

ON THE STABILITY OF SOME SYSTEMS OF EXPONENTIAL DIFFERENCE EQUATIONS

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Abstract. In this paper we prove the stability of the zero equilibria of two systems of difference equations of exponential type, which are some extensions of an one-dimensional biological model. The stability of these systems is investigated in the special case when one of the eigenvalues is equal to -1 and the other eigenvalue has absolute value less than 1, using centre manifold theory. In addition, we study the existence and uniqueness of positive equilibria, the attractivity and the global asymptotic stability of these equilibria of some related systems of difference equations.

Keywords: difference equations, asymptotic behaviour, global stability, centre manifold, biological dynamics.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

Difference equations and systems of difference equations containing exponential terms have numerous potential applications in biology. A large number of papers dealing with such or related equations have been published (see, e.g. [6, 16, 22–24]). In [34], the following model incorporating litter inhibition is discussed:

$$B_{t+1} = cN \frac{e^{a-bL_t}}{1 + e^{a-bL_t}}, \quad L_{t+1} = \frac{L_t^2}{L_t + d} + ckN \frac{e^{a-bL_t}}{1 + e^{a-bL_t}},$$

where B is the living biomass, L the litter mass, N the total soil nitrogen, t the time (measured in years) and constants $a, b, c, d > 0$ and $0 < k < 1$. In this model, the living biomass (B) is reduced below its equilibrium, by litter. Litter decay is determined by d , while litter production is k times the living biomass. The complexity of the grassland ecosystem makes its study interesting but complicated. In addition, in [18], the authors studied the boundedness and the persistence of the positive solutions, the existence,

the attractivity and the global asymptotic stability of the unique positive equilibrium, as well as the existence of periodic solutions of the following equation:

$$x_{n+1} = a \frac{x_n^2}{b + x_n} + c \frac{e^{k-dx_n}}{1 + e^{k-dx_n}},$$

where $a \in (0, 1)$, a, b, c, d, k are positive constants and x_0 is a positive real number.

Motivated by this discrete time model and recent studies of symmetric and close to symmetric systems of difference equations (see, e.g. [9, 11, 19, 25, 26]), in this paper, we will study the stability of the zero equilibria of the following systems:

$$x_{n+1} = a_1 \frac{y_n}{b_1 + y_n} + c_1 \frac{x_n e^{k_1 - d_1 x_n}}{1 + e^{k_1 - d_1 x_n}}, \quad y_{n+1} = a_2 \frac{x_n}{b_2 + x_n} + c_2 \frac{y_n e^{k_2 - d_2 y_n}}{1 + e^{k_2 - d_2 y_n}} \quad (1.1)$$

and

$$x_{n+1} = a_1 \frac{x_n}{b_1 + x_n} + c_1 \frac{y_n e^{k_1 - d_1 y_n}}{1 + e^{k_1 - d_1 y_n}}, \quad y_{n+1} = a_2 \frac{y_n}{b_2 + y_n} + c_2 \frac{x_n e^{k_2 - d_2 x_n}}{1 + e^{k_2 - d_2 x_n}}, \quad (1.2)$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, k_1, k_2$, are real constants and the initial values x_0 and y_0 are real numbers.

In addition, we will investigate the asymptotic behaviour of the positive solutions of the following systems of difference equations:

$$x_{n+1} = a_1 \frac{y_n^2}{b_1 + y_n} + c_1 \frac{e^{d_1 - k_1 x_n}}{1 + e^{d_1 - k_1 x_n}}, \quad y_{n+1} = a_2 \frac{e^{b_2 - k_2 x_n}}{1 + e^{b_2 - k_2 x_n}} + c_2 \frac{e^{d_2 - k_3 y_n}}{1 + e^{d_2 - k_3 y_n}}, \quad (1.3)$$

$$x_{n+1} = a_1 \frac{y_n^2}{b_1 + y_n} + c_1 \frac{e^{d_1 - k_1 x_n}}{1 + e^{d_1 - k_1 x_n}}, \quad y_{n+1} = a_2 \frac{x_n^2}{b_2 + x_n} + c_2 \frac{e^{d_2 - k_2 y_n}}{1 + e^{d_2 - k_2 y_n}}, \quad (1.4)$$

$$x_{n+1} = a_1 \frac{x_n^2}{b_1 + x_n} + c_1 \frac{e^{d_1 - k_1 y_n}}{1 + e^{d_1 - k_1 y_n}}, \quad y_{n+1} = a_2 \frac{e^{b_2 - k_2 x_n}}{1 + e^{b_2 - k_2 x_n}} + c_2 \frac{e^{d_2 - k_3 y_n}}{1 + e^{d_2 - k_3 y_n}}, \quad (1.5)$$

$$x_{n+1} = a_1 \frac{x_n^2}{b_1 + x_n} + c_1 \frac{e^{d_1 - k_1 y_n}}{1 + e^{d_1 - k_1 y_n}}, \quad y_{n+1} = a_2 \frac{y_n^2}{b_2 + y_n} + c_2 \frac{e^{d_2 - k_2 x_n}}{1 + e^{d_2 - k_2 x_n}}, \quad (1.6)$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, k_1, k_2, k_3$ are positive constants and the initial values x_0 and y_0 are positive.

The results of this paper could be used to create more elaborate biological models to facilitate understanding the underlying ecological mechanisms. The results obtained for the systems (1.1) and (1.2) provide conditions for stability of the zero equilibria of those systems. Those equilibria correspond to the physical situation where both quantities (x and y) vanish.

The asymptotic behaviour of positive solutions of scalar equations related to the previous systems is studied in [22]. For some related cyclic systems of difference equations see [9, 28, 31] and [32], as well as some three-dimensional systems (see, e.g., [19, 27, 33]). Finally we note that, since difference equations have several applications in applied sciences, there exists a rich bibliography concerning theory and applications of difference equations (see [1–34]).

2. STABILITY OF ZERO EQUILIBRIUM OF SYSTEM (1.1)

In the following, we prove the stability of the zero equilibrium of System (1.1), using Centre Manifold Theory.

Proposition 2.1. *Consider System (1.1) where a_1, b_1, b_2, c_2, k_1 , are real positive constants and a_2, c_1, k_2 are real negative constants such that*

$$c_2 < 1 + e^{-k_2}, \quad -\frac{1 + e^{k_1}}{e^{k_1}} < c_1 < -\frac{c_2 e^{k_2}(1 + e^{k_1})}{e^{k_1}(1 + e^{k_2})}, \quad (2.1)$$

$$\left(1 + \frac{c_1 e^{k_1}}{1 + e^{k_1}}\right) \left(1 + \frac{c_2 e^{k_2}}{1 + e^{k_2}}\right) = \frac{a_1 a_2}{b_1 b_2}, \quad (2.2)$$

$$d_2 > \max\{0, A_1, A_2, A_3\}, \quad (2.3)$$

where

$$A_1 = \frac{(1 + e^{k_2})^2}{c_2 \Gamma^2 e^{k_2}} \left(\frac{c_1 d_1 \Gamma e^{k_1}}{(1 + e^{k_1})^2} + \frac{a_1 \Gamma^3}{b_1^2} - \frac{a_2}{b_2^2} \right),$$

$$A_2 = \frac{(1 + e^{k_2})^2}{c_2 \Gamma \Delta e^{k_2}} \left(\frac{c_1 d_1 \Delta e^{k_1}}{(1 + e^{k_1})^2} + \frac{a_1 \Delta^2 \Gamma}{b_1^2} - \frac{a_2}{b_2^2} \right),$$

$$A_3 = \sqrt{\frac{2(1 + e^{k_2}(1 + c_2))(1 + e^{k_2})^2}{b_1^2 c_2 e^{k_2} (1 - e^{k_2})}}.$$

Then the zero equilibrium of (1.1) is stable.

