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IDEALS WITH LINEAR QUOTIENTS IN SEGRE PRODUCTS

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Abstract. We establish that the Segre product between a polynomial ring on a field K in m variables and the second squarefree Veronese subalgebra of a polynomial ring on K in n variables has the intersection degree equal to three. We describe a class of monomial ideals of the Segre product with linear quotients.

Keywords: monomial algebras, graded ideals, linear resolutions.

Mathematics Subject Classification: 13A30, 13D45.

1. INTRODUCTION

Let $A = K[x_1, \ldots, x_n]$ be the polynomial ring over the field K with the standard graduation and let I be a graded ideal of A generated in the same degree. The property for I to have linear quotients, introduced in [17], has been studied by many authors ([18, 19, 21]) and it implies that I has a linear resolution. Monomial subalgebras of A are of particular interest, since they are connected to the study of the subtended affine semigroup ([4, 16]). In this direction, in [18] the authors prove that the rth-Veronese subalgebra A' of A $(r \geq 2)$ has the maximal irrelevant ideal with linear quotients and, as a consequence, with a linear resolution. If $A^{(2)}$ is the 2nd squarefree Veronese subalgebra of A, the maximal ideal has linear quotients and it admits a linear resolution([1, 2]). More in general, let A and B be two homogeneous graded $K-$algebras and let $A * B$ be their Segre product $K[u_1, \ldots, u_N]$, where all generators have degree one. In [18] the notion of strongly Koszul algebra is introduced and the main consequence is that its maximal irrelevant ideal has linear quotients and a linear resolution. In particular if A and B are polynomial rings, $A * B$ is strongly Koszul and the maximal ideal has linear quotients and a linear resolution. If A and B are monomial algebras, the generators $u_1, \ldots, u_N$ are monomials and the degree of intersection can be investigated for $A * B$. In [15] a number $t > 0$ is called the intersection degree of a homogeneous monomial algebra $K[u_1, \ldots, u_N]$ if all colon ideals $(u_{i_j}) \cap (u_{i_k}), 1 \leq i_j, i_k \leq n$, are generated in degree less than or equal to t. A conjecture arising from computational arguments says that if A and B have intersection degree r and s respectively, the intersection degree t of the Segre product is less than or equal to $\max(r, s)$. The conjecture is true if A and B are polynomial rings with $r = s = t = 2$ and in general for A and B strongly Koszul algebras or any two Veronese subrings, in which case $A * B$ is a strongly Koszul algebra and $r = s = t = 2$. In this paper we consider the Segre product C of a polynomial ring B on a field $K$and the 2nd squarefree Veronese subring $A^{(2)}$ of $A = K[x_1, \dots, x_n]$, whose intersection degree is three for $n \geq 5$ ([1,2]). The aim is to compute the intersection degree of C. Then we consider ideals generated by subsets of the generators of the maximal irrelevant ideal of C. Clearly, in general they do not have linear quotients and now we will discuss the problem to find combinatorial conditions on the generators.

The plan of the paper is the following: In the first section, we consider the polynomials rings $A = K[x_1, \dots, x_n]$ and $B = K[y_1, \dots, y_m]$ with the standard graduation and the Segre product $B * A^{(2)}$ generated in degree one. We prove that the intersection degree of $B * A^{(2)}$ is equal to three, in other words we show that not all principal colon ideals of the maximal ideal of the Segre product are generated in degree one. We give the explicit description of the generators of degree one and degree two. In the second section, we focus our attention to monomial ideals of $B * A^{(2)}$, that admit linearly generated quotient ideals. We describe explicitly a class of ideals generated by a suitable subset of the set of the minimal generators of the K-algebra $B * A^{(2)}$, by applying a combinatorial condition on certain pairs of elements of the subset, which generalizes a condition given in [1] for $A^{(2)}$. In particular, we prove that the maximal irrelevant ideal of $B * A^{(2)}$ has linear quotients and a linear resolution, since it is a Koszul algebra ([3]).

2. INTERSECTION DEGREE

Let $A = K[x_1, \dots, x_n]$ and $B = K[y_1, \dots, y_m]$ be two polynomial rings in n and m variables respectively with coefficients in any field K. Let $A^{(2)} \subset A$ be the 2nd squarefree Veronese algebra of A and let $C = B *_S A^{(2)}$ be the Segre product of B and $A^{(2)}$. Then C is a standard $K-$algebra generated in degree one by the monomials $y_\alpha x_i x_j$, with $1 \leq \alpha \leq m$, $1 \leq i < j \leq n$. For convenience, we will indicate such a monomial by αij.

