image)

Case 17. Suppose that $G$ contains $F_7555$, $F_75555$, or $F_7556$ (Figure 20). Let $C$ be a hamiltonian cycle of $G - v_0$. By Lemma 2.1(a), $C$ does not contain the edges $v_1v_9$, $v_1v_2$, $v_2v_3$, and $v_3v_{10}$. Thus $C$ contains the edge $v_1v_7$. Subsequently, by Lemma 2.1(b), $C$ does not contain the path $[v_6, v_2, v_7]$ and by Lemma 2.1(c), $C$ contains neither the path $[v_6, v_2, v_5]$ nor the path $[v_5, v_2, v_4]$.


Proof of Claim 17a.

i. For $F_7555$ or $F_7556$, $C$ contains the path $[v_8, v_1, v_7]$, thus, by Lemma 2.1(b), $C$ does not contain the edge $v_2v_7$. Hence, $C$ contains the path $[v_6, v_2, v_4]$.

ii. For $F_75555$, $C$ contains the edge $v_1v_7$ as well as the edge $v_3v_4$. Moreover, $C$ does not contain the path $[v_7, v_2, v_4]$, because otherwise $C$ does not contain the edges $v_6v_7$, $v_6v_2$, $v_5v_2$, and $v_5v_4$, and by Lemma 2.1(d), $C$ does not contain also the edges $v_6v_{14}$ and $v_5v_{13}$; but then $C$ contains the edge $v_5v_6$ and, finally, $C - [v_1, v_7, v_2, v_4, v_3] + [v_1, v_0, v_3] - [v_0, v_5] + [v_6, v_7, v_2, v_4, v_3]$ is a hamiltonian cycle of $G$, a contradiction. Therefore, $C$ contains the path $[v_6, v_2, v_4]$ (or the symmetrical path $[v_5, v_2, v_7]$). □
Claim 17b. C contains the edge $v_3v_4$.

Proof of Claim 17b.

i. It is obvious for the configurations $F_7555$ and $F_75555$ (there are only two remaining edges incident with $v_3$).

ii. For $F_7556$, by Lemma 2.1(c), $C$ does not contain the path $[v_11, v_3, v_12]$, thus $C$ contains the edge $v_3v_4$. □

Now, $C$ contains the path $[v_6, v_2, v_4, v_3]$.

Claim 17c. $C$ contains the edge $v_5v_6$.

Proof of Claim 17c.

By Lemma 2.1(d), $C$ does not contain the edge $v_5v_{13}$, thus $C$ contains the edge $v_5v_6$. □

Finally, $C$ contains the path $[v_5, v_6, v_2, v_4, v_3]$ and the edge $v_1v_7$, hence by Lemma 2.1(g), $G$ is hamiltonian, a contradiction. □

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