Proof. The Jacobian matrix J_0 at the zero equilibrium for (1.1) is

$$J_0 = \begin{bmatrix} \frac{c_1 e^{k_1}}{1 + e^{k_1}} & \frac{a_1}{b_1} \\ \frac{a_2}{b_2} & \frac{c_2 e^{k_2}}{1 + e^{k_2}} \end{bmatrix}.$$

Calculating the eigenvalues of J_0 , using (2.1) and (2.2) we obtain

$$\lambda_1 = -1, \quad \lambda_2 = 1 + \frac{c_1 e^{k_1}}{1 + e^{k_1}} + \frac{c_2 e^{k_2}}{1 + e^{k_2}} \quad \text{and so} \quad |\lambda_2| < 1.$$

Now, the initial system can be written as

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = J_0 \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}, \quad (2.4)$$

where

$$f(x, y) = \frac{a_1 y}{y + b_1} - \frac{a_1 y}{b_1} + \frac{c_1 x e^{k_1 - d_1 x}}{1 + e^{k_1 - d_1 x}} - \frac{c_1 e^{k_1}}{1 + e^{k_1}} x,$$

$$g(x, y) = \frac{a_2 x}{x + b_2} - \frac{a_2 x}{b_2} + \frac{c_2 y e^{k_2 - d_2 y}}{1 + e^{k_2 - d_2 y}} - \frac{c_2 e^{k_2}}{1 + e^{k_2}} y.$$

We let now

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = T \begin{bmatrix} u_n \\ v_n \end{bmatrix},$$

where T is the matrix that diagonalizes J_0 defined by

$$T = \begin{bmatrix} 1 & 1 \\ \Gamma & \Delta \end{bmatrix},$$

where

$$\Gamma = -\frac{b_1(1 + e^{k_1}(1 + c_1))}{a_1(1 + e^{k_1})}, \quad \Delta = \frac{b_1(1 + e^{k_2}(1 + c_2))}{a_1(1 + e^{k_2})}. \quad (2.5)$$

Then, (2.4) can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \hat{f}(u_n, v_n) \\ \hat{g}(u_n, v_n) \end{bmatrix}, \quad (2.6)$$

where

$$\begin{aligned} & \hat{f}(u, v) \\ &= R \left(\Delta \left(\frac{a_1(\Gamma u + \Delta v)}{b_1 + \Gamma u + \Delta v} - \frac{a_1(\Gamma u + \Delta v)}{b_1} + \frac{c_1(u + v)e^{k_1 - d_1(u+v)}}{1 + e^{k_1 - d_1(u+v)}} - \frac{c_1(u + v)e^{k_1}}{1 + e^{k_1}} \right) \right. \\ & \quad \left. - \left(\frac{a_2(u + v)}{b_2 + u + v} - \frac{a_2(u + v)}{b_2} + \frac{c_2(\Gamma u + \Delta v)e^{k_2 - d_2(\Gamma u + \Delta v)}}{1 + e^{k_2 - d_2(\Gamma u + \Delta v)}} - \frac{c_2(\Gamma u + \Delta v)e^{k_2}}{1 + e^{k_2}} \right) \right), \\ & \hat{g}(u, v) \\ &= R \left(-\Gamma \left(\frac{a_1(\Gamma u + \Delta v)}{b_1 + \Gamma u + \Delta v} - \frac{a_1(\Gamma u + \Delta v)}{b_1} + \frac{c_1(u + v)e^{k_1 - d_1(u+v)}}{1 + e^{k_1 - d_1(u+v)}} - \frac{c_1(u + v)e^{k_1}}{1 + e^{k_1}} \right) \right. \\ & \quad \left. + \frac{a_2(u + v)}{b_2 + u + v} - \frac{a_2(u + v)}{b_2} + \frac{c_2(\Gamma u + \Delta v)e^{k_2 - d_2(\Gamma u + \Delta v)}}{1 + e^{k_2 - d_2(\Gamma u + \Delta v)}} - \frac{c_2(\Gamma u + \Delta v)e^{k_2}}{1 + e^{k_2}} \right) \end{aligned} \quad (2.7)$$

and

$$R = \frac{1}{\Delta - \Gamma}.$$

We now let $v = h(u)$ with $h(u) = \psi(u) + O(u^4)$, $\psi(u) = \eta u^2 + \theta u^3$, $\eta, \theta \in \mathbb{R}$. The use of this approximation is justified by Theorem 7 of [1]. Consequently, according to Theorem 8 of [1] and using (2.6), the study of the stability of the zero equilibrium of System (1.1) reduces to the study of the stability of the zero equilibrium of the equation:

$$u_{n+1} = -u_n + \hat{f}(u_n, \psi(u_n)) = G(u_n). \quad (2.8)$$

We now need to determine η (the coefficient in the Taylor expansion). From (2.6), we conclude that map h must satisfy the centre manifold equation (see [1, p. 34], [3, p. 243], [15], [16, p. 642] and [19]):

$$h(-u + \hat{f}(u, h(u))) - \lambda_2 h(u) - \hat{g}(u, h(u)) = 0.$$

Keeping the terms up to u^3 and using (2.7), we obtain

$$\eta = \frac{R}{1 - \lambda_2} \left(\frac{c_1 d_1 \Gamma e^{k_1}}{(1 + e^{k_1})^2} + \frac{a_1 \Gamma^3}{b_1^2} - \frac{a_2}{b_2^2} - \frac{c_2 d_2 \Gamma^2 e^{k_2}}{(1 + e^{k_2})^2} \right). \quad (2.9)$$

From (2.8) and (2.9) we obtain $G'(0) = -1$ and

$$\begin{aligned} G'''(0) = R \left(\Delta \left(-\frac{6c_1 d_1^2 e^{2k_1}}{(1 + e^{k_1})^3} + \frac{3c_1 d_1^2 e^{k_1}}{(1 + e^{k_1})^2} - \frac{12c_1 d_1 \eta e^{k_1}}{(1 + e^{k_1})^2} - \frac{12a_1 \Delta \Gamma \eta}{b_1^2} + \frac{6a_1 \Gamma^3}{b_1^3} \right) \right. \\ \left. - \left(\frac{6a_2}{b_2^3} - \frac{12a_2 \eta}{b_2^2} - \frac{12c_2 d_2 \Delta \Gamma e^{k_2} \eta}{(1 + e^{k_2})^2} - \frac{6c_2 d_2^2 \Gamma^3 e^{2k_2}}{(1 + e^{k_2})^3} + \frac{3c_2 d_2^2 \Gamma^3 e^{k_2}}{(1 + e^{k_2})^2} \right) \right). \end{aligned} \quad (2.10)$$

From (2.10) and since $R > 0$, we deduce that if the following inequalities hold, then, $G'''(0) > 0$:

$$\eta \left(-\frac{c_1 d_1 \Delta e^{k_1}}{(1 + e^{k_1})^2} - \frac{a_1 \Delta^2 \Gamma}{b_1^2} + \frac{a_2}{b_2^2} + \frac{c_2 d_2 \Delta \Gamma e^{k_2}}{(1 + e^{k_2})^2} \right) > 0, \quad (2.11)$$

$$\Gamma^3 \left(\frac{2a_1 \Delta}{b_1^3} + \frac{c_2 d_2^2 e^{k_2} (e^{k_2} - 1)}{(1 + e^{k_2})^3} \right) > 0, \quad (2.12)$$

$$\frac{c_1 d_1^2 \Delta e^{k_1} (1 - e^{k_1})}{(1 + e^{k_1})^3} - \frac{2a_2}{b_2^3} > 0. \quad (2.13)$$

Now, from (2.3), we have that $d_2 > A_1$ and so from (2.9) we conclude that $\eta < 0$. Inequality (2.11) holds, since $\eta < 0$ and from (2.3) we obtain $d_2 > A_2$. Moreover, from (2.1), we obtain $c_1 > -\frac{1+e^{k_1}}{e^{k_1}}$ and therefore from (2.5) we get $\Gamma < 0$. Hence, (2.12) holds, since from (2.3) we have $d_2 > A_3$. Finally, (2.13) is always true, since $c_1, a_2 < 0$ and $\Delta, k_1 > 0$.