In order to compute the intersection degree, we compute first all quotient ideals of principal ideals of C, whose generators are that ones of the maximal ideal m^* of C.

Theorem 2.1. *Let $C = B *_S A^{(2)}$ be the Segre product and let $m^* = (u_1, \dots, u_N)$, $N = m\binom{n}{2}$ the maximal ideal of C. Let $(u_r) : (u_s)$, $1 \leq r, s \leq N, r \neq s$, a colon ideal of generators of m^*, in the lexicographic order. Then we have:*

1. $(\alpha\, ij_1) : (\alpha\, ij_2) = (\beta k j_1, k \neq j_1, j_2, \beta \in \{1, \dots, m\})$,
2. $(\alpha_1 ij_1) : (\alpha_2 ij_2) = (\alpha_1 k j_1,\ k \neq j_1, j_2)$,
3. $(\alpha\, i_1 j) : (\alpha\, i_2 j) = (\beta i_1 k, k \neq i_1, i_2, \beta \in \{1, \dots, m\})$,
4. $(\alpha_1 i_1 j) : (\alpha_2 i_2 j) = (\alpha_1 i_1 k,\ k \neq i_1, i_2)$,

5. $(\alpha\, i_1 j) : (\alpha\, j j_2) = (\beta i_1 k,\, k \neq i_1, j_2,\, \beta \in \{1, \ldots, m\})$,
6. $(\alpha\, i j_1) : (\alpha\, i_2 i) = (\beta k j_1,\, k \neq j_1, i_2,\, \beta \in \{1, \ldots, m\})$,
7. $(\alpha_1 i_1 j) : (\alpha_2 j j_2) = (\alpha_1 i_1 k,\, k \neq i_1, j_2)$,
8. $(\alpha_1 i j_1) : (\alpha_2 i_2 i) = (\alpha_1 k j_1,\, k \neq i_2, j_1)$,
9. $(\alpha_1 i_1 j_1) : (\alpha_2 i_2 j_2) = (\alpha_1 i_1 j_1, (\alpha_1 i_1 s)(\beta j_1 s), \beta \in \{1, \ldots, m\}, s \neq i_1, j_1, i_2, j_2)$,
10. $(\alpha_1 ij) : (\alpha_2 ij) = (\alpha_1 kl, k \neq l)$

Proof.

1. Let $a \in (\alpha\, i j_1) : (\alpha\, i j_2)$ be a monomial generator of the colon ideal. Then

$$a\alpha\, i j_2 = b\alpha\, i j_1, \tag{2.1}$$

where a and b are semigroup elements and any factor of a is of type $\beta k j_1$, $\beta \in \{1, \ldots, m\}$, since any factor of a must contain j_1. We prove that $k \neq j_2$. Suppose $k = j_2$ and consider a decomposition of a:

$$a = (\beta_1 k_1 j_1)(\beta_1 k_1 j_1) \ldots (\beta_r k_r j_r).$$

If each factor of a contains j_2, then j_2 appears $(r+1)$ times in the first member of (2.1) and this implies it must appears $(r+1)$ times in the second member of (2.1), contradiction because j_2 appears at maximum r times only in b, being $i \neq j_2$. It follows there exists a factor $\beta_s k_s j_s$ of a such that $k_s \neq j_2$, hence we can suppose $k \neq j_2$. We write the decomposition of a as:

$$a = (\beta j_1 j_2)(\beta_s k_s j_s) \ldots$$

If $k_s \neq j_1$, we can write (2.3) as:

$$a = (\beta k_s j_1)(\beta_s j_s j_2) \ldots$$

If $k_s = j_1$, (2.3) can be written as:

$$a = (\beta j_1 j_2)(\beta_s j_1 j_s) \ldots \text{ with } j_s \neq j_1$$

that we rewrite as:

$$a = (\beta j_s j_1)(\beta_s j_1 j_2) \ldots$$

In any case a is a multiple of a factor of the type βkj, with $k \neq j_2$, $\beta \in \{1, \ldots, m\}$. Hence the assertion.

