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WIENER INDEX OF STRONG PRODUCT OF GRAPHS

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Abstract. The Wiener index of a connected graph G is the sum of distances between all pairs of vertices of G. The strong product is one of the four most investigated graph products. In this paper the general formula for the Wiener index of the strong product of connected graphs is given. The formula can be simplified if both factors are graphs with the constant eccentricity. Consequently, closed formulas for the Wiener index of the strong product of a connected graph G of constant eccentricity with a cycle are derived.

Keywords: Wiener index, graph product, strong product.

Mathematics Subject Classification: 05C12, 05C76.

1. INTRODUCTION

The *Wiener index* is a graph invariant based on distances in a graph and is one of the oldest molecular-graph-based structure-descriptors proposed by a chemist Harold Wiener in 1947 [17]. Starting from the middle of 1970s, the Wiener index gained much on popularity and, since then, new results related to it are constantly reported. For example, the Wiener index of recently very investigated chemical molecules called fullerenes and carbon nanotubes was calculated in [1,9]. In the mathematical literature the Wiener index seems to be first studied in 1976 [8]. For a survey and further bibliography on the Wiener index see for example [7].

The Wiener index found its first, simplest and most straightforward applications within modeling of the properties of acyclic molecules, so called alkanes. However, the vast majority of molecules of interest in chemistry are cyclic. There exists a plethora of types of cyclic molecules, and – as a consequence – very few general mathematical results are known for their Wiener indices. Mathematical research is purposeful only within classes of graphs having some common and uniform structural features.

On the other hand is the Wiener index closely related to another iconic mathematical concept: average distance. For a fix graph they differ only by a multiplication with a constant related to the order of a graph. The average distance is widely use in different areas of science such as physics, astronomy, engineering, social sciences and many others. If we restrict our self to the graph theory, there exists a connection between average distance and independence number [3, 4], domination number [5], generalized packing [6], to name just a few. Hence there also exists a connection between the Wiener index and mentioned graph concepts.

Graph products form a natural process how to obtain in an ordered way a larger representative from smaller objects of the same type. One can try also vice versa and start with a big object and try to decompose it into smaller objects that follow certain properties. Often, if this process is successful with the respect to some graph product, one can study the properties of smaller graphs and deriving with it some information about larger graphs. This comes very handy from time consuming point of view with studying bigger and bigger systems in recent years. The most studied graph products are the Cartesian product, the strong product, the direct product, and the lexicographic product, which are also called *standard products*, see the recent monograph [11] on them. In particular for Wiener index, the solution for the Cartesian product is long known see [10, 18]. In [18] also the lexicographic product was settled. For the strong and the direct product only results for special graphs exist, see [13, 16] and [14, 15], respectively. In this work we present a general formula for the Wiener index of the strong product.

The average distance of the strong product was treated in [2] by Casablanca *et al.* from the point of pairs of vertices at the same distance. We use another approach based on distances from a single vertices which yields different formulas presented in the next section together with some basic definitions. Both main results are then proven in the forthcoming sections. In the last section we present the use of our approach on the strong product of an arbitrary connected graph with a constant eccentricity and a cycle. The later shows the simplicity of our methods comparing with the methods in [13, 16]. In addition, we cover a small gap from [13] for the strong product of two cycles.

2. PRELIMINARIES AND MAIN RESULTS

Let G be a simple undirected graph. The *distance* $d_G(u, v)$ between vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G. The *eccentricity* of v, $\operatorname{ecc} v$, is the maximum distance between v and any other vertex in G.

We use for a graph G the standard notations $N_i^G[g]$ for the *i-th closed neighborhood* $\{g' \in V(G) : d_G(g, g') \leq i\}$, $N_i^G(g)$ for the *i-th open neighborhood* $N_i^G[g] - \{g\}$ and $S_i^G(g)$ for the *i-th sphere* $\{g' \in V(G) : d_G(g, g') = i\}$. For the cardinality of sets $N_i^G[g]$, $N_i^G(g)$ and $S_i^G(g)$ we use notations $n_i^G[g]$, $n_i^G(g)$ and $s_i^G(g)$, respectively. Therefore