So, we have shown, that if the conditions in the proposition hold, then $G'''(0) > 0$. Hence, for the Schwarzian derivative (see [3], and [13]), we have $Sf(0) < 0$. Therefore, from Theorem 8 of [1], the zero equilibrium of (1.1) is stable. \square

3. STABILITY OF ZERO EQUILIBRIUM OF SYSTEM (1.2)

In the following, we prove the stability of the zero equilibrium of System (1.2), using Centre Manifold Theory.

Proposition 3.1. *Consider System (1.2) where $b_1, b_2, c_1, c_2, k_1, k_2$ are real positive constants and a_1, a_2 are real negative constants such that*

$$-2 < \frac{a_1}{b_1} + \frac{a_2}{b_2} < 0, \quad -b_1 < a_1 < 0, \quad -b_2 < a_2 < 0. \quad (3.1)$$

Let

$$\begin{aligned}
A_1 &= \sqrt{\frac{2a_2(1+e^{k_1})^2(a_1+b_1)^4}{b_2^2(a_2+b_2)(1-e^{k_1})a_1^2e^{2k_1}}}, \\
A_2 &= \frac{a_2^2(a_1+b_1)^4(1+e^{k_1})^3(1+e^{k_2})(e^{k_2}-1)}{-2a_1b_1b_2^3e^{3k_1}e^{k_2}(a_2+b_2)}, \\
A_3 &= \left(1+\frac{a_1}{b_1}\right)\left(1+\frac{a_2}{b_2}\right)\frac{(1+e^{k_1})(1+e^{k_2})}{e^{k_1+k_2}}, \\
K_1 &= \frac{1+e^{k_1}}{b_2}\sqrt{\frac{2a_2}{(a_2+b_2)(1-e^{k_1})}}, \\
K_2 &= -\frac{a_1c_1e^{k_1}}{(a_1+b_1)^2}, \\
K_3 &= \sqrt{\frac{-2a_1(1+e^{k_2})^3(1+e^{k_1})(a_2+b_2)}{b_2c_1c_2b_1^3e^{k_1}e^{k_2}(e^{k_2}-1)}}, \\
K_4 &= -\frac{(a_1+b_1)^2a_2(1+e^{k_1})^2(1+e^{k_2})^2}{b_1^2b_2^2c_1^2c_2e^{2k_1}e^{k_2}}.
\end{aligned}$$

Suppose also that the following relations hold:

$$4(a_1+b_1)(a_2+b_2) < a_1a_2(e^{k_1}-1)(e^{k_2}-1), \quad A_1 < c_1 < \sqrt{\frac{A_2}{A_3}}, \quad c_2 \leq \frac{A_3}{c_1} \quad (3.2)$$

and

$$K_1 < d_1 < K_2, \quad (3.3)$$

$$K_3 < d_2 < K_4. \quad (3.4)$$

Then the zero equilibrium of (1.2) is stable.

Proof. Firstly, we note that $4(a_1+b_1)(a_2+b_2) < a_1a_2(e^{k_1}-1)(e^{k_2}-1)$ implies that $A_1 < \sqrt{\frac{A_2}{A_3}}$. Next, we can easily see that $K_1 < K_2$ is true, since from (3.2), we have $c_1 > A_1$. Moreover, using $c_1c_2 < A_3$ and $c_1^2 < \frac{A_2}{A_3}$ from (3.2), we have $c_1^3c_2 < A_2$ and therefore $K_3 < K_4$ is true.

Now, the Jacobian matrix J_0 at the zero equilibrium for (1.2) is

$$J_0 = \begin{bmatrix} \frac{a_1}{b_1} & \frac{c_1e^{k_1}}{1+e^{k_1}} \\ \frac{c_2e^{k_2}}{1+e^{k_2}} & \frac{a_2}{b_2} \end{bmatrix}.$$

Calculating the eigenvalues of J_0 , using (3.1) and (3.2) we obtain

$$\lambda_1 = -1, \quad \lambda_2 = 1 + \frac{a_1}{b_1} + \frac{a_2}{b_2} \quad \text{and so} \quad |\lambda_2| < 1.$$

Now, the initial system can be written as

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = J_0 \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}, \quad (3.5)$$

where

$$\begin{aligned} f(x, y) &= \frac{a_1 x}{x + b_1} - \frac{a_1 x}{b_1} + \frac{c_1 y e^{k_1 - d_1 y}}{1 + e^{k_1 - d_1 y}} - \frac{c_1 e^{k_1}}{1 + e^{k_1}} y, \\ g(x, y) &= \frac{a_2 y}{y + b_2} - \frac{a_2 y}{b_2} + \frac{c_2 x e^{k_2 - d_2 x}}{1 + e^{k_2 - d_2 x}} - \frac{c_2 e^{k_2}}{1 + e^{k_2}} x. \end{aligned}$$

We let now

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = T \begin{bmatrix} u_n \\ v_n \end{bmatrix},$$

where T is the matrix that diagonalizes J_0 defined by

$$T = \begin{bmatrix} \Gamma & \Delta \\ 1 & 1 \end{bmatrix},$$

where

$$\Gamma = -\frac{b_1 c_1 e^{k_1}}{(b_1 + a_1)(1 + e^{k_1})}, \quad \Delta = \frac{b_2 c_1 e^{k_1}}{(a_2 + b_2)(1 + e^{k_1})}. \quad (3.6)$$

Then, (3.5) can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \hat{f}(u_n, v_n) \\ \hat{g}(u_n, v_n) \end{bmatrix}, \quad (3.7)$$

where

$$\begin{aligned} &\hat{f}(u, v) \\ &= R \left(\frac{a_1(\Gamma u + \Delta v)}{b_1 + \Gamma u + \Delta v} - \frac{a_1(\Gamma u + \Delta v)}{b_1} + \frac{c_1(u + v)e^{k_1 - d_1(u+v)}}{1 + e^{k_1 - d_1(u+v)}} - \frac{c_1(u + v)e^{k_1}}{1 + e^{k_1}} \right. \\ &\quad \left. - \Delta \left(\frac{a_2(u + v)}{b_2 + u + v} - \frac{a_2(u + v)}{b_2} + \frac{c_2(\Gamma u + \Delta v)e^{k_2 - d_2(\Gamma u + \Delta v)}}{1 + e^{k_2 - d_2(\Gamma u + \Delta v)}} - \frac{c_2(\Gamma u + \Delta v)e^{k_2}}{1 + e^{k_2}} \right) \right), \\ &\hat{g}(u, v) \\ &= -R \left(\frac{a_1(\Gamma u + \Delta v)}{b_1 + \Gamma u + \Delta v} - \frac{a_1(\Gamma u + \Delta v)}{b_1} + \frac{c_1(u + v)e^{k_1 - d_1(u+v)}}{1 + e^{k_1 - d_1(u+v)}} - \frac{c_1(u + v)e^{k_1}}{1 + e^{k_1}} \right. \\ &\quad \left. - \Gamma \left(\frac{a_2(u + v)}{b_2 + u + v} - \frac{a_2(u + v)}{b_2} + \frac{c_2(\Gamma u + \Delta v)e^{k_2 - d_2(\Gamma u + \Delta v)}}{1 + e^{k_2 - d_2(\Gamma u + \Delta v)}} - \frac{c_2(\Gamma u + \Delta v)e^{k_2}}{1 + e^{k_2}} \right) \right) \end{aligned} \quad (3.8)$$