2. Let $a \in (\alpha_1\, i j_1) : (\alpha_2\, i j_2)$ be a monomial generator of the colon ideal. Then

$$a\alpha_2 i j_2 = b\alpha_1 i j_1, \tag{2.2}$$

where a and b are semigroup elements. We claim that $\alpha_1 k j_1$ is always a factor of a, with $k \neq j_1$. Consider a decomposition of a:

$$a = (\beta_1 k_1 j_1)(\beta_2 k_2 j_2) \ldots (\beta_r k_r j_r).$$

Suppose each factor contains j_2. Then j_2 appears $(r+1)$ times in the first member of (2.2) and contrary it can appear only s times in the second member of (2.2). Then there exists $\beta_l k_l j_l$ with $j_l \neq j_2$ and $k_l \neq j_2$. For the same reason, if we suppose that each factor contains α_2, we obtain a contradiction. Then there exists $\beta_t k_t j_t$ with $\beta_t \neq \alpha_2$. Consider

$$a = (\beta_2 j_1 j_2)(\beta_l k_l j_l)(\alpha_t k_t j_t)\ldots$$

We can rewrite it as:

$$a = (\alpha_1 k_l j_1)(\beta_l j_l j_2)(\alpha_2 k_n j_n)(\beta_t k_t j_t)\ldots$$

with $k_l \neq j_1$. In conclusion, in any case, the monomial $\alpha_1 k j_1$ is a factor of a, with $k \neq j_2$. Hence the assertion.

3. The proof is analogue to the proof of 1.

4. The proof is analogue to the proof of 2.

5. $(\alpha i_1 j) : (\alpha j j_2) = (\beta i_1 k, k \neq i_1, j_2, \beta \in \{1, \ldots, m\}), i_1 < j < j_2$. Let $a \in (\alpha i_1 j) : (\alpha j j_2)$. Then

$$a = \alpha j j_2 = b\alpha i_1 j \tag{2.3}$$

and $\beta i_1 k_1$ is a factor of a, $k \neq i_1$. To prove that $k_1 \neq j_2$, consider a decomposition of a in r factors $a = (\beta_1 i_1 k_1)(\beta_2 i_2 k_2)\ldots(\beta_r i_r k_r)$. Suppose each factor contains j_2. Then j_2 appears $(r+1)$ times in the first member of 2.3. Contradiction. Then we have a factor $\beta_s i_s k_s$ of a such that $k_s \neq j_2$ and, as consequence, $i_s \neq j_2$. Now we can write $a = (\beta_1 i_1 k_1)(\beta_2 i_2 k_2)\ldots(\beta_s i_s k_s)\ldots(\beta_r i_r k_r)$. If $k_s = j_1$, since $i_1 \neq j_1$, we can write $a = (\beta_1 i_1 k_1)(\beta_s i_s j_1)\ldots = (\beta_1 i_1 j_1)(\beta_s i_s k_1)\ldots = (\beta_1 i_s j_1)(\beta_s i_1 j_2)\ldots$ and in any case a is a multiple of a factor of the type $\beta i_1 k, k \neq j_1 j_2$.

6. Let $a \in (\alpha_1\, i_1 j_1) : (\alpha_2\, i_2 j_2)$ be a semigroup element which does not contain the factor $\alpha_1\, i_1 j_1$. Then $a\alpha_2\, i_2 j_2 = b\alpha_1\, i_1 j_1$. Let write a decomposition of a as:

$$a = (\alpha_1 i_1 s)(\beta j_1 t)\ldots,\ \beta \in \{1, \ldots, n\}.$$

If $s \neq t$, we can rewrite $a = (\alpha_1 i_1 j_1)(\beta s t)\ldots$ Contradiction, since a does not contain $\alpha_1 i_1 j_1$ as a factor.Then $s = t$ and for a we can write the previous decomposition as:

$$a = (\alpha_1 i_1 s)(\beta j_1 t)\ldots \text{ with } s \neq i_1 j_1, \beta \in \{1, \ldots, n\}.$$

It follows that we have a generator of degree two that is not a multiple of $\alpha_1 i_1 j_1$.