$$W_i^G(g) = \sum_{g' \in N_i^G(g)} d_G(g, g'), \quad W^G(g) = W_{\operatorname{ecc} g}^G(g) = \sum_{i=1}^{\operatorname{ecc} g} i s_i^G(g)$$

and the *Wiener index* of a graph G is

$$W(G) = \frac{1}{2} \sum_{g \in V(G)} W^G(g). \tag{2.1}$$

The *strong product* $G \boxtimes H$ of graphs G and H is a graph with the vertex set $V(G) \times V(H)$. Two vertices (g,h) and (g',h') of $V(G \boxtimes H)$ are adjacent if $gg' \in E(G)$ and $h = h'$ or if $g = g'$ and $hh' \in E(H)$ or if $gg' \in E(G)$ and $hh' \in E(H)$. It is easy to see that $G \boxtimes H$ is connected whenever both G and H are connected. The distance formula is

$$d_{G \boxtimes H}((u,v),(x,y)) = \max\{d_G(u,x), d_H(v,y)\}.$$

For more properties on the strong product we recommend the book [11].

In this work we prove the following general result on the Winer index of strong product of two graphs.

Theorem 2.1. *Let G and H be connected graphs. If* $\min = \min\{\operatorname{ecc} g, \operatorname{ecc} h\}$ *for $g \in V(G)$ and $h \in V(H)$, then $W(G \boxtimes H)$ equals to*

$$\frac{1}{2} \sum_{(g,h) \in V(G \boxtimes H)} W^G(g) \sum_{j=0}^{\operatorname{ecc} g} s_j^H(h) + \frac{1}{2} \sum_{(g,h) \in V(G \boxtimes H)} W^H(h) \sum_{j=0}^{\operatorname{ecc} h} s_j^G(g)$$
$$- \frac{1}{2} \sum_{(g,h) \in V(G \boxtimes H)} \left(\sum_{j=2}^{\operatorname{ecc} g} s_j^H(h) W_{j-1}^G(g) + \sum_{j=2}^{\operatorname{ecc} h} s_j^G(g) W_{j-1}^H(h) + \sum_{i=1}^{\min} i s_i^G(g) s_i^H(h) \right).$$

Theorem 2.1 can not be simplified in general. However if we add some additional conditions on factors, like equal eccentricity of vertices, one obtain much nicer result. If all vertices of G have the same eccentricity, then we say that G is a graph with the *constant eccentricity* $\operatorname{ecc} G := \operatorname{ecc} g$, $g \in V(G)$. All vertex transitive graphs clearly have this property and among other examples there are complete multipartite graphs $K_{n_1,\ldots,n_k}$, where $n_i > 1$ for every $i \in \{1,\ldots,k\}$ and $\operatorname{ecc} K_{n_1,\ldots,n_k} = 2$.

Theorem 2.2. *Let G and H be connected graphs with constant eccentricities* $\operatorname{ecc} G$ *and* $\operatorname{ecc} H$. *If* $\operatorname{ecc} G < \operatorname{ecc} H$, *then $W(G \boxtimes H)$ equals to*

$$\frac{1}{2}|V(G)|^2 \sum_{h \in V(H)} \sum_{j=\operatorname{ecc} G+1}^{\operatorname{ecc} H} j s_j^H(h)$$
$$+ \frac{1}{2} \sum_{(g,h) \in V(G \boxtimes H)} \sum_{j=1}^{\operatorname{ecc} G} \left[j s_j^G(g) n_j^H[h] + j s_j^H(h) n_j^G[g] - j s_j^G(g) s_j^H(h) \right]$$

and if $\operatorname{ecc} G = \operatorname{ecc} H$, *then $W(G \boxtimes H)$ equals to*

$$\frac{1}{2} \sum_{(g,h) \in V(G \boxtimes H)} \sum_{j=1}^{\operatorname{ecc} G} \left[j s_j^G(g) n_j^H[h] + j s_j^H(h) n_j^G[g] - j s_j^G(g) s_j^H(h) \right].$$