and

$$R = \frac{1}{\Gamma - \Delta}.$$

We now let $v = h(u)$ with $h(u) = \psi(u) + O(u^4)$, $\psi(u) = \eta u^2 + \theta u^3$, $\eta, \theta \in \mathbb{R}$. Using this approximation is justified by Theorem 7 of [1]. Consequently, according to Theorem 8 of [1] and using (3.7), the study of the stability of the zero equilibrium of System (1.2) reduces to the study of the stability of the zero equilibrium of the equation

$$u_{n+1} = -u_n + \hat{f}(u_n, \psi(u_n)) = G(u_n). \quad (3.9)$$

We now need to determine η (the coefficient in the Taylor expansion). From (3.7), we conclude that map h must satisfy the centre manifold equation (see [1, p. 34], [3, p. 243], [15], [16, p. 642] and [19]):

$$h(-u + \hat{f}(u, h(u))) - \lambda_2 h(u) - \hat{g}(u, h(u)) = 0.$$

Keeping the terms up to u^3 and using (3.8), we obtain

$$\eta = \frac{R}{1 - \lambda_2} \left(\frac{c_1 d_1 e^{k_1}}{(1 + e^{k_1})^2} + \frac{a_1 \Gamma^2}{b_1^2} - \Gamma \left(\frac{a_2}{b_2^2} + \frac{c_2 d_2 \Gamma^2 e^{k_2}}{(1 + e^{k_2})^2} \right) \right). \quad (3.10)$$

From (3.9) and (3.10) we obtain $G'(0) = -1$ and

$$\begin{aligned} G'''(0) = R \left(-\frac{6c_1 d_1^2 e^{2k_1}}{(1 + e^{k_1})^3} + \frac{3c_1 d_1^2 e^{k_1}}{(1 + e^{k_1})^2} - \frac{12c_1 d_1 \eta e^{k_1}}{(1 + e^{k_1})^2} - \frac{12a_1 \Delta \Gamma \eta}{b_1^2} + \frac{6a_1 \Gamma^3}{b_1^3} \right. \\ \left. - \Delta \left(\frac{6a_2}{b_2^3} - \frac{12a_2 \eta}{b_2^2} - \frac{12c_2 d_2 \Delta \Gamma e^{k_2} \eta}{(1 + e^{k_2})^2} - \frac{6c_2 d_2^2 \Gamma^3 e^{2k_2}}{(1 + e^{k_2})^3} + \frac{3c_2 d_2^2 \Gamma^3 e^{k_2}}{(1 + e^{k_2})^2} \right) \right). \end{aligned} \quad (3.11)$$

From (3.11) and since $R < 0$, we deduce that if the following inequalities hold, then, $G'''(0) > 0$:

$$\eta \left(-\frac{c_1 d_1 e^{k_1}}{(1 + e^{k_1})^2} - \frac{a_1 \Delta \Gamma}{b_1^2} + \frac{a_2 \Delta}{b_2^2} + \frac{c_2 d_2 \Delta^2 \Gamma e^{k_2}}{(1 + e^{k_2})^2} \right) < 0, \quad (3.12)$$

$$\Gamma^3 \left(\frac{2a_1}{b_1^3} + \frac{2c_2 d_2^2 \Delta e^{2k_2}}{(1 + e^{k_2})^3} - \frac{c_2 d_2^2 \Delta e^{k_2}}{(1 + e^{k_2})^2} \right) < 0, \quad (3.13)$$

$$-\frac{2c_1 d_1^2 e^{2k_1}}{(1 + e^{k_1})^3} + \frac{c_1 d_1^2 e^{k_1}}{(1 + e^{k_1})^2} - \frac{2\Delta a_2}{b_2^3} < 0. \quad (3.14)$$

Now, from the second inequalities of (3.3) and (3.4), we obtain from (3.10) that $\eta > 0$. In addition, from conditions (3.1) and equation (3.6) we obtain $\Gamma < 0$. Hence, (3.12) holds, since $\eta > 0$, $a_1, a_2, \Gamma < 0$ and $c_1, c_2, d_1, d_2 > 0$. Moreover, from the first inequality of (3.4) and since $\Gamma < 0$, we deduce that (3.13) is true. Finally, from the first inequality of (3.3), we obtain that (3.14) is also true.

So, we have shown, that if the conditions in the proposition hold, then $G'''(0) > 0$. Hence, for the Schwarzian derivative, we have $Sf(0) < 0$. Therefore, from Theorem 8 of [1], the zero equilibrium of (1.2) is stable. \square

4. GLOBAL BEHAVIOUR OF SOLUTIONS OF SYSTEM (1.3)

In this section, we will investigate the asymptotic behaviour of the positive solutions of (1.3). We will use the following theorem, which is essentially a slight modification of Theorem 1.16 of [5]. The proof of Theorem 1.16 of [5] can be easily adapted to this case.

Theorem 4.1. *Let f, g , with $f : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous functions, where $\mathbb{R}_+ = (0, \infty)$. Let a, A, b, B be positive numbers, such that $a < A$, $b < B$ and $f : [a, A] \times [b, B] \rightarrow [a, A]$, $g : [a, A] \times [b, B] \rightarrow [b, B]$. Consider the system of difference equations*

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \dots \quad (4.1)$$

Suppose that $f(x, y)$ is a non-increasing function with respect to x and a non-decreasing with respect to y . Moreover, suppose that $g(x, y)$ is a non-increasing function with respect to x and a non-increasing function with respect to y . Finally suppose that, if m, M, r, R are real numbers such that if $M = f(m, R)$, $m = f(M, r)$, $R = g(m, r)$, $r = g(M, R)$, then $m = M$ and $r = R$. Then the system of difference equations (4.1) has a unique positive equilibrium (\bar{x}, \bar{y}) and every positive solution of System (4.1) which satisfies $x_{n_0} \in [a, A]$, $y_{n_0} \in [b, B]$ tends to the unique equilibrium of (4.1).

We now prove the following result.

Proposition 4.2. *Consider System (1.3). Suppose that the following relations hold true:*

$$k_1 c_1 < 1, \quad k_3 c_2 < 1 \quad (4.2)$$

and

$$a_1 a_2 k_2 < (1 - k_1 c_1)(1 - k_3 c_2). \quad (4.3)$$

Then System (1.3) has a unique positive equilibrium and every positive solution of System (1.3) tends to the unique equilibrium of (1.3) as $n \rightarrow \infty$.

Proof. System (1.3) can be written as $x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_n, y_n)$, where

$$f(x, y) = a_1 \frac{y^2}{b_1 + y} + c_1 \frac{e^{d_1 - k_1 x}}{1 + e^{d_1 - k_1 x}} \quad (4.4)$$

and

$$g(x, y) = a_2 \frac{e^{b_2 - k_2 x}}{1 + e^{b_2 - k_2 x}} + c_2 \frac{e^{d_2 - k_3 y}}{1 + e^{d_2 - k_3 y}}. \quad (4.5)$$

We can now easily see that $f(x, y)$ is non-increasing in x and non-decreasing in y , while $g(x, y)$ is non-increasing in x and non-increasing in y .