The proof of cases $6, 7, 8$ is analogue to the proof of 5 and the proof of case 10 follows from 7 and 8. □

Corollary 2.2. *The intersection degree of the monomial algebra $B * A^{(2)}$ is equal to three for $n > 4$.*

Proof. Let $C = B * A^{(2)} = K[u_1, \ldots, u_N]$ be and let u_{i_j}, u_{i_k} be two any generators, $1 \leq i_j, i_k \leq N, i_j \neq i_k$. Consider the isomorphism of $C-$modules $(u_{i_j}) : (u_{i_k}) \to (u_{i_j} \cap u_{i_k})(-1)$. Let $a \in (u_{i_j}) : (u_{i_k})$ be such that $au_{j_k} \in (u_{i_j})$, then $au_{j_k} \in (u_{i_j} \cap u_{i_k})$. Since $\deg(u_{j_k}) = 1$ and $\deg(a) \leq 2$, $\deg(au_{j_k}) \leq 3$, hence the assertion follows. □

Remark 2.3. For $n = 4$, the intersection degree is two ([18]), $A^{(2)}$ is a strongly Koszul algebra and consequently the Segre product $B * A^{(2)}$ ([18]).

Remark 2.4. The intersection degree $t \geq 2$ of a homogeneous monomial algebra can increase arbitrarily ([18]). The behaviour of t under tensor product or Segre product of monomial algebras is almost predictable. For arbitrary subalgebras we can have different statements. From the combinatorial point of view, the d-th Veronese subring has the simplest possible semigroup. For the squarefree case the degree grows (see $d = 2$, for which is $t = 3$ for $n \geq 5$). Triangulations of the simplicial complexes subtended by $A^{(d)}$ can be studied ([12, 20]). For homogeneous semigroup rings arising from Grassmann varieties, Hankel varieties ([7, 9, 11, 20]) and their subvarieties ([8]), the problem is more difficult. For $G(1,3) = H(1,3)$ the intersection degree is 2. In fact its toric ring is strongly Koszul, being a quotient of the polynomial ring $K[[12],[13],[14],[23],[24],[34]]$ by the ideal generated by the binomial relation $[14][23] - [13][24]$, where $[ij]$ is the variable corresponding to the initial term for the diagonal order of the minor with columns i, j, $i < j$, of a 2×4 generic matrix. The semigroup ring of $\mathbb{G}(1,4)$ is a subring of $K[t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}]$, t_{ij} the generic entry of a 2×5- matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} \end{pmatrix}$$

and it is generated by the diagonal initial terms of ten 2×2 minors of the matrix. The semigroup of $H(1,4)$ is a subring of $K[t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}]$, generated by the diagonal initial terms of ten 2×2 minors of the Hankel matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \end{pmatrix}.$$

These considerations leave us with the problem to compute the intersection degree of the previous semigroup rings and to compare them.

3. MONOMIAL IDEALS WITH LINEAR QUOTIENTS

The aim of this section is to find classes of monomial ideals of the Segre product $C = B * A^{(2)}$ having linear quotients. In particular we consider monomial ideals generated by certain subsets of the set of generators of the maximal irrelevant ideal M of C, hence generated in degree one. We recall the definition of ideal with linear quotients, as introduced in [17].

Definition 3.1. Let R be a homogeneous $K-$algebra, K a field, finitely generated over K by elements of degree one, and let $I \subset R$ be a homogeneous ideal. I is said to have linear quotients if it has a system of generators $f_1, \ldots, f_t$, such that, for $j = 1, \ldots, t$, the colon ideals:

$$(f_1 + \ldots f_{j-1}) : f_j$$

have linear forms (notice that this property depends on the order of generators).

Proposition 3.2. *Suppose R a strongly Koszul k-algebra. Let $I \subset R$ be a homogeneous ideal generated by a subset of generators of the maximal irrelevant ideal of R. Then I has linear quotients and a linear resolution on R.*

Proof. The ideal I has linear quotients by definition of Strongly Koszul algebra. The proof is contained in [18, Theorem 1.2]. $\square$

In the following we introduce in the set of monomials of $K[x_1, \dots, x_n, y_1, \dots, y_m]$ the lexicographic order with the order on the variables $y_1 > \dots > y_m > x_1 > \dots > x_n$ and order on the generators of $A^{(2)}$ given by $x_1x_2 > x_1x_3 > \dots > x_{n-1}x_n$. Moreover, following [1,2] and notations of Section 1, we call "bad pair" a pair of monomials ij, kl in $A^{(2)}$ or $\alpha\, ij, \beta\, kl$ in C, with $i \neq k$ and $j \neq l$. If $\alpha \neq \beta$, we call such a pair a "strongly bad pair" of monomials.