3. PROOF OF THEOREM 2.1

We first compute $W^{G\boxtimes H}((g,h))$ which is $\sum_{i=1}^{\mathrm{ecc}\,(g,h)} is_i^{G\boxtimes H}(g,h)$. It is well known that $N_i^{G\boxtimes H}[g,h]$ is a subproduct of $N_i^G[g]\times N_i^H[h]$. Since

$$S_i^{G\boxtimes H}(g,h)=N_i^{G\boxtimes H}[g,h]-N_{i-1}^{G\boxtimes H}[g,h],$$

we have that

$$S_i^{G\boxtimes H}(g,h)=(S_i^G(g)\times N_i^H[h])\cup(N_i^G[g]\times S_i^H(h)).$$

Moreover, the intersection of both sets from above union is exactly $S_i^G(g)\times S_i^H(h)$. Hence we have that

$$\begin{aligned}W^{G\boxtimes H}((g,h)) &= \sum_{i=1}^{\mathrm{ecc}\,(g,h)} is_i^{G\boxtimes H}(g,h)\\ &= \sum_{i=1}^{\mathrm{ecc}\,(g,h)} i\left[s_i^G(g)n_i^H[h]+s_i^H(h)n_i^G[g]-s_i^G(g)s_i^H(h)\right].\end{aligned}$$

Let

$$\begin{aligned}A &= \sum_{i=1}^{\mathrm{ecc}\,(g,h)} is_i^G(g)n_i^H[h],\\ B &= \sum_{i=1}^{\mathrm{ecc}\,(g,h)} is_i^H(h)n_i^G[g]\end{aligned}$$

and

$$C=\sum_{i=1}^{\mathrm{ecc}\,(g,h)} is_i^G(g)s_i^H(h).$$

Next we take more closer look at the A, B and C. For this, notice that $n_i^G[g]=\sum_{j=0}^{i}s_j^G(g)$ and similar $n_i^H[h]=\sum_{j=0}^{i}s_j^H(h)$. For A we have that

$$\begin{aligned}A &= \sum_{i=1}^{\mathrm{ecc}\,(g,h)} is_i^G(g)n_i^H[h]=\sum_{i=1}^{\mathrm{ecc}\,g} is_i^G(g)\sum_{j=0}^{i}s_j^H(h)\\ &= \sum_{j=0}^{\mathrm{ecc}\,g} s_j^H(h)\sum_{i=j}^{\mathrm{ecc}\,g} is_i^G(g)=\sum_{j=0}^{\mathrm{ecc}\,g} s_j^H(h)\left(\sum_{i=1}^{\mathrm{ecc}\,g} is_i^G(g)-\sum_{i=1}^{j-1} is_i^G(g)\right)\\ &= \sum_{j=0}^{\mathrm{ecc}\,g} s_j^H(h)\left(W^G(g)-W_{j-1}^G(g)\right)=W^G(g)\sum_{j=0}^{\mathrm{ecc}\,g} s_j^H(h)-\sum_{j=2}^{\mathrm{ecc}\,g} s_j^H(h)W_{j-1}^G(g).\end{aligned}$$

Here $W_{-1}^G(g) = W_0^G(g) = 0$ and the sum can go up to $\operatorname{ecc} g$, since $s_i^G(g) = 0$ for every $i > \operatorname{ecc} g$. Notice also that the correctness of exchange of order of i and j (from the second to the third line of above computation) can be seen by depicting these in $\mathbb{Z} \times \mathbb{Z}$ where one coordinate represents i and the other j. Analogue computation can be done for B:

$$B = \sum_{i=1}^{\operatorname{ecc}(g,h)} i s_i^H(h) n_i^G[g] = W^H(h) \sum_{j=0}^{\operatorname{ecc} h} s_j^G(g) - \sum_{j=2}^{\operatorname{ecc} h} s_j^G(g) W_{j-1}^H(h).$$