We now show that $f : [a, A] \times [b, B] \rightarrow [a, A]$ and $g : [a, A] \times [b, B] \rightarrow [b, B]$, where

$$B = a_2 + c_2, \quad A = a_1 \frac{B^2}{b_1 + B} + c_1,$$

$$b = a_2 \frac{e^{b_2 - k_2 A}}{1 + e^{b_2 - k_2 A}} + c_2 \frac{e^{d_2 - k_3 B}}{1 + e^{d_2 - k_3 B}} \quad \text{and} \quad a = a_1 \frac{b^2}{b_1 + b} + c_1 \frac{e^{d_1 - k_1 A}}{1 + e^{d_1 - k_1 A}}.$$

Indeed, from (4.5), we can easily obtain that $g(x, y) \leq a_2 + c_2 = B$. Now, for $y \leq B$, from (4.4), we get $f(x, y) \leq a_1 \frac{B^2}{b_1 + B} + c_1 = A$. Moreover, for $x \leq A$ and $y \leq B$ from (4.5), we obtain

$$g(x, y) \geq a_2 \frac{e^{b_2 - k_2 A}}{1 + e^{b_2 - k_2 A}} + c_2 \frac{e^{d_2 - k_3 B}}{1 + e^{d_2 - k_3 B}} = b.$$

Finally, taking $x \leq A$, $y \geq b$ from (4.4), we conclude that

$$f(x, y) \geq a_1 \frac{b^2}{b_1 + b} + c_1 \frac{e^{d_1 - k_1 A}}{1 + e^{d_1 - k_1 A}} = a.$$

Now, let m, M, r, R be positive real numbers such that $M = f(m, R)$, $m = f(M, r)$, $R = g(m, r)$ and $r = g(M, R)$. Therefore, we obtain $M - m = f(m, R) - f(M, r)$ and $R - r = g(m, r) - g(M, R)$. So, we can write

$$M - m = f(m, R) - f(M, R) + f(M, R) - f(M, r)$$

and

$$R - r = g(m, r) - g(M, r) + g(M, r) - g(M, R).$$

Using the Mean Value Theorem, we obtain

$$M - m = f_x(\xi_1, R)(m - M) + f_y(M, \xi_2)(R - r)$$

and

$$R - r = g_x(\xi_3, r)(m - M) + g_y(M, \xi_4)(r - R)$$

for some $\xi_1, \xi_3 \in (m, M)$ and $\xi_2, \xi_4 \in (r, R)$. Therefore, we can write

$$|M - m| \leq |f_x(\xi_1, R)||m - M| + |f_y(M, \xi_2)||R - r| \quad (4.6)$$

and

$$|R - r| \leq |g_x(\xi_3, r)||m - M| + |g_y(M, \xi_4)||r - R|. \quad (4.7)$$

However, we have

$$f_x(\xi_1, R) = \frac{-c_1 k_1 e^{d_1 - k_1 \xi_1}}{(1 + e^{d_1 - k_1 \xi_1})^2}, \quad f_y(M, \xi_2) = a_1 \frac{2b_1 \xi_2 + \xi_2^2}{(b_1 + \xi_2)^2}, \quad (4.8)$$

$$g_x(\xi_3, r) = \frac{-a_2 k_2 e^{b_2 - k_2 \xi_3}}{(1 + e^{b_2 - k_2 \xi_3})^2}, \quad g_y(M, \xi_4) = \frac{-c_2 k_3 e^{d_2 - k_3 \xi_4}}{(1 + e^{d_2 - k_3 \xi_4})^2}. \quad (4.9)$$

From (4.6), (4.7), (4.8), (4.9), we obtain

$$|M - m|(1 - k_1 c_1) \leq a_1 |R - r|, \quad |R - r|(1 - k_3 c_2) \leq a_2 k_2 |M - m|. \quad (4.10)$$

Therefore, from (4.10), we conclude that

$$|M - m| \leq \frac{a_1 a_2 k_2}{(1 - k_1 c_1)(1 - k_3 c_2)} |M - m|. \quad (4.11)$$

Hence, from (4.3), (4.10) and (4.11), we obtain

$$M = m, \quad R = r.$$

Let now (x_n, y_n) be an arbitrary solution of (1.3). From the discussion above, it is obvious that $y_1 \leq B$. Then we can see that $x_2 \leq A$. In addition, we also have $y_2 \leq B$ and so we can get $y_3 \geq b$. Finally, since we also have $x_3 \leq A$, we obtain $x_4 \geq a$. Hence, we have shown that $x_n \in [a, A]$, $y_n \in [b, B]$, for all $n \geq 4$.

Therefore, from Theorem 4.1, System (1.3) has a unique positive equilibrium (\tilde{x}, \tilde{y}) and every positive solution of System (1.3) tends to the unique positive equilibrium as $n \rightarrow \infty$. This completes the proof of the proposition. \square

Proposition 4.3. *Consider System (1.3), where the conditions in Proposition 4.2 hold. In addition, suppose that the following relation holds true:*

$$k_1 k_3 c_1 c_2 + k_2 a_1 a_2 < 1. \quad (4.12)$$

Then the unique positive equilibrium (\tilde{x}, \tilde{y}) of System (1.3) is globally asymptotically stable.

Proof. First, we will prove that (\tilde{x}, \tilde{y}) is locally asymptotically stable. The linearised system of (1.3) is

$$w_{n+1} = Aw_n, \quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad w_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

where

$$\alpha = -\frac{c_1 k_1 e^{d_1 - k_1 \tilde{x}}}{(1 + e^{d_1 - k_1 \tilde{x}})^2}, \quad \beta = \frac{a_1 (\tilde{y}^2 + 2b_1 \tilde{y})}{(b_1 + \tilde{y})^2},$$

$$\gamma = -\frac{a_2 k_2 e^{b_2 - k_2 \tilde{x}}}{(1 + e^{b_2 - k_2 \tilde{x}})^2}, \quad \delta = -\frac{c_2 k_3 e^{d_2 - k_3 \tilde{y}}}{(1 + e^{d_2 - k_3 \tilde{y}})^2}.$$

The characteristic equation of A is $\lambda^2 - (\alpha + \delta)\lambda + \alpha\delta - \beta\gamma = 0$. From relation (4.12), we obtain $\alpha\delta - \beta\gamma < 1$. Moreover, from (4.2), we obtain $|\alpha + \delta| < 1 + \alpha\delta - \beta\gamma$. Therefore, from Theorem 1.3.4 of [13], we deduce that (\tilde{x}, \tilde{y}) is locally asymptotically stable. Using Proposition 4.2, we conclude that (\tilde{x}, \tilde{y}) is globally asymptotically stable. This completes the proof of the proposition. \square

5. GLOBAL BEHAVIOUR OF SOLUTIONS OF SYSTEM (1.4)

In this section, we will investigate the asymptotic behaviour of the positive solutions of (1.4). We will use the following theorem, which is essentially a slight modification of Theorem 1.16 of [5]. The proof of Theorem 1.16 of [5] can be easily adapted to this case.