Theorem 3.3. *Let $(u_1, \dots, u_t)$ be the ideal of $B * A^{(2)}$ generated by a sequence $\mathcal{L} = \{\alpha_1 i_1 j_1, \dots, \alpha_t i_t j_t\}$ of generators of M, with $u_1 > \dots > u_t$. Fixed $\alpha kl \in \mathcal{L}$, let*

$$\mathcal{L}_{\alpha kl} = \{\beta rs \in \mathcal{L} / \beta rs > \alpha kl \text{ and } rs > kl\}$$

and

$$\mathcal{L}'_{\alpha kl} = \{\beta rs \in \mathcal{L} / \beta rs < \alpha kl \text{ and } rs > kl\}.$$

Suppose that the sequence $\mathcal{L}$ satisfies the following properties (the property P in summary):

1. *for each bad pair $\alpha ij > \alpha kl$ in $\mathcal{L}$, $\alpha ik \in \mathcal{L}_{\alpha kl}$ or $\alpha il \in \mathcal{L}_{\alpha kl}$ or $\alpha kl \in \mathcal{L}_{\alpha jk}$ or $\alpha jl \in \mathcal{L}_{\alpha kl}$,*
2. *for each bad pair $\alpha ij > \beta kl$ in $\mathcal{L}$, with $ij > kl$, $\alpha ik \in \mathcal{L}_{\beta kl}$ or $\alpha il \in \mathcal{L}_{\beta kl}$ or $\alpha jk \in \mathcal{L}_{\beta kl}$ or $\alpha jl \in \mathcal{L}_{\beta kl}$,*
3. *for each bad pair $\alpha ij > \beta kl$ in $\mathcal{L}$, with $ij < kl$, or $\beta ki \in \mathcal{L}'_{\alpha ij}$ or $\beta kj \in \mathcal{L}'_{\alpha ij}$ or $\beta il \in \mathcal{L}'_{\alpha ij}$ or $\beta jl \in \mathcal{L}'_{\alpha ij}$.*

Then any colon ideal $(u_1, \dots, u_r) : (u_{r+1})$ is generated by a subset of $u_1, \dots, u_t$, $1 \leq r \leq t-1$.

Proof. Consider the colon ideal $J = (\alpha_1 i_1 j_1, \dots, \alpha_{q-1} i_{q-1} j_{q-1}) : \alpha_q i_q j_q$. Put $I = (\alpha_1 i_1 j_1, \dots, \alpha_{q-1} i_{q-1} j_{q-1})$. We want to prove that the colon ideal J is linear, that is it has generators of the semigroup ring. Let $a \in I : \alpha_q i_q j_q$ be a semigroup generator of the colon ideal I. Then $a \in \alpha_p i_p j_p : \alpha_q i_q j_q$ for some $p < q$. If $\alpha_p i_p j_p, \alpha_q i_q j_q$ is not a bad pair, a is linear. Suppose that $\alpha_p i_p j_p > \alpha_q i_q j_q$ is a bad pair.

$$\text{I case:} \qquad \alpha_p = \alpha_q = \alpha, \qquad \alpha\, i_p j_p > \alpha\, i_q j_q, \qquad i_p j_p > i_q j_q.$$

If a has not degree one, we have by Theorem 2.1,9:

$$a = (\alpha\, i_p k)(\beta\, j_p k), \qquad k \neq i_p, i_q, j_p, j_q,\ \beta \in \{1, \dots m\}.$$

Since the sequence satisfies the property P and $\alpha i_p i_q > \alpha i_q j_q$, one has

$$\alpha\, i_p i_q \in \mathcal{L}_{\alpha\, i_q j_q} \text{ or } \alpha\, i_p j_q \in \mathcal{L}_{\alpha\, i_q j_q}$$

$$\text{or } \alpha\, j_p i_q \in \mathcal{L}_{\alpha\, i_q j_q} \text{ or } \alpha\, j_p j_q \in \mathcal{L}_{\alpha\, i_q j_q}$$

If $k < j_q$

$$\alpha\, i_p k > \alpha\, i_p j_q \in \mathcal{L}_{\alpha\, i_q j_q} \subset \mathcal{L}.$$