In particular, notice that $\sum_{j=0}^{\operatorname{ecc} h} s_j^G(g) = |V(G)|$ whenever $\operatorname{ecc} h \geq \operatorname{ecc} g$ and analogue $\sum_{j=0}^{\operatorname{ecc} g} s_j^H(h) = |V(H)|$ whenever $\operatorname{ecc} h \leq \operatorname{ecc} g$. Also C can be adjusted as follows:

$$C = \sum_{i=1}^{\operatorname{ecc}(g,h)} i s_i^G(g) s_i^H(h) = \sum_{i=1}^{\min} i s_i^G(g) s_i^H(h),$$

since either $s_i^G(g) = 0$ or $s_i^H(h) = 0$ for every $i > \min$. We have proved the following:

$$\begin{aligned} W^{G\boxtimes H}((g,h)) = {} & W^G(g) \sum_{j=0}^{\operatorname{ecc} g} s_j^H(h) + W^H(h) \sum_{j=0}^{\operatorname{ecc} h} s_j^G(g) \\ & - \left(\sum_{j=2}^{\operatorname{ecc} g} s_j^H(h) W_{j-1}^G(g) + \sum_{j=2}^{\operatorname{ecc} h} s_j^G(g) W_{j-1}^H(h) + \sum_{i=1}^{\min} i s_i^G(g) s_i^H(h) \right) \end{aligned}$$

and the proof of Theorem 2.1 is completed by (2.1).

4. PROOF OF THEOREM 2.2

Let G and H be connected graphs with constant eccentricities $\operatorname{ecc} G$ and $\operatorname{ecc} H$ and let $\operatorname{ecc} G \leq \operatorname{ecc} H$. We split the general result of Theorem 2.1 in the following three parts for easier computation:

$$\begin{aligned} D &= \frac{1}{2} \sum_{(g,h)\in V(G\boxtimes H)} W^G(g) \sum_{j=0}^{\operatorname{ecc} g} s_j^H(h), \\ E &= \frac{1}{2} \sum_{(g,h)\in V(G\boxtimes H)} W^H(h) \sum_{j=0}^{\operatorname{ecc} h} s_j^G(g), \\ F &= \frac{1}{2} \sum_{(g,h)\in V(G\boxtimes H)} \left(\sum_{i=2}^{\operatorname{ecc} g} s_i^H(h) W_{i-1}^G(g) + \sum_{i=2}^{\operatorname{ecc} h} s_i^G(g) W_{i-1}^H(h) + \sum_{i=1}^{\min} i s_i^G(g) s_i^H(h) \right). \end{aligned}$$

Since every vertex of G has the constant eccentricity $\operatorname{ecc} G$, we can simplify D as follows:

$$D = \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)} W^G(g)\sum_{j=0}^{\operatorname{ecc} g} s_j^H(h) = \frac{1}{2}\left(\sum_{h\in V(H)}\sum_{j=0}^{\operatorname{ecc} G} s_j^H(h)\right)\left(\sum_{g\in V(G)} W^G(g)\right)$$
$$= W(G)\left(\sum_{h\in V(H)}\sum_{j=0}^{\operatorname{ecc} G} s_j^H(h)\right) = W(G)\sum_{h\in V(H)} n^H_{\operatorname{ecc} G}[h].$$

Also every vertex of H has constant eccentricity $\operatorname{ecc} H$ and we have symmetrically

$$E = W(H)\sum_{g\in V(G)} n^G_{\operatorname{ecc} H}[g].$$

Moreover, since $\operatorname{ecc} G \leq \operatorname{ecc} H$, we have that

$$E = W(H)\sum_{g\in V(G)} |V(G)| = W(H)|V(G)|^2.$$

Next we concentrate on F. Recall that $W_0^G(g) = 0 = W_0^H(h)$, that $s_i^G(g) = 0$ for $i > \operatorname{ecc} G$ and that $\operatorname{ecc} G \leq \operatorname{ecc} H$ and we can write

$$F = \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{i=1}^{\operatorname{ecc} G}\left[s_i^H(h)W_{i-1}^G(g) + s_i^G(g)W_{i-1}^H(h) + is_i^G(g)s_i^H(h)\right]$$
$$= \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{i=1}^{\operatorname{ecc} G}\Big[s_i^H(h)(W_{i-1}^G(g) + is_i^G(g))$$
$$+ s_i^G(g)(W_{i-1}^H(h) + is_i^H(h) - is_i^H(h))\Big]$$
$$= \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{i=1}^{\operatorname{ecc} G}\left[s_i^H(h)W_i^G(g) + s_i^G(g)W_i^H(h) - is_i^G(g)s_i^H(h)\right].$$