Theorem 5.1. *Let f, g , with $f : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous functions, where $\mathbb{R}_+ = (0, \infty)$. Let a, A, b, B be positive numbers, such that $a < A$, $b < B$ and $f : [a, A] \times [b, B] \rightarrow [a, A]$, $g : [a, A] \times [b, B] \rightarrow [b, B]$. Consider the system of difference equations*

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \dots \quad (5.1)$$

Suppose that $f(x, y)$ is a non-increasing function with respect to x and a non-decreasing with respect to y . Moreover, suppose that $g(x, y)$ is a non-decreasing function with respect to x and a non-increasing function with respect to y . Finally suppose that, if m, M, r, R are real numbers such that if $m = f(M, r)$, $M = f(m, R)$, $r = g(m, R)$, $R = g(M, r)$, then $m = M$ and $r = R$. Then the system of difference equations (5.1) has a unique positive equilibrium (\bar{x}, \bar{y}) and every positive solution of system (5.1) which satisfies $x_{n_0} \in [a, A]$, $y_{n_0} \in [b, B]$ tends to the unique equilibrium of (5.1).

We now prove the following result.

Proposition 5.2. *Consider system (1.4). Suppose that the following relations hold true:*

$$k_1 c_1 < 1, \quad k_2 c_2 < 1, \quad a_1, a_2 < 1 \quad (5.2)$$

and

$$a_1 a_2 < (1 - k_1 c_1)(1 - k_2 c_2). \quad (5.3)$$

Then System (1.4) has a unique positive equilibrium and every positive solution of the System (1.4) tends to the unique equilibrium of (1.4) as $n \rightarrow \infty$.

Proof. System (1.4) can be written as $x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_n, y_n)$, where

$$f(x, y) = a_1 \frac{y^2}{b_1 + y} + c_1 \frac{e^{d_1 - k_1 x}}{1 + e^{d_1 - k_1 x}} \quad (5.4)$$

and

$$g(x, y) = a_2 \frac{x^2}{b_2 + x} + c_2 \frac{e^{d_2 - k_2 y}}{1 + e^{d_2 - k_2 y}}. \quad (5.5)$$

We can now easily see that $f(x, y)$ is non-increasing in x and non-decreasing in y , while $g(x, y)$ is non-decreasing in x and non-increasing in y .

We now show that $f : [a, A] \times [b, B] \rightarrow [a, A]$ and $g : [a, A] \times [b, B] \rightarrow [b, B]$, where

$$A = \frac{a_1 c_2 + c_1}{1 - a_1 a_2} + \theta, \quad B = \frac{a_2 c_1 + c_2}{1 - a_1 a_2} + \theta,$$

$$a = c_1 \frac{e^{d_1 - k_1 A}}{1 + e^{d_1 - k_1 A}} \quad \text{and} \quad b = c_2 \frac{e^{d_2 - k_2 B}}{1 + e^{d_2 - k_2 B}}$$

for some $\theta \in (0, \infty)$. If $y \leq B$, from (5.4), we have

$$f(x, y) \leq a_1 y + c_1 \leq a_1 \left(\frac{a_2 c_1 + c_2}{1 - a_1 a_2} + \theta \right) + c_1 \leq A.$$

Next, if $x \leq A$, from (5.5), we get

$$g(x, y) \leq a_2 \left(\frac{a_1 c_2 + c_1}{1 - a_1 a_2} + \theta \right) + c_2 \leq B.$$

Moreover, if $x \leq A$, from (5.4), we obtain

$$f(x, y) \geq c_1 \frac{e^{d_1 - k_1 A}}{1 + e^{d_1 - k_1 A}} = a.$$

Finally, for $y \leq B$, from (5.5) we obtain

$$g(x, y) \geq c_2 \frac{e^{d_2 - k_2 B}}{1 + e^{d_2 - k_2 B}} = b.$$

In addition, working in the same way as in the proof of Proposition 4.2, we can show that there exist real numbers m, M, r, R such that if $m = f(M, r)$, $M = f(m, R)$, $r = g(m, R)$, $R = g(M, r)$, then $m = M$ and $r = R$.

Moreover, from (1.4), we can see that $x_{n+1} \leq a_1 y_n + c_1$, $y_{n+1} \leq a_2 x_n + c_2$. So, we get $x_{n+2} \leq a_1 a_2 x_n + a_1 c_2 + c_1$, $y_{n+2} \leq a_1 a_2 y_n + a_2 c_1 + c_2$. We now let $w_{n+2} = a_1 a_2 w_n + a_1 c_2 + c_1$ and $z_{n+2} = a_1 a_2 z_n + a_2 c_1 + c_2$ with $x_0 = w_0$, $y_0 = z_0$, $x_1 = w_1$ and $y_1 = z_1$. These two equations are essentially linear difference equations of first order, so solvable ones (many recent equations and systems are solved by using the equation such as [25–27, 29, 33]). Solving these equations, we obtain

$$w_n = A_1 (a_1 a_2)^{\frac{n}{2}} + A_2 (-1)^n (a_1 a_2)^{\frac{n}{2}} + \frac{a_1 c_2 + c_1}{1 - a_1 a_2}$$

and

$$z_n = B_1 (a_1 a_2)^{\frac{n}{2}} + B_2 (-1)^n (a_1 a_2)^{\frac{n}{2}} + \frac{a_2 c_1 + c_2}{1 - a_1 a_2},$$

respectively, where A_1, A_2, B_1, B_2 depend on the initial conditions. Now, since $x_0 = w_0$, $y_0 = z_0$, $x_1 = w_1$, $y_1 = z_1$, working inductively, we obtain that $x_n \leq w_n$, $y_n \leq z_n$, $n \in \mathbb{N}$, and therefore we conclude that for some $n_0 \in \mathbb{N}^+$ and $\theta \in (0, \infty)$, we have $x_n \leq \frac{a_1 c_2 + c_1}{1 - a_1 a_2} + \theta = A$ and $y_n \leq \frac{a_2 c_1 + c_2}{1 - a_1 a_2} + \theta = B$, $n \geq n_0$. Hence, we can easily see that $x_n \geq a$, $n > n_0$ and $y_n \geq b$, $n > n_0$. So, we have obtained that $x_n \in [a, A]$ and $y_n \in [b, B]$, $n > n_0$.

Therefore, using Theorem 5.1, we deduce that System (1.4) has a unique positive equilibrium (\tilde{x}, \tilde{y}) , and every positive solution of System (1.4) tends to the unique positive equilibrium as $n \rightarrow \infty$. This completes the proof of the proposition. \square

Proposition 5.3. *Consider System (1.4). If the conditions in Proposition 5.2 hold, then the unique positive equilibrium (\tilde{x}, \tilde{y}) of System (1.4) is globally asymptotically stable.*

Proof. Firstly, we will prove that (\tilde{x}, \tilde{y}) is locally asymptotically stable. The linearised system of (1.4) is

$$w_{n+1} = Aw_n, \quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad w_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix},$$

where

$$\alpha = -\frac{c_1 k_1 e^{d_1 - k_1 \tilde{x}}}{(1 + e^{d_1 - k_1 \tilde{x}})^2}, \quad \beta = \frac{a_1(\tilde{y}^2 + 2b_1 \tilde{y})}{(b_1 + \tilde{y})^2},$$

$$\gamma = \frac{a_2(\tilde{x}^2 + 2b_2 \tilde{x})}{(b_2 + \tilde{x})^2}, \quad \delta = -\frac{c_2 k_2 e^{d_2 - k_2 \tilde{y}}}{(1 + e^{d_2 - k_2 \tilde{y}})^2}.$$

The characteristic equation of A is $\lambda^2 - (\alpha + \delta)\lambda + \alpha\delta - \beta\gamma = 0$. Therefore, from relation (5.2), we obtain $\alpha\delta - \beta\gamma < 1$ and from (5.3), we obtain $|\alpha + \delta| < 1 + \alpha\delta - \beta\gamma$. Therefore, from Theorem 1.3.4 of [13], we deduce that (\tilde{x}, \tilde{y}) is locally asymptotically stable. Using Proposition 5.2, we conclude that (\tilde{x}, \tilde{y}) is globally asymptotically stable. This completes the proof of the proposition. \square

6. GLOBAL BEHAVIOUR OF SOLUTIONS OF SYSTEM (1.5)

In this section, we will investigate the asymptotic behaviour of the positive solutions of (1.5). We will use the following theorem, which is essentially a slight modification of Theorem 1.16 of [5]. The proof of Theorem 1.16 of [5] can be easily adapted to this case.