If $k > j_q$, being $i_p < j_q$,

$$\alpha\, i_p k > \alpha\, j_p j_q \in \mathcal{L}_{\alpha\, i_q j_q} \subset \mathcal{L}.$$

So in I we have the factor $\alpha\, i_p k$ of degree 1 and a is a multiple of a generator of I.

$$\text{II case:} \qquad \alpha_p > \alpha_q, \qquad \alpha\, i_p j_p > \beta\, i_q j_q, \qquad i_p j_p > i_q j_q.$$

If a has not degree one, we have:

$$a = (\alpha\, i_p k)(\beta\, j_p k), \qquad k \neq i_p, i_q, j_p, j_q \text{ and } \beta \in \{1, \ldots, m\}.$$

Since the sequence satisfies the property P, for $\alpha > \beta$, we have:

$$\alpha\, i_p i_q \in \mathcal{L}_{\beta\, i_q j_q} \text{ or } \alpha\, i_p j_q \in \mathcal{L}_{\beta\, i_q j_q} \text{ or}$$

$$\alpha\, j_p i_q \in \mathcal{L}_{\beta\, i_q j_q} \text{ or } \alpha\, j_p j_q \in \mathcal{L}_{\beta\, i_q j_q}.$$

If $k < j_q$

$$\alpha\, i_p k > \alpha\, i_p j_q > \beta\, i_p j_q \in \mathcal{L}_{\alpha\, i_q j_q} \subset \mathcal{L}.$$

If $k > j_q$

$$\alpha\, i_p k > \alpha\, j_p j_q > \beta\, j_p j_q \in \mathcal{L}_{\beta\, i_q j_p} \subset \mathcal{L}.$$

Then a is a multiple of a generator of I.

$$\text{III case:} \qquad \alpha_p > \alpha_q, \qquad \alpha\, i_p j_p > \beta\, i_q j_q, \qquad i_p j_p < i_q j_q.$$

In this case we have as a generator of the colon ideal $\alpha\, i_p j_p : \beta\, i_q j_q$

$$a = (\alpha\, i_p k)(\beta\, j_p k), \qquad k \neq i_p, i_q, j_p, j_q\ \beta \in \{1, \ldots, m\}, \tag{3.1}$$

(see Theorem 2.1).

Since the sequence satisfies the property P, for $\alpha > \beta$, we have by condition (3):

$$\beta i_q i_p \in \mathcal{L}^{'}_{\alpha i_p j_p} \text{ or } \beta i_q j_p \in \mathcal{L}^{'}_{\alpha i_p j_p}$$

$$\text{or } \beta i_p j_q \in \mathcal{L}^{'}_{\alpha i_p j_p} \text{ or } \beta j_p j_q \in \mathcal{L}^{'}_{\alpha i_p j_p}$$

If $k > j_p$, rewrite (3.1) as $a = (\beta\, i_p k)(\alpha\, j_p k)$. Then

$$\beta\, i_p\, k < \alpha\, i_p j_p \ \text{ and } \ i_p\, k > i_p j_p.$$

It follows

$$\beta\, i_p\, k \in \mathcal{L}'_{\alpha i_p j_p} \subset \mathcal{L} \text{ and } \beta\, i_p\, k \in \mathcal{L}.$$

If $k < j_p$

$$\beta k j_p < \alpha\, j_p j_q \text{ and } k j_p > j_p j_q.$$

Then

$$\beta k j_p \in \mathcal{L}'_{\alpha j_p j_p} \subset \mathcal{L} \text{ and } \beta j_p\, k \in \mathcal{L}.$$

In any case a is a multiple of a generator of the ideal I generated by the set $\mathcal{L}$. The proof of the remaining cases is analogue. $\square$

4. OPEN PROBLEMS

1) To find new classes of monomial ideals generated in degree 1 with a linear resolution. We proved in [13] that the monomial ideals of $B * A^{(2)}$ in Theorem 3.3 have a linear resolution.
2) To find monomial ideals of $B * A^{(2)}$ generated not in the same degree and candidate to be component-wise linear [16].
3) To study monomial ideals of $C = B^{(2)} * A^{(2)}$ that have linear quotients and a linear resolution. The Segre products $B * A^{(2)}$ and $B^{(2)} * A^{(2)}$ were already studied in [5, 6], where results on the subtended semigroup map are obtained.

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