We now split F to F_1, F_2 and F_3, where $F_3 = F - F_1 - F_2$ for

$$F_1 = \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{i=1}^{\operatorname{ecc} G} s_i^H(h)W_i^G(g),$$
$$F_2 = \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{i=1}^{\operatorname{ecc} G} s_i^G(g)W_i^H(h).$$

For F_1 we have that

$$\begin{aligned}
F_1 &= \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{i=1}^{\mathrm{ecc}\,G} s_i^H(h)\sum_{j=1}^{\mathrm{i}} js_j^G(g)\\
&= \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{j=1}^{\mathrm{ecc}\,G} js_j^G(g)\sum_{i=j}^{\mathrm{ecc}\,G} s_i^H(h)\\
&= \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{j=1}^{\mathrm{ecc}\,G} js_j^G(g)\left[n_{\mathrm{ecc}\,G}^H[h]-n_{j-1}^H[h]\right]\\
&= \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\left[n_{\mathrm{ecc}\,G}^H[h]W^G(g)-\sum_{j=1}^{\mathrm{ecc}\,G} js_j^G(g)n_{j-1}^H[h]\right]\\
&= W(G)\sum_{h\in V(H)} n_{\mathrm{ecc}\,G}^H[h]-\frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{j=1}^{\mathrm{ecc}\,G} js_j^G(g)n_{j-1}^H[h].
\end{aligned}$$

By comutativity of computation we get for F_2 that

$$F_2 = \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{j=1}^{\mathrm{ecc}\,G} js_j^H(h)\left[n_{\mathrm{ecc}\,G}^G[g]-n_{j-1}^G[g]\right],$$

which is a third line of computation of F_1. Notice that if $\mathrm{ecc}\,G < \mathrm{ecc}\,H$, we can not continue as in the case of computation of F_1. However we have that

$$\begin{aligned}
F_2 &= \frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{j=1}^{\mathrm{ecc}\,G} js_j^H(h)\left[|V(G)|-n_{j-1}^G[g]\right]\\
&= \frac{1}{2}|V(G)|^2\sum_{h\in V(H)}\sum_{j=1}^{\mathrm{ecc}\,G} js_j^H(h)-\frac{1}{2}\sum_{(g,h)\in V(G\boxtimes H)}\sum_{j=1}^{\mathrm{ecc}\,G} js_j^H(h)n_{j-1}^G[g].
\end{aligned}$$

Now we have

$$
\begin{aligned}
W(G\boxtimes H) &= D+E-F\\
&= W(H)|V(G)|^2 - \frac{1}{2}|V(G)|^2 \sum_{h\in V(H)} \sum_{j=1}^{\operatorname{ecc} G} j s_j^H(h)\\
&\quad + \frac{1}{2} \sum_{(g,h)\in V(G\boxtimes H)} \sum_{j=1}^{\operatorname{ecc} G} \left[j s_j^G(g) n_{j-1}^H[h] + j s_j^H(h) n_{j-1}^G[g] + j s_j^G(g) s_j^H(h)\right]\\
&= \frac{1}{2}|V(G)|^2 \sum_{h\in V(H)} \left[\sum_{j=1}^{\operatorname{ecc} H} j s_j^H(h) - \sum_{j=1}^{\operatorname{ecc} G} j s_j^H(h) \right]\\
&\quad + \frac{1}{2} \sum_{(g,h)\in V(G\boxtimes H)} \sum_{j=1}^{\operatorname{ecc} G} \left[j s_j^G(g) n_j^H[h] + j s_j^H(h) n_j^G[g] - j s_j^G(g) s_j^H(h)\right]\\
&= \frac{1}{2}|V(G)|^2 \sum_{h\in V(H)} \sum_{j=\operatorname{ecc} G+1}^{\operatorname{ecc} H} j s_j^H(h)\\
&\quad + \frac{1}{2} \sum_{(g,h)\in V(G\boxtimes H)} \sum_{j=1}^{\operatorname{ecc} G} \left[j s_j^G(g) n_j^H[h] + j s_j^H(h) n_j^G[g] - j s_j^G(g) s_j^H(h)\right]
\end{aligned}
$$

and the first statement of Theorem 2.2 is proved. However, if $\operatorname{ecc} G = \operatorname{ecc} H$, then the above result clearly simplifies to

$$
W(G\boxtimes H) = \frac{1}{2} \sum_{(g,h)\in V(G\boxtimes H)} \sum_{j=1}^{\operatorname{ecc} G} \left[j s_j^G(g) n_j^H[h] + j s_j^H(h) n_j^G[g] - j s_j^G(g) s_j^H(h)\right],
$$

which proves also the second statement of Theorem 2.2.