Theorem 6.1. *Let f, g , with $f : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous functions, where $\mathbb{R}_+ = (0, \infty)$. Let a, A, b, B be positive numbers, such that $a < A$, $b < B$ and $f : [a, A] \times [b, B] \rightarrow [a, A]$, $g : [a, A] \times [b, B] \rightarrow [b, B]$. Consider the system of difference equations*

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \dots \quad (6.1)$$

Suppose that $f(x, y)$ is a non-decreasing function with respect to x and a non-increasing with respect to y . Moreover, suppose that $g(x, y)$ is a non-increasing function with respect to x and a non-increasing function with respect to y . Finally suppose that, if m, M, r, R are real numbers such that if $M = f(M, r)$, $m = f(m, R)$, $R = g(m, r)$, $r = g(M, R)$, then $m = M$ and $r = R$. Then the system of difference equations (6.1) has a unique positive equilibrium (\bar{x}, \bar{y}) and every positive solution of system (6.1) which satisfies $x_{n_0} \in [a, A]$, $y_{n_0} \in [b, B]$ tends to the unique equilibrium of (6.1).

We now prove the following result.

Proposition 6.2. *Consider System (1.5). Suppose that the following relations hold true:*

$$k_3 c_2 < 1, \quad a_1 < 1 \quad (6.2)$$

and

$$k_1 c_1 a_2 k_2 < (1 - k_3 c_2)(1 - a_1). \quad (6.3)$$

Then System (1.5) has a unique positive equilibrium and every positive solution of System (1.5) tends to the unique equilibrium of (1.5) as $n \rightarrow \infty$.

Proof. System (1.5) can be written as $x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_n, y_n)$, where

$$f(x, y) = a_1 \frac{x^2}{b_1 + x} + c_1 \frac{e^{d_1 - k_1 y}}{1 + e^{d_1 - k_1 y}} \quad (6.4)$$

and

$$g(x, y) = a_2 \frac{e^{b_2 - k_2 x}}{1 + e^{b_2 - k_2 x}} + c_2 \frac{e^{d_2 - k_3 y}}{1 + e^{d_2 - k_3 y}}. \quad (6.5)$$

We can now easily see that $f(x, y)$ is non-decreasing in x and non-increasing in y , while $g(x, y)$ is non-increasing in x and non-increasing in y .

We now show that $f : [a, A] \times [b, B] \rightarrow [a, A]$ and $g : [a, A] \times [b, B] \rightarrow [b, B]$, where

$$A = \frac{c_1}{1 - a_1} + \theta, \quad B = a_2 + c_2,$$

$$a = c_1 \frac{e^{d_1 - k_1 B}}{1 + e^{d_1 - k_1 B}} \quad \text{and} \quad b = a_2 \frac{e^{b_2 - k_2 A}}{1 + e^{b_2 - k_2 A}} + c_2 \frac{e^{d_2 - k_3 B}}{1 + e^{d_2 - k_3 B}}$$

for some $\theta \in (0, \infty)$. If $x \leq A$, then from (6.4) we get

$$f(x, y) \leq a_1 x + c_1 \leq a_1 \left(\frac{c_1}{1 - a_1} + \theta \right) + c_1 \leq A.$$

Now, from (6.5), we can easily see that $g(x, y) \leq a_2 + c_2 = B$. So, if $y \leq B$, from (6.4), we obtain

$$f(x, y) \geq c_1 \frac{e^{d_1 - k_1 B}}{1 + e^{d_1 - k_1 B}} = a.$$

Finally, if $x \leq A$ and $y \leq B$, from (6.5) we obtain

$$g(x, y) \geq a_2 \frac{e^{b_2 - k_2 A}}{1 + e^{b_2 - k_2 A}} + c_2 \frac{e^{d_2 - k_3 B}}{1 + e^{d_2 - k_3 B}} = b.$$

In addition, working in the same way as in the proof of Proposition 4.2, we can show that there exist real numbers m, M, r, R such that if $M = f(M, r)$, $m = f(m, R)$, $R = g(m, r)$, $r = g(M, R)$, then $m = M$ and $r = R$.

Moreover, we have $x_{n+1} \leq a_1 x_n + c_1$, $n \in \mathbb{N}$ and we let z_n be the solution such that $z_0 = x_0$. Then, we have $z_n = (x_0 - \frac{c_1}{1 - a_1}) a_1^n + \frac{c_1}{1 - a_1}$, $n \in \mathbb{N}$. Now, since $z_0 = x_0$, and working inductively, we can show that $x_n \leq z_n$, $n \in \mathbb{N}$. Therefore, since $a_1 \in (0, 1)$, we get that for some $\theta \in (0, \infty)$ and $n_0 \in \mathbb{N}^+$, we have $x_n \leq \frac{c_1}{1 - a_1} + \theta = A$, $n \geq n_0$. Finally, we can also easily see that $x_n \geq a$, $n \geq 2$, and $y_n \leq B$, $n \geq 1$, and $y_n \geq b$, $n > n_0$. Hence, we have obtained that $x_n \in [a, A]$ and $y_n \in [b, B]$, $n > n_0$.

Therefore, using Theorem 6.1, System (1.5) has a unique positive equilibrium (\tilde{x}, \tilde{y}) , and every positive solution of System (1.5) tends to the unique positive equilibrium as $n \rightarrow \infty$. This completes the proof of the proposition. \square

Proposition 6.3. *Consider System (1.5). If the conditions in Proposition 6.2 hold, then the unique positive equilibrium (\tilde{x}, \tilde{y}) of System (1.5) is globally asymptotically stable.*

Proof. Firstly, we will prove that (\tilde{x}, \tilde{y}) is locally asymptotically stable. The linearised system of (1.5) is

$$w_{n+1} = Aw_n, \quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad w_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix},$$

where

$$\alpha = \frac{a_1(\tilde{x}^2 + 2b_1\tilde{x})}{(b_1 + \tilde{x})^2}, \quad \beta = -\frac{c_1k_1e^{d_1-k_1\tilde{y}}}{(1 + e^{d_1-k_1\tilde{y}})^2},$$

$$\gamma = -\frac{a_2k_2e^{b_2-k_2\tilde{x}}}{(1 + e^{b_2-k_2\tilde{x}})^2}, \quad \delta = -\frac{c_2k_3e^{d_2-k_3\tilde{y}}}{(1 + e^{d_2-k_3\tilde{y}})^2}.$$

The characteristic equation of A is $\lambda^2 - (\alpha + \delta)\lambda + \alpha\delta - \beta\gamma = 0$. Now, $\alpha\delta - \beta\gamma < 0 < 1$. Moreover, from relation (6.3) we obtain $|\alpha + \delta| < 1 + \alpha\delta - \beta\gamma$. Therefore, from Theorem 1.3.4 of [13], we deduce that (\tilde{x}, \tilde{y}) is locally asymptotically stable. Using Proposition 6.2, we conclude that (\tilde{x}, \tilde{y}) is globally asymptotically stable. This completes the proof of the proposition. \square

7. GLOBAL BEHAVIOUR OF SOLUTIONS OF SYSTEM (1.6)

In this section, we will investigate the asymptotic behaviour of the positive solutions of (1.6). We will use the following theorem, which is essentially a slight modification of Theorem 1.16 of [5]. The proof of Theorem 1.16 of [5] can be easily adapted to this case.