5. SOME SPECIAL CLASSES

Theorem 2.1 can be used to obtain the Wiener index of the strong product of a connected graph with a complete graph, the result already known [16] and therefore omitted here. In this section we apply Theorem 2.2 for the calculation of the strong product of a connected graph of constant eccentricity with a cycle. Only the Wiener index of two cycles can be found in literature [2, 13], which is a special subfamily of our family. The method used there is again a direct computation, while the use of Theorem 2.2 yields much more general result and more elegant proof. We also cover a small error from [13] for a case of the strong product of an odd and an even cycle.

Since it is relevant to us, we first give closed formulas for the Wiener and the hyper-Wiener index of cycles. Closed formulas for the Wiener index of cycles are well known:

$$
W(C_n) = \begin{cases} \frac{(n-1)n(n+1)}{8}, & n \text{ odd},\\ \frac{n^3}{8}, & n \text{ even}. \end{cases}
$$

The hyper-Wiener index for cyclic graphs was introduced by Klein *et al.* [12] as

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{2}\sum_{j=1}^{\operatorname{ecc} G} j^2 d(G,j),$$

where $d(G,j)$ is the number of pairs of vertices from G at distance j. A straightforward calculation yields the hyper-Wiener index of cycles

$$WW(C_n) = \begin{cases} \frac{n(n+1)(n-1)(n+3)}{48}, & n \text{ odd}, \\ \frac{n^2(n+1)(n+2)}{48}, & n \text{ even}. \end{cases}$$

The direct consequence of the definition is the relation

$$\sum_{g\in V(G)} \sum_{j=1}^{\operatorname{ecc} G} j^2 s_j^G(g) = 4WW(G) - 2W(G). \tag{5.1}$$

We have to distinguish two cases, one for the strong product of a graph G with the constant eccentricity with an odd cycle and another case for an even cycle.

5.1. $W(G \boxtimes C_{2k+1})$

To use Theorem 2.2 efficiently we must add an assumption that $\operatorname{ecc} G \le k$ when computing $W(G \boxtimes C_{2k+1})$. The reason for this is that $s_j^{C_{2k+1}}(v) = 2$ for every $1 \le j \le k$, but we have in general no information about $s_j^G(v)$.

Theorem 5.1. *Let G be a graph with the constant eccentricity* $\operatorname{ecc} G$. *If* $\operatorname{ecc} G \le k$, *then*

$$W(G \boxtimes C_{2k+1}) = (2k+1)\left[|V(G)|^2 \frac{k(k+1)}{2} + 2WW(G) - W(G)\right].$$

Proof. It is easy to see that $\operatorname{ecc} C_{2k+1} = k$ and that for every vertex v of C_{2k+1} $s_j^{C_{2k+1}}(v) = 2$ and $n_j^{C_{2k+1}}[v] = 2j+1$ for $j \in \{1,\dots,k\}$. Let G be a graph with the constant eccentricity $\operatorname{ecc} G \leq k$. For the computation of $W(G \boxtimes C_{2k+1})$ we need the following derivation:

$$\begin{aligned}
&\sum_{g\in V(G)} \sum_{j=1}^{\operatorname{ecc} G} j n_j^G[g] \\
&= \sum_{g\in V(G)} \sum_{j=1}^{\operatorname{ecc} G} j \left(1 + \sum_{i=1}^{j} s_i^G(g)\right) \\
&= \sum_{g\in V(G)} \sum_{j=1}^{\operatorname{ecc} G} j + \sum_{g\in V(G)} \sum_{i=1}^{\operatorname{ecc} G} s_i^G(g) \sum_{j=i}^{\operatorname{ecc} G} j \\
&= \frac{1}{2}|V(G)| \operatorname{ecc} G(\operatorname{ecc} G + 1) + \frac{1}{2} \sum_{g\in V(G)} \sum_{i=1}^{\operatorname{ecc} G} s_i^G(g)(\operatorname{ecc}^2 G + \operatorname{ecc} G - i^2 + i) \qquad (5.2)\\
&= \frac{1}{2}\operatorname{ecc} G(\operatorname{ecc} G + 1) \left[|V(G)| + |V(G)|^2 - |V(G)|\right] \\
&\quad + \frac{1}{2} \sum_{g\in V(G)} \sum_{i=1}^{\operatorname{ecc} G} \left[-i^2 s_i^G(g) + i s_i^G(g)\right] \\
&= \frac{1}{2}\operatorname{ecc} G(\operatorname{ecc} G + 1)|V(G)|^2 + W(G) - \frac{1}{2} \sum_{g\in V(G)} \sum_{i=1}^{\operatorname{ecc} G} i^2 s_i^G(g).
\end{aligned}$$

Using Theorem 2.2 and then derivation (5.2) and formula (5.1) we obtain

$$\begin{aligned}
W(G \boxtimes C_{2k+1}) &= \frac{1}{2}|V(G)|^2 \sum_{h\in V(C_{2k+1})} \left[\sum_{j=1}^{k} 2j - \sum_{j=1}^{\operatorname{ecc} G}\right] \\
&\quad + \frac{1}{2} \sum_{(g,h)\in V(G\boxtimes C_{2k+1})} \sum_{j=1}^{\operatorname{ecc} G} \left[j s_j^G(g)(2j+1) + 2j n_j^G[g] - 2j s_j^G(g)\right] \\
&= |V(G)|^2(2k+1)\left[\frac{k(k+1)}{2} - \frac{\operatorname{ecc} G(\operatorname{ecc} G + 1)}{2}\right] \\
&\quad + (2k+1)\left[\sum_{g\in V(G)} \sum_{j=1}^{\operatorname{ecc} G} \left[j^2 s_j^G(g) - \frac{1}{2} j s_j^G(g)\right]\right] \\
&\quad + (2k+1)\left[\frac{\operatorname{ecc} G(\operatorname{ecc} G + 1)}{2}|V(G)|^2 + W(G)\right. \\
&\qquad\qquad \left. - \frac{1}{2} \sum_{g\in V(G)} \sum_{i=1}^{\operatorname{ecc} G} i^2 s_i^G(g)\right]
\end{aligned}$$

$$= (2k+1)\left[|V(G)|^2\frac{k(k+1)}{2} + W(G) + \frac{1}{2}\sum_{g\in V(G)}\sum_{j=1}^{\operatorname{ecc} G}\left[j^2 s_j^G(g) - j s_j^G(g)\right]\right]$$

$$= (2k+1)\left[|V(G)|^2\frac{k(k+1)}{2} + \frac{1}{2}\sum_{g\in V(G)}\sum_{j=1}^{\operatorname{ecc} G} j^2 s_j^G(g)\right]$$

$$= (2k+1)\left[|V(G)|^2\frac{k(k+1)}{2} + 2WW(G) - W(G)\right],$$

which completes the proof. □

Next two corollaries follows from Theorem 5.1 and the implementation of closed formulas for the Wiener and the hyper-Wiener index for cycles.

Corollary 5.2. *Let k and ℓ be two positive integers. If $\ell \le k$, then*

$$W(C_{2\ell} \boxtimes C_{2k+1}) = \frac{1}{3}(2k+1)\ell^2\left(2k^2 + 2k + 2\ell^2 + 1\right).$$

Corollary 5.3. *Let k and ℓ be two positive integers. If $\ell \le k$, then*

$$W(C_{2\ell+1} \boxtimes C_{2k+1}) = \frac{1}{6}(2k+1)(2\ell+1)^2\left(3k^2 + 3k + \ell^2 + \ell\right).$$

5.2. $W(G \boxtimes C_{2\ell})$

The condition needed by odd cycles must be even stronger in the case of even cycles. Namely, we have $s_\ell^{C_{2\ell}}(v) = 1$ and $n_\ell^{C_{2\ell}}[v] = 2\ell$, which yields some complications computing $W(G\boxtimes C_{2\ell})$ when $\operatorname{ecc} G = \ell$. Therefore we restrict ourselves to graphs with the constant eccentricity $\operatorname{ecc} G < \ell$.