Theorem 7.1. *Let f, g , with $f : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous functions, where $\mathbb{R}_+ = (0, \infty)$. Let a, A, b, B be positive numbers, such that $a < A$, $b < B$ and $f : [a, A] \times [b, B] \rightarrow [a, A]$, $g : [a, A] \times [b, B] \rightarrow [b, B]$. Consider the system of difference equations*

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \dots \quad (7.1)$$

Suppose that $f(x, y)$ is a non-decreasing function with respect to x and a non-increasing function with respect to y . Moreover, suppose that $g(x, y)$ is a non-increasing function with respect to x and a non-decreasing function with respect to y . Finally suppose that, if m, M, r, R are real numbers such that if $M = f(M, r)$, $m = f(m, R)$, $R = g(m, R)$, $r = g(M, r)$, then $m = M$ and $r = R$. Then the system of difference equations (7.1) has a unique positive equilibrium (\bar{x}, \bar{y}) and every positive solution of System (7.1) which satisfies $x_{n_0} \in [a, A]$, $y_{n_0} \in [b, B]$ tends to the unique equilibrium of (7.1).

We now prove the following result.

Proposition 7.2. *Consider System (1.6). Suppose that the following relations hold true:*

$$a_1, a_2 < 1 \quad (7.2)$$

and

$$c_1c_2k_1k_2 < (1 - a_1)(1 - a_2). \quad (7.3)$$

Then System (1.6) has a unique positive equilibrium and every positive solution of System (1.6) tends to the unique equilibrium of (1.6) as $n \rightarrow \infty$.

Proof. System (1.6) can be written as $x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_n, y_n)$, where

$$f(x, y) = a_1 \frac{x^2}{b_1 + x} + c_1 \frac{e^{d_1 - k_1 y}}{1 + e^{d_1 - k_1 y}} \quad (7.4)$$

and

$$g(x, y) = a_2 \frac{y^2}{b_2 + y} + c_2 \frac{e^{d_2 - k_2 x}}{1 + e^{d_2 - k_2 x}}. \quad (7.5)$$

We can now easily see that $f(x, y)$ is non-decreasing in x and non-increasing in y , while $g(x, y)$ is non-increasing in x and non-decreasing in y .

We now show that $f : [a, A] \times [b, B] \rightarrow [a, A]$ and $g : [a, A] \times [b, B] \rightarrow [b, B]$, where

$$A = \frac{c_1}{1 - a_1} + \theta, \quad B = \frac{c_2}{1 - a_2} + \theta, \quad a = c_1 \frac{e^{d_1 - k_1 B}}{1 + e^{d_1 - k_1 B}} \quad \text{and} \quad b = c_2 \frac{e^{d_2 - k_2 A}}{1 + e^{d_2 - k_2 A}}$$

for some $\theta \in (0, \infty)$. If $x \leq A$, from (7.4) we have

$$f(x, y) \leq a_1 \left(\frac{c_1}{1 - a_1} + \theta \right) + c_1 \leq A$$

and if $y \leq B$, from (7.5) we obtain

$$g(x, y) \leq a_2 \left(\frac{c_2}{1 - a_2} + \theta \right) + c_2 \leq B.$$

Moreover, for $y \leq B$, from (7.4) we get

$$f(x, y) \geq c_1 \frac{e^{d_1 - k_1 B}}{1 + e^{d_1 - k_1 B}} = a$$

and for $x \leq A$, from (7.5) we obtain

$$g(x, y) \geq c_2 \frac{e^{d_2 - k_2 A}}{1 + e^{d_2 - k_2 A}} = b.$$

In addition, working in the same way as in the proof of Proposition 4.2, we can show that there exist real numbers m, M, r, R such that if $M = f(M, r)$, $m = f(m, R)$, $R = g(m, R)$, $r = g(M, r)$, then $m = M$ and $r = R$.

Moreover, we have $x_{n+1} \leq a_1 x_n + c_1$, $n \in \mathbb{N}$. We consider the difference equation $z_{n+1} = a_1 z_n + c_1$, $n \in \mathbb{N}$ and we let z_n be the solution such that $z_0 = x_0$. Then, we have

$$z_n = \left(x_0 - \frac{c_1}{1 - a_1} \right) a_1^n + \frac{c_1}{1 - a_1}, \quad n \in \mathbb{N}.$$

Now, since $z_0 = x_0$, and working inductively, we can show that $x_n \leq z_n$, $n \in \mathbb{N}$. Therefore, since $a_1 \in (0, 1)$ we get that for some $\theta \in (0, \infty)$ and $n_0 \in \mathbb{N}^+$, we have $x_n \leq A = \frac{c_1}{1 - a_1} + \theta$, $n \geq n_0$. In the same way, we can prove that for some $\theta \in (0, \infty)$

and $n_0 \in \mathbb{N}^+$, we have $y_n \leq B = \frac{c_2}{1-a_2} + \theta$, $n \geq n_0$. So, we can now easily see that $x_n \geq a$, $n > n_0$, and $y_n \geq b$, $n > n_0$. Hence, we have obtained that $x_n \in [a, A]$ and $y_n \in [b, B]$, $n > n_0$.

Therefore, using Theorem 7.1, System (1.6) has a unique positive equilibrium (\tilde{x}, \tilde{y}) , and every positive solution of System (1.6) tends to the unique positive equilibrium as $n \rightarrow \infty$. This completes the proof of the proposition. \square

Proposition 7.3. *Consider System (1.6). If the conditions in Proposition 7.2 hold, then the unique positive equilibrium (\tilde{x}, \tilde{y}) of System (1.6) is globally asymptotically stable.*

Proof. First, we will prove that (\tilde{x}, \tilde{y}) is locally asymptotically stable. The linearised system of (1.6) is

$$w_{n+1} = Aw_n, \quad A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad w_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix},$$

where

$$\alpha = \frac{a_1(\tilde{x}^2 + 2b_1\tilde{x})}{(b_1 + \tilde{x})^2}, \quad \beta = -\frac{c_1k_1e^{d_1-k_1\tilde{y}}}{(1 + e^{d_1-k_1\tilde{y}})^2},$$

$$\gamma = -\frac{c_2k_2e^{d_2-k_2\tilde{x}}}{(1 + e^{d_2-k_2\tilde{x}})^2}, \quad \delta = \frac{a_2(\tilde{y}^2 + 2b_2\tilde{y})}{(b_2 + \tilde{y})^2}.$$

The characteristic equation of A is $\lambda^2 - (\alpha + \delta)\lambda + \alpha\delta - \beta\gamma = 0$. Now, from (7.2), we obtain that $\alpha\delta - \beta\gamma < \alpha\delta < 1$. Moreover, from relation (7.3) we obtain $|\alpha + \delta| < 1 + \alpha\delta - \beta\gamma$. Therefore, from Theorem 1.3.4 of [13], we deduce that (\tilde{x}, \tilde{y}) is locally asymptotically stable. Using Proposition 7.2, we conclude that (\tilde{x}, \tilde{y}) is globally asymptotically stable. This completes the proof of the proposition. \square

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