Theorem 5.4. *Let G be a graph with the constant eccentricity* $\operatorname{ecc} G$*. If* $\operatorname{ecc} G < \ell$*, then*

$$W(G \boxtimes C_{2\ell}) = 2\ell\left[|V(G)|^2\frac{\ell^2}{2} + 2WW(G) - W(G)\right].$$

Proof. Let G be a graph with the constant eccentricity $\operatorname{ecc} G < \ell$. As already mentioned, we have two irregularities $s_\ell^{C_{2\ell}}(v) = 1$ and $n_\ell^{C_{2\ell}}[v] = 2\ell$, but otherwise we have

$\operatorname{ecc} C_{2\ell} = \ell$, $s_j^{C_{2\ell}}(v) = 2$ and that $n_j^{C_{2\ell}}[v] = 2j+1$ for every vertex v of $C_{2\ell}$ and every $j \in \{1,\ldots,\ell-1\}$. Similarly as in the odd case we get

$$\begin{aligned}
W(G \boxtimes C_{2\ell}) &= \frac{1}{2}|V(G)|^2 \sum_{h\in V(C_{2\ell})} \left[\sum_{j=1}^{\ell} 2j - \sum_{j=1}^{\operatorname{ecc} G} 2j - \ell\right] \\
&\quad + \frac{1}{2} \sum_{(g,h)\in V(G\boxtimes C_{2\ell})} \sum_{j=1}^{\operatorname{ecc} G} \left[j s_j^G(g)(2j+1) + 2j n_j^G[g] - 2j s_j^G(g)\right] \\
&= |V(G)|^2 2\ell \left[\frac{\ell(\ell+1)}{2} - \frac{\operatorname{ecc} G(\operatorname{ecc} G+1)}{2} - \frac{\ell}{2}\right] + \\
&\quad + 2\ell \left[\sum_{g\in V(G)} \sum_{j=1}^{\operatorname{ecc} G} \left[j^2 s_j^G(g) - \frac{1}{2} j s_j^G(g)\right]\right] \\
&\quad + 2\ell \left[\frac{\operatorname{ecc} G(\operatorname{ecc} G+1)}{2}|V(G)|^2 + W(G) - \frac{1}{2}\sum_{g\in V(G)} \sum_{i=1}^{\operatorname{ecc} G} i^2 s_i^G(g)\right] \\
&= 2\ell \left[|V(G)|^2 \frac{\ell^2}{2} + W(G) + \frac{1}{2}\sum_{g\in V(G)} \sum_{j=1}^{\operatorname{ecc} G} \left[j^2 s_j^G(g) - j s_j^G(g)\right]\right] \\
&= 2\ell \left[|V(G)|^2 \frac{\ell^2}{2} + \frac{1}{2}\sum_{g\in V(G)} \sum_{j=1}^{\operatorname{ecc} G} j^2 s_j^G(g)\right] \\
&= 2\ell \left[|V(G)|^2 \frac{\ell^2}{2} + 2WW(G) - W(G)\right],
\end{aligned}$$

which completes the proof. □

Next corollaries follows from Theorem 5.4 and the implementation of closed formulas for the Wiener and the hyper-Wiener index for cycles.

Corollary 5.5. *Let k and ℓ be two positive integers. If $k \le \ell$, then*

$$W(C_{2k} \boxtimes C_{2\ell}) = \frac{2}{3}\ell k^2 \left(6\ell^2 + 2k^2 + 1\right).$$

Proof. For $k < \ell$ Corollary 5.5 follows directly from Theorem 5.4 and the case of equal eccentricities is a straightforward use of Theorem 2.2. □

Corollary 5.6. *Let k and ℓ be two positive integers. If $k < \ell$, then*

$$W(C_{2k+1} \boxtimes C_{2\ell}) = \frac{1}{3}(2k+1)^2 \ell \left(k^2 + k + 3\ell^2\right).$$

Note that the last result was not covered in [13] since the author did not distinguish possibilities $\ell > k$ and $\ell \le k$ in the derivation of the closed formula for the Wiener index of $C_{2\ell} \boxtimes C_{2k+1}$.

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